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A posteriori error estimates of stabilized low-order mixed finite elements for the Stokes eigenvalue problem

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ABSTRACT

In this paper we obtain a priori and a posteriori error estimates for stabilized low-order mixed finite element methods for the Stokes eigenvalue problem. We prove the convergence of the method and a priori error estimates for the eigenfunctions and the eigenvalues. We define an error estimator of the residual type which can be computed locally from the approximate eigenpair and we prove that, up to higher order terms, the estimator is equivalent to the energy norm of the error. We also present some numerical tests which show the performance of the adaptive scheme.

1. Introduction

Adaptive procedures based on a posteriori error estimators have gained an enormous importance in the numerical approximation of partial differential equations. Several approaches, most of them focused on source problems, have been considered to construct estimators based on the residual equations (see [1,2] and their references). Moreover, for the standard Laplace eigenvalue problem a simple and clear analysis has been obtained in [3,4], and there are some similar results for other eigenvalue problems (see, for example, [5–8] and the references therein). However, there are few results concerning a posteriori error estimates for the Stokes eigenvalue problem. In [9] the authors present an a posteriori error analysis for the Stokes eigenvalue problem assuming that the schemes used in its finite element discretization are stable (as, for example, the mini elements).

Despite the fact that the lower-order mixed finite elements for the Stokes equations violate the inf–sup condition, it is well known that low-order velocity–pressure pairs have a relevant interest due to its simple and attractive computational aspects (see [10] and the references therein). There are many stabilized finite element methods to counteract the lack of stability (see, for example, [11–17]). In particular, Bochev, Dohrmann and Gunzburger proposed in [13] a new family of stabilized methods, for the source Stokes problem, and proved that this simple and useful approach is unconditionally stable. Based on this work, in [18] the authors introduce an a posteriori error indicator, for the source Stokes problem, and it yields global upper and lower bounds on the error of stabilized finite element methods.

In this work we prove the convergence of stabilized low-order mixed finite elements for the Stokes eigenvalue problem and we obtain optimal a priori error estimates for the eigenfunctions and the eigenvalues by using the spectral theory given

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in [19]. We define an a posteriori error estimator of the residual type which can be computed locally from the approximate eigenpair. We show its global reliability and local efficiency by proving that the estimator is equivalent to the energy norm of the error up to higher order terms. We also present some numerical tests which allow us to show the good performance of the error indicator and the adaptive algorithm.

The rest of the paper is organized as follows. In Section 2 we introduce the Stokes eigenvalue problem. In Section 3 we present the stabilized low-order mixed finite element method and obtain L^2 a priori error estimates. In Section 4 we prove the convergence for the eigenfunctions and the eigenvalues. In Section 5 we introduce the a posteriori error estimator and prove its equivalence with the energy norm of the error. In Section 6 we report some numerical examples which allow us to assess the performance of the adaptive scheme.

2. Statement of the problem

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded and polygonal domain with boundary $\Gamma := \partial\Omega$. For $\mu \geq 0$ we consider the Stokes eigenvalue problem: Find $(\mathbf{u}, p, \lambda)$, with $\mathbf{u} = (u_1, u_2) \neq 0$ and $\lambda \in \mathbb{R}$, such that

$$\begin{cases} -\mu\Delta\mathbf{u} + \nabla p = \lambda\mathbf{u} & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \Gamma, \end{cases} \tag{1}$$

which models the slow motion of an incompressible viscous fluid occupying Ω, where $\mathbf{u}$ is the fluid velocity and p is the pressure.

We will denote by boldface the spaces consisting of vector value functions. Let $\mathbf{V} := \mathbf{H}_0^1(\Omega)$ and $S := L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q = 0\}$. The norms and seminorms in $\mathbf{H}^m(D)$, with m an integer, are denoted by $\|\cdot\|_{m,D}$ and $|\cdot|_{m,D}$ respectively and $(\cdot,\cdot)_D$ denotes the inner product in $L^2(D)$ or $\mathbf{L}^2(D)$ for any subdomain $D \subset \Omega$. The domain subscript is dropped for the case $D = \Omega$.

Problem (1) can be written, after normalization for $\mathbf{u}$, in a variational form as follows:

Find $(\mathbf{u}, p, \lambda) \in (\mathbf{V}, S, \mathbb{R})$, with $\|\mathbf{u}\|_0 = 1$, such that

$$Q(\mathbf{u}, p, \mathbf{v}, q) = \lambda(\mathbf{u}, \mathbf{v}) \quad \forall(\mathbf{v}, q) \in (\mathbf{V}, S), \tag{2}$$

where

$$Q(\mathbf{u}, p, \mathbf{v}, q) = \mu \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v} - \int_\Omega p\nabla\cdot\mathbf{v} - \int_\Omega q\nabla\cdot\mathbf{u},$$

with $(\cdot,\cdot)$ the inner product in $\mathbf{L}^2(\Omega)$.

Now, it is clear that the symmetric bilinear form Q is continuous, i.e., for every $(\mathbf{u}, q), (\mathbf{v}, s) \in (\mathbf{V}, S)$

$$Q(\mathbf{u}, q, \mathbf{v}, s) \leq C(\|\mathbf{u}\|_1 + \|q\|_0)(\|\mathbf{v}\|_1 + \|s\|_0),$$

moreover it is known [20,21] that $Q(\mathbf{u}, q, \mathbf{v}, s)$ satisfies the following inf–sup condition with a positive constant β:

$$\sup_{(\mathbf{v},s)\in(\mathbf{V},S)} \frac{Q(\mathbf{u}, q, \mathbf{v}, s)}{\|\mathbf{v}\|_1 + \|s\|_0} \geq \beta(\|\mathbf{u}\|_1 + \|q\|_0) \quad \forall(\mathbf{u}, q) \in (\mathbf{V}, S), \tag{3}$$

and so the bilinear form Q is stable.

Now, from the spectral theory (see [19]) we know that the eigenvalue problem (2) has a positive eigenvalue sequence λ_j which we assume to be increasingly ordered:

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots \lim_{j\to+\infty} \lambda_j = +\infty,$$

and the associated eigenfunctions

$$(\mathbf{u}_1, p_1), (\mathbf{u}_2, p_2), \ldots, (\mathbf{u}_k, p_k), \ldots$$

with $(\mathbf{u}_i, \mathbf{u}_j) = \delta_{ij}$. For simplicity, we only consider simple eigenvalues in this paper.

3. Stabilized mixed finite element approximations

Let $\mathcal{T}_h$ be a family of triangulations of Ω such that any two triangles in $\mathcal{T}_h$ share at most a vertex or an edge. Let h stand for the mesh-size; namely $h = \max_{T\in\mathcal{T}_h} h_T$, with h_T being the diameter of the triangle T. We assume that the family of triangulations $\{\mathcal{T}_h\}$ satisfies a minimum angle condition, i.e., there exists a constant $\tau > 0$ such that $h_T/r_T \leq \tau$, where r_T is the diameter of the largest circle contained in T.

Let

$$P_1(\Omega) = \{u \in C(\Omega) |\ u|_T \in \mathcal{P}_1(T)\ \forall\, T \in \mathcal{T}_h\}.$$

We consider the pair

$$\mathbf{V}^h = \mathbf{P}_1 \cap \mathbf{H}_0^1(\Omega) \quad \text{and} \quad S^h = P_1 \cap L_0^2(\Omega). \tag{4}$$

As it is well known (see, for example, [22]) this finite element pair does not satisfy the discrete inf–sup condition. In this paper we consider the stabilized mixed methods for $(\mathbf{V}^h, S^h)$ (which is the lowest equal order C^0 pair) introduced by Bochev, Dohrmann and Gunzburger [13] for the Stokes source problem. Therefore, the discretization of our eigenvalue problem (2) is given by: Find $(\mathbf{u}_h, p_h, \lambda_h) \in (\mathbf{V}^h, S^h, \mathbb{R})$, with $\|\mathbf{u}_h\|_0 = 1$, such that

$$\tilde{Q}(\mathbf{u}_h, p_h, \mathbf{v}_h, q_h) = \lambda_h(\mathbf{u}_h, \mathbf{v}_h) \quad \forall(\mathbf{v}_h, q_h) \in (\mathbf{V}^h, S^h), \tag{5}$$

with

$$\tilde{Q}(\mathbf{u}_h, p_h, \mathbf{v}_h, q_h) = \mu \int_\Omega \nabla\mathbf{u}_h : \nabla\mathbf{v}_h - \int_\Omega p_h \nabla\cdot\mathbf{v}_h - \int_\Omega q_h \nabla\cdot\mathbf{u}_h - G(p_h, q_h),$$

and

$$G(p_h, q_h) = \int_\Omega (I - \Pi)(p_h)(I - \Pi)(q_h),$$

where $\Pi : L^2(\Omega) \to P_0(\Omega)$, with

$$P_0(\Omega) = \{u \in L^2(\Omega) |\ u|_T \in \mathcal{P}_0(T)\ \forall\, T \in \mathcal{T}_h\}$$

is given by

$$\Pi q|_T = \frac{1}{|T|}\int_T q.$$

The problem (5) is reduced to a generalized eigenvalue problem which attains a finite number of eigenpairs $(\lambda_{h,j}, (\mathbf{u}_{h,j}, p_{h,j}))$, $1 \le j \le N$, with positive eigenvalues. We assume the eigenvalues to be increasingly ordered:

$$0 < \lambda_{h,1} \le \cdots \le \lambda_{h,N},$$

and $(\mathbf{u}_{h,i}, \mathbf{u}_{h,j}) = \delta_{i,j}$, $1 \le i, j \le N$.

In order to simplify notation from now on we will drop the subindex j in λ_j, $\lambda_{h,j}$, $\mathbf{u}_j$, $\mathbf{u}_{h,j}$, p_j and $p_{h,j}$.

Our first goal is to prove that the solutions of the discrete eigenvalue problem (5) converge to those of the spectral problem (2). To do this, we will apply the classical spectral approximation theory from [19]. To that purpose, we first present some error estimates for the following Stokes source problem: Given $\mathbf{f} \in \mathbf{L}^2(\Omega)$, find $(\mathbf{u}, p) \in (\mathbf{V}, S)$ such that

$$Q(\mathbf{u}, p, \mathbf{v}, q) = (\mathbf{f}, \mathbf{v}) \quad \forall(\mathbf{v}, q) \in (\mathbf{V}, S). \tag{6}$$

We recall that, since the inf–sup condition holds, the problem (6) has a unique solution. Moreover, it is well known that the solution $(\mathbf{u}, p)$ belongs to $(\mathbf{H}^{1+r}(\Omega) \cap \mathbf{V}, H^r(\Omega) \cap S)$, where $r = 1$ if Ω is convex and $r < \frac{\pi}{\omega}$ (with ω being the largest inner angle of Ω) otherwise (see for example [23]). In the case Ω be a convex polygon we also have the following a priori estimates [24–26]

$$\|\mathbf{u}\|_2 + \|\nabla p\|_0 \le C\|\mathbf{f}\|_0. \tag{7}$$

The stabilized mixed finite element for the Stokes Problem (6) is given by: Find $(\mathbf{u}_h, p_h) \in (\mathbf{V}^h, S^h)$ such that

$$\tilde{Q}(\mathbf{u}_h, p_h, \mathbf{v}_h, q_h) = (\mathbf{f}, \mathbf{v}_h) \quad \forall(\mathbf{v}_h, q_h) \in (\mathbf{V}^h, S^h). \tag{8}$$

The following theorems of [13], give the a priori error estimates in the energy norm and convergence results.

