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Boundary value problems for anisotropic plates with internal line hinges

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Abstract This paper deals with the formulation of an analytical model for the dynamic behavior of anisotropic plates, with an arbitrarily located internal line hinge with elastic supports and piecewise smooth boundaries elastically restrained against rotation and translation among other complicating effects. The equations of motion and its associated boundary and transition conditions are derived using Hamilton's principle. By introducing an adequate change of variables, the energies that correspond to the different elastic restraints are handled in a general framework. The concept of transition conditions and the determination of the analytical expressions are presented. Analytical examples are worked out to illustrate the range of applications of the developed analytical model. One of the essential features of this work is to demonstrate how the commonly formal derivations used in the applications of the calculus of variations can be made rigorous.

1 Introduction

It is well known that the calculus of variations is concerned with the problem of extremizing functionals, a generalization of the problem of finding extremes of functions of several variables. This discipline has a long history of interaction with other fields of mathematics and physics, particularly with mechanics. For centuries, scientists tried to formulate laws of natural sciences as extremal problems called variational principles and to use the techniques of the calculus of variations as mathematical tools to derive and investigate the motion and equilibrium in nature. Its applications now embrace a great variety of disciplines, such us optimal control, economics, quantum mechanics, etc. Engineers and applied mathematicians increasingly used the techniques of calculus of variations to solve a large number of problems. Nevertheless, in this discipline, the "operator" δ has been assigned special properties and handled using heuristic procedures. A mechanical "δ-method" has been developed and extensively used, as can be observed in the current engineering literature. Commonly, the domain of definition of a functional and the space of admissible directions of the variation of this functional are not clearly stated; thus, most of the analytical manipulations, for instance involving integration by parts, are confusing and not mathematically precise. The desire to fill this gap gave rise to the writing of this paper.

On the other hand, the calculus of variations has called the attention of several mathematicians, who made important contributions to its development, and we have reached a stage where many technical details are hardly available to a non-mathematician. Nevertheless, the concepts needed to its application in solid mechanics can be easily established. For instance, the notion of variation of a functional is a straightforward generalization of the definition of the directional derivative of a real-valued function defined on a subset of $\mathbb{R}^n$. This definition is applied in the present paper together with a clear specification of the domain of the *action integral*, which

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corresponds to a plate with an arbitrarily located internal line hinge, and the space of admissible directions of the first variation of this functional. Also, a complete rigorous application of Hamilton's principle is developed for the derivation of equations of motion and its associated boundary and transition conditions. All the energy functional terms are expressed by using definite integrals. This approach allows the direct application of the fundamental lemma of the calculus of variations and provides a better understanding of the mathematical manipulations in the case of non-smooth boundaries and, particularly when an internal line hinge is present.

Substantial literature has been devoted to the formulation, by means of the calculus of variations of boundary value problems in the statics and dynamics of isotropic plates [1–10]. On the other hand, several books are devoted to the study of anisotropic plates including the determination of static, buckling and vibrations characteristics [11–14]. It is not the intention to review the literature consequently; only some of the published papers related to the present work will be cited. A great number of articles treated the dynamical behavior of plates with complicating effects [15–19]. Most of the available mathematical theories about plates deal with smooth boundaries. However, the majority of the plate problems that arise in practice are naturally posed in domains whose geometry is simple but not smooth. This is the case of quadrilateral and triangular plates. In [20] and [21], the calculus of variations was used to derive the boundary value problems that describe the static and dynamic behaviors of anisotropic plates with corner points. Also, there is only a very limited amount for plates with internal hinges [22,23].

The present paper deals with anisotropic plates with an arbitrarily located internal line hinge with elastic supports and piecewise smooth boundaries elastically restrained against rotation and translation, among other complicating effects.

This paper is organized in the following way. In Sects. 2 and 3, a detailed treatment of techniques of the calculus of variations to obtain the governing differential equations, the boundary conditions and the transitions conditions is presented. In Sect. 4, the transitions conditions are analyzed. Several plate problems are treated, by using the derived general mathematical model, in Sect. 5. Finally, Sect. 6 contains the conclusions of this paper.

2 The variation of the energy functional

Let us consider an anisotropic plate that, in the equilibrium position, covers the two-dimensional domain G, with piecewise smooth boundary ∂G elastically restrained against rotation and translation. The plate has an intermediate line hinge elastically restrained against rotation and translation, as it is shown in Fig. 1.

In order to analyze the transverse displacements of the system under study, we suppose that the vertical position of the plate at any time t is described by the function $w = w(x_1, x_2, t)$, where $(x_1, x_2) \in \bar{G}$, $\bar{G} = G \cup \partial G$ and that the domain G is divided into two parts $G^{(1)}$ and $G^{(2)}$(with boundaries $\partial G^{(1)}$ and $\partial G^{(2)}$, respectively) by the line $\Gamma^{(c)}$ (see Fig. 1).

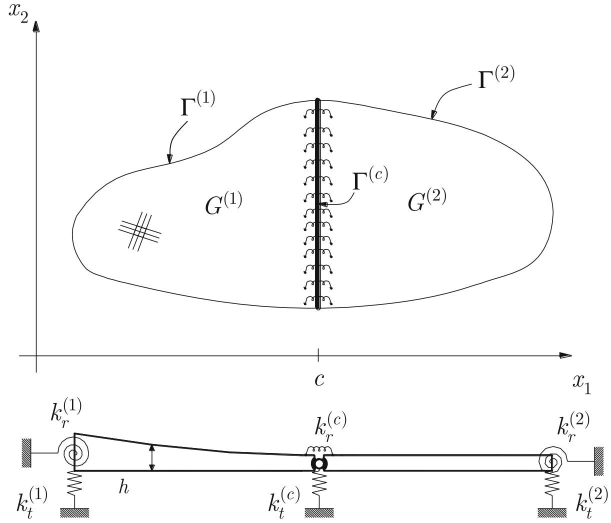

Fig. 1 Mechanical system under study

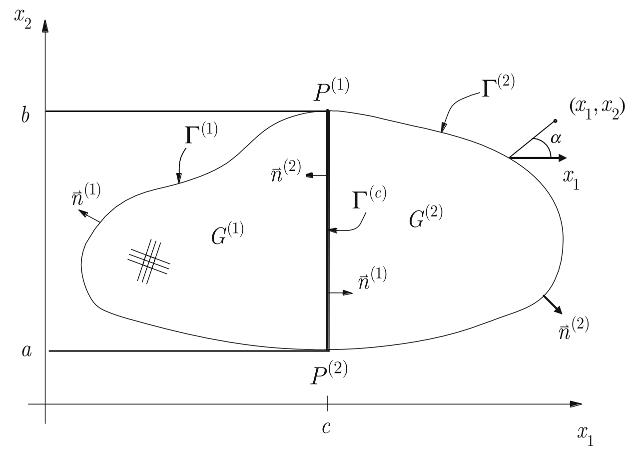

Fig. 2 Domains and boundaries

We assume that each subdomain $G^{(i)}$ consists of a finite number of smooth arcs and therefore possesses at most a finite number of corners. We consider here the case where different rigidities $D_{kl}^{(i)}(x_1, x_2)$ and mass density $\rho^{(i)}(x_1, x_2)h^{(i)}(x_1, x_2)$ of the anisotropic material correspond to the subdomains $G^{(1)}$ and $G^{(2)}$. The extreme points of the line $\Gamma^{(c)}$ divide the boundary curve ∂G into two arcs given by:

$$\Gamma^{(i)} = \partial G^{(i)} - \Gamma^{(c)}, \quad i = 1, 2 \quad \text{such that} \quad \partial G = \Gamma^{(1)} \cup \Gamma^{(2)}.$$

Let us assume that the boundary curve ∂G is described by a piecewise smooth path γ in $\mathbb{R}^2$ defined in the compact interval $[0, l]$, where $l = l(\partial G)$ is the length of the path γ. Then, γ is a continuous function $\gamma : [0, l] \to \mathbb{R}^2$; and the derivative γ' is continuous everywhere on $[0, l]$ except possibly at a finite number of points. The image of $[0, l]$ under γ (the graph of γ) is the boundary curve ∂G and will be denoted by $im(\gamma)$. We also assume that the curves $\Gamma^{(1)}$ and $\Gamma^{(2)}$ are described, respectively, by the piecewise smooth paths:

$$\gamma^{(i)} : \left[0, l^{(i)}\right] \to \mathbb{R}^2; \ \gamma^{(i)}(s) = \left(\gamma_1^{(i)}(s), \gamma_2^{(i)}(s)\right), s \in \left[0, l^{(i)}\right], \ i = 1, 2, \tag{1}$$

where s denotes the arc length measured from the point $P^{(i)}$ of the curve $\Gamma^{(i)}$ (see Fig. 2), $l^{(i)} = l(\Gamma^{(i)})$ is the length of the path $\gamma^{(i)}$ and the following is verified: $\gamma^{(1)}(l^{(1)}) = \gamma^{(2)}(0)$, $\gamma^{(1)}(0) = \gamma^{(2)}(l^{(2)})$.

Then, we have the function

$$\gamma(s) = \begin{cases} \gamma^{(1)}(s) & \text{if } s \in \left[0, l^{(1)}\right], \\ \gamma^{(2)}\left(s - l^{(1)}\right) & \text{if } s \in \left[l^{(1)}, l^{(1)} + l^{(2)}\right], \end{cases} \\ l = l^{(1)} + l^{(2)}, \tag{2}$$

where s denotes the arc length measured from the point $P^{(1)}$ of the curve ∂G.

That is, ∂G is described by the path γ given by (2) and the curves $\Gamma^{(1)}$ and $\Gamma^{(2)}$ are traced out by the paths $\gamma^{(1)}$ and $\gamma^{(2)}$ given by (1).

