Applied Acoustics 74 (2013) 325–334

Contents lists available at SciVerse ScienceDirect

Applied Acoustics

journal homepage: www.elsevier.com/locate/apacoust

Technical Note

Variational approach to vibrations of frames with inclined members

Ricardo Oscar Grossi *, Carlos Marcelo Albarracín

Facultad de Ingeniería, Universidad Nacional de Salta, Avenida Bolivia 5150, 4400 Salta, Argentina

ARTICLE INFO

Article history:
Received 23 March 2012
Received in revised form 29 June 2012
Accepted 17 July 2012

Keywords:
Vibrations
Frames
Inclined members
Elastic restrains
Functionals
Hamilton's principle

ABSTRACT

This paper deals with the use of calculus of variations to derive the boundary value problems which describe the dynamical behaviour of two and three-bar frames with inclined members, and with ends and intermediate points elastically restrained. The determination of exact eigenfrequencies and modes in the case of free vibrations is included. A rigorous and complete development is presented. First, a brief description of textbooks and papers previously published is included. Second the variational formulation of the problem is presented. Third, the Hamilton principle is rigorously stated and the corresponding boundary value problem is obtained. Finally, the method of separation of variables is used for the determination of the exact frequencies and mode shapes. In order to obtain an indication of the accuracy of the developed mathematical model, some cases available in the literature have been considered and a special code implementing the finite element method have been developed and used. New results are presented for different geometrical configurations and mechanical parameters.

1. Introduction

There are two basic approaches to obtaining the boundary problems that describe the dynamical behaviour of mechanical systems. One is based on Newton's law of motion; the other is to use Hamilton's principle to determine the trajectory of the system which minimises the action integral. The first approach is more intuitive but it is not adequate for systems where there are many degrees of freedoms, interacting members and deformable solids.

In structural dynamics, the variational approach is extremely important and Hamilton's principle provides a straightforward method for determining the boundary value problems that describe the dynamical behaviour of mechanical systems. The use of the mentioned principle and the techniques of the calculus of variations constitute an excellent procedure in the case of the determination of the equations of motion, the boundary conditions and the intermediate conditions of systems with many degrees of freedom and many interacting members. Thus, this approach is particularly useful for frame structures with inclined members, whose ends and intermediate points, are elastically restrained against rotation and translation.

Substantial literature has been devoted to the theory and applications of the calculus of variations. For instance, the textbooks [1–8] present the theoretical aspects of the mentioned discipline and some of these works include applications in the statics and dynamics of structural elements such as beams and plates.

On the other hand several textbooks and monographs deal with vibrating frames [9–15]. Also, there has been extensive research into the vibration of frame structures and many different configurations and complexities, have been treated. The majority of this work has been in the area of closed frames [16–25].

A number of papers have been published on vibrating planar frame structures with elastically restrained ends. Filipich and Laura [26,27] analysed the in-plane vibrations of portal frames with elastically restrained ends. Laura, Valerga and Filipich [28] dealt with the determination of the fundamental frequency of a frame elastically restrained at the ends, carrying concentrated masses. Albarracín and Grossi [29] dealt with the exact determination of eigenfrequencies of a frame which consists of a beam supported by a column, with intermediate elastic constraints and ends elastically restrained. Grossi and Albarracín [30] used the calculus of variations to derive the boundary problems that describe the dynamical behaviour of portal frames, with ends and intermediate points elastically restrained and determined the exact frequencies and mode shapes.

There is only a limited amount of information on the vibration of elastically restrained frames with inclined members. The book by Chin Hao Chang [15] includes the mechanics of elastic structures with inclined members. A model established for the vibration of inclined bars is applied to frames with inclined members. The cases of all joints hinged, two ends fixed and the central joint hinged, and all joints rigidly connected, are investigated. Nevertheless, the case of ends and intermediate points elastically restrained against rotation and translation has not been treated.

The aim of the present paper is to investigate the natural frequencies and mode shapes of two and three-bar frames with

* Corresponding author. Fax: +54 0387 4255351.
E-mail address: grossiro@unsa.edu.ar (R.O. Grossi).

0003-682X/$ - see front matter © 2012 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.apacoust.2012.07.014

Nomenclature

A_i	cross-sectional area of the ith beam
$D(F)$	space of admissible functions of functional F
$D_a(F)$	space of admissible directions
E_iI_i	flexural rigidity of the ith beam
F	Energy functional
l_i	length of the ith beam
$r_j^{(i)}$	spring constant of the jth rotational restraint
$\mathbb{R}$	set of real numbers
$t_j^{(i)}$	spring constant of the jth translational restraint
t	time
T_S	kinetic energy of the mechanical system
U_S	strain energy of the mechanical system
u_i	axial displacement along the ith beam
$\tilde{u}_i$	admissible direction at u_i
w_i	lateral deflection along the ith beam
$\tilde{w}_i$	admissible direction at w_i
x_i	coordinate along the ith beam axis
$X^{(G)}$, $Y^{(G)}$	global coordinates
$\mathbf{u}$	vector (u_1, u_2, u_3)
$\tilde{\mathbf{u}}$	vector $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$
$\mathbf{v}$	vector $(\mathbf{u}, \mathbf{w})$
$\tilde{\mathbf{v}}$	vector $(\tilde{\mathbf{u}}, \tilde{\mathbf{w}})$
$\mathbf{w}$	vector (w_1, w_2, w_3)
$\tilde{\mathbf{w}}$	vector $(\tilde{w}_1, \tilde{w}_2, \tilde{w}_3)$
α_i	angles of inclination
$\delta F(\mathbf{w}; \tilde{\mathbf{w}})$	variation of functional F
λ	non-dimensional frequency parameter $\sqrt[4]{(\rho_0 A_0 / E_0 I_0)\omega_i^2} l_0$
ω	circular frequency of the structure (rad/unit time)
Ω_i	$[0, l_i] \times [t_a, t_b]$
ρ_i	material density of the ith beam
U	$C^2(\Omega_1) \times C^2(\Omega_2) \times C^2(\Omega_3)$
W	$C^4(\Omega_1) \times C^4(\Omega_2) \times C^4(\Omega_3)$
$U \times W$	linear product space

inclined members with ends and intermediate points elastically restrained against rotation and translation, intending the developments within each section to be rigorous and complete. A condensed notation has been implemented by introducing several differential operators. The use of the proposed condensed notation avoids complicated and obscure formulae and allows including all the analytical details.

The vague "operator" δ which leads to the extensively used mechanical "δ method", constitutes a sometimes convenient shorthand tactic, but its lack of rigour can be a source of confusion and can arise as a disadvantage. Consequently, one of the motivations of the present paper is to present a rigorous procedure, by introducing the functional adequate to the formulation of the Hamilton's principle and the corresponding spaces of admissible functions and admissible directions.

Hamilton's principle requires that between times t_a and t_b, at which the positions of the mechanical system are known, it should execute a motion which makes stationary the functional given by the Hamilton's action integral

$$F(w) = \int_{t_a}^{t_b} (T_S(w) - U_S(w))\mathrm{d}t, \tag{1}$$

where $L = T_S - U_S$ is the well known Lagrangian for the motion. It must be noted that the frame is a structural system which cannot be studied using as admissible functions neither $w(\cdot, t) \in C^4[0, l]$ (used in beams) nor $w(\cdot, \cdot, t) \in C^4(\Omega)$, $\Omega \subseteq \mathbb{R}^2$ (used in plates). So, it is necessary to clearly determine the space of admissible functions D of the functional defined by (1) and the space D_a of admissible directions. The procedure adopted is particularly important in the determination of the analytical expressions of the corresponding natural boundary conditions and the intermediate conditions.

In the present paper the method of separation of variables is used for the determination of the exact frequencies and mode shapes. Tables are given for frequencies and two-dimensional plots are given for mode shapes in some selected cases. In order to obtain an indication of the accuracy of the developed mathematical model, some particular cases available in the literature has been considered, and a particular code with the application of the finite element method has been developed.

The present paper is organised in the following way. First the brief story stated above. In Section 2, the variational formulation of the problem is described. In Section 3 the Hamilton principle is rigorously stated by introducing adequate vectorial functions and a particular product space. Also the corresponding boundary value problem is obtained. In Section 4 the implementation of the method of separation of variables is described for the determination of the results of exact frequencies and mode shapes. The paper is concluded in Section 5 with some discussions.

2. Variational formulation of the problem

The frame considered is composed by three inclined beams of lengths l_1, l_2 and l_3 respectively as it is shown in Fig. 1, where the coordinate systems has been adopted to help the analytical developments. The angles of inclination are given by α_i, $i = 1, 2, 3$. The behaviours of the individual members of the frame are assumed to be governed by Euler's beam theory with the axial deformations effects included. In practice, the frame corners and ends may experience partial resistance to rotation and translation, and this situation can be modelled by considering ends and joints elastically restrained against rotation and translation. For this reason, the following elastic restraints have been included:

(1) Four rotational restraints characterised by the spring constants r_i, $i = 1, \ldots, 4$.
(2) Eight translational restraints characterised by the spring constants t_{ix}, t_{iy}, $i = 1, 2, 3, 4$.