Theorem 3.1. *Let $(\mathbf{V}^h, S^h)$ be the pair (4). Then, there exists a positive constant C whose value is independent of h such that*

$$\sup_{(\mathbf{v}_h, q_h)\in(\mathbf{V}^h, S^h)} \frac{\tilde{Q}(\mathbf{u}_h, p_h, \mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_1 + \|q_h\|_0} \ge C(\|\mathbf{u}_h\|_1 + \|p_h\|_0) \quad \forall(\mathbf{u}_h, p_h) \in (\mathbf{V}^h, S^h). \tag{9}$$

Theorem 3.2. *Let $(\mathbf{V}^h, S^h)$ be the pair (4), let $(\mathbf{u}, p)$ be the solution of the Stokes Problem (6), and let $(\mathbf{u}_h, p_h)$ be the solution of the stabilized problem (8). Then,*

$$\|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_0 \le C\{\inf_{\mathbf{v}\in V_h} \|\mathbf{u} - \mathbf{v}\|_1 + \inf_{q\in Q_h} \|p - q\|_0 + \|(I - \Pi)p\|_0\}. \tag{10}$$

Corollary 3.1. *Assume that $(\mathbf{u}, p) \in (\mathbf{H}_0^1(\Omega) \cap \mathbf{H}^{1+r}(\Omega), L_0^2(\Omega) \cap H^r(\Omega))$ solves the Stokes Problem (6), where $r = 1$ if Ω is convex and $r < \frac{\pi}{\omega}$ otherwise, and that $(\mathbf{u}_h, p_h)$ is the solution of the stabilized mixed problem (8). Then,*

$$\|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_0 \le Ch^r(\|\mathbf{u}\|_{1+r} + \|p\|_r).$$

Proof. The result is a consequence of Theorem 3.2, the estimates (4.1) and (5.1) of [13] and standard error estimates for interpolation (see for example [27]). □

Next, we obtain L^2 error estimates for the velocity which are fundamental for our spectral analysis.

Theorem 3.3. *Assume that Ω is convex. Let $(\mathbf{u}, p) \in (\mathbf{H}_0^1(\Omega)\cap\mathbf{H}^2(\Omega), L_0^2(\Omega)\cap H^1(\Omega))$ be the solution of the Stokes Problem (6) and $(\mathbf{u}_h, p_h)$ the solution of the stabilized mixed problem (8). Then,*

$$\|\mathbf{u}-\mathbf{u}_h\|_0 \le Ch^2(\|\mathbf{u}\|_2 + \|p\|_1).$$

Proof. From (6) and (8) we know that for any $(\mathbf{v}_h, q_h) \in (\mathbf{V}^h, S^h)$ the errors $\mathbf{e} = \mathbf{u}-\mathbf{u}_h$ and $\epsilon = p - p_h$ satisfy the following error equation:

$$\mu \int_\Omega \nabla\mathbf{e} : \nabla\mathbf{v}_h - \int_\Omega \epsilon\nabla\cdot\mathbf{v}_h - \int_\Omega q_h\nabla\cdot\mathbf{e} = -G(p_h, q_h). \tag{11}$$

Next, we consider the following auxiliary problem: Find $(\mathbf{\Phi}, \alpha) \in (\mathbf{V}, S)$ such that

$$\begin{cases} -\mu\Delta\mathbf{\Phi} + \nabla\alpha = \mathbf{e} & \text{in } \Omega, \\ \nabla\cdot\mathbf{\Phi} = 0 & \text{in } \Omega, \\ \mathbf{\Phi} = 0 & \text{on } \Gamma := \partial\Omega, \end{cases} \tag{12}$$

and by using (12) and integration by parts we obtain

$$\begin{aligned} \int_\Omega \mathbf{e}^2 &= \int_\Omega \mathbf{e}\cdot(-\mu\Delta\mathbf{\Phi} + \nabla\alpha) \\ &= -\mu\int_\Omega \mathbf{e}\cdot\Delta\mathbf{\Phi} + \int_\Omega \mathbf{e}\cdot\nabla\alpha \\ &= \mu\int_\Omega \nabla\mathbf{e} : \nabla\mathbf{\Phi} - \int_\Omega \nabla\cdot\mathbf{e}\alpha. \end{aligned}$$

A well known approximation result (see, for example, [20, page 217]) is that for every $u \in H^2(\Omega)$, there exists a function $I_1u \in P_1(\Omega)$ such that

$$\|u - I_1u\|_0 + h\|u - I_1u\|_1 \le Ch^2\|u\|_2. \tag{13}$$

On the other hand, the space $P_0(\Omega) = \{u \in L^2(\Omega)|\ \ u|_T \in \mathcal{P}_0(T)\ \forall\, T \in \mathcal{T}_h\}$ has the following approximation property [20, page 102]: for every $q \in H^1(\Omega)$, there exists $I_0q \in P_0(\Omega)$ such that

$$\|q - I_0q\|_0 \le Ch\|\nabla q\|_0. \tag{14}$$

We denote for $\mathbf{\Phi} = (\phi_1, \phi_2)$, $\mathbf{I}_1\mathbf{\Phi} = (I_1\phi_1, I_1\phi_2)$. Then,

$$\int_\Omega \mathbf{e}^2 = \mu\int_\Omega \nabla\mathbf{e} : \nabla(\mathbf{\Phi} - \mathbf{I}_1\mathbf{\Phi}) + \mu\int_\Omega \nabla\mathbf{e} : \nabla\mathbf{I}_1\mathbf{\Phi} - \int_\Omega \nabla\cdot\mathbf{e}(\alpha - I_0\alpha) - \int_\Omega \nabla\cdot\mathbf{e}\, I_0\alpha,$$

hence, by using the error equation (11), the fact that $\nabla\cdot\mathbf{\Phi} = 0$, the Hölder inequality, (13) and (14) we get

$$\begin{aligned} \int_\Omega \mathbf{e}^2 &= \mu\int_\Omega \nabla\mathbf{e} : \nabla(\mathbf{\Phi} - \mathbf{I}_1\mathbf{\Phi}) - \int_\Omega \nabla\cdot\mathbf{e}(\alpha - I_0\alpha) + \int_\Omega \epsilon\nabla\cdot\mathbf{I}_1\mathbf{\Phi} - G(p_h, I_0\alpha) \\ &= \mu\int_\Omega \nabla\mathbf{e} : \nabla(\mathbf{\Phi} - \mathbf{I}_1\mathbf{\Phi}) - \int_\Omega \nabla\cdot\mathbf{e}(\alpha - I_0\alpha) + \int_\Omega \epsilon\nabla\cdot(\mathbf{\Phi} - \mathbf{I}_1\mathbf{\Phi}) - G(p_h, I_0\alpha) \\ &\le Ch\|\nabla\mathbf{e}\|_0\|\mathbf{\Phi}\|_2 + Ch\|\nabla\cdot\mathbf{e}\|_0\|\alpha\|_1 + Ch\|\epsilon\|_0\|\mathbf{\Phi}\|_2 + \|(I-\Pi)p_h\|_0\|(I-\Pi)I_0\alpha\|_0 \\ &\le Ch(\|\mathbf{\Phi}\|_2 + \|\nabla\alpha\|_0)(\|\mathbf{e}\|_1 + \|\epsilon\|_0) + \|(I-\Pi)p_h\|_0\|(I-\Pi)I_0\alpha\|_0. \end{aligned}$$

From the a priori estimates (7) we can assume that

$$\|\mathbf{\Phi}\|_2 + \|\nabla\alpha\|_0 \le C\|\mathbf{e}\|_0, \tag{15}$$

and therefore, from Corollary 3.1, we get

$$\int_\Omega \mathbf{e}^2 \le Ch^2\|\mathbf{e}\|_0(\|\mathbf{u}\|_2 + \|p\|_1) + \|(I-\Pi)p_h\|_0\|(I-\Pi)I_0\alpha\|_0.$$

As was shown in [13], for every $p \in H^1(\Omega)$, the operator Π satisfies:

$$\begin{aligned} &\|(I-\Pi)p\|_0 \le Ch\|\nabla p\|_0, \\ &\|\Pi p\|_0 \le C\|p\|_0. \end{aligned}$$

Then,

$$\begin{aligned} \|(I-\Pi)p_h\|_0 &\le \|(I-\Pi)(p-p_h)\|_0 + \|(I-\Pi)p\|_0 \\ &\le C(\|p-p_h\|_0 + h\|p\|_1) \le Ch(\|\mathbf{u}\|_2 + \|p\|_1), \end{aligned}$$

and

$$\|(I-\Pi)I_0\alpha\|_0 \le \|(I-\Pi)\alpha\|_0 + \|(I-\Pi)(\alpha - I_0\alpha)\|_0 \le C(h\|\alpha\|_1 + \|\alpha - I_0\alpha\|_0)$$
$$\le Ch\|\nabla\alpha\|_0 \le Ch\|\mathbf{e}\|_0.$$

So,

$$\int_\Omega \mathbf{e}^2 \le Ch^2\|\mathbf{e}\|_0(\|\mathbf{u}\|_2 + \|p\|_1), \tag{16}$$

and the result follows. □

Remark 3.1. We can use the same arguments given in the last proof, in the case in which Ω is not convex. Let ω be the largest inner angle of Ω and $r < \frac{\pi}{\omega} < 1$. And now, let $(\mathbf{\Phi}, \alpha) \in (\mathbf{H}^{1+r}(\Omega) \cap \mathbf{V}, H^r(\Omega) \cap S)$ be the solution of the Stokes Problem (12), if we assume, in addition, that there exists a positive constant C such that the following a priori estimates holds:

$$\|\mathbf{\Phi}\|_{1+r} + \|\alpha\|_r \le C\|\mathbf{e}\|_0, \tag{17}$$

then

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \le Ch^{2r}(\|\mathbf{u}\|_{1+r} + \|p\|_r). \tag{18}$$

4. Spectral approximation

In this section, by using the classical spectral approximation theory given in [19] (see also [28,29]), we obtain the convergence of the eigenfunctions and eigenvalues with optimal order. Let $\mathbf{W} = (\mathbf{V}, S)$ with $\|(\mathbf{u}, p)\|_{\mathbf{W}} = \|\mathbf{u}\|_1 + \|p\|_0$. As we mentioned in the previous section, if $\mathbf{f} \in \mathbf{L}^2(\Omega)$ there exists a unique $(\mathbf{u}, p) \in \mathbf{W}$ such that

$$Q(\mathbf{u}, p, \mathbf{v}, q) = (\mathbf{f}, \mathbf{v}) \quad \forall(\mathbf{v}, q) \in \mathbf{W}. \tag{19}$$

Then, for any $\mathbf{F} = (\mathbf{f}, \sigma) \in \mathbf{W}$ we can define the operator $T : \mathbf{W} \to \mathbf{W}$ as

$$T\mathbf{F} = (\mathbf{u}, p),$$

and, for any $\mathbf{f} \in \mathbf{L}^2(\Omega)$ we can also define $R : \mathbf{L}^2(\Omega) \to \mathbf{L}^2(\Omega)$ as

$$R\mathbf{f} = \mathbf{u},$$

where $(\mathbf{u}, p)$ denotes the corresponding solution of (19).