It is well known [24,25] that if $f : im(\beta) \to \mathbb{R}$ is a continuous function defined on the image Γ of a piecewise smooth path $\beta : [c, d] \to \mathbb{R}^2$, the curvilinear integral of f along Γ is given by:

$$\int_{\Gamma} f(x_1, x_2)\,\mathrm{d}s = \int_c^d (f \circ \beta)(r)\left\|\beta'(r)\right\|\mathrm{d}r, \tag{3}$$

where $(f \circ \beta)(r) = f(\beta(r))$ and the norm $\left\|\beta'(r)\right\|$ is given by $\left(\beta_1'^2(r) + \beta_2'^2(r)\right)^{1/2}$. In the case of a real continuous function f defined on the image of the path γ (given by (2)), i.e. the boundary curve $\partial\Omega$, the definition (3) when s is taken as the parameter r leads to:

$$\int_{\partial G} f\left(x_1, x_2\right) \mathrm{d}s = \int_0^l (f \circ \gamma)(s) \left\|\gamma'(s)\right\| \mathrm{d}s = \int_0^l (f \circ \gamma)(s) \mathrm{d}s \tag{4}$$

and

$$\int_{\partial G} f\left(x_1, x_2\right) \mathrm{d}s = \int_{\Gamma^{(1)}} f\left(x_1, x_2\right) \mathrm{d}s + \int_{\Gamma^{(2)}} f\left(x_1, x_2\right) \mathrm{d}s. \tag{5}$$

The additive property (5) will prove valuable when defining functions and functionals over $\partial\Omega$, since they can be established independently for $\Gamma^{(1)}$ and $\Gamma^{(2)}$ by using Eq. (1). Thus, we assume that the rotational rigidities of the elastic restrains along the boundary are given by the functions: $k_r^{(i)} : im(\gamma^{(i)}) \to \mathbb{R}, \quad i = 1, 2$ and $k_r^{(c)} : im(\gamma^{(c)}) \to \mathbb{R}$, where $\gamma^{(c)}$ is the path that describes the line $\Gamma^{(c)}$. In the same manner, the translational rigidities are given by the functions

$$k_t^{(i)} : im(\gamma^{(i)}) \to \mathbb{R},\ i = 1, 2 \quad \text{and} \quad k_t^{(c)} : im(\gamma^{(c)}) \to \mathbb{R}.$$

At time t, the kinetic energy of the plate is given by

$$E_K(w) = \frac{1}{2} \sum_{i=1}^{2} \int_{G^{(i)}} \rho^{(i)} h^{(i)} \left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x, \tag{6}$$

where $x = (x_1, x_2)$, $\mathrm{d}x = \mathrm{d}x_1 \mathrm{d}x_2$, $w = w(x, t)$, $\rho^{(i)} h^{(i)} = \left(\rho^{(i)} h^{(i)}\right)(x)$.

On the other hand, at time t, the total potential energy due to the elastic deformation of the plate deformed by a load of density $q = q(x, t)$ acting on $\bar{G}$, the deformation of the elastic restraints on the boundary ∂G and the elastic restraints at the intermediate line $\Gamma^{(c)}$ is given by

$$\begin{aligned}
E_D(w) = \frac{1}{2} \sum_{i=1}^{2} \Bigg\{ \int_{G^{(i)}} \Bigg[& D_{11}^{(i)} \left(\frac{\partial^2 w}{\partial x_1^2}\right)^2 + 2 D_{12}^{(i)} \frac{\partial^2 w}{\partial x_1^2} \frac{\partial^2 w}{\partial x_2^2} + D_{22}^{(i)} \left(\frac{\partial^2 w}{\partial x_2^2}\right)^2 \\
& + 4 \frac{\partial^2 w}{\partial x_1 \partial x_2} \left(D_{16}^{(i)} \frac{\partial^2 w}{\partial x_1^2} + D_{26}^{(i)} \frac{\partial^2 w}{\partial x_2^2}\right) + 4 D_{66}^{(i)} \left(\frac{\partial^2 w}{\partial x_1 \partial x_2}\right)^2 - 2 q^{(i)} w \Bigg] \mathrm{d}x \\
& + \int_{\Gamma^{(i)}} k_t^{(i)}(x)\, w^2(x, t)\, \mathrm{d}s + \int_{\Gamma^{(i)}} k_r^{(i)}(x) \left(\frac{\partial w}{\partial \vec{n}^{(i)}}(x, t)\right)^2 \mathrm{d}s \Bigg\} \\
& + \frac{1}{2} \int_{\Gamma^{(c)}} k_t^{(c)}(x)\, w^2(x, t)\, \mathrm{d}s + \frac{1}{2} \int_{\Gamma^{(c)}} k_r^{(c)}(x) \left[\frac{\partial w}{\partial x_1}\right]^2 \mathrm{d}s,
\end{aligned} \tag{7}$$

where $w = w(x, t)$, $D_{kl}^{(i)} = D_{kl}^{(i)}(x)$ and $\partial w / \partial \vec{n}^{(i)}$ is the directional derivative of w with respect to the outward normal unit vector $\vec{n}^{(i)}$ to the curve $\Gamma^{(i)}$. The outward unit normal vector $\vec{n}$ to the boundary ∂G is given by

$$\vec{n}(x) = \begin{cases} \vec{n}^{(1)}(x) & \text{if } x \in \Gamma^{(1)}, \\ \vec{n}^{(2)}(x) & \text{if } x \in \Gamma^{(2)}, \end{cases} \tag{8}$$

where $\vec{n}^{(i)}(x) = \left(n_1^{(i)}(x), n_2^{(i)}(x)\right)$, $i = 1, 2$ and

$$\begin{aligned}
& n_1^{(i)}(x)\Big|_{x \in \Gamma^{(i)}} = n_1^{(i)}\left(\gamma_1^{(i)}(s), \gamma_2^{(i)}(s)\right) = \cos\alpha \left(\gamma_1^{(i)}(s), \gamma_2^{(i)}(s)\right), \\
& n_2^{(i)}(x)\Big|_{x \in \Gamma^{(i)}} = n_2^{(i)}\left(\gamma_1^{(i)}(s), \gamma_2^{(i)}(s)\right) = \sin\alpha \left(\gamma_1^{(i)}(s), \gamma_2^{(i)}(s)\right), \\
& s \in \left[0, l^{(i)}\right], \quad i = 1, 2,
\end{aligned} \tag{9.1, 2}$$

where α denotes the angle made by the outward normal $\vec{n}^{(i)}$ to $\Gamma^{(i)}$ with the positive x_1 in the point $\left(\gamma_1^{(i)}(s), \gamma_2^{(i)}(s)\right)$ as it is shown in Fig. 2. We assume that α is a continuous function of s except possibly at a finite number of corners. The symbol $[\partial w/\partial x_1]$, used in (7), denotes the difference

$$\left[\frac{\partial w}{\partial x_1}\right] = \left.\frac{\partial w}{\partial x_1}\right|_{(+)} (c, x_2, t) - \left.\frac{\partial w}{\partial x_1}\right|_{(-)} (c, x_2, t)\,. \tag{10}$$

Since w is a continuous function, the lateral derivatives in (10) are defined as

$$\left.\frac{\partial w}{\partial x_1}\right|_{(-)} (c, x_2, t) = \lim_{\substack{x_1 \to c \\ x_1 < c}} \frac{w(x_1, x_2, t) - w(c, x_2, t)}{x_1 - c},$$

$$\left.\frac{\partial w}{\partial x_1}\right|_{(+)} (c, x_2, t) = \lim_{\substack{x_1 \to c \\ x_1 > c}} \frac{w(x_1, x_2, t) - w(c, x_2, t)}{x_1 - c}.$$

Finally, we have

$$q(x,t) = \begin{cases} q^{(1)}(x,t) & \text{if } x \in \bar{G}^{(1)}, \\ q^{(2)}(x,t) & \text{if } x \in \bar{G}^{(2)}. \end{cases}$$

It can be observed that the strain energy due to the rotational restraint of the internal line hinge in (7) is computed by

$$\frac{1}{2}\int_{\Gamma^{(c)}} k_r^{(c)}(x) \left(\left.\frac{\partial w}{\partial x_1}\right|_{(+)} (c, x_2, t) - \left.\frac{\partial w}{\partial x_1}\right|_{(-)} (c, x_2, t) \right)^2 \mathrm{d}s,$$

which implies that the distributed springs are connected at points of $G^{(1)}$ and at points of $G^{(2)}$.

It is convenient from now on to introduce a change of variables in order to deal with the points that correspond to the curves $\Gamma^{(1)}$ and $\Gamma^{(2)}$. Let us consider the new variables (y_1, y_2) where y_1 is a distance measured from the boundary and along the normal to ∂G and y_2 is the arc length measured from the point $P^{(1)}$ of the boundary ∂G (see Fig. 2). More specifically, given a point (x_1, x_2), the new variable y_2 is obtained by determining first the shortest normal to ∂G through (x_1, x_2) and then the intersection of this normal with ∂G. The distance from (x_1, x_2) to ∂G along the normal determines the new variable y_1. It must be noted that $y_1 > 0$ if $(x_1, x_2) \notin \bar{G}$ and $y_1 \le 0$ if $(x_1, x_2) \in \bar{G}$. It is assumed that through each point (x_1, x_2) there be a uniquely determined shortest normal to ∂G. We can, without loss of generality, impose this condition since the new variables y_1, y_2 will be used only in the evaluation of functions on ∂G, thus for $y_1 = 0$. Since the boundary curve ∂G is described by the path given by (2), let us consider the following functions:

$$g^{(i)} : \Omega^{(i)} \to \mathbb{R}^2, \Omega^{(i)} \subset \mathbb{R}^2, \quad u : A \to \mathbb{R}, A \subset \mathbb{R}^2, \quad g^{(i)}\left(\Omega^{(i)}\right) \subset A, \quad i = 1, 2$$

where

$$\Omega^{(i)} = \left\{(y_1, y_2), \quad y_1 \in \left[a^{(i)}, \infty\right), \quad y_2 \in \left[0, l^{(i)}\right]\right\}, \quad a^{(i)} \le 0 \tag{11}$$

and

$$\begin{aligned} &(y_1, y_2) \xrightarrow{g^{(i)}} (x_1, x_2) \xrightarrow{u^{(i)}} z = u^{(i)}\left[g_1^{(i)}(y_1, y_2), g_2^{(i)}(y_1, y_2)\right] = \tilde{u}^{(i)}(y_1, y_2), \\ &(y_1, y_2) \in \Omega^{(i)}, \quad i = 1, 2, \end{aligned}$$

where $\tilde{u}^{(i)} = u^{(i)} \circ g^{(i)}$ denotes the composition of $u^{(i)}$ and $g^{(i)}$. Throughout this paper, this notation will be used for different composite functions. The function u is given by:

$$u(x) = \begin{cases} u^{(1)}(x) & \text{if } x \in \bar{G}^{(1)}, \\ u^{(2)}(x) & \text{if } x \in \bar{G}^{(2)}. \end{cases}$$

We have the occasion below to employ u replaced by w, the function which is defined only on $\bar{G}$, is also dependent on the variable t and describes the vertical position of the plate. The components of functions $g^{(i)}$ are defined by:

$$\begin{aligned} &g_1^{(i)}(y_1, y_2) = \gamma_1^{(i)}(y_2) + y_1 \cos\alpha\left(\gamma_1^{(i)}(y_2), \gamma_2^{(i)}(y_2)\right), \\ &g_2^{(i)}(y_1, y_2) = \gamma_2^{(i)}(y_2) + y_1 \sin\alpha\left(\gamma_1^{(i)}(y_2), \gamma_2^{(i)}(y_2)\right), \\ &(y_1, y_2) \in \Omega^{(i)}, \quad i = 1, 2, \end{aligned} \tag{12.1, 2}$$

where α is the angle defined in (9.1, 2). These expressions can now be rewritten as

$$g_j^{(i)}(y_1, y_2) = \gamma_j^{(i)}(y_2) + y_1 \tilde{n}_j^{(i)}(y_2), \quad (y_1, y_2) \in \Omega^{(i)}, \quad i, j = 1, 2, \tag{13}$$

where $\tilde{n}_j^{(i)} = n_j^{(i)} \circ \gamma^{(i)}$.

