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Variational Approach of Timoshenko Beams with Internal Elastic Restraints

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Abstract: An exact approach for free transverse vibrations of a Timoshenko beam with ends elastically restrained against rotation and translation and arbitrarily located internal restraints is presented. The calculus of variations is used to obtain the equations of motion, the boundary conditions and the transitions conditions which correspond to the described mechanical system. The derived differential equations are solved individually for each segment of the beam with the corresponding boundary and transitions conditions. The derived mathematical formulation generates as particular cases, and several mathematical models are used to simulate the presence of cracks. Some cases available in the literature and the presence of some errors are discussed. New results are presented for different end conditions and restraint conditions in the intermediate elastic constraints with their corresponding modal shapes.

Key words: Calculus of variations, Timoshenko beams, elastically restrained ends, exact result.

1. Introduction

The behavior of the natural frequencies of beams has been extensively analyzed to detect the presence, size and location of cracks. Several approaches have been implemented to model cracks in Euler-Bernoulli and Timoshenko beams. The most relevant techniques of crack detection were usually based on changes in natural frequencies, but approaches based on measuring dynamics flexibility or comparison of mode shapes, have also been implemented. Due to the great amount of information, it is not the intention to review the literature and in consequence, only some relevant papers will be cited. Ostachowicz and Krawczuk [1] and Farghaly [2] modeled the crack as a continuous flexibility using the displacement field in the vicinity of the crack modeled with fracture mechanics methods. Dimarogonas [3] presented a state of the art review. Chondros et al. [4] developed a consistent cracked beam vibration theory. Shifrin and Ruotolo [5] analyzed a beam with an arbitrary number of cracks by representing cracks as massless springs and using a continuous mathematical model of the beam in transverse vibration. Chondros et al. [6] modeled a continuous simply supported beam with a breathing crack for the prediction of changes in transverse vibration. Zheng and Fan [7] tackled the problem of beams with arbitrary number of cracks with a new Fourier series method. Li [8, 9] presented a model of massless rotational spring adopted to analyze the free vibrations of multi-step uniform and non-uniforms beams, with an arbitrary number of cracks and concentrated masses. Fernández-Sáez and Navarro [10] implemented the method of flexibility influence functions to approximate the fundamental frequency for bending vibrations of cracked Euler-Bernoulli beams with different boundary conditions. Lele and Maiti [11] presented the method of detection of location of cracks in beams based on frequency measurements and Timoshenko beam theory. Lin et al. [12] proposed a solution for simply supported cracked

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Timoshenko beams by deriving a closed form solution. Khiem and Lien [13] formulated a multi-crack detection method for beam by natural frequencies in the form of a non-linear optimization problem. Ruotolo and Surace [14] proposed the extension of the smooth function method for bending vibrations to the calculation of longitudinal natural frequencies of a vibrating isotropic bar with an arbitrary finite number of symmetric transverse open cracks. Binici [15] proposed a method to determine eigenfrequency changes of axially loaded beams where cracks are modeled as rotational springs. Hsu [16] applied the differential quadrature method to solve the eigenvalue problems of a clamped-free and a hinged-hinged Bernoulli-Euler beams on elastic foundation with a single edge crack, axial loading and excitation force. Loya et al. [17] derived exact and perturbation solutions for the natural frequencies for vibrations of cracked Timoshenko beams. Khaji et al. [18] developed an analytical approach for crack identification procedure in uniform Timoshenko beams with a crack model where the cracked section of the beam was modeled as a local flexibility that can be regarded as a rotational spring. Zhou [19] studied the free vibration of multi-spam Timoshenko beams by the Rayleigh-Ritz method.

The main purpose of the present paper is to present an approach based on modeling a cracked beam as two segments connected by a general elastic restraint, composed by several rotational and translational springs. It is also assumed that the beam ends are elastically restrained against rotation and translation (Fig. 1).

The method of separation of variables is used for the determination of the exact natural frequencies and mode shapes. In order to obtain an indication of the accuracy of the developed mathematical model, some cases available in the literature have been considered and comparisons of numerical results are included. Particularly, a comparison of numerical values with those obtained by Khaji et al. [18] is included.