On the other hand, we also know that there exists a unique $(\mathbf{u}_h, p_h) \in (\mathbf{V}^h, S^h) \subseteq \mathbf{W}$ such that

$$\tilde{Q}(\mathbf{u}_h, p_h, \mathbf{v}_h, q_h) = (\mathbf{f}, \mathbf{v}_h) \quad \forall(\mathbf{v}_h, q_h) \in (\mathbf{V}^h, S^h). \tag{20}$$

Then, for any $\mathbf{F} = (\mathbf{f}, \sigma) \in \mathbf{W}$ we can define the operator $T_h : \mathbf{W} \to \mathbf{W}$ as

$$T_h\mathbf{F} = (\mathbf{u}_h, p_h),$$

and, for any $\mathbf{f} \in \mathbf{L}^2(\Omega)$ we can also define $R_h : \mathbf{L}^2(\Omega) \to \mathbf{L}^2(\Omega)$ as

$$R_h\mathbf{f} = \mathbf{u}_h,$$

where $(\mathbf{u}_h, p_h)$ denotes the corresponding solution of (20).

Let $\lambda \neq 0$, we observe that $\mathbf{f}$ is an eigenfunction of R of eigenvalue λ if and only if $(\mathbf{f}, p, 1/\lambda)$ is a solution of (2) for some $p \in S$, and $\mathbf{F} = (\mathbf{f}, p)$ is an eigenfunction of T of eigenvalue λ if and only if $(\mathbf{f}, p, 1/\lambda)$ is a solution of (2). In the same way, let $\lambda_h \neq 0$, then $\mathbf{f}_h$ is an eigenfunction of R_h of eigenvalue λ_h if and only if $(\mathbf{f}_h, p_h, 1/\lambda_h)$ is a solution of (5) for some $p_h \in S^h$, and $\mathbf{F} = (\mathbf{f}_h, p_h)$ is an eigenfunction of T_h of eigenvalue λ_h if and only if $(\mathbf{f}_h, p_h, 1/\lambda_h)$ is a solution of (5).

In order to use the spectral approximation theory, stated in [19], we are going to prove that the operators T, R, T_h, R_h are bounded and compact; T_h converge to T and R_h converge to R as h goes to zero.

In what follows, we assume Ω is convex.

We observe that R and T are bounded operators; in fact, by (7) and using the Poincaré inequality we have

$$\|\mathbf{u}\|_0 \le C\|\mathbf{f}\|_0,$$
$$\|(\mathbf{u}, p)\|_{\mathbf{W}} = \|\mathbf{u}\|_1 + \|p\|_0 \le C\|\mathbf{f}\|_0 \le C\|\mathbf{F}\|_{\mathbf{W}}.$$

It is also clear that R_h and T_h are bounded operators, i.e., using Corollary 3.1, Theorem 3.3 and the fact that R and T are bounded, we have that

$$\|\mathbf{u}_h\|_0 \le \|\mathbf{u}_h - \mathbf{u}\|_0 + \|\mathbf{u}\|_0 \le C\|\mathbf{f}\|_0,$$
$$\|\mathbf{u}_h\|_1 + \|p_h\|_0 \le \|\mathbf{u}_h - \mathbf{u}\|_1 + \|p_h - p\|_0 + \|\mathbf{u}\|_1 + \|p\|_0 \le C\|\mathbf{f}\|_0 \le C\|\mathbf{F}\|_{\mathbf{W}}.$$

From the error estimates for the Stokes source problem given by Corollary 3.1 and Theorem 3.3, and (7), we obtain that, for all $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $\mathbf{F} \in \mathbf{W}$,

$$\|R\mathbf{f} - R_h\mathbf{f}\|_0 = \|\mathbf{u} - \mathbf{u}_h\|_0 \le Ch^2\|\mathbf{f}\|_0,$$
$$\|T\mathbf{F} - T_h\mathbf{F}\|_{\mathbf{W}} = \|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_0 \le Ch\|\mathbf{f}\|_0 \le Ch\|\mathbf{F}\|_{\mathbf{W}},$$

then $R_h \to R$ and $T_h \to T$ in norm when h goes to 0.

We observe that T and R are compact operators since, for any Hilbert space $\mathcal{X}$, the space of compact operators in $\mathcal{X}$ is close in $\mathcal{B}(\mathcal{X})$, where $\mathcal{B}(\mathcal{X}) = \{L : \mathcal{X} \to \mathcal{X}, L \text{ linear and continuous}\}$.

Now, we are in condition to present the next theorem.

Theorem 4.1. *Assume that Ω is convex. Given an eigenpair $(\mathbf{u}, p, \lambda) \in (\mathbf{V}, S, \mathbb{R})$ solution of (2), with $\|\mathbf{u}\|_0 = 1$. Then, there exists a discrete eigenpair $(\mathbf{u}_h, p_h, \lambda_h) \in (\mathbf{V}^h, S^h, \mathbb{R})$ solution of (5), with $\|\mathbf{u}_h\|_0 = 1$, such that*

$$|\lambda - \lambda_h| \le Ch^2,$$
$$\|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_0 \le Ch,$$
$$\|\mathbf{u} - \mathbf{u}_h\|_0 \le Ch^2.$$

Proof. Let $(\mathbf{u}, p, \lambda)$, $\|\mathbf{u}\|_0 = 1, \lambda \neq 0$ be the solution of (2), by Remarks 7.3 and 7.4 from [19] we have, for small h, that there exists $(\mathbf{u}_h, p_h, \lambda_h)$, $\|\mathbf{u}_h\|_0 = 1, \lambda^h \neq 0$ such that

$$|\lambda - \lambda_h| \le C\|R - R_h\| \le Ch^2,$$
$$\|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_0 \le C\|T - T_h\| \le Ch,$$
$$\|\mathbf{u} - \mathbf{u}_h\|_0 \le C\|R - R_h\| \le Ch^2. \quad \square$$

Remark 4.1. We can use the same arguments given in the last proof, in the case in which Ω is not convex. Let ω be the largest inner angle of Ω and $r < \frac{\pi}{\omega} < 1$. If we assume, in addition, that the solution $(\mathbf{u}, p)$ of (19) satisfies the following a priori estimate

$$\|\mathbf{u}\|_{1+r} + \|p\|_r \le C\|\mathbf{f}\|_0$$

for any $\mathbf{f} \in \mathbf{L}^2(\Omega)$ then, the estimate (18) holds and we obtained

$$|\lambda - \lambda_h| \le Ch^{2r},$$
$$\|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_0 \le Ch^r,$$
$$\|\mathbf{u} - \mathbf{u}_h\|_0 \le Ch^{2r}.$$

The next lemma gives an expression for the difference between the eigenvalue λ and its approximation λ_h and it gives, in particular, a relationship between the eigenvalues error and the error for eigenfunctions in norm.

Lemma 4.1. *Given $(\mathbf{u}, p, \lambda) \in (\mathbf{V}, S, \mathbb{R})$ solution of (2), with $\|\mathbf{u}\|_0 = 1$ and $(\mathbf{u}_h, p_h, \lambda_h) \in (\mathbf{V}^h, S^h, \mathbb{R})$ solution of (5), with $\|\mathbf{u}_h\|_0 = 1$. Let $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$ and $\epsilon = p - p_h$. Then,*

$$\lambda_h - \lambda = Q(\mathbf{e}, \epsilon, \mathbf{e}, \epsilon) - \lambda\|\mathbf{u} - \mathbf{u}_h\|_0^2 - G(p_h, p_h).$$

Proof. From (2) and (5) we have

$$Q(\mathbf{u}, p, \mathbf{u}, p) = \lambda\|\mathbf{u}\|_0^2$$
$$\tilde{Q}(\mathbf{u}_h, p_h, \mathbf{u}_h, p_h) = \lambda_h\|\mathbf{u}_h\|_0^2.$$

Then

$$\begin{aligned} \lambda + \lambda_h &= \lambda\|\mathbf{u}\|_0^2 + \lambda_h\|\mathbf{u}_h\|_0^2 = Q(\mathbf{u}, p, \mathbf{u}, p) + \tilde{Q}(\mathbf{u}_h, p_h, \mathbf{u}_h, p_h) \\ &= Q(\mathbf{u}, p, \mathbf{u}, p) + Q(\mathbf{u}_h, p_h, \mathbf{u}_h, p_h) - G(p_h, p_h). \end{aligned}$$

If we observe that

$$Q(\mathbf{e}, \epsilon, \mathbf{e}, e) = Q(\mathbf{u}, p, \mathbf{u}, p) + Q(\mathbf{u}_h, p_h, \mathbf{u}_h, p_h) - 2Q(\mathbf{u}, p, \mathbf{u}_h, p_h),$$

and using (2) we get

$$\begin{aligned} \lambda + \lambda_h &= Q(\mathbf{e}, \epsilon, \mathbf{e}, e) + 2Q(\mathbf{u}, p, \mathbf{u}_h, p_h) - G(p_h, p_h) \\ &= Q(\mathbf{e}, \epsilon, \mathbf{e}, e) + 2\lambda \int_\Omega \mathbf{u} \cdot \mathbf{u}_h - G(p_h, p_h) \\ &= Q(\mathbf{e}, \epsilon, \mathbf{e}, e) - \lambda(\|\mathbf{u} - \mathbf{u}_h\|_0^2 - \|\mathbf{u}\|_0^2 - \|\mathbf{u}_h\|_0^2) - G(p_h, p_h) \\ &= Q(\mathbf{e}, \epsilon, \mathbf{e}, e) - \lambda\|\mathbf{u} - \mathbf{u}_h\|_0^2 + 2\lambda - G(p_h, p_h), \end{aligned}$$

and the result holds. $\square$

5. A posteriori error analysis

In this section we introduce an error indicator and show its equivalence, up to higher order terms, with the error norm. First, we introduce some notations that we will use in the definition and the analysis of the error estimator. For any $T \in \mathcal{T}_h$

we denote by $\mathcal{E}(T)$ and $\mathcal{N}(T)$ the set of its edges and vertices respectively, and let

$$\mathcal{E}^h := \bigcup_{T\in\mathcal{T}_h} \mathcal{E}(T), \qquad \mathcal{N}^h := \bigcup_{T\in\mathcal{T}_h} \mathcal{N}(T).$$

Given an $\ell \in \mathcal{E}^h$ we denote by $\mathcal{N}(\ell)$ the set of its vertices. For $T \in \mathcal{T}_h$ and $\ell \in \mathcal{E}^h$ we define

$$\omega_T := \bigcup_{\mathcal{N}(T)\cap\mathcal{N}(T')\neq\emptyset} T', \qquad \omega_\ell := \bigcup_{\mathcal{N}(\ell)\cap\mathcal{N}(T')\neq\emptyset} T'.$$

Remark 5.1. The minimal angle condition implies that the ratio $h_T/|\ell|$, for any $T \in \mathcal{T}_h$ and $\ell \in \mathcal{E}(T)$, the ratio $h_T/h_{T'}$, for any $T, T' \in \mathcal{T}_h$ with $\mathcal{N}(T)\cap\mathcal{N}(T') \neq \emptyset$ are bounded from below and from above by constants which only depend on τ.