The points of the curves $\Gamma^{(1)}$ and $\Gamma^{(2)}$ are determined from (13) as:

$$x_j = g_j^{(i)}(0, y_2) = \gamma_j^{(i)}(y_2), \quad y_2 \in \left[0, l^{(i)}\right], \quad x_j \in \Gamma^{(i)}, \quad i, j = 1, 2. \tag{14}$$

These functions also give natural parametric representations for the first two curvilinear integrals of the functional (7). Thus, from Eqs. (4) and (14), it follows that

$$\begin{aligned} \int_{\Gamma^{(i)}} k_t^{(i)}(x)\, w^2(x, t)\, \mathrm{d}s &= \int_0^{l^{(i)}} \left(k_t^{(i)} \circ g^{(i)}\right)(0, y_2)\left(\left(w \circ g^{(i)}\right)(0, y_2, t)\right)^2 \mathrm{d}y_2 \\ &= \int_0^{l^{(i)}} \tilde{k}_t^{(i)}(0, y_2)\, \tilde{w}^2(0, y_2, t)\mathrm{d}y_2 \end{aligned} \tag{15}$$

and

$$\begin{aligned} &\int_{\Gamma^{(i)}} k_r^{(i)}(x)\left(\frac{\partial w}{\partial \vec{n}^{(i)}}(x, t)\right)^2 \mathrm{d}s \\ &= \int_0^{l^{(i)}} \left(k_r^{(i)} \circ g^{(i)}\right)(0, y_2)\left(\left(\frac{\partial w}{\partial \vec{n}^{(i)}} \circ g^{(i)}\right)(0, y_2, t)\right)^2 \mathrm{d}y_2. \end{aligned} \tag{16}$$

In Eq. (15), the expression $\left(w \circ g^{(i)}\right)(0, y_2, t)$ denotes the composition given by $w\left(g_1^{(i)}(0, y_2), g_2^{(i)}(0, y_2), t\right)$ and similarly for $\left(\frac{\partial w}{\partial \vec{n}^{(i)}} \circ g^{(i)}\right)(0, y_2, t)$ in Eq. (16). By the formula of derivation of composite functions, we have

$$\frac{\partial \tilde{w}}{\partial y_1}(y_1, y_2, t) = \sum_{j=1}^{2} \frac{\partial w}{\partial x_j}(x_1, x_2, t)\frac{\partial g_j^{(i)}}{\partial y_1}(y_1, y_2),$$

where $x_k = g_k^{(i)}(y_1, y_2)$, $k = 1, 2$ and $(y_1, y_2) \in \Omega^{(i)}$, $i = 1, 2$. From (13), it follows that

$$\frac{\partial g_j^{(i)}}{\partial y_1}(y_1, y_2) = \tilde{n}_j^{(i)}(y_2), \quad i, j = 1, 2 \tag{17}$$

and

$$\frac{\partial \tilde{w}}{\partial y_1}(y_1, y_2, t) = \sum_{j=1}^{2} \frac{\partial w}{\partial x_j}(x_1, x_2, t)\tilde{n}_j^{(i)}(y_2), \quad x_k = g_k^{(i)}(y_1, y_2), \quad k = 1, 2,$$
$$(y_1, y_2) \in \Omega^{(i)}, \quad i = 1, 2. \tag{18}$$

It must be noted that in (18) the condition $(y_1, y_2) \in \Omega^{(i)}$ clearly indicates which restriction of w corresponds. In particular, we have

$$\left.\frac{\partial \tilde{w}}{\partial y_1}(0, y_2, t)\right|_{y_2 \in [0, l^{(i)}]} = \sum_{j=1}^{2} \frac{\partial w}{\partial x_j}\left(g_1^{(i)}(0, y_2), g_2^{(i)}(0, y_2), t\right)\tilde{n}_j^{(i)}(y_2), \quad i = 1, 2. \tag{19}$$

The directional derivative involved in (16) is commonly determined by

$$\left.\frac{\partial w}{\partial \vec{n}^{(i)}}(x, t)\right|_{x \in \Gamma^{(i)}} = \sum_{j=1}^{2} \left.\frac{\partial w}{\partial x_j}(x, t)\right|_{x \in \Gamma^{(i)}} \tilde{n}_j^{(i)}(y_2)$$
$$= \sum_{j=1}^{2} \frac{\partial w}{\partial x_j}\left(g_1^{(i)}(0, y_2), g_2^{(i)}(0, y_2), t\right)\tilde{n}_j^{(i)}(y_2), \quad i = 1, 2. \tag{20}$$

From Eqs. (19) and (20), we have

$$\left(\frac{\partial w}{\partial \vec{n}^{(i)}} \circ g^{(i)}\right)(0, y_2, t) = \left.\frac{\partial \tilde{w}}{\partial y_1}(0, y_2, t)\right|_{y_2 \in [0, l^{(i)}]}, \quad i = 1, 2. \tag{21}$$

By substituting (21) into (16), we obtain

$$\int_{\Gamma^{(i)}} k_r^{(i)}(x)\left(\frac{\partial w}{\partial \vec{n}^{(i)}}(x, t)\right)^2 \mathrm{d}s = \int_0^{l^{(i)}} \tilde{k}_r^{(i)}(0, y_2)\left(\frac{\partial \tilde{w}}{\partial y_1}(0, y_2, t)\right)^2 \mathrm{d}y_2. \tag{22}$$

A simple parametric representation of the last two terms of functional (7) can be obtained by using

$$\gamma^{(c)}(x_2) = \left(\gamma_1^{(c)}(x_2), \gamma_2^{(c)}(x_2)\right) = (c, x_2), \; x_2 \in [a, b], \tag{23}$$

where $a = \gamma_2^{(1)}(l^{(1)}), \; b = \gamma_2^{(1)}(0)$ (see Fig. 2). Thus, we have

$$\int_{\Gamma^{(c)}} k_t^{(c)}(x)\, w^2(x, t)\, \mathrm{d}s = \int_a^b k_t^{(c)}(c, x_2)\, w^2(c, x_2, t)\mathrm{d}x_2, \tag{24}$$

$$\int_{\Gamma^{(c)}} k_r^{(c)}(x)\left[\frac{\partial w}{\partial x_1}\right]^2 \mathrm{d}s = \int_a^b k_r^{(c)}(c, x_2)\left(\left.\frac{\partial w}{\partial x_1}\right|_{(+)}(c, x_2, t) - \left.\frac{\partial w}{\partial x_1}\right|_{(-)}(c, x_2, t)\right)^2 \mathrm{d}x_2. \tag{25}$$

Hamilton's principle requires that between times t_0 and t_1, at which the positions of the mechanical system are known, it should execute a motion that makes stationary the functional $F(w) = \int_{t_0}^{t_1}(E_K - E_D)\,\mathrm{d}t$, on the space of admissible functions. In consequence, from (6) and (7), it follows that the action integral to be considered is given by

$$F(w)=\frac{1}{2}\int_{t_0}^{t_1}\left\{\sum_{i=1}^{2}\left[\int_{G^{(i)}}\left(\rho^{(i)}h^{(i)}\left(\frac{\partial w}{\partial t}\right)^2-D_{11}^{(i)}\left(\frac{\partial^2 w}{\partial x_1^2}\right)^2\right.\right.\right.$$
$$-2D_{12}^{(i)}\frac{\partial^2 w}{\partial x_1^2}\frac{\partial^2 w}{\partial x_2^2}-D_{22}^{(i)}\left(\frac{\partial^2 w}{\partial x_2^2}\right)^2-4\frac{\partial^2 w}{\partial x_1\partial x_2}\left(D_{16}^{(i)}\frac{\partial^2 w}{\partial x_1^2}+D_{26}^{(i)}\frac{\partial^2 w}{\partial x_2^2}\right)$$
$$\left.-4D_{66}^{(i)}\left(\frac{\partial^2 w}{\partial x_1\partial x_2}\right)^2+2q^{(i)}w\right)\mathrm{d}x-\int_0^{l^{(i)}}\tilde{k}_t^{(i)}(0,y_2)\,\tilde{w}^2(0,y_2,t)\mathrm{d}y_2$$
$$\left.\left.-\int_0^{l^{(i)}}\tilde{k}_r^{(i)}(0,y_2)\left(\frac{\partial\tilde{w}}{\partial y_1}(0,y_2,t)\right)^2\mathrm{d}y_2\right]-R\right\}\mathrm{d}t, \tag{26}$$

where

$$R=\int_a^b k_t^{(c)}(c,x_2)\,w^2(c,x_2,t)\mathrm{d}x_2$$
$$+\int_a^b k_r^{(c)}(c,x_2)\left(\left.\frac{\partial w}{\partial x_1}\right|_{(+)}(c,x_2,t)-\left.\frac{\partial w}{\partial x_1}\right|_{(-)}(c,x_2,t)\right)^2\mathrm{d}x_2.$$

The condition of stationary functional for (26) requires that

$$\delta F(w;v)=0,\quad \forall v\in D_a, \tag{27}$$

where $\delta F(w;v)$ is the first variation of F at w in the direction v and D_a is the space of admissible directions at w for the domain D of this functional.

The definition of the variation of F at w in the direction v is given as a generalization of the definition of the directional derivative of a real-valued function defined on a subset of $\mathbb{R}^n$, [10]. Consequently, the definition of the first variation of F at w in the direction v is given by

$$\delta F(w;v)=\left.\frac{\mathrm{d}F}{\mathrm{d}\varepsilon}(w+\varepsilon v)\right|_{\varepsilon=0}. \tag{28}$$

In order to make the mathematical developments required by the application of the techniques of the calculus of variations, we assume that

$$\rho^{(i)}h^{(i)}\in C\left(\bar{G}^{(i)}\right),\ q^{(i)}(\bullet,t)\in C\left(\bar{G}^{(i)}\right),\ D_{kl}^{(i)}\in C^2\left(\bar{G}^{(i)}\right), w(x,\bullet)\in C^2[t_0,t_1],$$
$$w(\bullet,t)\in C\left(\bar{G}\right),\ \text{and}\ \ w(\bullet,t)|_{\bar{G}^{(i)}}\in C^4\left(\bar{G}^{(i)}\right),\ \bar{G}^{(i)}=G^{(i)}\cup\partial G^{(i)},\quad i=1,2,$$

where the following notation has been used: $C^k(S)$ denotes the set of all real functions $u:G\to\mathbb{R}$ that have continuous partial derivatives of orders $m=0,1,2,\ldots,k$, and $C^k\left(\bar{S}\right)$ denotes the set of all $u\in C^k(S)$ for which all partial derivatives of order $m=0,1,2,\ldots,k$, can be extended continuously to the closure $\bar{S}$ of S. It must be noted that as a consequence of the presence of the line hinge, the derivative $\partial w/\partial x_1$ and the corresponding derivatives of greater order do not necessarily exist in the domain G, so it is necessary to impose the conditions $w(\bullet,t)|_{\bar{G}^{(i)}}\in C^4\left(\bar{G}^{(i)}\right),\ i=1,2$.

