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A robust economic MPC for changing economic criterion

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Summary

This note presents a robust economic model predictive control controller suitable for changing economic criterion. The proposal ensures feasibility under any change of the economic criterion, thanks to the use of artificial variables and a relaxed terminal constraint, and robustness in presence of additive bounded disturbances. The resulting robust formulation considers a nominal prediction model and restricted constraints (in order to account for the effect of additive disturbances). The controlled system under the proposed controller is shown to be input-to-state stable in the sense that it is asymptotically steered to an invariant region around the best admissible steady state. An illustrative example shows the benefits and the properties of the proposed controller.

KEYWORDS

economic optimization, model predictive control, robust control

1 | INTRODUCTION

Model predictive control (MPC) has its roots in optimal control and is one of the most used advanced control strategies in the process industries, chemical plants, and oil refineries since the 1980s. The success of this control technique is due to its control problem formulation, the usage of a model to predict the future behavior of the system to be controlled, the ability of handling large multivariable systems subject to explicit constraints on states and inputs, and the capability of ensuring stability.[1,2]

The goal of many advanced control systems is to guide a process to a target setpoint rapidly and reliably, but in the last years, a particular MPC formulation called economic MPC has gained popularity. There are a lot of different formulations in literature, each of them devoted to study/analyze some aspects of this interesting control technique.[3-6] Some methods include the economic objective as a terminal cost,[7,8] but other ones optimize the dynamic economic performance directly.[3,6,9] In both cases, the control objective is not only to stabilize the controlled system in a particular steady state but also to optimize some economic performance criterion. The main idea in these formulations is to consider an economic objective, possibly the real-time optimizer (RTO) cost function, as part of the cost of the MPC controller. For the second case of strategies, one of the main challenges has been showing the existence of a Lyapunov function in order to prove stability. This goal has been firstly achieved under the assumption of strong duality,[10] and, later, under the (relaxed) dissipativity assumption,[3,11] with the latter being not just sufficient but also a necessary condition for the closed-loop stability.[12,13] Asymptotic stability has also been proved in the case of using terminal cost (with[14] and without[15] terminal constraints) and in the case of Lyapunov-based MPC controllers.[16]

When making an economic formulation of MPC, it is important to take into account that possible changes in economic criteria may occur.[17,18] This means that a change can occur in the state to which the controller must drive the system, ie, in the economically optimal stationary state, producing, because of stability requirements, a possible loss of feasibility.

It should be noted that the changes, which may occur during the operation of a plant, may be due to, for instance: (i) market fluctuations, which causes changes in the cost function and in the prices that parameterized this function, and (ii) changes in disturbances estimation or constraints due to data reconciliation algorithms.

A formulation of the economic MPC, which contemplates changes in the economic parameter, is presented in the work of Liu et al.[19] In that work, a controller is designed with a slightly modified cost function, to which it is added, following the fundamentals presented in the work of Zhao et al,[14] a terminal cost function, and a relaxed terminal constraint, which requires the terminal state at the end of the horizon to be *any* admissible equilibrium point. As a consequence, this controller ensures the following properties: (i) it guarantees feasibility under any change of the economic criterion; (ii) it ensures economic optimality; (iii) it provides a larger domain of attraction than standard economic MPC. Moreover, asymptotic stability is proved resorting to a Lyapunov function.

The aim of this paper is to propose a robust formulation of the economic MPC presented in the work of Liu et al[19] for the case of additive disturbance. The proposed robust strategy uses the nominal prediction model, restricting the constraints to properly account for the disturbance effects.[20] As a result, a robust economic MPC is obtained, which ensures recursive feasibility under any change of the economic criterion or parameter, with robust constraints satisfaction. On the other hand, since the controlled system is subject to nonvanishing disturbances, it is demonstrated that the proposed controller is input-to-state stable (ISS). This implies that it is asymptotically directed to a region around the best permissible steady state, characterized as the minimum robust positive invariant (RPI) set.