The longitudinal and transverse displacements of the beams are related to their local axes at any time t and described respectively by the functions

$$u_i(x_i, t), w_i(x_i, t); \quad x_i \in [0, l_i], \quad i = 1, 2, 3. \tag{2}$$

In the joints of connection, there exist constrained conditions which lead to the following compatibility equations (see Fig. 1):

$$\cos\alpha_1 u_1(l_1, t) - \sin\alpha_1 w_1(l_1, t) = \cos\alpha_2 u_2(0, t) - \sin\alpha_2 w_2(0, t), \tag{3}$$

$$\sin\alpha_1 u_1(l_1, t) + \cos\alpha_1 w_1(l_1, t) = \sin\alpha_2 u_2(0, t) + \cos\alpha_2 w_2(0, t), \tag{4}$$

$$\frac{\partial w_1}{\partial x_1}(l_1, t) = \frac{\partial w_2}{\partial x_2}(0, t), \tag{5}$$

$$\cos\alpha_2 u_2(l_2, t) - \sin\alpha_2 w_2(l_2, t) = \cos\alpha_3 u_3(0, t) - \sin\alpha_3 w_3(0, t), \tag{6}$$

$$\sin\alpha_2 u_2(l_2, t) + \cos\alpha_2 w_2(l_2, t) = \sin\alpha_3 u_3(0, t) + \cos\alpha_3 w_3(0, t), \tag{7}$$

$$\frac{\partial w_2}{\partial x_2}(l_2, t) = \frac{\partial w_3}{\partial x_3}(0, t). \tag{8}$$

At time t, the kinetic energy of the mechanical system under study is given by

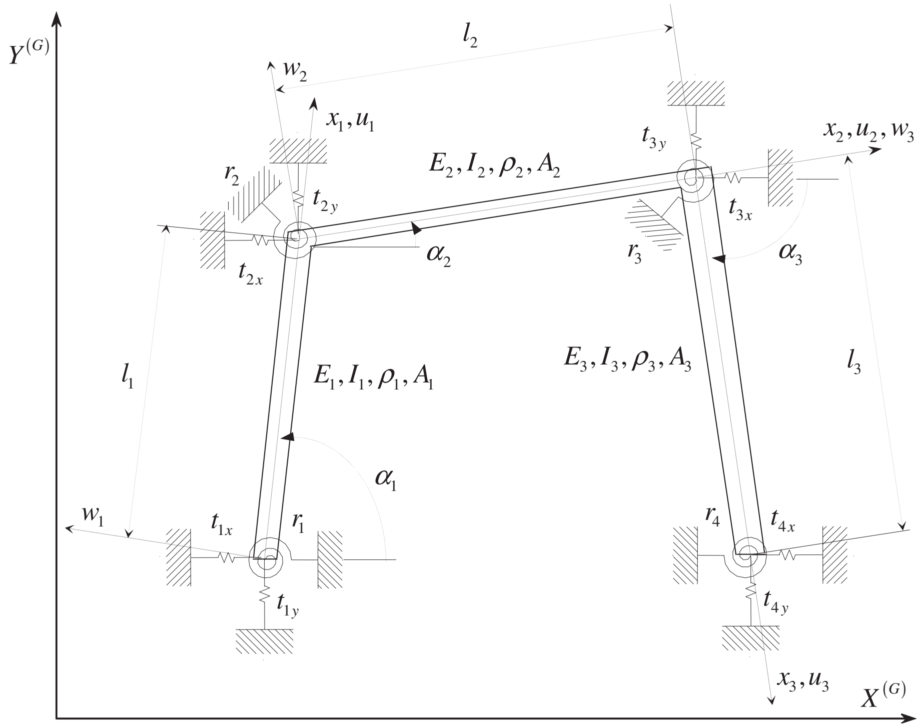

Fig. 1. Frame with inclined members and with ends and intermediate points elastically restrained.

$$T_S = \frac{1}{2}\sum_{i=1}^{3}\int_0^{l_i}\rho_i A_i\left(\left(\frac{\partial u_i}{\partial t}(x_i,t)\right)^2 + \left(\frac{\partial w_i}{\partial t}(x_i,t)\right)^2\right)\mathrm{d}x_i, \tag{9}$$

where $\rho_i A_i$ denotes the mass per unit length of the ith member of the frame. The total potential energy due to the elastic deformation of the beams, the springs at the ends restraints and the springs at the intermediate restraints is given by:

$$\begin{aligned} U_S = {} & \frac{1}{2}\sum_{i=1}^{3}\int_0^{l_i}\left(E_i A_i\left(\frac{\partial u_i}{\partial x_i}(x_i,t)\right)^2 + E_i I_i\left(\frac{\partial^2 w_i}{\partial x_i^2}(x_i,t)\right)^2\right)\mathrm{d}x_i \\ & + \frac{1}{2}\sum_{i=1}^{3}\left[k_{11}^{(i)}u_i^2(0,t) + k_{22}^{(i)}w_i^2(0,t) + k_{33}^{(i)}\left(\frac{\partial w_i}{\partial x_i}(0,t)\right)^2\right. \\ & \left. + 2k_{12}^{(i)}u_i(0,t)w_i(0,t)\right] \\ & + \frac{1}{2}\left[k_{11}^{(4)}u_3^2(l_3,t) + k_{22}^{(4)}w_3^2(l_3,t) + k_{33}^{(4)}\left(\frac{\partial w_3}{\partial x_3}(l_3,t)\right)^2\right. \\ & \left. + 2k_{12}^{(4)}u_3(l_3,t)w_3(l_3,t)\right], \end{aligned} \tag{10}$$

where $E_i I_i$ denotes the flexural rigidity of the ith member of the frame and the rigidities $k_{ij}^{(l)}$ generated by the elastic restraints are defined in Appendix A. So, the energy functional to be considered is given by

$$\begin{aligned} F = {} & \frac{1}{2}\sum_{i=1}^{3}\int_{t_a}^{t_b}\int_0^{l_i}\rho_i A_i\left[\left(\frac{\partial u_i}{\partial t}(x_i,t)\right)^2 + \left(\frac{\partial w_i}{\partial t}(x_i,t)\right)^2\right]\mathrm{d}x_i\mathrm{d}t - \frac{1}{2}\sum_{i=1}^{3}\int_{t_a}^{t_b} \\ & \times\int_0^{l_i}\left[E_i A_i\left(\frac{\partial u_i}{\partial x_i}(x_i,t)\right)^2 + E_i I_i\left(\frac{\partial^2 w_i}{\partial x_i^2}(x_i,t)\right)^2\right]\mathrm{d}x_i\mathrm{d}t \\ & - \frac{1}{2}\sum_{i=1}^{3}\int_{t_a}^{t_b}\left[k_{11}^{(i)}u_i^2(0,t) + k_{22}^{(i)}w_i^2(0,t) + k_{33}^{(i)}\left(\frac{\partial w_i}{\partial x_i}(0,t)\right)^2 + 2k_{12}^{(i)}u_i(0,t)w_i(0,t)\right]\mathrm{d}t \\ & - \frac{1}{2} \\ & \times\int_{t_a}^{t_b}\left[k_{11}^{(4)}u_3^2(l_3,t) + k_{22}^{(4)}w_3^2(l_3,t) + k_{33}^{(4)}\left(\frac{\partial w_3}{\partial x_3}(l_3,t)\right)^2 + 2k_{12}^{(4)}u_3(l_3,t)w_3(l_3,t)\right]\mathrm{d}t. \end{aligned} \tag{11}$$

3. Product spaces and the concept of variation of the energy functional

A simple examination of (11) reveals that it defines a functional F which depends on the six real functions

$$u_i, \quad w_i, \quad i = 1, 2, 3,$$

so, it is convenient to introduce the following vectorial functions:

$$\mathbf{v} = (\mathbf{u}, \mathbf{w}), \quad \mathbf{u} = (u_1, u_2, u_3), \quad \mathbf{w} = (w_1, w_2, w_3). \tag{12}$$

Now we recall the definition of product vector spaces, see for instance [31,32].

Definition. Let V_1, V_2, …, V_n, be linear spaces. The product space

$$V = V_1 \times V_2 \times \cdots \times V_n$$

consists of all the n-tuples $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ where

$$u_i \in V_i \quad \text{for} \quad i = 1, 2, \ldots, n.$$

The algebraic operations:

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n), \quad \forall \mathbf{u}, \mathbf{v} \in V,$$
$$a\mathbf{u} = (au_1, au_2, \ldots, au_n), \quad \forall a \in \mathbb{R}, \quad \forall \mathbf{u} \in V,$$

transforms the space V in a linear space.