The procedure proposed in this work made it possible to detect an error in two coefficients presented in Ref. [17]. A comparison with the corrected eigenvalues is included. Since the developed algorithm can be applied to a wide range of the different elastic restraint conditions, a great number of problems were solved but results are presented for only a few cases.

This paper is organized in the following way: In Section 2, a summary of the treatment of techniques of the calculus of variations to obtain the governing differential equations and the boundary and transition conditions is presented; In Section 3, the exact solution obtained with the method of separation of variables is presented; In Section 4, comparison studies with some cases available in the literature and new results are presented for the natural frequency and their corresponding modal shapes; Finally, Section 5 contains the conclusions of this paper.

2. Theory and Formulations

Let us consider a uniform Timoshenko beam of length l, which has elastically restrained ends, and also is constrained at an intermediate point with a general elastic restraint composed by several rotational and translational springs, as shown in Fig. 1.

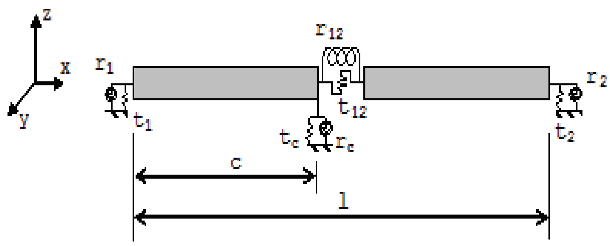


Fig. 1 Beam model description.

The beam system is made up of two different spans, which correspond to the intervals [0, c] and [c, l], respectively. It is assumed that the ends and the intermediate point c are elastically restrained against translation and/or rotation. The rotational restraints are characterized by the spring constants r_1, r_2, r_{12} and r_c and the translational restraints by the spring constants t_1, t_2, t_{12} and t_c. Adopting the adequate values of the parameters r_i and t_i, $i = 1, 2$, all the possible combinations of classical end conditions, (i.e., clamped, pinned, sliding and free) can be generated. On the other hand, adopting the adequate values of the parameters r_c, r_{12}, t_{12} and t_c different constraints at the intermediate point $x = c$ can be generated.

According to Timoshenko beam theory, two independent variables: the transverse deflection w and the normal rotational angle ϕ due to bending are used to describe the deformation of the beam.

Hamilton's principle requires that between times t_a and t_b at which the positions are known, the motion will make stationary the action integral $F(u) = \int_{t_a}^{t_b} L\mathrm{d}t$ on the space of admissible functions, where the Lagrangian L is given by $L = T - U$, where U is the elastic strain energy due to the beam and to the elastic restraints at any instant t; and T is the kinetic energy of the beam at any instant t [20]. In consequence, the energy functional to be considered is given by

$$\begin{aligned} I(w,\phi) = \frac{1}{2}\int_{t_a}^{t_b}\Bigg\{\sum_{i=1}^{2}\int_{D_i}\Bigg[&\rho_i(x)A_i(x)\left(\frac{\partial w(x,t)}{\partial t}\right)^2 \\ &+\rho_i(x)I_i(x)\left(\frac{\partial \phi(x,t)}{\partial t}\right)^2 - E_i(x)I_i(x)\left(\frac{\partial \phi(x,t)}{\partial x}\right)^2 \\ &-\kappa_i G_i(x)A_i(x)\left(\frac{\partial w(x,t)}{\partial x} - \phi(x,t)\right)^2 + q(x,t)w(x,t)\Bigg]dx \\ &-r_1\phi^2(0^+,t) - t_1 w^2(0^+,t) - r_c\left[\phi(c^-,t)\right]^2 \\ &-r_{12}\left[\phi(c^+,t) - \phi(c^-,t)\right]^2 - t_c\left[w(c^-,t)\right]^2 \\ &-t_{12}\left[w(c^+,t) - w(c^-,t)\right]^2 - r_2\phi^2(l^-,t) - t_2 w^2(l^-,t)\Bigg\}dt \end{aligned} \tag{1}$$

where

$$D_1 = (0,c),\ D_2 = (c,l)$$

$q(x,t)$ is the distributed load, E_i is the Young's modulus, G_i is the transverse shear modulus, I_i is the moment of inertia, A_i is the area of the cross-section, k_i is the shear correction factor and ρ_i is the mass per unit volume. The index $i = 1, 2$ denotes the i-th span.