In view of all these observations and since Hamilton's principle requires that at times t_0 and t_1 the positions are known, the space D is given by

$$\begin{aligned} D = \Big\{ & w;\, w(x,\bullet) \in C^2[t_0,t_1],\, w(\bullet,t) \in C\left(\bar{G}\right),\ w(\bullet,t)|_{\bar{G}^{(i)}} \in C^4\left(\bar{G}^{(i)}\right), \\ & i=1,2,\ w(x,t_0),\, w(x,t_1)\ \text{prescribed}\Big\}. \end{aligned} \tag{29}$$

The only admissible directions v at $w \in D$ are those for which $w + \varepsilon v \in D$ for all sufficiently small ε and $\delta F(w;v)$ exists. In consequence, and in view of (29), v is an admissible direction at w for D if, and only if, $v \in D_a$ where

$$\begin{aligned} D_a = \Big\{ & v;\, v(x,\bullet) \in C^2[t_0,t_1],\, v(\bullet,t) \in C\left(\bar{G}\right),\ v(\bullet,t)|_{\bar{G}^{(i)}} \in C^4\left(\bar{G}^{(i)}\right), \\ & i=1,2,\ v(x,t_0) = v(x,t_1) = 0, \quad \forall x \in \bar{G}\Big\}. \end{aligned} \tag{30}$$

Performing the derivative (28) with F given by (26), together with the classical decomposition of symmetric terms, we have

$$\begin{aligned} \delta F(w;v) = \int_{t_0}^{t_1} \Bigg\{ \sum_{i=1}^{2} \Bigg[\int_{G^{(i)}} \Bigg(& \rho^{(i)} h^{(i)} \frac{\partial w}{\partial t}\frac{\partial v}{\partial t} - D_{11}^{(i)} \frac{\partial^2 w}{\partial x_1^2}\frac{\partial^2 v}{\partial x_1^2} - D_{12}^{(i)}\left(\frac{\partial^2 w}{\partial x_1^2}\frac{\partial^2 v}{\partial x_2^2} + \frac{\partial^2 w}{\partial x_2^2}\frac{\partial^2 v}{\partial x_1^2}\right) \\ & - D_{22}^{(i)} \frac{\partial^2 w}{\partial x_2^2}\frac{\partial^2 v}{\partial x_2^2} - 2D_{16}^{(i)}\left(\frac{\partial^2 w}{\partial x_1 \partial x_2}\frac{\partial^2 v}{\partial x_1^2} + \frac{1}{2}\frac{\partial^2 w}{\partial x_1^2}\frac{\partial}{\partial x_1}\left(\frac{\partial v}{\partial x_2}\right) + \frac{1}{2}\frac{\partial^2 w}{\partial x_1^2}\frac{\partial}{\partial x_2}\left(\frac{\partial v}{\partial x_1}\right)\right) \\ & - 2D_{26}^{(i)}\left(\frac{\partial^2 w}{\partial x_1 \partial x_2}\frac{\partial^2 v}{\partial x_2^2} + \frac{1}{2}\frac{\partial^2 w}{\partial x_2^2}\frac{\partial}{\partial x_1}\left(\frac{\partial v}{\partial x_2}\right) + \frac{1}{2}\frac{\partial^2 w}{\partial x_2^2}\frac{\partial}{\partial x_2}\left(\frac{\partial v}{\partial x_1}\right)\right) - 2D_{66}^{(i)}\left(\frac{\partial^2 w}{\partial x_1 \partial x_2}\frac{\partial}{\partial x_1}\left(\frac{\partial v}{\partial x_2}\right)\right. \\ & \left. + \frac{\partial^2 w}{\partial x_2 \partial x_1}\frac{\partial}{\partial x_2}\left(\frac{\partial v}{\partial x_1}\right)\right) + q^{(i)} v \Bigg) \mathrm{d}x - \int_0^{l^{(i)}} \tilde{k}_t^{(i)}(0,y_2)\, \tilde{w}(0,y_2,t)\, \tilde{v}(0,y_2,t) \mathrm{d}y_2 \\ & - \int_0^{l^{(i)}} \tilde{k}_r^{(i)}(0,y_2) \frac{\partial \tilde{w}}{\partial y_1}(0,y_2,t) \frac{\partial \tilde{v}}{\partial y_1}(0,y_2,t) \mathrm{d}y_2 \Bigg] - \mathrm{RV} \Bigg\} \mathrm{d}t, \end{aligned} \tag{31}$$

where

$$\mathrm{RV} = \int_a^b k_t^{(c)}(c,x_2)\, w(c,x_2,t)\, v(c,x_2,t) \mathrm{d}x_2 + \int_a^b k_r^{(c)}(c,x_2) \left[\frac{\partial w}{\partial x_1}\right]\left[\frac{\partial v}{\partial x_1}\right] \mathrm{d}x_2, \tag{32}$$

with $\left[\frac{\partial v}{\partial x_1}\right] = \left.\frac{\partial v}{\partial x_1}\right|_{(+)}(c,x_2,t) - \left.\frac{\partial v}{\partial x_1}\right|_{(-)}(c,x_2,t)$.

Let us consider the first term of (31). Since $w(x,\bullet),\, v(x,\bullet) \in C^2[t_0,t_1]$, we can integrate by parts with respect to t, and if we apply the conditions imposed in (30), i.e.: $v(x,t_0) = v(x,t_1) = 0, \forall x \in \bar{G}$, we obtain

$$\begin{aligned} \int_{t_0}^{t_1} \int_{G^{(i)}} \rho^{(i)} h^{(i)} \frac{\partial w}{\partial t}\frac{\partial v}{\partial t} \mathrm{d}x \mathrm{d}t &= \int_{G^{(i)}} \rho^{(i)} h^{(i)} \frac{\partial w}{\partial t} v \Bigg|_{t_0}^{t_1} \mathrm{d}x, \\ - \int_{t_0}^{t_1} \int_{G^{(i)}} \rho^{(i)} h^{(i)} \frac{\partial^2 w}{\partial t^2} v \mathrm{d}x \mathrm{d}t &= - \int_{t_0}^{t_1} \int_{G^{(i)}} \rho^{(i)} h^{(i)} \frac{\partial^2 w}{\partial t^2} v \mathrm{d}x \mathrm{d}t. \end{aligned} \tag{33}$$

To transform the terms of (31) that are multiplied by a coefficient $D_{kl}^{(i)}$, we employ the well-known Green's formula:

$$\int_G u(x)\frac{\partial v}{\partial x_j}(x)\mathrm{d}x = \int_{\partial G} u(x)\,v(x)\,n_j(x)\,\mathrm{d}s - \int_G v(x)\frac{\partial u}{\partial x_j}(x)\mathrm{d}x, \qquad (34)$$
$$j = 1, 2, u, v \in C^{(1)}(\bar{G}),$$

where n_j denotes the $j - th$ component of the outward unit normal to the boundary ∂G and the curve that corresponds to the boundary is a piecewise smooth Jordan curve [24]. We have, upon applying (34) twice by succession and using (33)

$$\begin{aligned}\delta F(w; v) = \int_{t_0}^{t_1}\Bigg\{\sum_{i=1}^{2}\Bigg[\int_{G^{(i)}}\Bigg(&-\rho^{(i)}h^{(i)}\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2 M_1^{(i)}}{\partial x_1^2} + \frac{\partial^2 M_2^{(i)}}{\partial x_2^2} + 2\frac{\partial^2 H_{12}^{(i)}}{\partial x_1 \partial x_2}\\ &+ q^{(i)}\Bigg) v\mathrm{d}x + \int_{\partial G^{(i)}} M_{12}^{(i)}\mathrm{d}s - \int_{\partial G^{(i)}}\left(\sum_{j=1}^{2} N_j^{(i)} n_j^{(i)}\right) v\mathrm{d}s\\ &- \int_0^{l^{(i)}} \tilde{k}_t^{(i)}(0, y_2)\,\tilde{w}(0, y_2, t)\,\tilde{v}(0, y_2, t)\mathrm{d}y_2\\ &- \int_0^{l^{(i)}} \tilde{k}_r^{(i)}(0, y_2)\frac{\partial \tilde{w}}{\partial y_1}(0, y_2, t)\frac{\partial \tilde{v}}{\partial y_1}(0, y_2, t)\mathrm{d}y_2\Bigg] - \mathrm{RV}\Bigg\}\mathrm{d}t, \end{aligned} \qquad (35)$$

where RV is given by (32) and

$$M_1^{(i)} = -\left(D_{11}^{(i)}\frac{\partial^2 w}{\partial x_1^2} + D_{12}^{(i)}\frac{\partial^2 w}{\partial x_2^2} + 2D_{16}^{(i)}\frac{\partial^2 w}{\partial x_1 \partial x_2}\right), \qquad (36)$$

$$M_2^{(i)} = -\left(D_{22}^{(i)}\frac{\partial^2 w}{\partial x_2^2} + D_{12}^{(i)}\frac{\partial^2 w}{\partial x_1^2} + 2D_{26}^{(i)}\frac{\partial^2 w}{\partial x_1 \partial x_2}\right), \qquad (37)$$

$$H_{12}^{(i)} = -\left(D_{16}^{(i)}\frac{\partial^2 w}{\partial x_1^2} + D_{26}^{(i)}\frac{\partial^2 w}{\partial x_2^2} + 2D_{66}^{(i)}\frac{\partial^2 w}{\partial x_1 \partial x_2}\right), \qquad (38)$$

$$N_1^{(i)} = \frac{\partial M_1^{(i)}}{\partial x_1} + \frac{\partial H_{12}^{(i)}}{\partial x_2}, \qquad (39)$$

$$N_2^{(i)} = \frac{\partial M_2^{(i)}}{\partial x_2} + \frac{\partial H_{12}^{(i)}}{\partial x_1}, \qquad (40)$$

$$M_{12}^{(i)} = M_1^{(i)}\frac{\partial v}{\partial x_1}n_1^{(i)} + M_2^{(i)}\frac{\partial v}{\partial x_2}n_2^{(i)} + H_{12}^{(i)}\left(\frac{\partial v}{\partial x_1}n_2^{(i)} + \frac{\partial v}{\partial x_2}n_1^{(i)}\right). \qquad (41)$$

3 The determination of the boundary value problem

According to the condition of stationary functional (27), the variation (35) must vanish for the function w corresponding to the actual motion of the plate for all admissible directions v, and in particular for all admissible v, satisfying on the whole contours $\partial G^{(i)}$ the conditions:

$$v(x,t)|_{\partial G^{(i)}} = 0, \quad \left.\frac{\partial v(x,t)}{\partial x_1}\right|_{\partial G^{(i)}} = 0, \quad \left.\frac{\partial v(x,t)}{\partial x_2}\right|_{\partial G^{(i)}} = 0, \quad i = 1, 2. \qquad (42.1\text{–}3)$$