It is worth pointing out that the notion of product space allows to clearly specify the degree of smoothness of the functions involved in (11). For this purpose, we will make the following assumptions:

$$u_i(x_i,t) \in C^2(\Omega_i) \quad \text{and} \quad w_i(x_i,t) \in C^4(\Omega_i) \quad \text{where} \quad \Omega_i = [0, l_i] \times [t_a, t_b], i = 1, 2, 3.$$

Now if we introduce the spaces

$$U = C^2(\Omega_1) \times C^2(\Omega_2) \times C^2(\Omega_3) \tag{13}$$

and

$$W = C^4(\Omega_1)\times C^4(\Omega_2)\times C^4(\Omega_3), \tag{14}$$

it is immediately that $\mathbf{u}\in U$, $\mathbf{w}\in W$ and $\mathbf{v}\in U\times W$. Consequently, the functional defined by (11) depends on $\mathbf{v}$ and it can be written $F = F(\mathbf{v})$. It is quite easy to check that the product space $U\times W$ can be transformed in a linear space with the following operations:

$$(\mathbf{u}^{(1)},\mathbf{w}^{(1)}) + (\mathbf{u}^{(2)},\mathbf{w}^{(2)}) = (\mathbf{u}^{(1)}+\mathbf{u}^{(2)},\mathbf{w}^{(1)}+\mathbf{w}^{(2)}),\quad c(\mathbf{u},\mathbf{w}) = (c\mathbf{u},c\mathbf{w}),$$

where $\mathbf{u}^{(1)},\mathbf{u}^{(2)}\in U$, $\mathbf{w}^{(1)},\mathbf{w}^{(2)}\in W$ and $c\in\mathbb{R}$. Now it is possible to clearly specify the domain of admissible functions which is given by

$$D(F) = \{\mathbf{v};\mathbf{v}\in U\times W,\ \text{fulfils the compatibility conditions (3)–(8) and has prescribed values at } t=t_a \text{ and } t=t_b\}. \tag{15}$$

Since the domain of the functional $F = F(\mathbf{v})$ has been determined, it is possible to define the variation of the functional by

$$\delta F(\mathbf{v};\tilde{\mathbf{v}}) = \frac{\mathrm{d}F}{\mathrm{d}\varepsilon}(\mathbf{v}+\varepsilon\tilde{\mathbf{v}})\bigg|_{\varepsilon=0}, \tag{16}$$

where $\delta F(\mathbf{v};\tilde{\mathbf{v}})$ denotes the variation of functional F in the point $\mathbf{v}\in D(F)$ and in the direction $\tilde{\mathbf{v}}\in D_a(F)$. The admissible directions $\tilde{\mathbf{v}}$ at $\mathbf{v}\in D(F)$ are those for which $\mathbf{v}+\varepsilon\tilde{\mathbf{v}}\in D(F)$, $\forall$ sufficiently small ε and exists $\delta F(\mathbf{v};\tilde{\mathbf{v}})$. In consequence, in view of (15), $\tilde{\mathbf{v}}$ is an admissible direction at $\mathbf{v}$ for $D(F)$ if and only if $\tilde{\mathbf{v}}\in D_a(F)$ where

$$D_a(F) = \{\tilde{\mathbf{v}};\tilde{\mathbf{v}}\in U\times W,\ \text{fulfils the compatibility conditions (3)–(8) and } \tilde{\mathbf{v}}(x_i,t_a)=\tilde{\mathbf{v}}(x_i,t_b)=0,\ \forall x_i\in[0,l_i],\ i=1,2,3\}. \tag{17}$$

The application of (16) to the functional defined by (11) leads to a rather lengthy algebraic procedure which can be condensed by introducing the following differential operators:

$$P_i(x,t) = b_i\frac{\partial^2 w_j}{\partial x_j^2}(x,t) + c_i\frac{\partial^2 w_{j-1}}{\partial x_{j-1}^2}(x+l_{j-1},t) - k_{33}^{(i)}\frac{\partial w_j}{\partial x_j}(x,t), \tag{18}$$

$$Q_i(x,t) = d_i\frac{\partial u_j}{\partial x_j}(x,t) + e_i\frac{\partial^3 w_{j-1}}{\partial x_{j-1}^3}(x+l_{j-1},t) + f_i\frac{\partial u_{j-1}}{\partial x_{j-1}}(x+l_{j-1},t) - k_{11}^{(i)}u_j(x,t) - k_{12}^{(i)}w_j(x,t), \tag{19}$$

$$R_i(x,t) = -b_i\frac{\partial^3 w_j}{\partial x_j^3}(x,t) - k_{22}^{(i)}w_j(x,t) - k_{12}^{(i)}u_j(x,t) + g_i\frac{\partial^3 w_{j-1}}{\partial x_{j-1}^3}(x+l_{j-1},t) + h_i\frac{\partial u_{j-1}}{\partial x_{j-1}}(x+l_{j-1},t), \tag{20}$$

where $j = i$ if $i = 1, 2, 3$ and $j = i-1$ if $i = 4$. Thus we have (see Appendix A for more details):

$$\begin{aligned}\delta F(\mathbf{v};\tilde{\mathbf{v}}) = &-\sum_{i=1}^{3}\int_{t_a}^{t_b}\int_0^{l_i}\left[\left(\rho_iA_i\frac{\partial^2 u_i}{\partial t^2}-E_iA_i\frac{\partial^2 u_i}{\partial x_i^2}\right)\tilde{u}_i+\left(\rho_iA_i\frac{\partial^2 w_i}{\partial t^2}+E_iI_i\frac{\partial^4 w_i}{\partial x_i^4}\right)\tilde{w}_i\right]\mathrm{d}x_i\mathrm{d}t\\ &+\sum_{i=1}^{4}\int_{t_a}^{t_b}\frac{\partial\tilde{w}_i}{\partial x_i}(a_i,t)P_i(a_i,t)\mathrm{d}t+\sum_{i=1}^{4}\int_{t_a}^{t_b}\tilde{u}_i(a_i,t)Q_i(a_i,t)\mathrm{d}t\\ &+\sum_{i=1}^{4}\int_{t_a}^{t_b}\tilde{w}_i(a_i,t)R_i(a_i,t)\mathrm{d}t,\end{aligned} \tag{21}$$

where $a_i = 0,\ i = 1,2,3,\ a_4 = l_3,\ \tilde{w}_4 = \tilde{w}_3,\ \tilde{u}_4 = \tilde{u}_3$.

Now the stationary condition which corresponds to Hamilton's principle takes the following rigorous form:

$$\delta F(\mathbf{v};\tilde{\mathbf{v}}) = 0,\quad \forall\tilde{\mathbf{v}}\in D_a(F). \tag{22}$$

Restricting attention to those $\tilde{\mathbf{v}}\in D_a(F)$ which verify the conditions

$$\tilde{w}_i(a_i,t) = \tilde{u}_i(a_i,t) = \frac{\partial\tilde{w}_i}{\partial x_i}(a_i,t) = 0,\quad \forall t\in(t_a,t_b),\quad i = 1,\ldots,4, \tag{23}$$

we have from (21) and (22) that

$$\delta F(\mathbf{v};\tilde{\mathbf{v}}) = \sum_{i=1}^{3}\int_{t_a}^{t_b}\int_0^{l_i}\left[\left(\rho_iA_i\frac{\partial^2 u_i}{\partial t^2}-E_iA_i\frac{\partial^2 u_i}{\partial x_i^2}\right)\tilde{u}_i+\left(\rho_iA_i\frac{\partial^2 w_i}{\partial t^2}+E_iI_i\frac{\partial^4 w_i}{\partial x_i^4}\right)\tilde{w}_i\right]\mathrm{d}x_i\mathrm{d}t = 0,\quad \forall\tilde{\mathbf{v}}\in D_a(F). \tag{24}$$

Since $\tilde{u}_i$ and $\tilde{w}_i$ are arbitrary smooth functions and verify the conditions (23), the fundamental lemma of the calculus of variations can be applied to Eq. (24) to conclude that the functions u_i and w_i must respectively satisfy the following differential equations:

$$E_iA_i\frac{\partial^2 u_i}{\partial x_i^2}(x_i,t) - \rho_iA_i\frac{\partial^2 u_i}{\partial t^2}(x_i,t) = 0,\quad \forall x_i\in(0,l_i),\quad i = 1,2,3,\quad \forall t\in[t_a,t_b], \tag{25}$$

$$E_iI_i\frac{\partial^4 w_i}{\partial x_i^4}(x_i,t) + \rho_iA_i\frac{\partial^2 w_i}{\partial t^2}(x_i,t) = 0,\quad \forall x_i\in(0,l_i),\quad i = 1,2,3,\quad \forall t\in[t_a,t_b]. \tag{26}$$

Now it is possible to remove the restrictions (23), and since the functions u_i must satisfy Eq. (25) and the functions w_i Eq. (26), the expression (21) reduces to

$$\begin{aligned}\delta F(\mathbf{v};\tilde{\mathbf{v}}) = &\sum_{i=1}^{4}\int_{t_a}^{t_b}\frac{\partial\tilde{w}_i}{\partial x_i}(a_i,t)P_i(a_i,t)\mathrm{d}t\\ &+\sum_{i=1}^{4}\int_{t_a}^{t_b}\tilde{u}_i(a_i,t)Q_i(a_i,t)\mathrm{d}t\\ &+\sum_{i=1}^{4}\int_{t_a}^{t_b}\tilde{w}_i(a_i,t)R_i(a_i,t)\mathrm{d}t.\end{aligned} \tag{27}$$