The notations c^- and c^+ imply the use of lateral limits. It can be observed that the strain energy due to the rotational restraint r_{12} is computed by means of the expression:

$$r_{12}\left(\phi(c^+,t) - \phi(c^-,t)\right)^2$$

which implies that the spring is connected at the right end of the first span and at the left end of the second span. Meanwhile the expression $r_c\left(\phi(c^-,t)\right)^2$ indicates that the rotational spring is connected at the right end of the first span and is connected to a fixed wall. An analogue situation arises for the translational elastic restrictions t_{12} and t_c. For a rigorous application of the calculus of variations, it is assumed that $t_{12} \to \infty$. Notice that this implies that w is continuous in $x = c$. Then, this spring is considered heuristically in order to compare values with those of Ref. [17]. The application of the techniques of the calculus of variations Ref. [20] leads to the conclusion that the functions w and ϕ must satisfy the following differential equations:

$$\begin{aligned} &-\frac{\partial}{\partial t}\left(\rho_i(x)A_i(x)\frac{\partial w(x,t)}{\partial t}\right) \\ &+\frac{\partial}{\partial x}\left[\kappa_i G_i(x)A_i(x)\left(\frac{\partial w(x,t)}{\partial x} - \phi(x,t)\right)\right] + q(x,t) = 0 \\ &-\frac{\partial}{\partial t}\left(\rho_i(x)I_i(x)\frac{\partial \phi(x,t)}{\partial t}\right) + \frac{\partial}{\partial x}\left(E_i(x)I_i(x)\frac{\partial \phi(x,t)}{\partial x}\right) \\ &+\kappa_i G_i(x)A_i(x)\left(\frac{\partial w(x,t)}{\partial x} - \phi(x,t)\right) = 0 \\ &\forall x \in D_i,\ i = 1,2,\ t \geq 0 \end{aligned} \tag{2}$$

The mentioned procedure also yields the following boundary and transition conditions:

$$t_1 w(0^+,t) = \kappa_1 G_1(0^+)A_1(0^+)\left(\frac{\partial w(0^+,t)}{\partial x} - \phi(0^+,t)\right) \tag{3}$$

$$r_1\phi(0^+,t) = E_1(0^+)I_1(0^+)\frac{\partial \phi(0^+,t)}{\partial x} \tag{4}$$

$$\begin{aligned} t_c w(c^-,t) = &\\ = \kappa_2 G_2(c^+)A_2(c^+)&\left(\frac{\partial w(c^+,t)}{\partial x} - \phi(c^+,t)\right) \\ -\kappa_1 G_1(c^-)A_1(c^-)&\left(\frac{\partial w(c^-,t)}{\partial x} - \phi(c^-,t)\right) \end{aligned} \tag{5}$$

$$\begin{aligned} r_{12}\left(\phi(c^+,t) - \phi(c^-,t)\right) - r_c\phi(c^-,t) = \\ = E_1(c^-)I_1(c^-)\frac{\partial \phi(c^-,t)}{\partial x} \end{aligned} \tag{6}$$

$$w(c^+,t) = w(c^-,t) \tag{7}$$

$$r_{12}\left(\phi(c^+,t) - \phi(c^-,t)\right) = E_2(c^+)I_2(c^+)\frac{\partial \phi(c^+,t)}{\partial x} \tag{8}$$

$$t_2 w(l^-,t) = = -\kappa_2 G_2(l^-) A_2(l^-) \left(\frac{\partial w(l^-,t)}{\partial x} - \phi(l^-,t) \right) \tag{9}$$

$$r_2 \phi(l^-,t) = -E_2(l^-) I_2(l^-) \frac{\partial \phi(l^-,t)}{\partial x} \tag{10}$$

Since the domain of definition of the problem is $D = (0,l)$ and this is an open interval in $\mathbb{R}$, the boundary is given by two points, i.e., $\partial D = \{0,l\}$. Consequently, c is an interior point and Eqs. (5)-(8) are the transition conditions. So, Eqs. (3), (4), (9) and (10) correspond to the boundary conditions.

In the present paper, it is possible to simulate the crack with an internal elastic rotational constrain adopting $0 < r_{12} < \infty$, $t_{12} = \infty$, and $t_c = r_c = 0$.