In this case, since the functions $\tilde{v}$ and $\partial\tilde{v}/\partial y_1$ verify analog conditions, the curvilinear integrals and the one-dimensional definite integrals in Eq. (35) vanish, and only the double integrals remain:

$$\delta F\,(w;v)=\int_{t_0}^{t_1}\left[\sum_{i=1}^{2}\int_{G^{(i)}}\left(-\rho^{(i)}h^{(i)}\frac{\partial^2 w}{\partial t^2}+\frac{\partial^2 M_1^{(i)}}{\partial x_1^2}+\frac{\partial^2 M_2^{(i)}}{\partial x_2^2}+2\frac{\partial^2 H_{12}^{(i)}}{\partial x_1\partial x_2}+q^{(i)}\right)v\mathrm{d}x\right]\mathrm{d}t. \tag{43}$$

Since v is an arbitrary smooth function satisfying conditions (42), we have from the fundamental lemma of the calculus of variations that the restrictions of the function w to $G^{(1)}$ and to $G^{(2)}$ must, respectively, satisfy the following differential equations:

$$\begin{aligned}&\frac{\partial^2}{\partial x_1^2}\left(D_{11}^{(i)}\frac{\partial^2 w}{\partial x_1^2}+D_{12}^{(i)}\frac{\partial^2 w}{\partial x_2^2}+2D_{16}^{(i)}\frac{\partial^2 w}{\partial x_1\partial x_2}\right)\\&+\frac{\partial^2}{\partial x_2^2}\left(D_{12}^{(i)}\frac{\partial^2 w}{\partial x_1^2}+D_{22}^{(i)}\frac{\partial^2 w}{\partial x_2^2}+2D_{26}^{(i)}\frac{\partial^2 w}{\partial x_1\partial x_2}\right)\\&+\frac{\partial^2}{\partial x_1\partial x_2}\left(2D_{16}^{(i)}\frac{\partial^2 w}{\partial x_1^2}+2D_{26}^{(i)}\frac{\partial^2 w}{\partial x_2^2}+4D_{66}^{(i)}\frac{\partial^2 w}{\partial x_1\partial x_2}\right)\\&+\rho^{(i)}h^{(i)}\frac{\partial^2 w}{\partial t^2}-q^{(i)}=0,\quad \forall x\in G^{(i)},\ i=1,2,\ \forall t\geq 0.\end{aligned} \tag{44}$$

The fourth-order partial differential equations (44) describe the dynamical behavior of the vibrating plate. If we set $q\equiv 0$ so that there is no external forces acting on the plate, the equations (44) reduce to the equations of free vibrations of the anisotropic plate. On the other hand, if we set $\partial^2 w/\partial t^2\equiv 0$ and it is assumed that all variables are independent of time, the equations (44) reduce to the equations that describe the statical behavior of the mentioned plate when a load of density $q=q(x)$ is applied on $\bar{G}$. Next, we remove the conditions (42.1–3), and since the restrictions of w must satisfy (44), the functional (35) reduces to

$$\begin{aligned}\delta F\,(w;v)=\int_{t_0}^{t_1}\Bigg\{&\sum_{i=1}^{2}\Bigg[\int_{\partial G^{(i)}}M_{12}^{(i)}(x,t)\,\mathrm{d}s-\int_{\partial G^{(i)}}\left(\sum_{j=1}^{2}N_j^{(i)}(x,t)\,n_j^{(i)}(x)\right)v\,(x,t)\mathrm{d}s\\&-\int_0^{l^{(i)}}\tilde{k}_t^{(i)}(0,y_2)\,\tilde{w}\,(0,y_2,t)\,\tilde{v}\,(0,y_2,t)\mathrm{d}y_2\\&-\int_0^{l^{(i)}}\tilde{k}_r^{(i)}(0,y_2)\frac{\partial\tilde{w}}{\partial y_1}(0,y_2,t)\frac{\partial\tilde{v}}{\partial y_1}(0,y_2,t)\mathrm{d}y_2\Bigg]\\&-\int_a^b k_t^{(c)}(c,x_2)\,w\,(c,x_2,t)\,v\,(c,x_2,t)\mathrm{d}x_2-\int_a^b k_r^{(c)}(c,x_2)\left[\frac{\partial w}{\partial x_1}\right]\left[\frac{\partial v}{\partial x_1}\right]\mathrm{d}x_2\Bigg\}\,\mathrm{d}t.\end{aligned} \tag{45}$$

Since $\partial G^{(i)}=\Gamma^{(i)}\cup\Gamma^{(c)}$, we first consider the curvilinear integral

$$\int_{\Gamma^{(i)}}M_{12}^{(i)}(x,t)\,\mathrm{d}s, \tag{46}$$

where $M_{12}^{(i)}$ is given by (41). In the manner of achieving (18), we have

$$\frac{\partial \tilde{v}}{\partial y_1}(y_1, y_2, t) = \sum_{j=1}^{2} \frac{\partial v}{\partial x_j}(x_1, x_2, t) \frac{\partial g_j^{(i)}}{\partial y_1}(y_1, y_2),$$
$$\frac{\partial \tilde{v}}{\partial y_2}(y_1, y_2, t) = \sum_{j=1}^{2} \frac{\partial v}{\partial x_j}(x_1, x_2, t) \frac{\partial g_j^{(i)}}{\partial y_2}(y_1, y_2), \qquad (47.1, 2)$$

where $x_k = g_k^{(i)}(y_1, y_2)$, $k = 1, 2$ and $(y_1, y_2) \in \Omega^{(i)}$, $i = 1, 2$. From Eqs. (12.1, 2) and the well-known relations

$$\frac{\mathrm{d}\gamma_1^{(i)}}{\mathrm{d}y_2}(y_2) = -\tilde{n}_2^{(i)}(y_2), \quad \frac{\mathrm{d}\gamma_2^{(i)}}{\mathrm{d}y_2}(y_2) = \tilde{n}_1^{(i)}(y_2), \qquad (48)$$

it follows:

$$\frac{\partial g_j^{(i)}}{\partial y_2}(y_1, y_2) = (-1)^j \tilde{n}_{3-j}^{(i)}(y_2) F(y_1, y_2), \quad j = 1, 2, \qquad (49.1, 2)$$

where

$$F(y_1, y_2) = 1 + y_1 \frac{\mathrm{d}\tilde{\alpha}}{\mathrm{d}y_2}(y_2), \quad \tilde{\alpha} = \alpha \circ \gamma^{(i)}, \quad i = 1, 2. \qquad (50)$$

From Eqs. (17), (47.1, 2), (49.1, 2) and (50), we obtain

$$\frac{\partial \tilde{v}}{\partial y_1}(y_1, y_2, t) = \sum_{j=1}^{2} \frac{\partial v}{\partial x_j}(x_1, x_2, t) \tilde{n}_j^{(i)}(y_2),$$
$$\frac{\partial \tilde{v}}{\partial y_2}(y_1, y_2, t) = F(y_1, y_2) \sum_{j=1}^{2} (-1)^i \frac{\partial v}{\partial x_j}(x_1, x_2, t) \tilde{n}_{3-j}^{(i)}(y_2), \quad i = 1, 2. \qquad (51.1, 2)$$

If we solve the equations (51.1, 2) for $\partial v / \partial x_1$ and $\partial v / \partial x_2$, we get

$$\frac{\partial v}{\partial x_1}(x_1, x_2, t) = \sum_{j=1}^{2} (-1)^{j+1} \frac{\partial \tilde{v}}{\partial y_j}(y_1, y_2, t) G_j(y_1, y_2) \tilde{n}_j^{(i)}(y_2),$$
$$\frac{\partial v}{\partial x_2}(x_1, x_2, t) = \sum_{j=1}^{2} \frac{\partial \tilde{v}}{\partial y_j}(y_1, y_2, t) G_j(y_1, y_2) \tilde{n}_{3-j}^{(i)}(y_2), \qquad (52.1, 2)$$

with $x_k = g_k^{(i)}(y_1, y_2)$, $k = 1, 2$, $(y_1, y_2) \in \Omega^{(i)}$, $i = 1, 2$ and

$$G_j(y_1, y_2) = \begin{cases} 1 & \text{if } j = 1, \\ (F(y_1, y_2))^{-1} & \text{if } j = 2. \end{cases} \qquad (53)$$

The expressions of the second partial derivatives of $\tilde{w}$ are obtained differentiating (18) with respect to y_1 or y_2, and it follows that in abbreviated form:

$$\tilde{w}_{y_1 y_1}(y, t) = w_{x_1 x_1}(x, t) \tilde{n}_1^{(i)2}(y_2) + w_{x_2 x_2}(x, t) \tilde{n}_2^{(i)2}(y_2)$$
$$+ 2 w_{x_1 x_2}(x, t) \left(\tilde{n}_1^{(i)} \tilde{n}_2^{(i)} \right)(y_2), \qquad (54)$$

and in analogous form, we get $\tilde{w}_{y_2 y_2}$ and $\tilde{w}_{y_1 y_2}$. Since (52.1, 2) hold for any differentiable function, we can obtain w_{x_1} and w_{x_2} simply replacing v by w, and if we solve the equation (54) and those which correspond to $\tilde{w}_{y_2 y_2}$ and $\tilde{w}_{y_1 y_2}$ for $w_{x_1 x_1}$, $w_{x_2 x_2}$ and $w_{x_1 x_2}$, we obtain:

$$\begin{aligned} w_{x_1 x_1}(x,t) &= \tilde{w}_{y_1 y_1}(y,t)\,\tilde{n}_1^{(i)2}(y_2) + \tilde{w}_{y_2 y_2}(y,t)\,G_2^2(y)\,\tilde{n}_2^{(i)2}(y_2) \\ &\quad - 2\tilde{w}_{y_1 y_2}(y,t)\,G_2(y)\left(\tilde{n}_1^{(i)}\tilde{n}_2^{(i)}\right)(y_2) + \tilde{w}_{y_1}(y,t)\,G_2(y)\,\tilde{\alpha}'(y_2)\,\tilde{n}_2^{(i)2}(y_2) \\ &\quad + \tilde{w}_{y_2}(y,t)\left[2G_2^2(y)\,\tilde{\alpha}'(y_2)\left(\tilde{n}_1^{(i)}\tilde{n}_2^{(i)}\right)(y_2) - y_1 G_2^3(y)\,\tilde{\alpha}''(y_2)\,\tilde{n}_2^{(i)2}(y_2)\right], \end{aligned} \tag{55}$$

and in analogous form we get $w_{x_2 x_2}$ and $w_{x_1 x_2}$. Then, substituting Eq. (55) and those which correspond to $w_{x_2 x_2}$ and $w_{x_1 x_2}$ with $y_1 = 0$ into (36)–(38), we obtain the expressions of $\tilde{M}_1^{(i)} = \tilde{M}_1^{(i)}(y_1, y_2, t)$, $\tilde{M}_2^{(i)} = \tilde{M}_2^{(i)}(y_1, y_2, t)$ and $\tilde{H}_{12}^{(i)} = \tilde{H}_{12}^{(i)}(y_1, y_2, t)$. Further, in an analogous form, we can obtain, on direct although lengthy analytical procedure, the expressions of $\tilde{N}_1^{(i)} = \tilde{N}_1^{(i)}(y_1, y_2, t)$ and $\tilde{N}_2^{(i)} = \tilde{N}_2^{(i)}(y_1, y_2, t)$.