Since the functions $\tilde{w}_i(a_i,t),\ \tilde{u}_i(a_i,t),\ \frac{\partial\tilde{w}_i}{\partial x_i}(a_i,t),\ \forall t\in(t_a,t_b),\ i = 1,\ldots,4$, are smooth and arbitrary, the stationary condition (22) leads to the corresponding natural boundary conditions. To reduce the number of defining variables and parameters the following non-dimensional variables and parameters are introduced:

$$X = X_i = x_i/l_i,\quad i = 1,\ldots,3, \tag{28a}$$

$$R_i = \frac{r_il_i}{E_iI_i},\quad i = 1,\ldots,4,\quad l_4 = l_3,\quad E_4I_4 = E_3I_3, \tag{28b}$$

$$T_{ix} = \frac{t_{ix}l_i^3}{E_iI_i},\quad T_{iy} = \frac{t_{iy}l_i^3}{E_iI_i},\quad i = 1,2, \tag{28c}$$

$$T_{ix} = \frac{t_{ix}l_{i-1}^3}{E_{i-1}I_{i-1}},\quad T_{iy} = \frac{t_{iy}l_{i-1}^3}{E_{i-1}I_{i-1}},\quad i = 3,4, \tag{28d}$$

$$L_{ij} = \frac{l_i}{l_j},\quad A_{ij} = \frac{A_i}{A_j},\quad (i,j)\in\{(1,0),(2,0),(3,0),(2,1),(3,2)\}, \tag{28e}$$

$$E_{ij} = \frac{E_i}{E_j},\quad I_{ij} = \frac{I_i}{I_j},\quad (i,j)\in\{(1,0),(2,0),(3,0),(1,2),(2,3)\}, \tag{28f}$$

$$J_0 = \frac{I_0}{A_0l_0^2},\quad J_i = \frac{I_i}{A_il_i^2},\quad \rho_{i0} = \frac{\rho_i}{\rho_0},\quad i = 1,2,3, \tag{28g}$$

where l_0, ρ_0, A_0, E_0 and I_0 are generic parameters.

By assuming separable solutions in the form

$$u_i(X,t) = U_i(X)\cos\omega t, \tag{29}$$

$$w_i(X,t) = W_i(X)\cos\omega t,\quad i = 1,2,3,\quad X\in[0,1],\quad t\geqslant 0, \tag{30}$$

the natural boundary conditions and the compatibility conditions can be written as:

$$\frac{d^2W_1}{dX^2}(0) - K_{33}^{(1)}\frac{dW_1}{dX}(0) = 0, \quad (31)$$

$$\frac{dU_1}{dX}(0) - J_1K_{11}^{(1)}U_1(0) - J_1K_{12}^{(1)}W_1(0) = 0, \quad (32)$$

$$\frac{d^3W_1}{dX^3}(0) + K_{22}^{(1)}W_1(0) + K_{12}^{(1)}U_1(0) = 0, \quad (33)$$

$$\frac{d^2W_3(1)}{dX^2} + K_{33}^{(4)}\frac{dW_3(1)}{dX_3} = 0, \quad (34)$$

$$\frac{dU_3}{dX}(1) + J_3K_{11}^{(4)}U_3(1) + J_3K_{12}^{(4)}W_3(1) = 0, \quad (35)$$

$$\frac{d^3W_3}{dX^3}(1) - K_{22}^{(4)}W_3(1) - K_{12}^{(4)}U_3(1) = 0, \quad (36)$$

$$\cos\alpha_1U_1(1) - \sin\alpha_1W_1(1) - \cos\alpha_2U_2(0) + \sin\alpha_2W_2(0) = 0, \quad (37)$$

$$\sin\alpha_1U_1(1) + \cos\alpha_1W_1(1) - \sin\alpha_2U_2(0) - \cos\alpha_2W_2(0) = 0, \quad (38)$$

$$L_{21}\frac{dW_1}{dX}(1) - \frac{dW_2}{dX}(0) = 0, \quad (39)$$

$$\cos\alpha_2U_2(1) - \sin\alpha_2W_2(1) - \cos\alpha_3U_3(0) + \sin\alpha_3W_3(0) = 0, \quad (40)$$

$$\sin\alpha_2U_2(1) + \cos\alpha_2W_2(1) - \sin\alpha_3U_3(0) - \cos\alpha_3W_3(0) = 0, \quad (41)$$

$$L_{32}\frac{dW_2}{dX}(1) - \frac{dW_3}{dX}(0) = 0, \quad (42)$$

$$\frac{d^2W_2}{dX^2}(0) - K_{33}^{(2)}\frac{dW_2}{dX}(0) - E_{12}I_{12}L_{21}^2\frac{d^2W_1}{dX^2}(1) = 0, \quad (43)$$

$$\frac{d^2W_3}{dX^2}(0) - K_{33}^{(3)}\frac{dW_3}{dX}(0) - E_{23}I_{23}L_{32}^2\frac{d^2W_2}{dX^2}(1) = 0, \quad (44)$$

$$\frac{A_{21}}{L_{21}E_{12}J_1}\frac{dU_2}{dX}(0) - \frac{K_{11}^{(2)}}{L_{21}^3E_{12}I_{12}}U_2(0) - \frac{K_{12}^{(2)}}{L_{21}^3E_{12}I_{12}}W_2(0) + \sin(\alpha_2 - \alpha_1)\frac{d^3W_1}{dX^3}(1) - \frac{G_1}{J_1}\frac{dU_1}{dX}(1) = 0, \quad (45)$$

$$\frac{A_{32}}{L_{32}E_{23}J_2}\frac{dU_3}{dX}(0) - K_{11}^{(3)}U_3(0) - K_{12}^{(3)}W_3(0) - \sin(\alpha_2 - \alpha_3) \times \frac{d^3W_2}{dX^3}(1) - \frac{G_2}{J_2}\frac{dU_2}{dX}(1) = 0, \quad (46)$$

$$\frac{d^3W_2}{dX^3}(0) + K_{22}^{(2)}W_2(0) + K_{12}^{(2)}U_2(0) - \cos(\alpha_2 - \alpha_1)E_{12}I_{12}L_{21}^3 \times \frac{d^3W_1}{dX^3}(1) + G_3\frac{E_{12}L_{21}}{A_{21}J_2}\frac{dU_1}{dX}(1) = 0, \quad (47)$$

$$-\frac{d^3W_3}{dX^3}(0) - L_{32}^3E_{23}I_{23}K_{22}^{(3)}W_3(0) - L_{32}^3E_{23}I_{23}K_{12}^{(3)}U_3(0) + \cos(\alpha_2 - \alpha_3)E_{23}I_{23}L_{32}^3\frac{d^3W_2}{dX^3}(1) - G_4\frac{E_{23}L_{32}}{A_{32}J_3}\frac{dU_2}{dX}(1) = 0, \quad (48)$$

where the coefficients $K_{ij}^{(l)}$ are obtained from the expression of $k_{ij}^{(l)}$ replacing $t_{ix}$and t_{iy} respectively by T_{ix} and T_{iy}. The coefficients G_i are listed in Appendix A. Now, Eqs. (25) and (26) reduce to

$$\frac{d^2U_i}{dX^2}(X) + \lambda^4C_{2i}U_i(X) = 0, \quad \forall X \in (0,1), \quad i = 1,2,3, \quad (49)$$

$$\frac{d^4W_i}{dX^4}(X) - \lambda^4C_{1i}W_i(X) = 0, \quad \forall X \in (0,1), \quad i = 1,2,3, \quad (50)$$

where

$$C_{1i} = (\rho_{i0}A_{i0})(E_{i0}I_{i0})^{-1}L_{i0}^4, \quad C_{2i} = (\rho_{i0}A_{i0})(E_{i0}A_{i0})^{-1}J_0L_{i0}^2 \quad \text{and} \quad \lambda^4 = \frac{\rho_0A_0}{E_0I_0}\omega^2l_0^4.$$

In conclusion, the boundary problem which describes the natural vibrations of the mechanical system under study is given by Eqs. (49) and (50) and the boundary or compatibility conditions (31)–(48). It must be noted that the angles of inclination must verify the conditions $\alpha_1 \neq 0$ and $\alpha_2 \neq n\pi/2$, $n = 1, 2, \ldots$.

4. Numerical results

Using the well-known separation of variables method, the solutions of Eqs. (49) and (50) are respectively assumed to be of the form:

$$U_i(X) = d_1^{(i)}\cos\left(\lambda^2\sqrt{C_{2i}}X\right) + d_2^{(i)}\sin\left(\lambda^2\sqrt{C_{2i}}X\right), \quad (51)$$

$$W_i(X) = c_1^{(i)}\cosh\left(\lambda\sqrt[4]{C_{1i}}X\right) + c_2^{(i)}\sinh\left(\lambda\sqrt[4]{C_{1i}}X\right) + c_3^{(i)} \times \cos\left(\lambda\sqrt[4]{C_{1i}}X\right) + c_4^{(i)}\sin\left(\lambda\sqrt[4]{C_{1i}}X\right), \quad i = 1,2,3. \quad (52)$$

Replacing Eqs. (51) and (52) into the conditions (31)–(48), we obtain a set of eighteen homogeneous equations in the constants $c_j^{(i)}$ and $d_j^{(i)}$. Since the system is homogeneous, for existence of a nontrivial solution, the determinant of coefficients must be equal to zero. This procedure yields the frequency equation:

$$G(\mathbf{R}, \mathbf{T}_x, \mathbf{T}_y, \lambda) = 0, \quad (53)$$

where the components of $\mathbf{R}$, $\mathbf{T}_x$ and $\mathbf{T}_y$ are given by Eqs. (28b–d).