In Ref. [17], the proposed crack model includes a rotational and a translational elastic constrains and their values vary with the crack size. To represent this model in present work one must adopt $0 < r_{12} < \infty$, $0 < t_{12} < \infty$, $t_c = r_c = 0$, and Eq. (5) must be replaced with

$$t_{12}\left(w(c^+,t) - w(c^-,t) \right) = = \kappa_1 G_1(c^-) A_1(c^-) \left(\frac{\partial w(c^-,t)}{\partial x} - \phi(c^-,t) \right) \tag{11}$$

and Eq. (7) with

$$t_{12}\left(w(c^+,t) - w(c^-,t) \right) = = \kappa_2 G_2(c^+) A_2(c^+) \left(\frac{\partial w(c^+,t)}{\partial x} - \phi(c^+,t) \right) \tag{12}$$

3. Exact Solution

Using the well-known method of separation of variables, when the mass per unit length and the flexural rigidity at the *i*-th span are constant, we assume as solutions of Eq. (2) the expressions

$$w_i(x,t) = \sum_{n=1}^{\infty} W_{i,n}(x) \cos(\omega t),\ i = 1,2 \tag{13}$$

$$\phi_i(x,t) = \sum_{n=1}^{\infty} \Phi_{i,n}(x) \cos(\omega t),\ i = 1,2 \tag{14}$$

where $W_{i,n}(x)$ are the corresponding *n*-th modes of natural vibration.

In the case that $q(x,t) \equiv 0$, and considering that $E_i = E$, $I_i = I$, $A_i = A$, $G_i = G$, $\kappa_i = \kappa$, $\rho_i = \rho$, $i = 1, 2$, the general solutions of the differential Eq. (2) considering Eqs. (13) and (14) are given by

$$W_{1,n}(x) = C_1 \cosh(\beta_1 x) + C_2 \sinh(\beta_1 x) + C_3 \cos(\beta_2 x) + C_4 \sin(\beta_2 x) \tag{15}$$

$$\Phi_{1,n}(x) = C_1 m_1 \sinh(\beta_1 x) + C_2 m_1 \cosh(\beta_1 x) + C_3 m_2 \sin(\beta_2 x) - C_4 m_2 \cos(\beta_2 x) \tag{16}$$

$$W_{2,n}(x) = C_5 \cosh(\beta_1 x) + C_6 \sinh(\beta_1 x) + C_7 \cos(\beta_2 x) + C_8 \sin(\beta_2 x) \tag{17}$$

$$\Phi_{2,n}(x) = C_5 m_1 \sinh(\beta_1 x) + C_6 m_1 \cosh(\beta_1 x) + C_7 m_2 \sin(\beta_2 x) - C_8 m_2 \cos(\beta_2 x) \tag{18}$$

where

$$m_1 = \frac{\Omega^2 s + \beta_1^2}{\beta_1} \tag{19}$$

$$m_2 = \frac{\Omega^2 s - \beta_2^2}{\beta_2} \tag{20}$$

$$\beta_1 = \sqrt{\sqrt{a^2 - b} - a} \tag{21}$$

$$\beta_2 = \sqrt{\sqrt{a^2 - b} + a} \tag{22}$$

$$a = \frac{\Omega^2 (r + s)}{2} \tag{23}$$

$$b = \Omega^2 \left(\Omega^2 rs - 1 \right) \tag{24}$$

with $r = \dfrac{I}{Al^2}$, $s = \dfrac{E}{\kappa G} r$.

The coefficient $\Omega = \sqrt{\dfrac{\rho A}{EI}} \omega l^2$ is the dimensionless natural frequency parameter.

Substituting Eqs. (15)-(18) into the boundary and transition conditions given by Eqs. (3)-(10), we obtain a set of eight homogeneous equations in the constants C_i. Since the system is homogeneous for existence of a non-trivial solution, the determinant of coefficients must be equal to zero. This procedure yields the frequency equation:

$$G\left(T_i, R_i, T_c, R_c, T_{12}, R_{12}, \Omega, c\right) = 0 \tag{25}$$

where $T_i = \dfrac{t_i l^3}{EI}$, $T_c = \dfrac{t_c l^3}{EI}$, $T_{12} = \dfrac{t_{12} l^3}{EI}$, $R_i = \dfrac{r_i l}{EI}$, $R_c = \dfrac{r_c l}{EI}$, $R_{12} = \dfrac{r_{12} l}{EI}$, $i = 1, 2$. The values of the frequency parameter were obtained with the classical bisection method.