Let $(\mathbf{u}, p, \lambda)$ and $(\mathbf{u}_h, p_h, \lambda_h)$ be as in Theorem 4.1, we define $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$ and $\epsilon = p - p_h$. From (2) and (5) we know that for any $(\mathbf{v}, q) \in (\mathbf{V}, S)$ the errors $\mathbf{e}$ and ϵ satisfy

$$Q(\mathbf{e}, \epsilon, \mathbf{v}_h, q_h) = (\lambda\mathbf{u} - \lambda_h\mathbf{u}_h, \mathbf{v}_h) - G(p_h, q_h) \quad \forall(\mathbf{v}_h, q_h) \in (\mathbf{V}^h, S^h). \tag{21}$$

On the other hand, if $(\mathbf{v}, q) \in (\mathbf{V}, S)$, using the definition of Q, integration by parts and that $\Delta\mathbf{u}_h = 0$ holds in any $T \in \mathcal{T}_h$, we obtain

$$\begin{aligned} Q(\mathbf{e}, \epsilon, \mathbf{v}, q) &= \lambda(\mathbf{u}, \mathbf{v}) - Q(\mathbf{u}_h, p_h, \mathbf{v}, q) \\ &= \lambda(\mathbf{u}, \mathbf{v}) - \mu\int_\Omega \nabla\mathbf{u}_h : \nabla\mathbf{v} + \int_\Omega \nabla\cdot\mathbf{u}_h q + \int_\Omega \nabla\cdot\mathbf{v} p_h \\ &= \int_\Omega \lambda\mathbf{u}\cdot\mathbf{v} + \sum_{T\in\mathcal{T}_h}\left\{\mu\int_T \triangle\mathbf{u}_h\cdot\mathbf{v} - \mu\int_{\partial T}\frac{\partial\mathbf{u}_h}{\partial n}\cdot\mathbf{v} + \int_T \nabla\cdot\mathbf{u}_h q - \int_T \nabla p_h\cdot\mathbf{v} + \int_{\partial T} p_h n\cdot\mathbf{v}\right\} \\ &= \int_\Omega \lambda\mathbf{u}\cdot\mathbf{v} + \sum_{T\in\mathcal{T}_h}\left\{-\mu\int_{\partial T}\frac{\partial\mathbf{u}_h}{\partial n}\cdot\mathbf{v} + \int_T \nabla\cdot\mathbf{u}_h q - \int_T \nabla p_h\cdot\mathbf{v} + \int_{\partial T} p_h n\cdot\mathbf{v}\right\}, \end{aligned}$$

since p_h is continuous

$$\sum_{T\in\mathcal{T}_h}\int_{\partial T} p_h n\cdot\mathbf{v} = 0,$$

then

$$Q(\mathbf{e}, \epsilon, \mathbf{v}, q) = \int_\Omega (\lambda\mathbf{u} - \lambda_h\mathbf{u}_h)\cdot\mathbf{v} + \sum_{T\in\mathcal{T}_h}\left\{\int_T (\lambda_h\mathbf{u}_h - \nabla p_h)\cdot\mathbf{v} + \int_T \nabla\cdot\mathbf{u}_h q - \mu\int_{\partial T}\frac{\partial\mathbf{u}_h}{\partial n}\cdot\mathbf{v}\right\}. \tag{22}$$

We denote by $\mathcal{E}^h_\Omega$ the set of all interior edges. For each $\ell \in \mathcal{E}^h_\Omega$ we choose a unit normal vector $\mathbf{n}_\ell$ and denote the two triangles sharing this edge T_{in} and T_{out}, with $\mathbf{n}_\ell$ pointing outwards T_{in}. For $\mathbf{u}_h \in \mathbf{V}^h$ we set

$$\left[\frac{\partial\mathbf{u}_h}{\partial n}\right]_\ell := \nabla(\mathbf{u}_h|_{T_{out}})\cdot\mathbf{n}_\ell - \nabla(\mathbf{u}_h|_{T_{in}})\cdot\mathbf{n}_\ell.$$

Then, for any $(\mathbf{v}, q) \in (\mathbf{V}, S)$ the error equation can be written as:

$$Q(\mathbf{e}, \epsilon, \mathbf{v}, q) = \int_\Omega (\lambda\mathbf{u} - \lambda_h\mathbf{u}_h)\cdot\mathbf{v} + \sum_{T\in\mathcal{T}_h}\left\{\int_T (\lambda_h\mathbf{u}_h - \nabla p_h)\cdot\mathbf{v} + \int_T \nabla\cdot\mathbf{u}_h q - \mu\frac{1}{2}\sum_{\ell\in\mathcal{E}(T)\cap\mathcal{E}^h_\Omega}\int_\ell\left[\frac{\partial\mathbf{u}_h}{\partial n}\right]_\ell\cdot\mathbf{v}\right\}. \tag{23}$$

Now, the local error indicator is defined as follows

$$\eta_T^2 = h_T^2\|\lambda_h\mathbf{u}_h - \nabla p_h\|_{0,T}^2 + \|\nabla\cdot\mathbf{u}_h\|_{0,T}^2 + \frac{1}{4}\mu^2\sum_{\ell\in\mathcal{E}(T)\cap\mathcal{E}^h_\Omega}|\ell|\left\|\left[\frac{\partial\mathbf{u}_h}{\partial n}\right]_\ell\right\|_{0,\ell}^2, \tag{24}$$

and the global error indicator is given by

$$\eta = \left(\sum_{T\in\mathcal{T}_h}\eta_T^2\right)^{\frac{1}{2}}. \tag{25}$$

We denote by $I_h : \mathbf{H}^1_0(\Omega) \to \mathbf{P}_1(\Omega)\cap\mathbf{H}^1_0(\Omega)$ the Clément interpolation operator (see [30]) that satisfies, for $T \in \mathcal{T}_h$ and $\ell \in \mathcal{E}^h$,

$$\begin{aligned} \|\mathbf{u} - I_h\mathbf{u}\|_{0,T} &\le Ch_T\|\mathbf{u}\|_{1,\omega_T}, \\ \|\mathbf{u} - I_h\mathbf{u}\|_{0,\ell} &\le C|\ell|^{1/2}\|\mathbf{u}\|_{1,\omega_\ell}. \end{aligned} \tag{26}$$

Then, given $(\mathbf{v}, q) \in (\mathbf{V}, S)$ using the error equation (21) we have that

$$Q(\mathbf{e}, \epsilon, I_h\mathbf{v}, 0) = \int_\Omega (\lambda\mathbf{u} - \lambda_h\mathbf{u}_h)\cdot I_h\mathbf{v},$$

and by using the error equation (23) we get

$$\begin{aligned} Q(\mathbf{e}, \epsilon, \mathbf{v}, q) &= Q(\mathbf{e}, \epsilon, I_h\mathbf{v}, 0) + Q(\mathbf{e}, \epsilon, \mathbf{v} - I_h\mathbf{v}, q) \\ &= \int_\Omega (\lambda\mathbf{u} - \lambda_h\mathbf{u}_h)\cdot I_h\mathbf{v} + \int_\Omega (\lambda\mathbf{u} - \lambda_h\mathbf{u}_h)\cdot(\mathbf{v} - I_h\mathbf{v}) + \sum_{T\in\mathcal{T}_h}\left\{\int_T (\lambda_h\mathbf{u}_h - \nabla p_h)\cdot(\mathbf{v} - I_h\mathbf{v})\right. \\ &\quad \left. + \int_T \nabla\cdot\mathbf{u}_h q - \mu\frac{1}{2}\sum_{\ell\in\mathcal{E}(T)\cap\mathcal{E}_\Omega^h}\int_\ell \left[\frac{\partial\mathbf{u}_h}{\partial n}\right]_\ell\cdot(\mathbf{v} - I_h\mathbf{v})\right\} \\ &= \int_\Omega (\lambda\mathbf{u} - \lambda_h\mathbf{u}_h)\cdot\mathbf{v} + \sum_{T\in\mathcal{T}_h}\left\{\int_T (\lambda_h\mathbf{u}_h - \nabla p_h)\cdot(\mathbf{v} - I_h\mathbf{v})\right. \\ &\quad \left. + \int_T \nabla\cdot\mathbf{u}_h q - \mu\frac{1}{2}\sum_{\ell\in\mathcal{E}(T)\cap\mathcal{E}_\Omega^h}\int_\ell \left[\frac{\partial\mathbf{u}_h}{\partial n}\right]_\ell\cdot(\mathbf{v} - I_h\mathbf{v})\right\}. \end{aligned}$$

Hence, using the Hölder inequality and (26)

$$\begin{aligned} Q(\mathbf{e}, \epsilon, \mathbf{v}, q) &\le \|\lambda\mathbf{u} - \lambda_h\mathbf{u}_h\|_{0,\Omega}\|\mathbf{v}\|_{0,\Omega} + C\sum_{T\in\mathcal{T}_h}\left\{h_T\|\lambda_h\mathbf{u}_h - \nabla p_h\|_{0,T}\|\nabla\mathbf{v}\|_{0,\omega_T}\right. \\ &\quad \left. + \|\nabla\cdot\mathbf{u}_h\|_{0,T}\|q\|_{0,T} + \mu\frac{1}{2}\sum_{\ell\in\mathcal{E}T\cap\mathcal{E}_\Omega^h}|\ell|^{1/2}\left\|\left[\frac{\partial\mathbf{u}_h}{\partial n}\right]_\ell\right\|_{0,\ell}\|\nabla\mathbf{v}\|_{0,\omega_\ell}\right\} \\ &\le C\left\{\|\lambda\mathbf{u} - \lambda_h\mathbf{u}_h\|_{0,\Omega} + \sum_{T\in\mathcal{T}_h}\left\{h_T\|\lambda_h\mathbf{u}_h - \nabla p_h\|_{0,T} + \|\nabla\cdot\mathbf{u}_h\|_{0,T}\right.\right. \\ &\quad \left.\left. + \mu\frac{1}{2}\sum_{\ell\in\mathcal{E}(T)\cap\mathcal{E}_\Omega^h}|\ell|^{1/2}\left\|\left[\frac{\partial\mathbf{u}_h}{\partial n}\right]_\ell\right\|_{0,\ell}\right\}\right\}(\|\mathbf{v}\|_{1,\Omega} + \|q\|_{0,\Omega}). \end{aligned}$$