In order to obtain an indication of the accuracy of the developed mathematical model, the classical problems analysed in Refs. [9,29,30] have been solved. In all the cases an excellent agreement has been observed.

Since the algorithm developed allows and great number of configurations and mechanical characteristics and the number of cases is prohibitively large, results are presented for only a few cases. All the numerical results have been obtained with the following generic parameters: $\rho_0 = \rho_1$, $E_0 = E_1$, $I_0 = I_1$, $A_0 = A_1$ and $l_0 = l_1$.

Table 1a depicts values of coefficients λ_i, $i = 1, 2, \ldots, 5$ where $\lambda_i^4 = \frac{\rho_0A_0}{E_0I_0}\omega_i^2l_0^4$, for a frame with rigidly clamped ends and one inclined member. The geometric and mechanical parameters values are given by:

$$E_{12} = E_{23} = 1, \quad I_{12} = I_{23} = 1, \quad A_{21} = A_{32} = 1, \quad J_1 = 1/30000,$$
$$L_{10} = 1, \alpha_1 = \pi/2, \quad \alpha_3 = -\pi/2$$

and the remaining values are given in Table 1b except the values of $\alpha_2, J_2, J_3, L_{21}, L_{32}$ and L_{20} which vary with L_{30}, and are depicted in Table 1c. It can be observed that the values of λ_i, $i = 1, 2, \ldots, 5$ decrease when the values of the parameter L_{30} increase. The first five mode

Table 1a
First five exact values of the frequency parameter λ for a frame with one inclined member and ends rigidly clamped, for different values of L_{30}.

L_{30}	λ_1	λ_2	λ_3	λ_4	λ_5
1	1.51405646	2.02926280	3.39508597	4.19589016	4.59485951
6/5	1.41721109	2.01115388	3.28370976	3.78347996	4.42294976
7/5	1.34001905	1.99328729	3.03178225	3.51625304	4.35387959
8/5	1.27532429	1.97142612	2.72980005	3.36862073	4.26535142
9/5	1.21867577	1.94462321	2.46273729	3.23723455	4.07086141

shapes which correspond to the case $L_{30} = 9/5$ are presented in Fig. 2a–e.

Table 2a depicts values of coefficients λ_i, $i = 1, 2, \ldots, 5$ for a frame with rigidly clamped ends and one inclined member. The geometric and mechanical parameters values are given by:

$$L_{21} = 2,\ E_{12} = E_{23} = 1,\ I_{12} = I_{23} = 1,\ A_{21} = A_{32} = 1,\ J_1 = 1/30000,$$
$$J_2 = 1/120000,\ L_{10} = 1,\ L_{20} = 2,\ \alpha_1 = \pi/2,\ \alpha_2 = 0,$$

and the remaining values are given in Table 2b except the values of J_3, L_{32} and L_{30} which vary with α_3 and are depicted in Table 2c. The first five mode shapes which correspond to the case $\alpha_3 = -\pi/4$ are presented in Fig. 3a–e.

Table 3a depicts values of coefficients λ_i, $i = 1, 2, \ldots, 5$ for a frame with a fixed geometry and two intermediate points elastically restrained. The geometric and mechanical parameters values are given by:

$$R_1 = R_4 = \infty,\ R_3 = 0,\ T_{1x} = T_{4x} = T_{1y} = T_{4y} = \infty,\ T_{2x} = T_{2y} = 0,$$
$$L_{21} = \sqrt{2},\ L_{32} = \sqrt{2}/2,\ E_{12} = E_{23} = 1,\ I_{12} = I_{23} = 1,\ A_{21} = A_{32} = 1,$$

and the remaining values are given in Table 3b. It can be observed that the values of λ_i, $i = 1, 2, \ldots, 5$ increase with the increment of the values of the parameters T_{3x}, T_{3y}, and R_2. The first five mode shapes which correspond to the case $T_{3x} = T_{3y} = R_2 = 100$, are presented in Fig. 4a–e.

Table 4a depicts values of coefficients λ_i, $i = 1, 2, \ldots, 5$ for a classical portal frame with ends elastically restrained against rotation. The geometric and mechanical parameters values are given by:

$$R_2 = R_3 = 0, \quad L_{21} = 2, \quad L_{32} = 1/2, \quad E_{12} = E_{23} = 1,$$
$$I_{12} = I_{23} = 1, \quad A_{21} = A_{32} = 1,$$

and the remaining values are given in Table 4b. In this case, the values of λ_i, $i = 1, 2, \ldots, 5$ increase with the increment of the values of the parameters R_1 and R_4.

Comparison of Tables 3a and 4a shows that the translational restraints generally have greater influence on the frequencies than the rotational restraints.

The model established for the vibration of a frame with three inclined members can be applied to a two-bar frame. For such a frame it is sufficient to adopt the condition $\alpha_2 = \alpha_3$.

Table 5a depicts values of coefficients λ_i, $i = 1, 2, \ldots, 5$ for a two-bar frame with ends elastically restrained against rotation and translation. The geometric and mechanical parameters values are given by:

$$L_{21} = 1/2, \quad L_{32} = 1, \quad E_{12} = E_{23} = 1, \quad I_{12} = I_{23} = 1,$$
$$A_{21} = A_{32} = 1,$$

and the remaining values are given in Table 5b. In this case, the values of λ_i, $i = 1, 2, \ldots, 5$ increase with the increment of the values of K. The first five mode shapes which correspond to the case $K = 1$ and $K = 1000$ are respectively presented in Fig. 5a–e and Fig. 6a–e.

For all the cases depicted by Tables 1a–5a, the finite element method was implemented and a special code developed where 25 Euler beam elements have been employed to divide each beam. The numerical values obtained with this method, coincide at least

Table 1b
Values of characteristic parameters R_i, T_{ix}, T_{iy}, I_{i0}, A_{i0}, E_{i0} and ρ_{i0} which correspond to the case depicted in Table 1a.

i	R_i	T_{ix}	T_{iy}	I_{i0}	A_{i0}	E_{i0}	ρ_{i0}
1	∞	∞	∞	1	1	1	1
2	0	0	0	1	1	1	1
3	0	0	0	1	1	1	1
4	∞	∞	∞	–	–	–	–

Table 1c
Values of the characteristic parameters α_2, J_2, J_3, L_{21}, L_{32} and L_{20} which vary with L_{30} and correspond to the case depicted in Table 1a.

L_{30}	α_2	$L_{20} = L_{21}$	J_2	J_3	L_{32}
1	0.00E+0	2	1/120000	1/30000	1/2
6/5	7.60E−3	$\sqrt{101}/5$	1/121200	1/43200	$6\sqrt{101}/101$
7/5	7.73E−3	$2\sqrt{26}/5$	1/124800	1/55800	$7\sqrt{26}/52$
8/5	7.77E−3	$\sqrt{109}/5$	1/130800	1/76800	$8\sqrt{109}/109$
9/5	7.79E−3	$2\sqrt{29}/5$	1/139200	1/97200	$9\sqrt{29}/58$

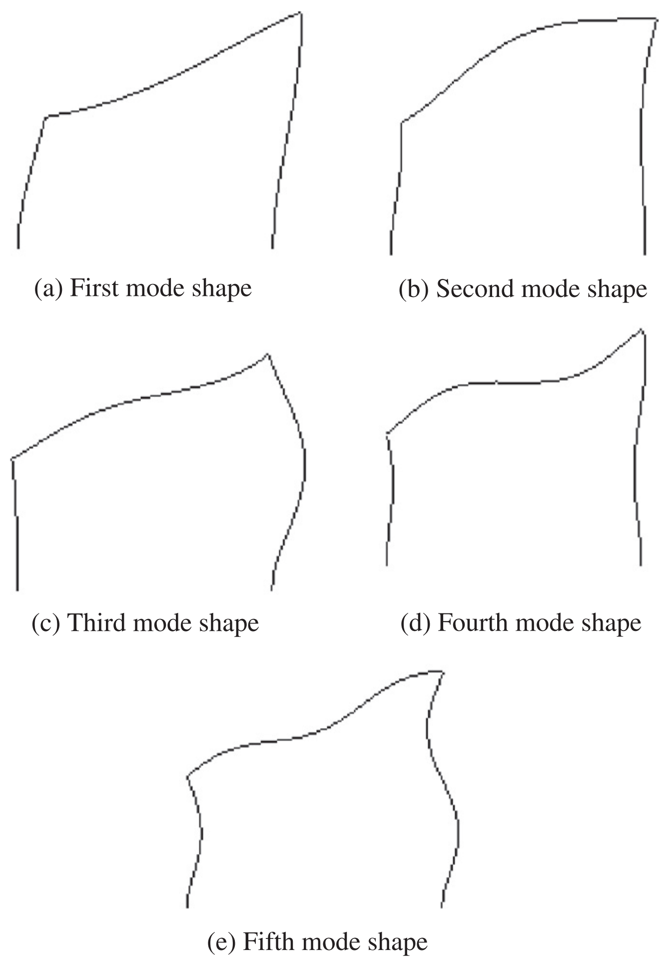

Fig. 2. Mode shapes of the frame with the same parameters used in Table 1a in the case $L_{30} = 9/5$: (a) first mode shape, (b) second mode shape, (c) third mode shape, (d) fourth mode shape, (e) fifth mode shape.