This work is organized as follows. After the introduction, the problem is stated in Section 2. Section 3 presents the proposed robust controller as well as a stability analysis. Section 3.2 analyzes the main properties of the proposed controller, whereas in Section 4, an illustrative example based on a simulation of a four tanks systems is presented. Finally, Section 5 gives some conclusions of this work.

1.1 | Notation

Consider $a \in \mathbb{R}^{n_a}$ and $b \in \mathbb{R}^{n_b}$, the vector made from stacking both vectors is defined as $(a, b) \stackrel{\Delta}{=} [a', b']' \in \mathbb{R}^{n_a+n_b}$, where $'$ denotes the transpose operator; for a set $\Gamma \subset \mathbb{R}^{n_a+n_b}$, the projection of Γ onto a is defined as $\text{Proj}_a(\Gamma) = \{a \in \mathbb{R}^{n_a} : \exists b \in \mathbb{R}^{n_b}, (a, b) \in \Gamma\}$. A matrix $\mathbf{0}_{n,m} \in \mathbb{R}^{n\times m}$ denotes a matrix of zeros, and $I_n \in \mathbb{R}^{n\times n}$ denotes the identity matrix. Given two sets $\mathcal{U}$ and $\mathcal{V}$, such that $\mathcal{U} \subset \mathbb{R}^n$ and $\mathcal{V} \subset \mathbb{R}^n$, the Minkowski sum is defined by $\mathcal{U} \oplus \mathcal{V} \stackrel{\Delta}{=} \{u + v : u \in \mathcal{U}, v \in \mathcal{V}\}$, and the Pontryagin set difference is $\mathcal{U} \ominus \mathcal{V} \stackrel{\Delta}{=} \{u : u \oplus \mathcal{V} \subseteq \mathcal{U}\}$; given a matrix $M \in \mathbb{R}^{p\times n}$, the set $M\mathcal{U} \subset \mathbb{R}^p$ is defined as $M\mathcal{U} \stackrel{\Delta}{=} \{Mu : u \in \mathcal{U}\}$; for a given λ, $\lambda\mathcal{U} \stackrel{\Delta}{=} (\lambda I_n)\mathcal{U}$.

2 | PROBLEM STATEMENT

This section states the problem, the robustness approach that is used, and the economic performance to be optimized.

Consider a system described by a linear time-invariant discrete time model

$$x^+ = Ax + Bu + w, \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the current control vector, x^+ is the successor state, and $w \in \mathbb{R}^n$ is an unknown but bounded state disturbance. In what follows, $x(k)$, $u(k)$, and $w(k)$ denote the state, the manipulable variable, and the disturbance, respectively, at sampling time k.

System (1) is subject to constraints on state and input

$$(x(k), u(k)) \in \mathcal{Z}, \tag{2}$$

for all $k \geq 0$, where $\mathcal{Z} = \mathcal{X} \times \mathcal{U}$ is a compact convex polyhedron containing the origin in its interior.

Define also the plant nominal model, given by (1), neglecting the disturbance input w

$$\bar{x}^+ = A\bar{x} + B\bar{u}. \tag{3}$$

The solution of this system for a given sequence of control inputs $\bar{\mathbf{u}} = \{\bar{u}(0), \ldots, \bar{u}(j-1)\}$, and an initial state $\bar{x}$ is denoted as $\bar{x}(j) = \phi(j; \text{x}, \bar{\mathbf{u}})$, $j \in \mathbb{I}_{\geq 1}$, where $\bar{x}(0) = \phi(0; \bar{x}, \bar{\mathbf{u}})$.

The following assumption holds.

Assumption 1. (i) The pair (A, B) is controllable, and the state is measured at each sampling time. (ii) The uncertainty vector w is bounded and lies in a compact convex polyhedron, $\mathcal{W}$, containing the origin in its interior.