4. Comparison Studies and New Numerical Results

The terminology to be used throughout the remainder of the paper for describing the boundary conditions of the beam considered will now be introduced. In all tables and figures, the symbols F, S and C denote free, simply supported and clamped ends, respectively, and for example in the designation SC, the first symbol indicates that the boundary condition at $x = 0$ is a simply supported end and the second symbol indicates that at $x = l$ the beam is clamped.

In order to establish the accuracy and applicability of the approach developed and discussed in the previous sections, numerical results were computed for a number of problems for which comparison values were available in the literature. Additionally, new numerical results were generated for elastically restrained ends with an internal hinge.

Through all the present analysis, beams were modeled with shear correction factor k = 5/6 and Poisson's ratio $\mu = 0.3$.

A comparison study of the first four values of the dimensionless frequency parameter $\boldsymbol{\Omega}$ with those of Ref. [19] is presented in Table 1. A SS and a CC beam with a rigid intermediate support located at $c / l = 0.4$ for $h / l = 0.1$ are considered. An excellent agreement of numerical values can be observed.

Also in the present work, a comparison with the model used in Ref. [18] is presented. The discontinuity in the slope of the beam was modeled as

Table 1 Comparison study of the first four values of the frequency parameter $\boldsymbol{\Omega}$ of a two-span Timoshenko beam ($T_c \to \infty$, $T_{12} \to \infty$ and $R_{12} \to \infty$) located at c/l = 0.4 for h/l = 0.1 with those obtained in Ref. [19].

BC	Ref.	Ω_1	Ω_2	Ω_3	Ω_4
SS	Present work	31.3371	66.9552	103.9196	185.3183
	Reference	31.3370	66.9554	103.9200	185.3186
CC	Present work	44.8970	89.3751	120.2982	202.0519
	Reference	44.8968	89.3762	120.3006	202.0673

$$\left(\frac{\partial w_2}{\partial x} - \frac{\partial w_1}{\partial x}\right)\Bigg|_{x=c} = \theta \cdot \frac{\partial \phi_2}{\partial x}\Bigg|_{x=c} \tag{26}$$

where $\theta = 6\pi\eta^2 f(\eta)\left(h/l\right)$ is the non-dimensional crack sectional flexibility and depends on the extension of the crack, $\eta = a / h$ is the crack depth ratio where a is the crack depth. The function f adopted is given by

$$\begin{aligned} f(\eta) &= 0.6384 - 1.035\eta + 3.7201\eta^2 - 5.1773\eta^3 \\ &+7.553\eta^4 - 7.332\eta^5 + 2.4909\eta^6 \end{aligned} \tag{27}$$

To perform a comparison of frequency results with those obtained in Ref. [18], the relationship between the non-dimensional rigidity of the rotational spring R_{12} and the non-dimensional crack sectional flexibility was determined by

$$R_{12} = \frac{1}{\theta} \tag{28}$$

Tables 2 and 3 depict the results of the first four natural frequencies for a SS and SC beam with η equal to 0.20, 0.35 and 0.70, $c/l = 0.5$ and $h / l = 0.25$. Table 2 shows the results obtained in Ref. [18] and Table 3 shows the results obtained in the present work. The methodology proposed in Refs. [17, 18] has also been

Table 2 First four values of the frequency parameter $\boldsymbol{\Omega}$ of a two-span Timoshenko beam ($T_c = 0$, $T_{12} \to \infty$ and $0 < R_{12} < \infty$) located at c/l = 0.5 for h/l = 0.25 obtained from Ref. [18].