Table 2a
First five exact values of the frequency coefficient λ for a frame with rigidly clamped ends and one inclined member, for different values of α_3.

α_3	λ_1	λ_2	λ_3	λ_4	λ_5
$-\pi/4$	1.44215592	2.01495355	3.04552404	3.62061372	4.42211965
$-3\pi/8$	1.52216252	2.03384784	3.35909676	4.03838813	4.52411483
$-\pi/2$	1.51405646	2.02926280	3.39508597	4.19589016	4.59485951
$-5\pi/8$	1.42440808	2.03930622	3.38256465	4.03291179	4.46872486
$-3\pi/4$	1.21392548	2.07675290	3.08866802	3.55039980	4.34065004

Table 2b
Values of characteristic parameters R_i, T_{ix}, T_{iy}, I_{i0}, A_{i0}, E_{i0} and ρ_{i0} which correspond to the case depicted in Table 2a.

i	R_i	T_{ix}	T_{iy}	I_{i0}	A_{i0}	E_{i0}	ρ_{i0}
1	∞	∞	∞	1	1	1	1
2	0	0	0	1	1	1	1
3	0	0	0	1	1	1	1
4	∞	∞	∞	–	–	–	–

Table 2c
Values of the characteristic parameters J_3, L_{30} and L_{32} which vary with α_3 and correspond to the case depicted in Table 2a.

α_3	J_3	L_{30}	L_{32}
$-\pi/4$	1.67E−5	1.414	0.707
$-3\pi/8$	2.85E−5	1.082	0.541
$-\pi/2$	3.33E−5	1.000	0.500
$-5\pi/8$	2.85E−5	1.082	0.541
$-3\pi/4$	1.67E−5	1.414	0.707

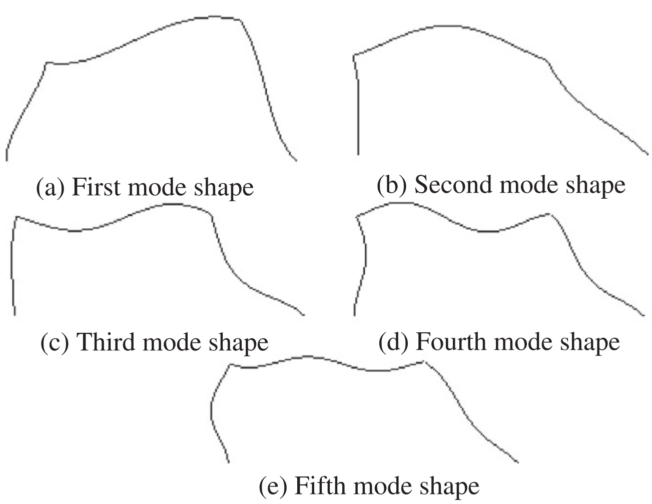

Fig. 3. Mode shapes of the frame with the same parameters used in Table 2a in the case $\alpha_3 = -\pi/4$: (a) first mode shape, (b) second mode shape, (c): third mode shape, (d) fourth mode shape, (e) fifth mode shape.

Table 3a
First five exact values of the frequency coefficient λ for a frame with a fixed geometry and two intermediate points elastically restrained.

$T_{3x} = T_{3y} = R_2$	λ_1	λ_2	λ_3	λ_4	λ_5
0.1	2.05141270	2.74674211	4.15813108	4.40725873	5.56447003
1.0	2.05672993	2.76782191	4.17877712	4.41378971	5.56682495
10.0	2.10631121	2.87768329	4.27928093	4.49634494	5.58487356
100.0	2.46585465	2.99012833	4.32884668	4.75083373	5.63290958
1000.0	3.01195249	3.69232241	4.35373620	4.86254731	5.73338619

Table 3b
Values of the characteristic parameters J_i, α_i, L_{i0}, I_{i0}, A_{i0}, E_{i0} and ρ_{i0} which correspond to the case depicted in Table 3a.

i	J_i	α_i	L_{i0}	I_{i0}	A_{i0}	E_{i0}	ρ_{i0}
1	1/60000	$\pi/4$	1	1	1	1	1
2	1/120000	0	$\sqrt{2}$	1	1	1	1
3	1/60000	$-\pi/4$	1	1	1	1	1

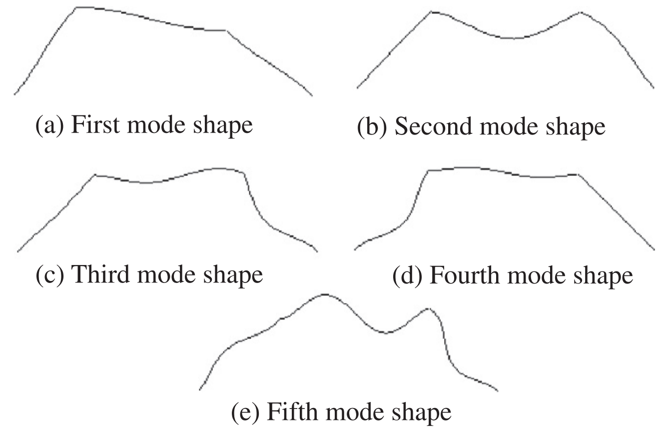

Fig. 4. Mode shapes of the frame with the same parameters used in Table 3a in the case $T_{3x} = T_{3y} = R_2 = 100$: (a) first mode shape, (b) second mode shape, (c) third mode shape, (d) fourth mode shape, (e) fifth mode shape.

Table 4a
First five exact values of the frequency coefficient λ for a classical portal frame with ends elastically restrained against rotation, for different values of the parameters R_1 and R_4.

$R_1 = R_4$	λ_1	λ_2	λ_3	λ_4	λ_5
0	1.00735667	1.89604323	2.80366323	3.04479119	3.25339525
1	1.17056652	1.90684805	2.80616330	3.05528110	3.29161526
10	1.40024036	1.93009678	2.81123764	3.07324651	3.39089853
100	1.47149835	1.93941103	2.81315615	3.07893735	3.43797790
1000	1.48027594	1.94062957	2.81340241	3.07962756	3.44445612
10000	1.48117443	1.94075519	2.81342774	3.07969804	3.44512809

Table 4b
Values of the characteristic parameters T_{ix}, T_{iy}, J_i, α_i, L_{i0}, I_{i0}, A_{i0}, E_{i0} and ρ_{i0} which correspond to the case depicted in Table 4a.

i	T_{ix}	T_{iy}	J_i	α_i	L_{i0}	I_{i0}	A_{i0}	E_{i0}	ρ_{i0}
1	100	100	1/30000	$\pi/2$	1	1	1	1	1
2	0	0	1/120000	0	2	1	1	1	1
3	0	0	1/30000	$-\pi/2$	1	1	1	1	1
4	100	100	–	–	–	–	–	–	–

Table 5a
First five exact values of the frequency coefficient λ for a two-beams frame with ends elastically restrained against rotation and translation. The restraint parameter vary with K, where $T_{1x} = T_{1y} = T_{4x} = T_{4y} = R_1 = R_4 = K$.