According to the adopted parametric representation, the first term of (46) is given by

$$\begin{aligned} &\int_{\Gamma^{(i)}} M_1^{(i)}(x,t)\,\frac{\partial v}{\partial x_1}(x,t)\,n_1^{(i)}(x)\,\mathrm{d}s \\ &= \int_0^{l^{(i)}} \tilde{M}_1^{(i)}(0, y_2, t)\left(\sum_{j=1}^{2}(-1)^{j+1}\frac{\partial \tilde{v}}{\partial y_j}(0, y_2, t)\,\tilde{n}_j^{(i)}(y_2)\right)\tilde{n}_1^{(i)}(y_2)\mathrm{d}y_2, \end{aligned} \tag{56}$$

where the expression of $v_{x_1}(x,t)$ has been obtained from (52.1, 2) and (53), taking into account from (50) that $F(0, y_2) = 1$. Operating in a similar fashion for the rest of the terms from (46), we obtain:

$$\begin{aligned} &\int_{\Gamma^{(i)}} M_{12}^{(i)}(x,t)\,\mathrm{d}s \\ &= \int_0^{l^{(i)}} \left(\tilde{P}^{(i)}(0, y_2, t)\,\frac{\partial \tilde{v}}{\partial y_1}(0, y_2, t) + \tilde{R}^{(i)}(0, y_2, t)\,\frac{\partial \tilde{v}}{\partial y_2}(0, y_2, t)\right)\mathrm{d}y_2, \end{aligned} \tag{57}$$

where

$$\begin{aligned} \tilde{P}^{(i)}(0, y_2, t) &= \tilde{M}_1^{(i)}(0, y_2, t)\left(\tilde{n}_1^{(i)}(y_2)\right)^2 + \tilde{M}_2^{(i)}(0, y_2, t)\left(\tilde{n}_2^{(i)}(y_2)\right)^2 \\ &\quad + 2\tilde{H}_{12}^{(i)}(0, y_2, t)\,\tilde{n}_1^{(i)}(y_2)\,\tilde{n}_2^{(i)}(y_2), \end{aligned} \tag{58}$$

$$\begin{aligned} \tilde{R}^{(i)}(0, y_2, t) &= \left(\tilde{M}_2^{(i)}(0, y_2, t) - \tilde{M}_1^{(i)}(0, y_2, t)\right)\tilde{n}_1^{(i)}(y_2)\,\tilde{n}_2^{(i)}(y_2) \\ &\quad + \tilde{H}_{12}^{(i)}(0, y_2, t)\left(\left(\tilde{n}_1^{(i)}(y_2)\right)^2 - \left(\tilde{n}_2^{(i)}(y_2)\right)^2\right). \end{aligned} \tag{59}$$

After integration by parts of the second term of the integrand in (57), we obtain

$$\begin{aligned} &\int_0^{l^{(i)}} \tilde{R}^{(i)}(0, y_2, t)\,\frac{\partial \tilde{v}}{\partial y_2}(0, y_2, t)\,\mathrm{d}y_2 \\ &= \tilde{R}^{(i)}(0, y_2, t)\,\tilde{v}(0, y_2, t)\,\Big|_0^{l^{(i)}} - \int_0^{l^{(i)}} \frac{\partial \tilde{R}^{(i)}}{\partial y_2}(0, y_2, t)\,\tilde{v}(0, y_2, t)\mathrm{d}y_2. \end{aligned} \tag{60}$$

On the other hand, it is immediate that the second term in (45), evaluated along $\Gamma^{(i)}$, is given by

$$\begin{aligned}
&\int_{\Gamma^{(i)}} \left(\sum_{j=1}^{2} N_j^{(i)}(x,t)\, n_j^{(i)}(x) \right) v(x,t)\mathrm{d}s \\
&= \int_0^{l^{(i)}} \left(\sum_{j=1}^{2} \left(N_j^{(i)} \circ g^{(i)} \right)(0,y_2,t) \left(n_j^{(i)} \circ g^{(i)} \right)(0,y_2) \right) \left(v \circ g^{(i)} \right)(0,y_2,t)\,\mathrm{d}y_2 \\
&= \int_0^{l^{(i)}} \left(\sum_{j=1}^{2} \tilde{N}_j^{(i)}(0,y_2,t)\, \tilde{n}_j^{(i)}(y_2) \right) \tilde{v}(0,y_2,t)\,\mathrm{d}y_2. \qquad (61)
\end{aligned}$$

Now let us consider the curvilinear integrals over $\Gamma^{(c)}$. From the parametric representation (23), using (39)–(40) and $\vec{n}^{(1)} = (1,0)$, we obtain

$$\int_{\Gamma^{(c)}} \sum_{j=1}^{2} N_j^{(1)}(x,t)\, n_j^{(1)}(x)\, v(x,t)\mathrm{d}s = \int_a^b \left(N_1|_{(-)}.v \right)(c,x_2,t)\,\mathrm{d}x_2, \qquad (62)$$

where $N_1|_{(-)}(c,x_2,t)$ is given by (39) with $i = 1$ and the derivatives replaced by the corresponding left lateral derivatives. Similarly, from (41), upon integrating by parts, we have

$$\begin{aligned}
\int_{\Gamma^{(c)}} M_{12}^{(1)}\mathrm{d}s = \int_a^b &\left[\left(M_1|_{(-)} \cdot \left.\frac{\partial v}{\partial x_1}\right|_{(-)} \right)(c,x_2,t) \right. \\
&\left. - \left(\left.\frac{\partial H_{12}}{\partial x_2}\right|_{(-)} .v \right)(c,x_2,t) \right] \mathrm{d}x_2 + \left(H_{12}|_{(-)}.v \right)(c,x_2,t)\Big|_a^b. \qquad (63)
\end{aligned}$$

A path that describes the line $\Gamma^{(c)}$, when it is considered as a part of the boundary $\partial G^{(2)}$, is given by

$$\gamma^{(c)}(x_2) = \left(\gamma_1^{(c)}(x_2), \gamma_2^{(c)}(x_2) \right) = (c, b - x_2),\ \ x_2 \in [0, b-a], \qquad (64)$$

and since $\vec{n}^{(2)} = (-1,0)$, in the manner of achieving (62)–(63), we have:

$$\int_{\Gamma^{(c)}} \sum_{j=1}^{2} N_j^{(2)} n_j^{(2)} v\mathrm{d}s = -\int_0^{b-a} \left(N_1|_{(+)}.v \right)(c, b-x_2, t)\,\mathrm{d}x_2, \qquad (65)$$

$$\begin{aligned}
\int_{\Gamma^{(c)}} M_1^{(2)}\mathrm{d}s = \int_0^{b-a} &\left[-\left(M_1|_{(+)} \cdot \left.\frac{\partial v}{\partial x_1}\right|_{(+)} \right)(c, b-x_2, t) \right. \\
&\left. + \left(\left.\frac{\partial H_{12}}{\partial x_2}\right|_{(+)} .v \right)(c, b-x_2, t) \right] \mathrm{d}x_2 - \left(H_{12}|_{(+)}.v \right)(c, b-x_2, t)\Big|_0^{b-a}. \qquad (66)
\end{aligned}$$

Then, substitution of (57), (60)–(63) and (65)–(66) into (45) gives

$$\delta F(w;v)=\int_{t_0}^{t_1}\left\{\sum_{i=1}^{2}\left[\int_{0}^{l^{(i)}}\left(\left(-\sum_{j=1}^{2}\tilde N_j^{(i)}(0,y_2,t)\,\tilde n_j^{(i)}(y_2)-\frac{\partial\tilde R^{(i)}}{\partial y_2}(0,y_2,t)\right.\right.\right.\right.$$
$$\left.-\tilde k_t^{(i)}(0,y_2)\,\tilde w(0,y_2,t)\right)\tilde v(0,y_2,t)$$
$$\left.+\left(\tilde P^{(i)}(0,y_2,t)-\tilde k_r^{(i)}(0,y_2)\frac{\partial\tilde w}{\partial y_1}(0,y_2,t)\right)\frac{\partial\tilde v}{\partial y_1}(0,y_2,t)\right)\mathrm{d}y_2$$
$$\left.\left.+\left(\tilde R^{(i)}\tilde v\right)(0,y_2,t)\Big|_0^{l^{(i)}}\right]+S\right\}\mathrm{d}t \tag{67}$$

where

$$S=\int_a^b\left[-\left(\left(N_1|_{(-)}+\left.\frac{\partial H_{12}}{\partial x_2}\right|_{(-)}\right)v\right)(c,x_2,t)+\left(M_1|_{(-)}\left.\frac{\partial v}{\partial x_1}\right|_{(-)}\right)(c,x_2,t)\right]\mathrm{d}x_2$$
$$+\int_0^{b-a}\left[\left(\left(N_1|_{(+)}+\left.\frac{\partial H_{12}}{\partial x_2}\right|_{(+)}\right)v\right)(c,b-x_2,t)-\left(M_1|_{(+)}\left.\frac{\partial v}{\partial x_1}\right|_{(+)}\right)(c,b-x_2,t)\right]\mathrm{d}x_2$$
$$-\int_a^b k_t^{(c)}(c,x_2)\,w(c,x_2,t)\,v(c,x_2,t)\mathrm{d}x_2-\int_a^b k_r^{(c)}(c,x_2)\left[\frac{\partial w}{\partial x_1}\right]\left[\frac{\partial v}{\partial x_1}\right]\mathrm{d}x_2$$
$$+\left(H_{12}|_{(-)}.v\right)(c,x_2,t)\Big|_a^b-\left(H_{12}|_{(+)}.v\right)(c,b-x_2,t)\Big|_0^{b-a}. \tag{68}$$

Since we can independently choose v and its derivatives and the interval $[t_0,t_1]$ is arbitrary, the condition of stationary functional (27) applied to (67) leads, in the manner for achieving Eq. (44), to the following natural boundary conditions that establish requirements on the bending moments and on the shear forces, respectively:

$$\tilde k_r^{(i)}(0,y_2)\frac{\partial\tilde w}{\partial y_1}(0,y_2,t)=\tilde P^{(i)}(0,y_2,t),\; y_2\in\left[0,l^{(i)}\right],\quad i=1,2, \tag{69.1, 2}$$

$$\tilde k_t^{(i)}(0,y_2)\,\tilde w(0,y_2,t)=-\sum_{j=1}^{2}\tilde N_j^{(i)}(0,y_2,t)\,\tilde n_j^{(i)}(y_2)-\frac{\partial\tilde R^{(i)}}{\partial y_2}(0,y_2,t),$$
$$y_2\in\left[0,l^{(i)}\right],\quad i=1,2, \tag{70.1, 2}$$

where $\tilde P^{(i)}$ and $\tilde R^{(i)}$ are, respectively, given by (58) and (59).