BC	η	R_{12}	Ω_1	Ω_2	Ω_3	Ω_4
SS	0.20	9.6689	8.2760	29.6610	52.1525	80.6253
	0.35	2.9396	7.1126	29.6610	48.9134	80.6253
	0.70	0.5185	4.2726	29.6610	43.9407	80.6253
SC	0.20	9.6689	12.0286	33.0248	54.3153	81.7510
	0.35	2.9396	11.1045	32.9459	51.0196	81.7050
	0.70	0.5185	9.1955	32.7702	46.0750	81.6194

Table 3 First four values of the frequency parameter $\boldsymbol{\Omega}$ of a two-span Timoshenko beam ($T_c = 0$, $T_{12} \to \infty$ and $0 < R_{12} < \infty$) located at c/l = 0.5 for h/l = 0.25 (present work).

BC	η	R_{12}	Ω_1	Ω_2	Ω_3	Ω_4
SS	0.20	9.6689	8.2733	29.6509	52.1349	80.5979
	0.35	2.9396	7.1102	29.6509	48.8968	80.5979
	0.70	0.5185	4.2711	29.6509	43.9256	80.5979
SC	0.20	9.6689	12.0246	33.0135	54.2969	81.7232
	0.35	2.9396	11.1007	32.9348	51.0023	81.6774
	0.70	0.5185	9.1919	32.7591	46.0592	81.5917

implemented and the following coefficients have been derived: $m_1 = \dfrac{\Omega^2 s + \beta_1^{\,2}}{\beta_1}$ and $m_2 = \dfrac{\Omega^2 s - \beta_2^{\,2}}{\beta_2}$. Unfortunately, it seems that the expressions of the m_i parameters have been incorrectly derived in Ref. [17].

Table 4 provides a comparison between the results obtained with the procedure proposed in Ref. [17] and the results obtained in the present work with the corresponding formulae. The results were obtained for a beam with E = 72 GPa, G = 27 GPa, c/l = 0.25, h/l = 0.25 and SS boundary condition. To represent the crack model proposed in Ref. [17], the values of the elastic constrains in the transition conditions were adopted as $t_{12} = \dfrac{l}{h}\dfrac{s}{r}\dfrac{\kappa GA}{q}$, $r_{12} = \dfrac{l}{h}\dfrac{EI}{\varphi}$, $t_c = 0$ and $r_c = 0$. The parameters that vary with crack depth were given by

$$q = \left(\frac{\eta}{1-\eta}\right)^2 \left(-0.22 + 3.82\,\eta + 1.54\,\eta^2 - 14.64\,\eta^3 + 9.60\,\eta^4\right) \tag{29}$$

$$\varphi = 2\left(\frac{\eta}{1-\eta}\right)^2 \left(5.93 - 19.69\,\eta + 37.14\,\eta^2 - 35.84\,\eta^3 + 13.12\,\eta^4\right) \tag{30}$$

In Table 4 the crack depth ratio considered was $\eta =$ 0.85.

Table 4 Comparison study of the first five values of the frequency parameter *Ω* which correspond to the crack model proposed in Ref. [17] and the present work with the crack model proposed in Ref. [17].

Mode sequence	Ref. [17]	Present work
1	2.18	2.31
2	17.57	18.99
3	46.76	41.70
4	79.93	60.29
5	81.00	88.38

Table 5 depicts the first four values of the frequency parameter *Ω* and their corresponding modal shapes of a two-span elastically restrained Timoshenko beam with $T_c = R_c = R_{12} = 0$ and $T_{12} \to \infty$, located at $c/l =$ 0.25 with $h/l =$0.3, $T_1 \to \infty$, $T_2 \to \infty$, and $R_1 = R_2 =$ 10.

Table 6 depicts the first four values of the frequency parameter *Ω* and their corresponding modal shapes of a two-span elastically restrained Timoshenko beam with $T_c = R_c = R_{12} = 0$ and $T_{12} \to \infty$, located at $c/l =$ 0.6 with $h/l =$ 0.3, $T_1 \to \infty$, $T_2 \to \infty$, and $R_1 = R_2 =$ 1,000.

Table 7 depicts the first four values of the frequency parameter *Ω* and their corresponding modal shapes of a two-span elastically restrained Timoshenko beam with $T_c = R_c = R_{12} = 0$ and $T_{12} \to \infty$, located at $c/l =$ 0.6 with $h/l =$ 0.3, $T_1 \to \infty$, $T_2 = 0$, $R_1 = 1000$ and $R_2 = 10$.