K	λ_1	λ_2	λ_3	λ_4	λ_5
0	0.00000000	1.87059351	2.80025968	4.07815885	4.42332371
1	1.72257177	2.45065136	2.90116036	4.33543228	4.61859014
10	2.42449159	3.14301012	3.55770052	4.76792675	5.06750404
100	3.45649311	4.01623701	4.57459083	5.32647698	6.06926303
1000	3.87182675	4.60882552	5.92481310	6.78226633	7.64381353
10000	3.91843903	4.70916362	7.01271047	7.72026488	9.19332081

Table 5b
Values of the characteristic parameters T_{ix}, T_{iy}, R_i, J_i, α_i, L_{i0}, I_{i0}, A_{i0}, E_{i0} and ρ_{i0} which correspond to the case depicted in Table 5a.

i	T_{ix}	T_{iy}	R_i	J_i	α_i	L_{i0}	I_{i0}	A_{i0}	E_{i0}	ρ_{i0}
1	K	K	K	1/10000	$\pi/4$	1	1	1	1	1
2	10	100	0	1/2500	$-\pi/4$	1/2	1	1	1	1
3	0	0	0	1/2500	$-\pi/4$	1/2	1	1	1	1
4	K	K	K	–	–	–	–	–	–	–

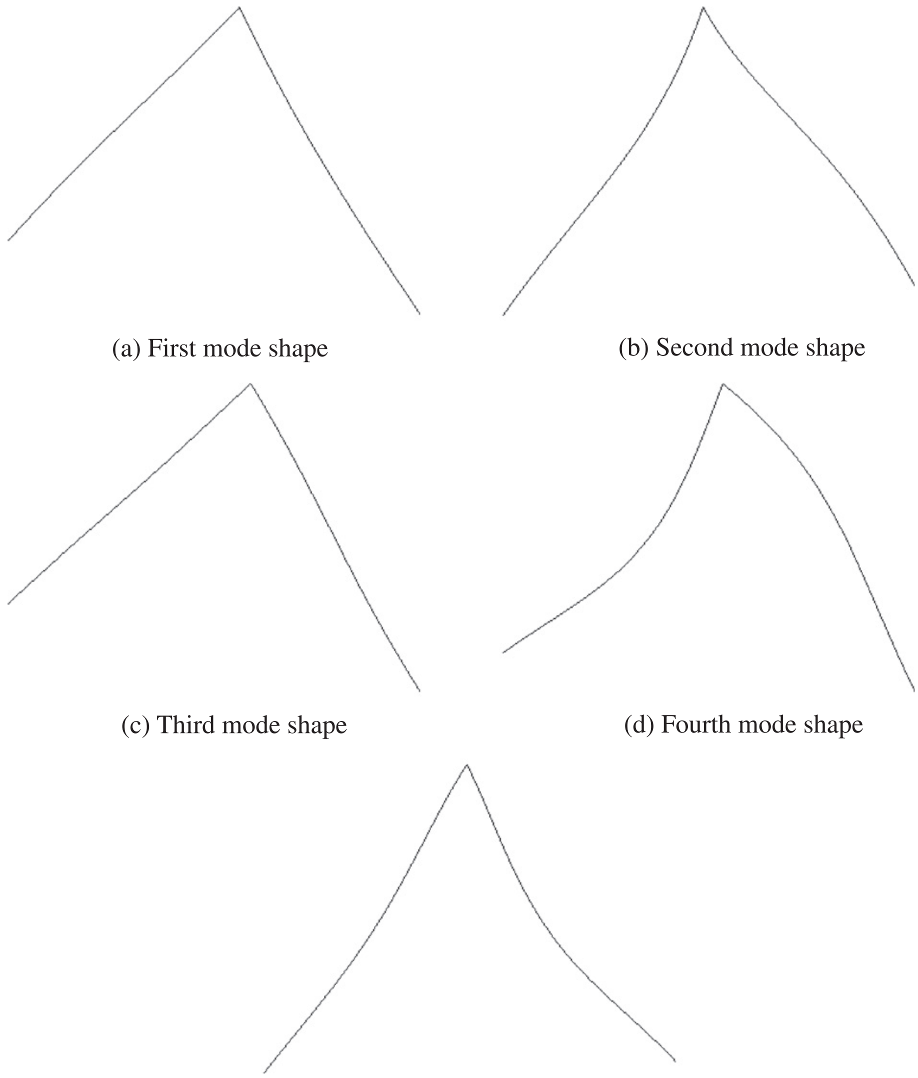

Fig. 5. Mode shapes of the frame with the same parameters used in Table 5a in the case $K = 1$: (a) first mode shape, (b) second mode shape, (c) third mode shape, (d) fourth mode shape, (e) fifth mode shape.

up to four decimal digits with the exact values, so they have not been included in the tables.

5. Conclusions

In this paper the calculus of variations was used to derive the boundary value problems which describe the dynamical behaviour of two and three-bar frames with inclined members, and with ends and intermediate points elastically restrained against rotation and translation.

A rigorous procedure was used by introducing a functional adequate to the formulation of the Hamilton's principle and the corresponding spaces of functions. The stationary condition for the functional F defined by (11) on the space of admissible functions has been clearly stated by defining the space of admissible functions $D(F)$ and the space $D_a(F)$ of admissible directions. It has been shown that the introduction of the vectorial functions (12), the spaces (13), (14) and the product space $U \times W$, leads to the rigorous definition of the variation of the functional F by means of the derivative (16).

A simple, computationally efficient and accurate approach has been developed for the determination of natural frequencies and modal shapes of free vibration of the frames described above. Numerical results for the first five natural frequencies in tabular form and the corresponding mode shapes were included. In order to obtain an indication of the accuracy of the mathematical model obtained, some cases available in the literature have been considered and the finite element method has also been implemented. In all cases excellent agreements have been determined. The algorithm is very general and it is attractive regarding its versatility in handling boundary and intermediate conditions. Besides, it allows taking into account a great variety of complicating effects and the adoption of different generic parameters l_0, ρ_0, A_0, E_0 and I_0, leads to different analytical expressions for the restraint parameters and the frequency coefficients.

(a) First mode shape

(b) Second mode shape

(c) Third mode shape

(d) Fourth mode shape

(e) Fifth mode shape

Fig. 6. Mode shapes of the frame with the same parameters used in Table 5a in the case $K = 1000$: (a) first mode shape, (b) second mode shape, (c) third mode shape, (d) fourth mode shape, (e) fifth mode shape.

Acknowledgment

The present study has been partially sponsored by CIUNSA PROJECT No. 1899.

Appendix A

Rigidity coefficients of the functional defined by (10) and algebraic procedure to obtain the expression (21).

As a consequence of the existence of the angles of inclination α_i, $i = 1, 2, 3$, the rigidity coefficients t_{ix} and t_{iy} generate the following coefficients included in Eq. (10):

$$k_{11}^{(i)} = t_{ix}\cos^2\alpha_i + t_{iy}\sin^2\alpha_i,$$

$$k_{12}^{(i)} = k_{21}^{(i)} = -t_{ix}\cos\alpha_i\sin\alpha_i + t_{iy}\sin\alpha_i\cos\alpha_i,$$

$$k_{22}^{(i)} = t_{ix}\sin^2\alpha_\mathrm{i} + t_{iy}\cos^2\alpha_\mathrm{i},$$

$$\begin{aligned}k_{33}^{(i)} &= r_i, \quad i = 1,2,3, k_{11}^{(4)} = t_{4x}\cos^2\alpha_3 + t_{4y}\sin^2\alpha_3, k_{12}^{(4)} = k_{21}^{(4)}\\ &= -t_{4x}\cos\alpha_3\sin\alpha_3 + t_{4y}\sin\alpha_3\cos\alpha_3, k_{22}^{(4)}\\ &= t_{4x}\sin^2\alpha_3 + t_{4y}\cos^2\alpha_3, k_{33}^{(4)} = r_4.\end{aligned}$$

The application of (16) to (11) leads to

$$\begin{aligned}\delta F(\mathbf{v};\tilde{\mathbf{v}}) = &\sum_{i=1}^{3}\int_{t_a}^{t_b}\int_0^{l_i}\left[\rho_i A_i\frac{\partial u_i}{\partial t}(x_i,t)\frac{\partial\tilde{u}_i}{\partial t}(x_i,t) + \rho_i A_i\frac{\partial w_i}{\partial t}(x_i,t)\right.\\ &\left.\frac{\partial\tilde{w}_i}{\partial t}(x_i,t)\right]\mathrm{d}x_i\mathrm{d}t - \sum_{i=1}^{3}\int_{t_a}^{t_b}\int_0^{l_i}\left[E_iA_i\frac{\partial u_i}{\partial x_i}(x_i,t)\frac{\partial\tilde{u}_i}{\partial x_i}(x_i,t) + E_iI_i\frac{\partial^2 w_i}{\partial x_i^2}(x_i,t)\right.\\ &\left.\frac{\partial^2\tilde{w}_i}{\partial x_i^2}(x_i,t)\right]\mathrm{d}x_i\mathrm{d}t - \sum_{i=1}^{3}\int_{t_a}^{t_b}\left[k_{11}^{(i)}u_i(0,t)\tilde{u}_i(0,t) + k_{22}^{(i)}w_i(0,t)\tilde{w}_i(0,t)\right.\\ &\left.+k_{33}^{(i)}\frac{\partial w_i}{\partial x_i}(0,t)\frac{\partial\tilde{w}_i}{\partial x_i}(0,t) + k_{12}^{(i)}u_i(0,t)\tilde{w}_i(0,t) + k_{12}^{(i)}w_i(0,t)\tilde{u}_i(0,t)\right]\mathrm{d}t\\ &-\int_{t_a}^{t_b}\left[k_{11}^{(4)}u_3(l_3,t)\tilde{u}_3(l_3,t) + k_{22}^{(4)}w_3(l_3,t)\tilde{w}_3(l_3,t) + k_{33}^{(4)}\frac{\partial w_3}{\partial x_3}(l_3,t)\right.\\ &\left.\frac{\partial\tilde{w}_3}{\partial x_3}(l_3,t) + k_{12}^{(4)}u_3(l_3,t)\tilde{w}_n(l_3,t) + k_{12}^{(4)}w_3(l_3,t)\tilde{u}_3(l_3,t)\right]\mathrm{d}t.\end{aligned}$$