Then, we obtain $\forall(\mathbf{v}, q) \in (\mathbf{V}, S)$

$$\begin{aligned} \frac{Q(\mathbf{e}, \epsilon, \mathbf{v}, q)}{\|\mathbf{v}\|_{1,\Omega} + \|q\|_{0,\Omega}} &\le C\left\{\|\lambda\mathbf{u} - \lambda_h\mathbf{u}_h\|_{0,\Omega} + \sum_{T\in\mathcal{T}_h}\left\{\|\lambda_h\mathbf{u}_h - \nabla p_h\|_{0,T}h_T\right.\right. \\ &\quad \left.\left. + \|\nabla\cdot\mathbf{u}_h\|_{0,T} + \mu\frac{1}{2}\sum_{\ell\in\mathcal{E}(T)\cap\mathcal{E}_\Omega^h}|\ell|^{1/2}\left\|\left[\frac{\partial\mathbf{u}_h}{\partial n}\right]_\ell\right\|_{0,\ell}\right\}\right\}, \end{aligned}$$

and therefore, by using the inf–sup condition (3), we conclude that

$$\begin{aligned} \|\mathbf{e}\|_1 + \|\epsilon\|_0 &\le C\left\{\|\lambda\mathbf{u} - \lambda_h\mathbf{u}_h\|_{0,\Omega} + \sum_{T\in\mathcal{T}_h}\left\{h_T\|\lambda_h\mathbf{u}_h - \nabla p_h\|_{0,T}\right.\right. \\ &\quad \left.\left. + \|\nabla\cdot\mathbf{u}_h\|_{0,T} + \mu\frac{1}{2}\sum_{\ell\in\mathcal{E}(T)\cap\mathcal{E}_\Omega^h}|\ell|^{1/2}\left\|\left[\frac{\partial\mathbf{u}_h}{\partial n}\right]_\ell\right\|_{0,\ell}\right\}\right\}, \end{aligned}$$

and so the following estimate holds:

Theorem 5.1. *Let $(\mathbf{u}, p, \lambda)$ and $(\mathbf{u}_h, p_h, \lambda_h)$ be as in Theorem 4.1. Let $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$, $\epsilon = p - p_h$ and η be as in (25). There exists a positive constant C such that*

$$\|\mathbf{e}\|_1 + \|\epsilon\|_0 \le C\left(\eta + \|\lambda\mathbf{u} - \lambda_h\mathbf{u}_h\|_{0,\Omega}\right).$$

We observe that, in view of Theorem 4.1 and Remark 4.1, the term $\|\lambda\mathbf{u} - \lambda_h\mathbf{u}_h\|_{0,\Omega}$ is a higher order term and so, the previous theorem proves the reliability of the error estimator.

Remark 5.2. From Lemma 4.1

$$|\lambda_h - \lambda| \leq |Q(\mathbf{e}, \epsilon, \mathbf{e}, \epsilon)| + |\lambda|\|\mathbf{u} - \mathbf{u}_h\|_0^2 + G(p_h, p_h),$$

using the fact that Q is continuous

$$|\lambda_h - \lambda| \leq C(\|\mathbf{e}\|_1 + \|\epsilon\|_0)^2 + |\lambda|\|\mathbf{u} - \mathbf{u}_h\|_0^2 + G(p_h, p_h),$$

and using Theorem 5.1 then

$$\begin{aligned} |\lambda_h - \lambda| &\leq C(\eta + \|\lambda\mathbf{u} - \lambda_h\mathbf{u}_h\|_{0,\Omega})^2 + |\lambda|\|\mathbf{u} - \mathbf{u}_h\|_0^2 + G(p_h, p_h) \\ &\leq C(\eta^2 + G(p_h, p_h) + \|\lambda\mathbf{u} - \lambda_h\mathbf{u}_h\|_{0,\Omega}^2 + |\lambda|\|\mathbf{u} - \mathbf{u}_h\|_0^2). \end{aligned}$$

Observe that $\|\lambda\mathbf{u} - \lambda_h\mathbf{u}_h\|_{0,\Omega}^2 + |\lambda|\|\mathbf{u} - \mathbf{u}_h\|_0^2$ is a high order term.

In order to guarantee that the error indicator is efficient to guide an adaptive refinement scheme, our next goal is to prove that η_T is bounded by the H^1 norm of the error on a neighborhood of T, up to higher order terms.

For $T \in \mathcal{T}_h$, let b_T be the standard cubic bubble given by

$$b_T := \begin{cases} \lambda_1^T\lambda_2^T\lambda_3^T, & \text{in } T, \\ 0, & \text{in } \Omega \setminus T, \end{cases}$$

where λ_1^T, λ_2^T and λ_3^T denote the barycentric coordinates of T.

For $\ell \in \mathcal{E}_\Omega$, we denote by T_1 and T_2 the two triangles sharing ℓ and we enumerate the vertices of T_1 and T_2 so that the vertices of ℓ are numbered first. Then we consider the piecewise quadratic edge bubble function b_ℓ defined by

$$b_\ell := \begin{cases} \lambda_1^{T_i}\lambda_2^{T_i}, & \text{in } T_i,\ i = 1, 2, \\ 0, & \text{in } \Omega \setminus T_1 \cup T_2. \end{cases}$$

From the inequalities (3.3) and (3.4) of [3] one can see that there exists a constant C, which only depends on the regularity of the element T, such that for every $\pi \in P_1(T)$

$$\begin{aligned} &\|b_T\pi\|_{0,T} \leq \|\pi\|_{0,T} \leq C\|b_T^{1/2}\pi\|_{0,T}, \\ &|b_T\pi|_{1,T} \leq C\frac{1}{h_T}\|\pi\|_{0,T}. \end{aligned} \tag{27}$$

The following lemma provides an upper estimate for the first term in the definition of η_T (cf. (24)).

Lemma 5.1. *Let $(\mathbf{u}, p, \lambda)$ and $(\mathbf{u}_h, p_h, \lambda_h)$ be as in Theorem 4.1. Let $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$ and $\epsilon = p - p_h$. Then, there exists a positive constant C such that*

$$h_T\|\lambda_h\mathbf{u}_h - \nabla p_h\|_{0,T} \leq C\{\|\mathbf{e}\|_{1,T} + \|\epsilon\|_{0,T} + h_T\|\lambda\mathbf{u} - \lambda_h\mathbf{u}_h\|_{0,T}\}.$$

Proof. We define

$$\mathbf{v}_T = h_T^2 b_T(\lambda_h\mathbf{u}_h - \alpha\mathbf{u}_h - \nabla p_h).$$

Then, using the inverse estimates (27) and the error equation (22) we get

$$\begin{aligned} h_T^2\|\lambda_h\mathbf{u}_h - \nabla p_h\|_{0,T}^2 &\leq C\int_T(\lambda_h\mathbf{u}_h - \nabla p_h)\cdot\mathbf{v}_T \\ &= C\left\{Q(\mathbf{e}, \epsilon, \mathbf{v}_T, 0) - \int_T(\lambda\mathbf{u} - \lambda_h\mathbf{u}_h)\cdot\mathbf{v}_T\right\} \\ &\leq C\{(\|\mathbf{e}\|_{1,T} + \|\epsilon\|_{0,T})\|\mathbf{v}_T\|_{1,T} + \|\lambda\mathbf{u} - \lambda_h\mathbf{u}_h\|_{0,T}\|\mathbf{v}_T\|_{0,T}\} \\ &\leq C\{(\|\mathbf{e}\|_{1,T} + \|\epsilon\|_{0,T})h_T\|\lambda_h\mathbf{u}_h - \nabla p_h\|_{0,T} + \|\lambda\mathbf{u} - \lambda_h\mathbf{u}_h\|_{0,T}h_T^2\|\lambda_h\mathbf{u}_h - \nabla p_h\|_{0,T}\}, \end{aligned}$$

and so

$$h_T\|\lambda_h\mathbf{u}_h - \nabla p_h\|_{0,T} \leq C\{\|\mathbf{e}\|_{1,T} + \|\epsilon\|_{0,T} + h_T\|\lambda\mathbf{u} - \lambda_h\mathbf{u}_h\|_{0,T}\}. \quad \square$$

Next, we prove an upper estimate for the second term in the definition of η_T.

Lemma 5.2. *Let $(\mathbf{u}, p, \lambda)$ and $(\mathbf{u}_h, p_h, \lambda_h)$ be as in Theorem 4.1. Let $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$ and $\epsilon = p - p_h$. Then, there exists a positive constant C such that*

$$\|\nabla\cdot\mathbf{u}_h\|_{0,T} \leq C\left(\|\mathbf{e}\|_{1,T} + \|\epsilon\|_{0,T}\right).$$

Proof. We define

$$q_T = b_T(\nabla\cdot\mathbf{u}_h),$$

then using the inverse estimate (27) and the error equation in (23) we obtain

$$\begin{aligned}\|\nabla\cdot\mathbf{u}_h\|_{0,T}^2 &\le C\int_T(\nabla\cdot\mathbf{u}_h)q_T\\ &= CQ(\mathbf{e},\epsilon,0,q_T)\\ &\le C(\|\mathbf{e}\|_{1,T}+\|\epsilon\|_{0,T})\|q_T\|_{0,T}\\ &\le C(\|\mathbf{e}\|_{1,T}+\|\epsilon\|_{0,T})\|\nabla\cdot\mathbf{u}_h\|_{0,T},\end{aligned}$$

from which we conclude the proof. □

Finally, we estimate the last term of η_T.