It must be noted from (67) that if the boundary ∂G is smooth, the continuity conditions

$$\tilde R^{(1)}\left(0,l^{(1)},t\right)\tilde v\left(0,l^{(1)},t\right)=\tilde R^{(2)}(0,0,t)\,\tilde v(0,0,t),$$
$$\tilde R^{(1)}(0,0,t)\,\tilde v(0,0,t)=\tilde R^{(2)}\left(0,l^{(2)},t\right)\tilde v\left(0,l^{(2)},t\right)$$

lead to

$$\sum_{i=1}^{2}\tilde R^{(i)}(0,y_2,t)\,\tilde v(0,y_2,t)\Big|_0^{l^{(i)}}=0. \tag{71}$$

The equations (69.1, 2)–(70.1, 2) are the boundary conditions along ∂G. Adopting the adequate values for the parameters $\tilde{k}_r^{(i)}$ and $\tilde{k}_t^{(i)}$ all the classical boundary supports (i.e.: clamped, simply supported and free) can be generated. In view of (69.1, 2)–(70.1, 2) and (71), the variation given by (67) reduces to

$$\delta F\left(w; v\right) = \int_{t_0}^{t_1} S \mathrm{d}t, \tag{72}$$

where S is given by (68). Finally, in the manner of achieving (69.1, 2)–(70.1, 2) and using the property

$$\int_0^{b-a} f\left(b - x_2\right) \mathrm{d}x_2 = \int_a^b f\left(x_2\right) \mathrm{d}x_2$$

from (72), we obtain:

$$k_r^{(c)}\left(c, x_2\right)\left(\left.\frac{\partial w}{\partial x_1}\right|_{(+)}\left(c, x_2, t\right) - \left.\frac{\partial w}{\partial x_1}\right|_{(-)}\left(c, x_2, t\right)\right) = -\left.M_1\right|_{(-)}\left(c, x_2, t\right), \quad x_2 \in [a, b], \tag{73}$$

$$k_r^{(c)}\left(c, x_2\right)\left(\left.\frac{\partial w}{\partial x_1}\right|_{(+)}\left(c, x_2, t\right) - \left.\frac{\partial w}{\partial x_1}\right|_{(-)}\left(c, x_2, t\right)\right) = -\left.M_1\right|_{(+)}\left(c, x_2, t\right), \quad x_2 \in [a, b], \tag{74}$$

$$\begin{aligned} k_t^{(c)}\left(c, x_2\right) w\left(c, x_2, t\right) = &-\left.N_1\right|_{(-)}\left(c, x_2, t\right) - \left.\frac{\partial H_{12}}{\partial x_2}\right|_{(-)}\left(c, x_2, t\right) \\ &+ \left.N_1\right|_{(+)}\left(c, x_2, t\right) + \left.\frac{\partial H_{12}}{\partial x_2}\right|_{(+)}\left(c, x_2, t\right), x_2 \in [a, b], \end{aligned} \tag{75}$$

and the relations

$$\left.H_{12}\right|_{(-)}\left(c, b, t\right) = -\left.H_{12}\right|_{(+)}\left(c, b, t\right), \left.H_{12}\right|_{(-)}\left(c, a, t\right) = -\left.H_{12}\right|_{(+)}\left(c, a, t\right). \tag{76.1, 2}$$

If the boundary ∂G is composed of a finite number of smooth arcs and therefore has at most a finite number of corner points, the condition (71) is no longer valid. To be definite, let us assume that the curve $\Gamma^{(1)}$ consists of two smooth arcs $\Gamma^{(1),1}$ and $\Gamma^{(1),2}$ of lengths $l^{(1),1}$ and $l^{(1),2}$, respectively, and that has a corner point of coordinates (y_1, y_2) given by $\left(0, l^{(1),1}\right)$. Let us suppose that $\Gamma^{(1),1}$ is represented parametrically by the function $\gamma^{(1),1}(s)$, $s \in \left[0, l^{(1),1}\right]$ and $\Gamma^{(1),2}$ by $\gamma^{(1),2}(s)$, $s \in \left[l^{(1),1}, l^{(1)}\right]$, $\quad l^{(1)} = l^{(1),1} + l^{(1),2}$. From Eq. (60) with $i = 1$ we get

$$\begin{aligned} \int_0^{l^{(1)}} \tilde{R}^{(1)}\left(0, y_2, t\right) \frac{\partial \tilde{v}}{\partial y_2}\left(0, y_2, t\right) \mathrm{d}y_2 = &\left.\tilde{R}^{(1),1}\left(0, y_2, t\right) \tilde{v}\left(0, y_2, t\right)\right|_0^{l^{(1),1}} \\ &+ \left.\tilde{R}^{(1),2}\left(0, y_2, t\right) \tilde{v}\left(0, y_2, t\right)\right|_{l^{(1),1}}^{l^{(1)}} - \int_0^{l^{(1)}} \frac{\partial \tilde{R}^{(1)}}{\partial y_2}\left(0, y_2, t\right) \tilde{v}\left(0, y_2, t\right) \mathrm{d}y_2, \end{aligned} \tag{77}$$

where $\tilde{R}^{(1),i}$ denotes the expression $\tilde{R}^{(1)}$ defined on $\Gamma^{(1),i}$. Taking into account (69.1, 2)–(70.1, 2) and (73)–(77), the variation (67) reduces to

$$\begin{aligned} \delta F\left(w; v\right) = \int_{t_0}^{t_1} \Bigg[&\left.\left(\tilde{R}^{(1),1} \tilde{v}\right)\left(0, y_2, t\right)\right|_0^{l^{(1),1}} + \left.\left(\tilde{R}^{(1),2} \tilde{v}\right)\left(0, y_2, t\right)\right|_{l^{(1),1}}^{l^{(1)}} \\ &+ \left.\left(\tilde{R}^{(2)} \tilde{v}\right)\left(0, y_2, t\right)\right|_0^{l^{(2)}} \Bigg] \mathrm{d}t, \end{aligned} \tag{78}$$

and adopting directions $\tilde{v}$ such that $\tilde{v} \neq 0$ in the corner point and $\tilde{v} = 0$ along the curve $\Gamma^{(2)}$ (where the function $\tilde{R}^{(2)}\tilde{v}$ is defined), the condition of stationary functional (27) leads to:

$$\tilde{R}^{(1),1}\left(0, l^{(1),1}, t\right)\tilde{v}\left(0, l^{(1),1}, t\right) = \tilde{R}^{(1),2}\left(0, l^{(1),1}, t\right)\tilde{v}\left(0, l^{(1),1}, t\right),$$

and since $\tilde{v}$ is continuous, we obtain

$$\tilde{R}^{(1),1}\left(0, l^{(1),1}, t\right) = \tilde{R}^{(1),2}\left(0, l^{(1),1}, t\right). \tag{79}$$

It must be noted that the condition $v \neq 0$ in the corner point is encountered, for instance, when the plate along the arcs $\Gamma^{(1),1}$ and $\Gamma^{(1),2}$ is free or elastically restrained against translation. In Sect. 5.1, the Eq. (79) is used to obtain the corner conditions that correspond to an anisotropic rectangular plate.

4 The transition conditions

Since the domain of definition of the problem is G and this is an open set in $\mathbb{R}^2$, given by $G = G^{(1)} \cup G^{(2)} \cup \Gamma^{(c)}$ with boundary $\partial G = \partial G^{(1)} \cup \partial G^{(2)} - \Gamma^{(c)}$, only the Eqs. (69.1, 2)–(70.1, 2) correspond to the boundary conditions. All the points of the line $\Gamma^{(c)}$ are interior points of G, and the equations formulated on $\Gamma^{(c)}$ can be called *transition conditions*. Then, (73)–(75) correspond to the transition conditions of the problem. Since $w(\bullet, t) \in C\left(\bar{G}\right)$, there exists continuity of deflection at the points (c, x_2), and this generate the transition condition

$$w\left(c^-, x_2, t\right) = w\left(c^+, x_2, t\right) = w\left(c, x_2, t\right), \quad x_2 \in [a, b],$$

where $w\left(c^-, x_2, t\right)$ denotes the limit from the left and $w\left(c^+, x_2, t\right)$ the limit from the right. If (73) and (74) are summed and then subtracted, more symmetric equations can be obtained, and the set of all the transitions conditions of the problem can be expressed as:

$$w\left(c^-, x_2, t\right) = w\left(c^+, x_2, t\right) = w\left(c, x_2, t\right), \quad x_2 \in [a, b]. \tag{80}$$

$$k_r^{(c)}(c, x_2)\left(\left.\frac{\partial w}{\partial x_1}\right|_{(+)}(c, x_2, t) - \left.\frac{\partial w}{\partial x_1}\right|_{(-)}(c, x_2, t)\right)$$
$$= -\frac{1}{2}\left(M_1|_{(+)}(c, x_2, t) + M_1|_{(-)}(c, x_2, t)\right), \quad x_2 \in [a, b], \tag{81}$$
$$M_1|_{(+)}(c, x_2, t) - M_1|_{(-)}(c, x_2, t) = 0, \quad x_2 \in [a, b], \tag{82}$$

$$\begin{aligned} k_t^{(c)}(c, x_2)\, w(c, x_2, t) = &- N_1|_{(-)}(c, x_2, t) - \left.\frac{\partial H_{12}}{\partial x_2}\right|_{(-)}(c, x_2, t) \\ &+ N_1|_{(+)}(c, x_2, t) + \left.\frac{\partial H_{12}}{\partial x_2}\right|_{(+)}(c, x_2, t), \quad x_2 \in [a, b]. \end{aligned} \tag{83}$$

Different situations can be generated by substituting values and/or limiting values of the restraint parameters $k_r^{(c)}$ and $k_t^{(c)}$ in (81) and (83).

It is well known that for a differential equation of order $2m$, the boundary conditions containing the function w and derivatives of w of orders not greater than $m-1$ are called *stable* or *geometric,* and those containing derivatives of orders higher than $m-1$ are called *unstable* or *natural,* [9]. In consequence, if $0 \leq \tilde{k}_r^{(i)}(0, y_2) < \infty$, $0 \leq \tilde{k}_t^{(i)}(0, y_2) < \infty$, the boundary conditions (69)–(70) are all unstable. If this classification is extended to the transition conditions, we conclude that if $0 \leq k_r^{(c)}(c, x_2) < \infty$, $0 \leq k_t^{(c)}(c, x_2) < \infty$, the conditions (81) and (83) are unstable. Obviously, the condition (80) is stable and (82) unstable.

The above classification is particularly important when using the Ritz method since we must choose a sequence of functions v_i which constitutes a base in the space of homogeneous stable boundary conditions. So, in this case, there is no need to subject the functions v_i to the natural boundary and transition conditions.