Using integration by parts we obtain

$$\delta F(\mathbf{v};\tilde{\mathbf{v}}) = -\sum_{i=1}^{3}\int_{t_a}^{t_b}\int_0^{l_i}\left[\rho_i A_i\frac{\partial^2 u_i}{\partial t^2}\tilde{u}_i + \rho_i A_i\frac{\partial^2 w_i}{\partial t^2}\tilde{w}_i - E_i A_i\frac{\partial^2 u_i}{\partial x_i^2}\tilde{u}_i + E_i I_i\frac{\partial^4 w_i}{\partial x_i^4}\tilde{w}_i\right]\mathrm{d}x_i\mathrm{d}t + \sum_{i=1}^{3}\int_{t_a}^{t_b}\left[E_i I_i\frac{\partial^2 w_i(0,t)}{\partial x_i^2}\frac{\partial \tilde{w}_i(0,t)}{\partial x_i} - E_i I_i\frac{\partial^2 w_i(l_i,t)}{\partial x_i^2}\frac{\partial \tilde{w}_i(l_i,t)}{\partial x_i} + E_i I_i\frac{\partial^3 w_i(l_i,t)}{\partial x_i^3}\tilde{w}_i(l_i,t) - E_i I_i\frac{\partial^3 w_i(0,t)}{\partial x_i^3}\tilde{w}_i(0,t) + E_i A_i\frac{\partial u_i}{\partial x_i}(0,t)\tilde{u}_i(0,t) - E_i A_i\frac{\partial u_i}{\partial x_i}(l_i,t)\tilde{u}_i(l_i,t)\right]\mathrm{d}t - \sum_{i=1}^{3}\int_{t_a}^{t_b}\left[k_{11}^{(i)}u_i(0,t)\tilde{u}_i(0,t) + k_{22}^{(i)}w_i(0,t)\tilde{w}_i(0,t) + k_{33}^{(i)}\frac{\partial w_i}{\partial x_i}(0,t)\frac{\partial \tilde{w}_i}{\partial x_i}(0,t) + k_{12}^{(i)}u_i(0,t)\tilde{w}_i(0,t) + k_{12}^{(i)}w_i(0,t)\tilde{u}_i(0,t)\right]\mathrm{d}t - \int_{t_a}^{t_b}\left[k_{11}^{(4)}u_3(l_3,t)\tilde{u}_3(l_3,t) + k_{22}^{(4)}w_3(l_3,t)\tilde{w}_3(l_3,t) + k_{33}^{(4)}\frac{\partial w_3}{\partial x_3}(l_3,t)\frac{\partial \tilde{w}_3}{\partial x_3}(l_3,t) + k_{12}^{(4)}u_3(l_3,t)\tilde{w}_3(l_3,t) + k_{12}^{(4)}w_3(l_3,t)\tilde{u}_3(l_3,t)\right]\mathrm{d}t.$$

Because the admissible directions satisfy the compatibility Eqs. (37), (38), (40) and (41) it is possible to collect terms adequately and the use of the differential operators (18)–(20) evaluated at (a_i, t) with the following coefficients:

$$b_i = E_i I_i,\ c_1 = 0,\ c_2 = -b_1,\ c_3 = -b_2,\ b_4 = -b_3,\ c_4 = 0,\ d_i = E_i A_i,\ i = 1,2,3,\ d_4 = -d_3,$$
$$e_1 = 0,\ e_2 = \sin(\alpha_2-\alpha_1)E_1 I_1,\ e_3 = -\sin(\alpha_2-\alpha_3)E_2 I_2,\ f_1 = 0,\ f_2 = -G_1 E_1 A_1,$$
$$f_3 = -G_2 E_2 A_2,\ e_4 = -d_3,\ e_4 = 0,\ f_4 = 0,\ g_1 = 0,\ h_1 = 0,\ g_2 = \cos(\alpha_2-\alpha_1)E_1 I_1,$$
$$h_2 = -G_3 E_1 A_1,\ g_3 = \cos(\alpha_2-\alpha_3)E_2 I_2,\ h_3 = -G_4 E_2 A_2,\ g_4 = 0,\ h_4 = 0.$$

In all cases, is $j = i$ if $i = 1, 2, 3$ and $j = i - 1$ if $i = 4$.

$$G_1 = \frac{\sin\alpha_2 - \cos\alpha_1\sin(\alpha_2-\alpha_1)}{\sin\alpha_1},\quad G_2 = \frac{\cos\alpha_3 - \sin\alpha_2\sin(\alpha_2-\alpha_3)}{\cos\alpha_2},$$
$$G_3 = \frac{\cos\alpha_2 - \cos\alpha_1\cos(\alpha_2-\alpha_1)}{\sin\alpha_1},\quad G_4 = \frac{\sin\alpha_2\cos(\alpha_2-\alpha_3) - \sin\alpha_3}{\cos\alpha_2}.$$

References

[1] Troutman JL. Variational calculus and optimal control. New York: Springer-Verlag; 1996.
[2] Sagan H. Calculus of variations. New York: Mc Graw Hill Book Co.; 1969.
[3] Weinstock R. Calculus of variations with applications to physics and engineering. New York: Dover; 1974.
[4] Gelfand IM, Fomin SV. Calculus of variations. New Jersey: Prentice Hall, Englewood Cliffs; 1963.
[5] Dym C, Shames I. Solid mechanics: a variational approach. New York: Mc Graw Hill Book Company; 1973.
[6] Kantorovich L, Krylov V. Approximate methods of higher analysis. New York: Interscience Publishers; 1964.
[7] Hildebrand F. Methods of applied mathematics. New Jersey: Prentice Hall; 1965.
[8] Elsgoltz L. Calculus of variations. Mass: Addison-Wesley Publishing Co. Reading; 1962.
[9] Blevins RD. Formulas for natural frequency and mode shape. New York: Van Nostrand Reinhold; 1979.
[10] Karnovsky IA, Lebed OI. Free vibrations of beams and frames. New York: Mc Graw Hill Engineering Reference; 2004.
[11] Clough RW, Penzien J. Dynamics of structures. New York: Mc Graw Hill Book Company; 1975.
[12] Nowacki W. Dynamics of elastic systems. New York: John Wiley and Sons; 1963.
[13] Craig R. Structural dynamics. An introduction to computer methods. New York: John Wiley and Sons; 1997.
[14] Timoshenko S, Young DH. Vibration problems in engineering. New Jersey: D. Van Nostrand Company, Inc.; 1956.
[15] Chang CH. Mechanics of elastic structures with inclined members. Berlin: Springer-Verlag, The Netherlands; 2005.
[16] Oguamanam DC, Hansen JS, Heppler GR. Vibrations of arbitrarily oriented two-member open frames with tip mass. J Sound Vib 1998;209:651–69.
[17] Heppler GR, Oguamanam DC, Hansen JS. Vibrations of a two-member open frame. J Sound Vib 2003;263:299–317.
[18] Chen WH, Wu CW. A spline wavelets element method for frame structures vibration. Comput Mech 1995;16:11–21.
[19] Lee HP, Ng TY. In-plane vibrations of planar frames structures. J Sound Vib 1994;172:420–7.
[20] Baychev I. Finite element method for frames with variable characteristics. Mech Res Commun 1996;23:213–20.
[21] Chui PPT, Chan SL. Vibration and deflection characteristics of semi-rigid jointed frames. Eng Struct 1997;19:1001–10.
[22] Mead DJ. Free vibrations of self-strained assemblies of beams. J Sound Vib 2002;249:101–27.
[23] Lin HP, Ro J. Vibration analysis of planar serial-frame structures. J Sound Vib 2003;262:1113–31.
[24] Wang X, Wang Y, Zhou Y. Applications of a new differential quadrature element method to free vibrational analysis of beams and frames structures. J Sound Vib 2004;269:1133–41.
[25] Yuan S, Kangsheng Y, Williams FW. Second order mode-finding method in dynamic stiffness matrix method. J Sound Vib 2004;269:689–708.
[26] Filipich CP, Laura PA. In-plane vibrations of portal frames with end supports elastically restrained against rotation and translation. J Sound Vib 1987;117:467–73.
[27] Filipich CP, Valerga de Greco BH, Laura PA. A note on the analysis of symmetric mode of vibrations of portal frames. J Sound Vib 1987;117:198–201.
[28] Laura PA, Valerga de Greco BH, Filipich CP. In-plane vibrations of frames carrying concentrated masses. J Sound Vib 1987;117:447–58.
[29] Albarracín CM, Grossi RO. Vibrations of elastically restrained frames. J Sound Vib 2005;285:467–76.
[30] Grossi RO, Albarracín CM. A variational approach to vibrating frames. J Multi-body Dynam 2007;221:247–59.
[31] Zeidler E. Applied functional analysis: main principles and their applications, vol. 109. New York: Springer-Verlag; 1995.
[32] Reddy JN. Applied functional analysis and variational methods in engineering. New York: Mc Graw-Hill Book Co.; 1986.