Lemma 5.3. *Let* $(\mathbf{u},p,\lambda)$ *and* $(\mathbf{u}_h,p_h,\lambda_h)$ *be as in Theorem 4.1. Let* $\mathbf{e}=\mathbf{u}-\mathbf{u}_h$ *and* $\epsilon=p-p_h$*. Then, there exists a positive constant C such that, if* $\ell\in\mathcal{E}_\Omega^h$*, then*

$$\mu|\ell|^{1/2}\left\|\left[\frac{\partial\mathbf{u}_h}{\partial n}\right]_\ell\right\|_{0,\ell}\le C\{\|\mathbf{e}\|_{1,\omega_\ell}+\|\epsilon\|_{0,\omega_\ell}+|\ell|\|\lambda\mathbf{u}-\lambda_h\mathbf{u}_h\|_{0,\omega_\ell}\}.$$

Proof. Let

$$J_\ell=\left[\frac{\partial\mathbf{u}_h}{\partial n}\right]_\ell\quad\text{and}\quad\mathbf{v}_\ell=b_\ell J_\ell.$$

As in (29) and (30) of [31], we have that

$$\begin{aligned}&\|\mathbf{v}_\ell\|_{0,\omega_\ell}^2\le C|\ell|\|b_\ell^{1/2}J_\ell\|_{0,\ell}^2,\\ &|\mathbf{v}_\ell|_{1,\omega_\ell}^2\le C\frac{1}{|\ell|}\|b_\ell^{1/2}J_\ell\|_{0,\ell}^2\end{aligned}\tag{28}$$

by Lemma 2.4 of [32]

$$\|J_\ell\|_{0,\ell}\le C\|b_\ell^{1/2}J_\ell\|_{0,\ell}.\tag{29}$$

Now, integrating by parts, using that $\Delta\mathbf{u}^h=0$ in any $T\in\mathcal{T}_h$, the continuity of Q, the inverse estimates (28) and (29), and Lemma 5.1, we can infer that

$$\begin{aligned}\mu\|b_\ell^{1/2}J_\ell\|_{0,\ell}^2 &= \mu\int_\ell\left[\frac{\partial\mathbf{u}_h}{\partial n}\right]_\ell\cdot\mathbf{v}_\ell=-\mu\int_{\omega_\ell}\nabla\mathbf{u}_h:\nabla\mathbf{v}_\ell\\ &= Q(\mathbf{e},\epsilon,\mathbf{v}_\ell,0)-\int_{\omega_\ell}(\lambda\mathbf{u}-\lambda_h\mathbf{u}_h)\cdot\mathbf{v}_\ell-\int_{\omega_\ell}(\lambda_h\mathbf{u}_h-\nabla p_h)\cdot\mathbf{v}_\ell\\ &\le C\left(\|\mathbf{e}\|_{1,\omega_\ell}+\|\epsilon\|_{0,\omega_\ell}\right)\|\mathbf{v}_\ell\|_{1,\omega_\ell}+\|\lambda\mathbf{u}-\lambda_h\mathbf{u}_h\|_{0,\omega_\ell}\|\mathbf{v}_\ell\|_{0,\omega_\ell}+\|\lambda_h\mathbf{u}_h-\nabla p_h\|_{0,\omega_\ell}\|\mathbf{v}_\ell\|_{0,\omega_\ell}\\ &\le C\Big\{(\|\mathbf{e}\|_{1,\omega_\ell}+\|\epsilon\|_{0,\omega_\ell})|\ell|^{-1/2}+\|\lambda\mathbf{u}-\lambda_h\mathbf{u}_h\|_{0,\omega_\ell}|\ell|^{1/2}\Big\}\|b_\ell^{1/2}J_\ell\|_{0,\ell}.\end{aligned}$$

Then, from this estimates and (29) we obtain

$$\mu|\ell|^{1/2}\left\|\left[\frac{\partial\mathbf{u}_h}{\partial n}\right]_\ell\right\|_{0,l}\le C\{\|\mathbf{e}\|_{1,\omega_\ell}+\|\epsilon\|_{0,\omega_\ell}+|\ell|\|\lambda\mathbf{u}-\lambda_h\mathbf{u}_h\|_{0,\omega_\ell}\}.\quad\square$$

Now we may conclude the efficiency of the error indicator up to higher order terms.

Theorem 5.2. *Let* $(\mathbf{u},p,\lambda)$ *and* $(\mathbf{u}_h,p_h,\lambda_h)$ *be as in Theorem 4.1. Let* $\mathbf{e}=\mathbf{u}-\mathbf{u}_h$, $\epsilon=p-p_h$ *and* η_T *as in (24). Then, there exists a positive constant C such that for all* $T\in\mathcal{T}_h$

$$\eta_T\le C\{\|\mathbf{e}\|_{1,\omega_T}+\|\epsilon\|_{0,\omega_T}+h_T\|\lambda\mathbf{u}-\lambda_h\mathbf{u}_h\|_{0,\omega_T}\}.$$

Proof. It is an immediate consequence of Lemmas 5.1–5.3. □

6. Numerical examples

In this section we present some numerical tests which allow us to assess the performance of the adaptive refinement strategy based on the error indicator defined in (24). Since the exact solution is unknown, we present some indices as in [9].

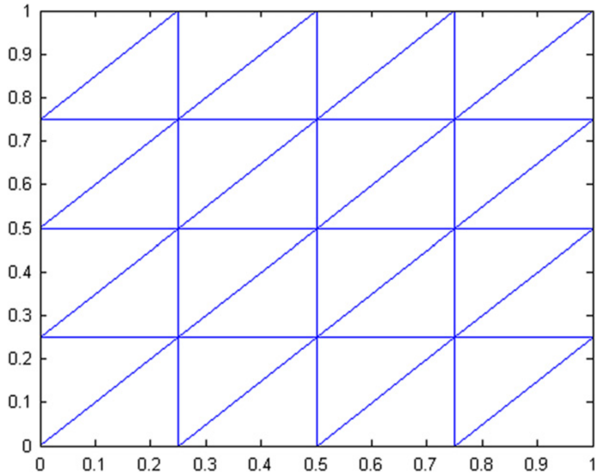

Fig. 1. Initial mesh in the square domain.

Table 1
Indices for the first eigenvalue for $\mu = 1$ in the square domain with adaptive refinement.

Number of nodes	η	$\lambda_{h,1}$	$\lvert\lambda-\lambda_{h,1}\rvert/\lambda$
25	1.1606	70.5906	0.3485
47	0.6058	63.9388	0.2214
57	0.5658	59.8504	0.1433
75	0.4738	57.9427	0.1069
81	0.4312	57.1718	0.0922
159	0.2619	55.0325	0.0513
169	0.2468	54.8278	0.0474
229	0.1781	53.9491	0.0306
251	0.1649	53.7902	0.0276
307	0.1444	53.5926	0.0238
499	0.0917	53.0972	0.0143
589	0.0723	52.9283	0.0111
625	0.0690	52.8929	0.0104
751	0.0613	52.7828	0.0083
1113	0.0455	52.6551	0.0059

We consider the problem (1) in two different domains: the square domain $\Omega = (0, 1) \times (0, 1)$ and the L-shaped domain $\Omega = (-1, 1) \times (-1, 1) \setminus [-1, 0] \times [-1, 0]$. In adaptive refinement we use, in all the examples, the maximum strategy to mark the triangles to be refined, i.e., all the triangles T with $\eta_T \geq \theta\eta_{\mathrm{M}}$ are marked to be refined, where

$$\eta_{\mathrm{M}} := \max\{\eta_K \mid K \in \mathcal{T}_h\},$$

and $\theta \in (0, 1)$ is a parameter. We take, in all tests, $\theta = 0.7$.

In all the numerical examples we only present the approximation of the first eigenvalue, in the case of the square domain just for simplicity and in the other cases because it is well known that when the domain is not convex the first eigenfunction is always singular.

We denote by N the number of degrees of freedom.

6.1. Test 1: square domain

In this case we take $\mu = 1$, and we consider the reference value $\lambda = 52.3447$ as in [17]. The initial mesh is shown in Figs. 1 and 2 shows different meshes obtained with the adaptive algorithm.

In order to show the stability and efficiency of the method for the considered problem, we also present in Fig. 3 the velocity streamlines and the pressure level lines, obtained with the most refined mesh. We can observe that the density of the refinement corresponds with the solution behavior.

Tables 1 and 2 show the results for the first eigenvalue using adaptive and uniform refinement, respectively. We observe similar accuracy with adaptive refinement and uniform refinement.

Figs. 4 and 5 show plots of $\log(\eta)$ and $\log|\lambda - \lambda_{h,1}|$ versus $\log(N^{1/2})$, where a linear dependence can be clearly observed for sufficiently large values of N.

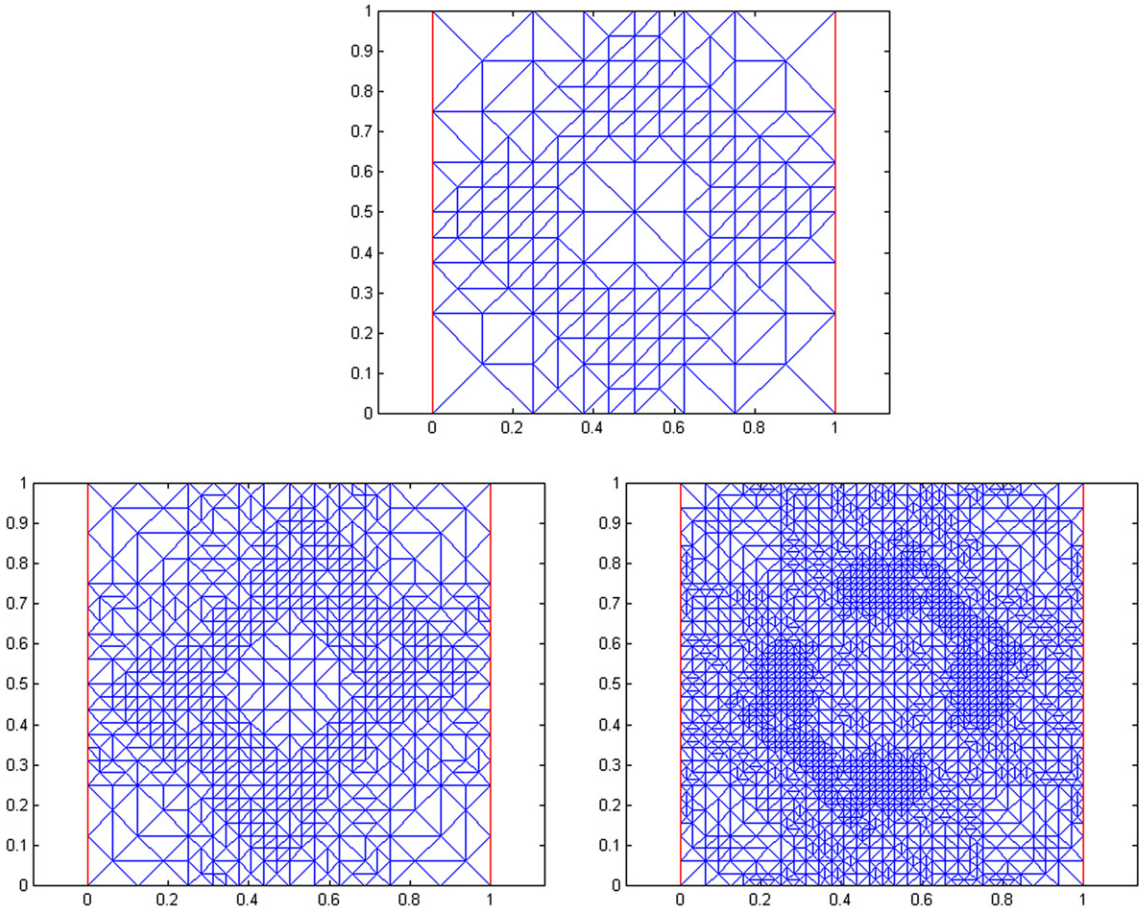

Fig. 2. Meshes after 6, 11 and 16 refinements for $\mu = 1$ in the square domain.