Table 8 depicts the first four values of the frequency parameter *Ω* and their corresponding modal shapes of a two-span elastically restrained Timoshenko beam with $R_{12} = 0$, $T_c \to \infty$, $R_c = 100$, and $T_{12} \to \infty$, located at $c/l =$ 0.7 with $h/l =$ 0.1, $T_c \to \infty$, $R_c = 0$, $T_{12} \to \infty$, $R_{12} = 100$, $T_1 \to \infty$, $R_1 \to \infty$, $T_2 = R_2 = 0$.

Table 5 First four values of the frequency parameter *Ω* and their corresponding modal shapes of an elastically restrained beam with a free internal hinge located at *c*/*l* = 0.25 with *h*/*l* = 0.3, *T*₁ → ∞, *T*₂ → ∞, and *R*₁ = *R*₂ = 10.

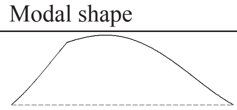
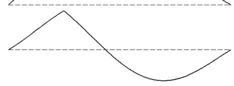
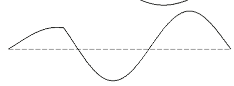
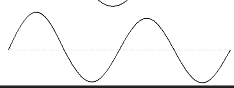
Mode sequence	*Ω*	Modal shape
1	3.5185	
2	4.8021	
3	6.8277	
4	8.4264	

Table 6 First four values of the frequency parameter *Ω* and their corresponding modal shapes of an elastically restrained beam with a free internal hinge located at *c*/*l* = 0.6 with *h*/*l* = 0.3, *T*₁ → ∞, *T*₂ → ∞, and *R*₁ = *R*₂ = 1000.

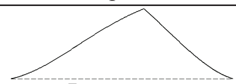
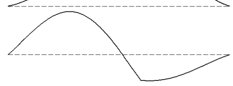
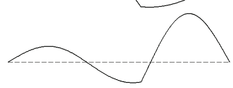
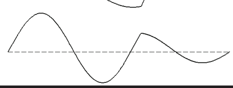
Mode sequence	*Ω*	Modal shape
1	3.4776	
2	5.3272	
3	6.7365	
4	7.8887	

Table 7 First four values of the frequency parameter Ω and their corresponding modal shapes of an elastically restrained beam with a free internal hinge located at c/l = 0.6 with h/l = 0.3, $T_1 \to \infty$, $T_2 = 0$, $R_1 = 1000$ and $R_2 = 10$.

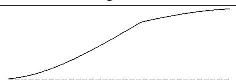
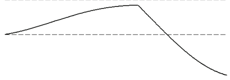
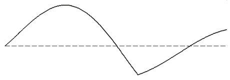
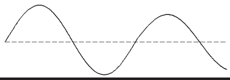
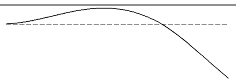
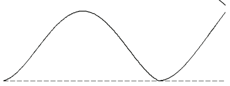
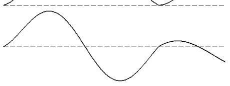
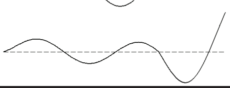
Mode sequence	Ω	Modal shape
1	2.0804	
2	3.9720	
3	5.3897	
4	7.6500	

Table 8 First four values of the frequency parameter Ω and their corresponding modal shapes of an elastically restrained beam with an internal hinge elastically restrained located at c/l = 0.7 with h/l = 0.1, $T_c \to \infty$, $R_c = 0$, $T_{12} \to \infty$, $R_{12} = 100$, $T_1 \to \infty$, $R_1 \to \infty$, $T_2 = R_2 = 0$.

Mode sequence	Ω	Modal shape
1	4.2467	
2	6.2195	
3	9.5334	
4	12.1018	

5. Conclusions

The free transverse vibrations of a Timoshenko beam with ends elastically restrained against rotation and translation, and arbitrarily located internal restrictions against rotation and translation are studied. For this purpose, an exact solution proposal was developed for the determination of natural frequencies. The algorithm is very general and it is characterized by its accuracy. The general intermediate elastic restraints implemented allow the analysis of several types of crack models. Close agreement with results presented by previous investigators is demonstrated for some examples and for a crack model.

These results obtained may provide useful information for structural designers and engineers. The algorithms developed can be easily extended to a beam with a greater number of intermediate points elastically restrained against rotation and translation.

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