5 Analytical examples

5.1 Anisotropic plate elastically restrained against rotation

If the anisotropic plate is elastically restrained only against rotation, letting $\tilde{k}_t^{(i)}(0, y_2) \to \infty$ in (70.1, 2), we get

$$\tilde{w}(0, y_2, t) = 0, \; y_2 \in \left[0, l^{(i)}\right], \quad i = 1, 2. \tag{84}$$

The above boundary conditions imply that the restrictions of the function w to $G^{(1)}$ and to $G^{(2)}$ vanish along the corresponding boundary curve. By replacing the expression of $\tilde{P}^{(i)}$ into (69.1, 2), we get:

$$\begin{aligned}\tilde{k}_r^{(i)}(0, y_2)\frac{\partial \tilde{w}}{\partial y_1}(0, y_2, t) &= \frac{\partial^2 \tilde{w}}{\partial y_1^2}(0, y_2, t)\sum_{j=1}^{3} A_j^{(i)} + \frac{\partial^2 \tilde{w}}{\partial y_2^2}(0, y_2, t)\sum_{j=1}^{3} B_j^{(i)} \\ &\quad + \frac{\partial^2 \tilde{w}}{\partial y_1 \partial y_2}(0, y_2, t)\sum_{j=1}^{2} C_j^{(i)} + \frac{\partial \tilde{w}}{\partial y_1}(0, y_2, t)\frac{d\tilde{\alpha}}{dy_2}(y_2)\sum_{j=1}^{3} B_j^{(i)} \\ &\quad - \frac{\partial \tilde{w}}{\partial y_2}(0, y_2, t)\frac{d\tilde{\alpha}}{dy_2}(y_2)\sum_{j=1}^{2} C_j^{(i)}, \quad y_2 \in \left[0, l^{(i)}\right], \quad i = 1, 2,\end{aligned} \tag{85}$$

where

$$\begin{aligned}
A_1^{(i)} &= E_1^{(i)}\left(\tilde{n}_1^{(i)}\right)^2, \; A_2^{(i)} = E_2^{(i)}\left(\tilde{n}_2^{(i)}\right)^2, \; A_3^{(i)} = E_3^{(i)}\tilde{n}_1^{(i)}\tilde{n}_2^{(i)}, \\
B_1^{(i)} &= E_1^{(i)}\left(\tilde{n}_2^{(i)}\right)^2, \; B_2^{(i)} = E_2^{(i)}\left(\tilde{n}_1^{(i)}\right)^2, \; B_3^{(i)} = -A_3^{(i)}, \\
C_1^{(i)} &= 2\left(E_2^{(i)} - E_1^{(i)}\right)\tilde{n}_1^{(i)}\tilde{n}_2^{(i)}, \quad C_2^{(i)} = E_3^{(i)}\left(\left(\tilde{n}_1^{(i)}\right)^2 - \left(\tilde{n}_2^{(i)}\right)^2\right), \\
E_1^{(i)} &= -\left(D_{11}^{(i)}\left(\tilde{n}_1^{(i)}\right)^2 + D_{12}^{(i)}\left(\tilde{n}_2^{(i)}\right)^2 + 2D_{16}^{(i)}\tilde{n}_1^{(i)}\tilde{n}_2^{(i)}\right), \\
E_2^{(i)} &= -\left(D_{12}^{(i)}\left(\tilde{n}_1^{(i)}\right)^2 + D_{22}^{(i)}\left(\tilde{n}_2^{(i)}\right)^2 + 2D_{26}^{(i)}\tilde{n}_1^{(i)}\tilde{n}_2^{(i)}\right), \\
E_3^{(i)} &= -2\left(D_{16}^{(i)}\left(\tilde{n}_1^{(i)}\right)^2 + D_{26}^{(i)}\left(\tilde{n}_2^{(i)}\right)^2 + 2D_{66}^{(i)}\tilde{n}_1^{(i)}\tilde{n}_2^{(i)}\right).
\end{aligned}$$

Thus, the boundary conditions that correspond to this case are given by Eqs. (84) and (85).

Now, let us consider a rectangular plate with

$$\begin{aligned}
G &= \{(x_1, x_2), x_1 \in (0, a), x_2 \in (0, b)\}, \\
G^{(1)} &= \{(x_1, x_2), x_1 \in (0, c), x_2 \in (0, b)\}, \\
G^{(2)} &= \{(x_1, x_2), x_1 \in (c, a), x_2 \in (0, b)\}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
&\frac{d\tilde{\alpha}}{dy_2}(y_2) \equiv 0, \quad E_1^{(i)} = -D_{11}^{(i)}\left(\tilde{n}_1^{(i)}\right)^2 - D_{12}^{(i)}\left(\tilde{n}_2^{(i)}\right)^2, \\
&\quad E_2^{(i)} = -D_{12}^{(i)}\left(\tilde{n}_1^{(i)}\right)^2 - D_{22}^{(i)}\left(\tilde{n}_2^{(i)}\right)^2, \quad E_3^{(i)} = -2\left(D_{16}^{(i)}\left(\tilde{n}_1^{(i)}\right)^2 + D_{26}^{(i)}\left(\tilde{n}_2^{(i)}\right)^2\right),
\end{aligned}$$

with $\tilde{n}_1^{(i)} = \pm 1, \tilde{n}_2^{(i)} = 0$ for the sides parallel to x_2 and $\tilde{n}_1^{(i)} = 0, \tilde{n}_2^{(i)} = \pm 1$ for the sides parallel to x_1. In consequence, from (85) it follows that the boundary condition that corresponds to the first side of the boundary

$$\Gamma^{(1)} = \{(x_1, b), x_1 \in [0, c]\} \cup \{(0, x_2), x_2 \in [0, b]\} \cup \{(x_1, 0), x_1 \in [0, c]\}$$

is given by:

$$\begin{aligned}
&\tilde{k}_r^{(1),1}(0, y_2)\,\frac{\partial \tilde{w}}{\partial y_1}(0, y_2, t)\\
&\quad = -D_{22}^{(1)}(0, y_2)\,\frac{\partial^2 \tilde{w}}{\partial y_1^2}(0, y_2, t) - D_{12}^{(1)}(0, y_2)\,\frac{\partial^2 \tilde{w}}{\partial y_2^2}(0, y_2, t)\\
&\qquad + 2D_{26}^{(1)}(0, y_2)\,\frac{\partial^2 \tilde{w}}{\partial y_1 \partial y_2}(0, y_2, t), \quad y_2 \in \left[0, l_1^{(1)}\right], \quad l_1^{(1)} = c,
\end{aligned}$$

and analog expressions correspond to the sides $\Gamma^{(1),2}$ and $\Gamma^{(1),3}$. Using (51a) (with $\tilde{v}$ replaced by $\tilde{w}$) and the expressions of $\tilde{w}_{y_1 y_1}$, $\tilde{w}_{y_2 y_2}$ and $\tilde{w}_{y_1 y_2}$, these boundary conditions can be expressed in the original variables. Employing the same technique, we can obtain the boundary conditions which correspond to $G^{(2)}$.

It must be noted that in this case, the boundary is composed of four smooth arcs and has four corner points. Let us consider in the (x_1, x_2) variables the corner point $(0, b)$, and the Eq. (79), which is now given by $R^{(1),1}(0, b, t) = R^{(1),2}(0, b, t)$ and by virtue of Eq. (59), reduces to

$$H_{12}^{(1),1}(0, b, t) = H_{12}^{(1),1}(0, b, t)\,.$$

Then, from (38), this expression gives rise to the following corner condition:

$$\begin{aligned}
&D_{16}^{(1)}(x)\,\frac{\partial^2 w}{\partial x_1^2}(x, t) + D_{26}^{(1)}(x)\,\frac{\partial^2 w}{\partial x_2^2}(x, t) + 2D_{66}^{(1)}(x)\,\frac{\partial^2 w}{\partial x_1 \partial x_2}(x, t) = 0,\\
&x = (0, b)\,.
\end{aligned}$$

In an analog form, we can obtain the remaining corner conditions. The transition conditions are directly given by Eqs. (80)–(83).

5.2 Isotropic circular plate elastically restrained against rotation

Let us consider an isotropic circular plate whose boundary is elastically restrained against rotation. From (14), we have

$$x_j\big|_{\Gamma^{(i)}} = g_j^{(i)}(0, y_2) = \gamma_j^{(i)}(y_2) = x_{j0} + a\tilde{n}_j(y_2)\,, \quad y_2 \in [0, \pi a]\,, \quad i, j = 1, 2,$$

where x_{10}, x_{20} are the coordinates of the center of the circle. The case of isotropic plate is obtained replacing the coefficients:

$$D_{11}^{(i)} = D_{22}^{(i)} = D^{(i)}, \quad D_{12}^{(i)} = \mu D^{(i)}, \quad D_{16}^{(i)} = D_{26}^{(i)} = 0, \quad D_{66}^{(i)} = \frac{D^{(i)}}{2}(1 - \mu)\,.$$

In this case, we have

$$\begin{aligned}
E_1 &= -D^{(i)}\left(\left(\tilde{n}_1^{(i)}\right)^2 + \mu\left(\tilde{n}_2^{(i)}\right)^2\right),\\
E_2 &= -D^{(i)}\left(\mu\left(\tilde{n}_1^{(i)}\right)^2 + \left(\tilde{n}_2^{(i)}\right)^2\right),\\
E_3 &= -2D^{(i)}(1 - \mu)\,\tilde{n}_1^{(i)}\tilde{n}_2^{(i)}.
\end{aligned}$$

Finally, since we have the relations $y_2 = a\tilde{\alpha}$ and $y_1 = r - a$, it follows that $\mathrm{d}\tilde{\alpha}/\mathrm{d}y_2 = a^{-1}$, and from (69.1, 2)–(70.1, 2), we obtain the boundary conditions expressed in polar coordinates:

$$\begin{aligned}
&w(a, \alpha, t) = 0, \quad \alpha \in [0, 2\pi]\,,\\
&k_r^{(i)}(a, \alpha)\,\frac{\partial w}{\partial r}(a, \alpha, t) = -D^{(i)}\left(\frac{\partial^2 w}{\partial r^2}(a, \alpha, t) + \frac{\mu}{a^2}\frac{\partial^2 w}{\partial \alpha^2}(a, \alpha, t) + \frac{\mu}{a}\frac{\partial w}{\partial r}(a, \alpha, t)\right),\\
&\quad i = 1, 2, \alpha \in [0, 2\pi]\,.
\end{aligned}$$

6 Concluding remarks

This paper presents the formulation of an analytical model for the dynamic behavior of anisotropic plates, with an arbitrarily located internal line hinge with elastic supports and piecewise smooth boundaries elastically restrained against rotation and translation. The equations of motion and its associated boundary and transition conditions were derived using Hamilton's principle. By introducing an adequate change of variables, the energies that correspond to the different elastic restraints were handled in a rigorous framework. The concept of transition conditions was introduced, and the corresponding analytical expressions were derived.

It has also been demonstrated that the presence of corner points, when the boundary is free or elastically restrained against translation, generates additional conditions at those points. The proposed mathematical manipulation offers an enhancement and generalization of the approaches that appear in the engineering literature and fills the gap existing as a consequence of the commonly used formal arguments. Finally, analytical examples were worked out to illustrate the range of applications of the developed analytical model.

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