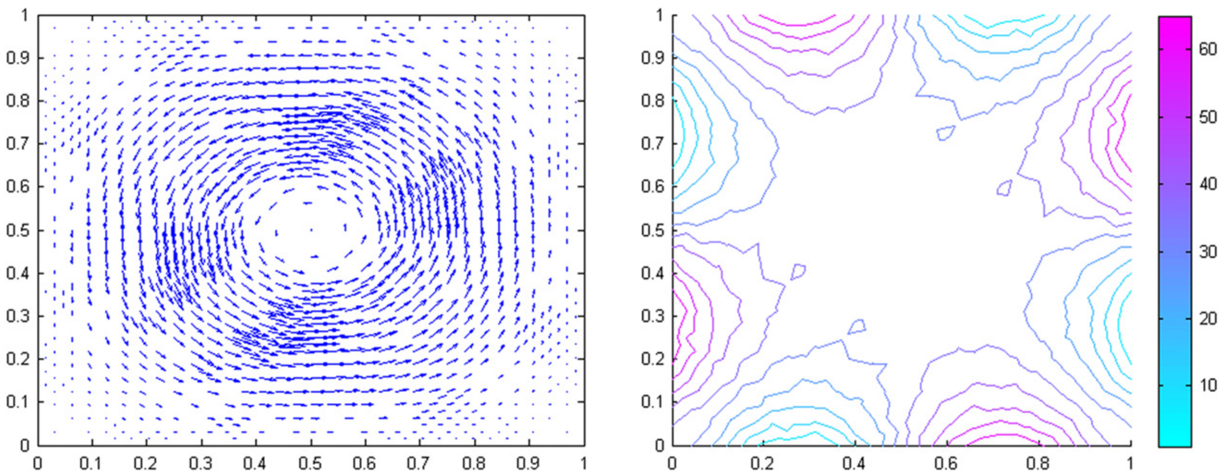

Fig. 3. Velocity streamlines and pressure contours.

6.2. Test 2: L-shaped domain

In this second test we consider an L-shaped domain so, as it is well known, we are now dealing with a singular solution. The initial mesh is shown in Fig. 6.

First of all, we take $\mu = 1$ and the corresponding reference value $\lambda = 32.2$ which has been obtained by extrapolation.

The behavior of the adaptive algorithm can be appreciated in Fig. 7 in which we observe a more dominant refinement closer to the singularity.

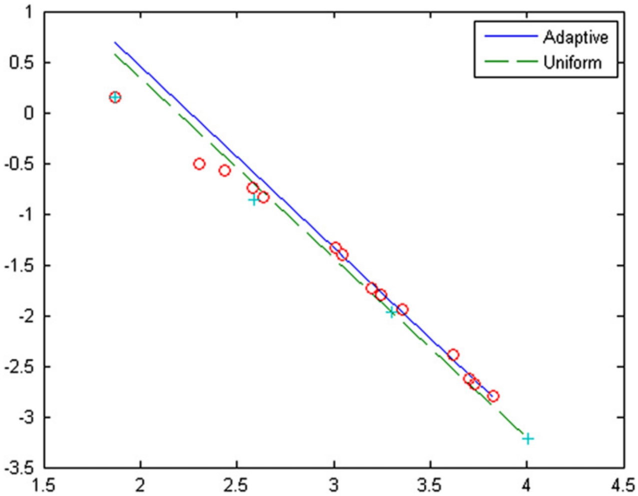

Fig. 4. Error curves for η in the square domain for $\mu = 1$.

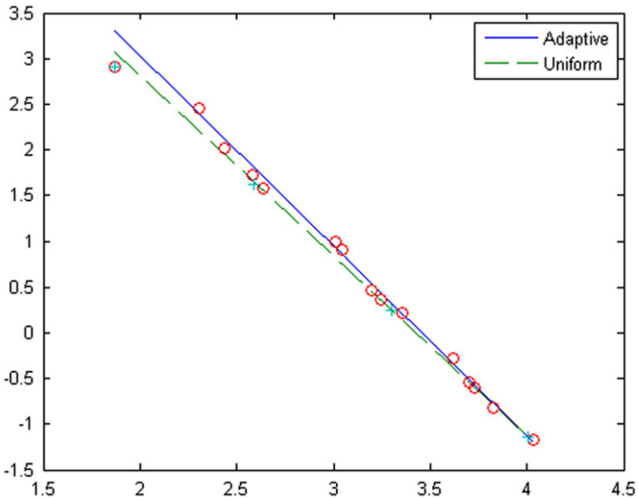

Fig. 5. Error curves for $\lambda_{h,1}$ in the square domain for $\mu = 1$.

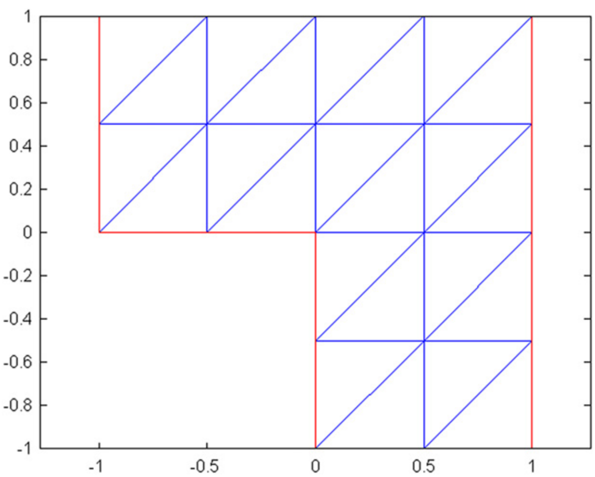

Fig. 6. Initial mesh in the L-shaped domain.

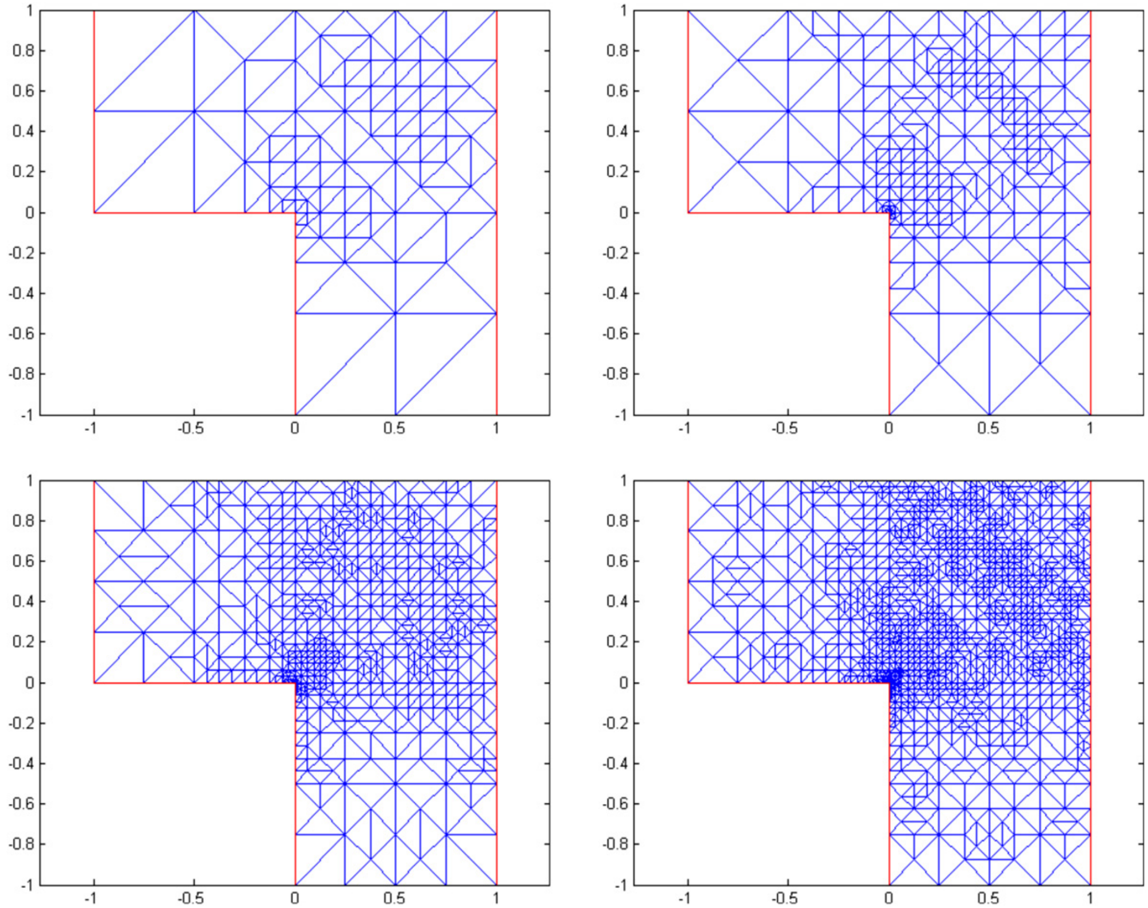

Fig. 7. Meshes after 5, 10, 15 and 20 refinements for $\mu = 1$ in the L-shaped domain with adaptive refinement.

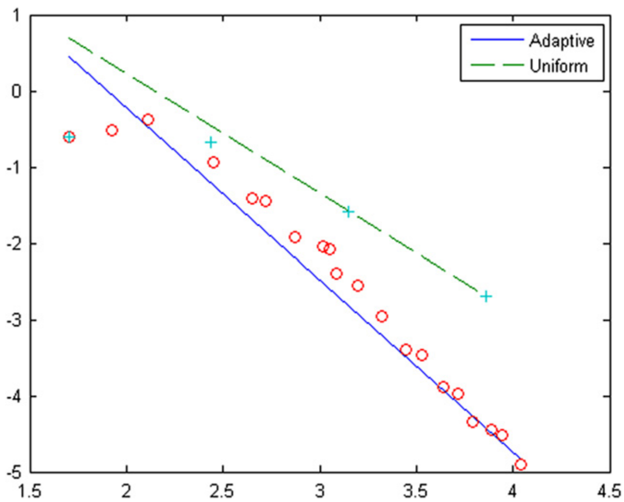

Fig. 8. Error curves for η in the L domain for $\mu = 1$.

Tables 3 and 4 show the results for the first eigenvalue using adaptive and uniform refinement respectively. We observe that the adaptive refinement requires fewer number of nodes than the uniform refinement to obtain the same error estimator η and relative error $|\lambda - \lambda_{h,1}|/\lambda$.

Figs. 8 and 9 show plots of $\log(\eta)$ and $\log|\lambda - \lambda_{h,1}|$ versus $\log(N^{1/2})$. We also observe a linear dependence.

In order to show the behavior of our adaptive algorithm for small values of μ, we consider the problem (1) with $\mu = 0.1$ and the corresponding reference value $\lambda = 3.2$ which has been obtained by extrapolation. Fig. 10 shows different meshes

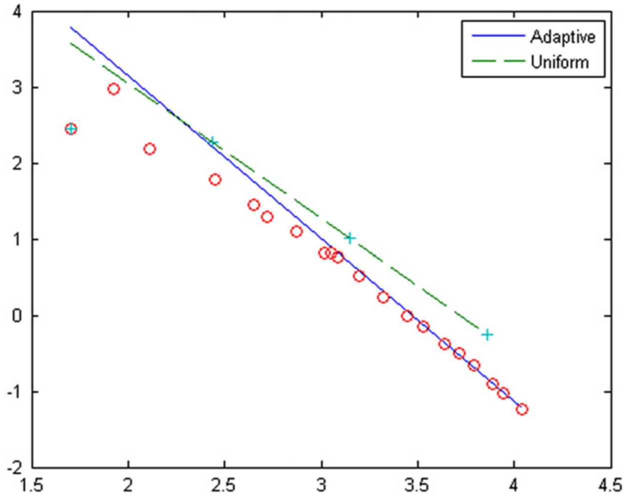

Fig. 9. Error curves for $\lambda_{h,1}$ in the L domain for $\mu = 1$.

Table 2
Indices for the first eigenvalue for $\mu = 1$ in the square domain with uniform refinement.

Number of nodes	η	$\lambda_{h,1}$	$\lvert\lambda - \lambda_{h,1}\rvert/\lambda$
25	1.1606	70.5906	0.3485
81	0.4224	57.3950	0.0964
289	0.1400	53.6201	0.0243
1089	0.0402	52.6637	0.0060

Table 3
Indices for the first eigenvalue for $\mu = 1$ in the L-shaped domain with adaptive refinement.

Number of nodes	η	$\lambda_{h,1}$	$\lvert\lambda - \lambda_{h,1}\rvert/\lambda$
21	0.5467	43.68	0.3578
28	0.5918	51.59	0.6035
35	0.6910	41.02	0.2750
60	0.3899	38.15	0.1857
85	0.2425	36.44	0.1326
96	0.2332	35.80	0.1129
127	0.1476	35.14	0.0923
164	0.1295	34.44	0.0706
174	0.1251	34.43	0.0702
187	0.0911	34.31	0.0665
229	0.0782	33.84	0.0518
291	0.0520	33.43	0.0392
369	0.0338	33.16	0.0306
430	0.0310	33.03	0.0267
533	0.0204	32.86	0.0214
613	0.0188	32.77	0.0187
716	0.0130	32.68	0.0160
852	0.0116	32.57	0.0126
950	0.0109	32.52	0.0110
1153	0.0074	32.46	0.0090

Table 4
Indices for the first eigenvalue for $\mu = 1$ in the L-shaped domain with uniform refinement.

Number of nodes	η	$\lambda_{h,1}$	$\lvert\lambda - \lambda_{h,1}\rvert/\lambda$
21	0.5467	43.68	0.3578
65	0.5061	41.81	0.2997
225	0.2039	34.89	0.0846
833	0.0671	32.95	0.0242

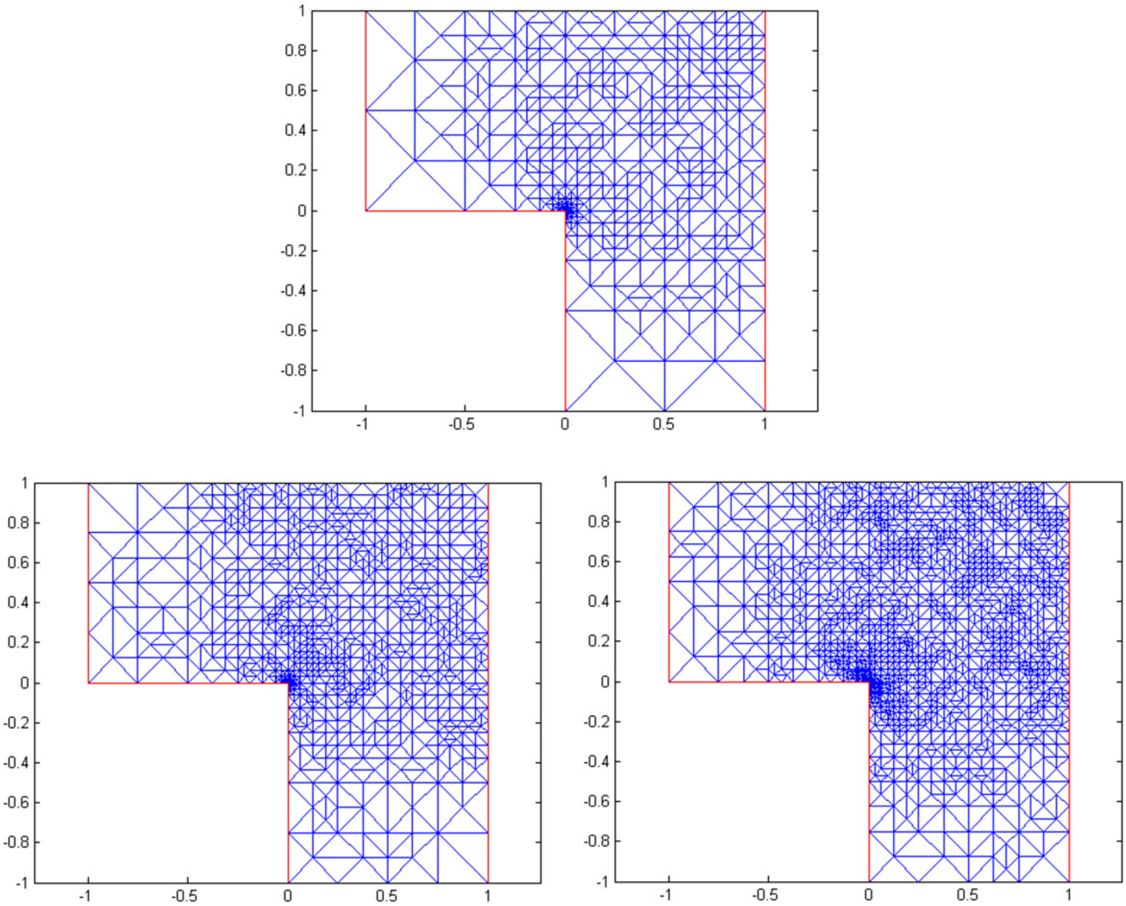

Fig. 10. Meshes after 6, 8 and 10 refinements for $\mu = 0.1$ in the L-shaped domain.

Table 5
Indices for the first eigenvalue for $\mu = 0.1$ in the L-shaped domain with adaptive refinement.

Number of nodes	η	$\lambda_{h,1}$	$\lvert\lambda-\lambda_{h,1}\rvert/\lambda$
21	0.8306	10.4243	2.2576
37	1.2518	6.8408	1.1377
54	1.4436	4.8451	0.5140
61	1.0599	4.5338	0.4168
75	0.8229	4.1831	0.3072
95	0.6493	3.7900	0.1844
143	0.4012	3.5707	0.1158
167	0.3549	3.4864	0.0895
204	0.2813	3.4440	0.0762
236	0.2518	3.4166	0.0676
293	0.1852	3.3731	0.0540
360	0.1533	3.3372	0.0428
441	0.1197	3.3135	0.0354
540	0.1047	3.2950	0.0297
629	0.0781	3.2833	0.0260
735	0.0690	3.2703	0.0219
970	0.0487	3.2566	0.0176

obtained with the adaptive algorithm in which we observe the typical adaptive behavior again: a more density refinement closer to the singularity.

Tables 5 and 6 present the indices of the numerical solutions for adaptive and uniform refinement. In this case, the adaptive refinement requires fewer number of nodes than the uniform refinement to obtain the same error indicator η and relative error $|\lambda - \lambda_{h,1}|/\lambda$ too.

Figs. 11 and 12 show plots of $\log(\eta)$ and $\log|\lambda - \lambda_{h,1}|$ versus $\log(N^{1/2})$ where a linear dependence can be clearly observed for sufficiently large values of N.

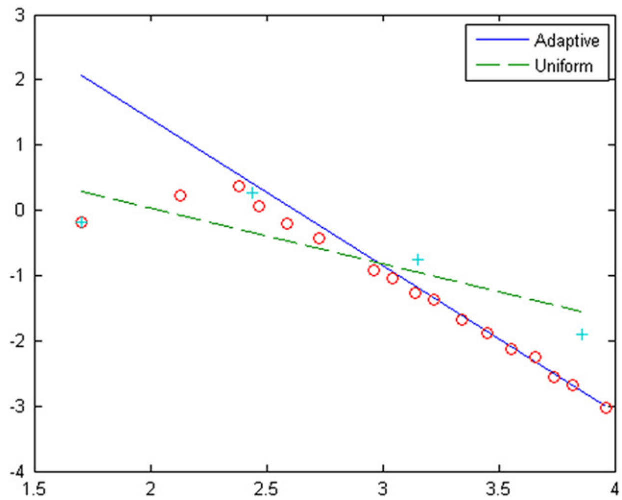

Fig. 11. Error curves for η in the L domain for $\mu = 0.1$.

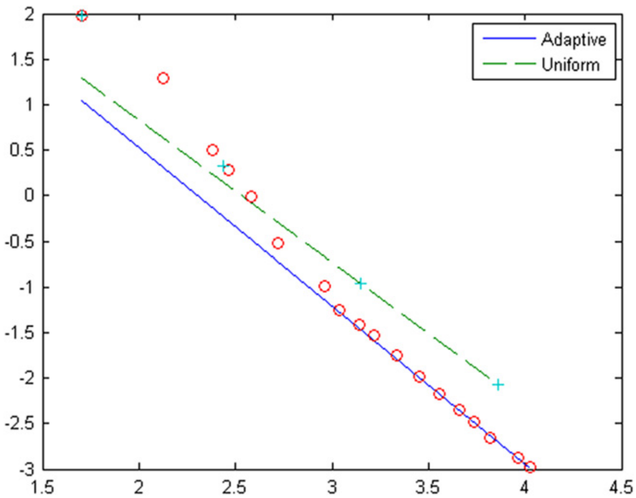

Fig. 12. Error curves for $\lambda_{h,1}$ in the L domain for $\mu = 0.1$.

Table 6
Indices for the first eigenvalue for $\mu = 0.1$ in the L-shaped domain with uniform refinement.

Number of nodes	η	$\lambda_{h,1}$	$\lvert\lambda-\lambda_{h,1}\rvert/\lambda$
21	0.8306	10.4243	2.2576
65	1.2969	4.5833	0.4323
225	0.4715	3.5809	0.1190
833	0.1490	3.3265	0.0395

Acknowledgments

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