2.1 | Robust approach

The main goal of this work is to design a robust economic MPC controller. To this aim, following the ideas presented in the works of Jiang et al[20] and Wen et al,[21] it is proposed here to use a nominal model for predictions and to restrict the constraints at any step of the prediction horizon to take into account the effect of the disturbance.

First, the idea is to prestabilize the nominal system using a state feedback control gain K such that $A_K = A + BK$ has all its eigenvalues in the interior of the unit circle. The controlled nominal system is then given by

$$\begin{aligned} \bar{x}(k+1) &= A_K\bar{x}(k) + Bc(k) \\ \bar{u}(k) &= K\bar{x}(k) + c(k), \end{aligned} \tag{4}$$

where $c(k)$ can be seen as the difference between the applied control input $\bar{u}(k)$ and the nominal feedback, ie, $c(k) \stackrel{\Delta}{=} \bar{u}(k) - K\bar{x}(k)$.

The second step is to reduce the constraint sets. To accomplish this goal, we need to define the so-called reachable sets, which contain the response of the prestabilized autonomous system (ie, (4) whith $c(k) = 0$) because of the uncertainty.

Definition 1. The reachable set in j steps, $\mathcal{R}_j$, is given by

$$\mathcal{R}_j = \bigoplus_{i=0}^{j-1} A_K^i \mathcal{W}. \tag{5}$$

This is the set of states of the nominal closed-loop prestabilized autonomous system that are reachable in j steps from the origin under the disturbance w. This set satisfies the following properties:

i. It is given by the recursion $\mathcal{R}_j \oplus A_K^j \mathcal{W} = \mathcal{R}_{j+1}$, with $\mathcal{R}_1 = \mathcal{W}$.
ii. $A_K^j \mathcal{R}_j \oplus \mathcal{W} = \mathcal{R}_{j+1} = \mathcal{R}_j \oplus A_K^j \mathcal{W}$.
iii. $\mathcal{R}_j \subseteq \mathcal{R}_{j+1}$.
iv. The sequence of sets $\mathcal{R}_j$ has a limit $\mathcal{R}_\infty$ as $j \to \infty$, and $\mathcal{R}_\infty$ is an RPI set for the prestabilized autonomous system. Note that $\mathcal{R}_j$ is not robust invariant for a finite j.
v. $\mathcal{R}_\infty$ is the minimal RPI set.

Definition 2. [Robust positively invariant (RPI) set[22]] A set Ω is called an RPI set for the uncertain system $x(k+1) = A_K x(k) + w(k)$, with $w(k) \in \mathcal{W}$, if $A_K\Omega \oplus \mathcal{W} \subseteq \Omega$.

We can now define the sets of **restricted constraints** as

$$\bar{\mathcal{X}}_j \triangleq \mathcal{X} \ominus \mathcal{R}_j \tag{6}$$

$$\bar{\mathcal{U}}_j \triangleq \mathcal{U} \ominus K\mathcal{R}_j. \tag{7}$$

To ensure that these sets are nonempty, it must hold that $\mathcal{R}_\infty \subset \mathcal{X}$ and $K\mathcal{R}_\infty \subset \mathcal{U}$. Moreover, it is important to note that the computation of such sets is made off-line, so it has no practical effects on the MPC problem.

Remark 1. The control gain K plays an important role in the proposed robust approach since it determines the dynamic of the closed-loop system in presence of disturbances, and hence, it must ensure that $K\mathcal{R}_\infty \subset \mathcal{U}$ holds. Moreover, this parameter can be chosen in order to obtain a less conservative (ie, tighter) set $\mathcal{R}_\infty$.

2.2 | Economic objective

First, we make an equilibrium characterization. Thus, if we consider the joint variable $(\bar{x}, \bar{u})$, the state and input equilibrium subspace, associated to the nominal model (3), is given by $\mathcal{N}([A - I_n \;\; B])$, where $\mathcal{N}$ is the null space of a matrix, that is,

$$[A - I_n \;\; B]\begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} = \mathbf{0}_{n,1}.$$

Defining $\bar{\mathcal{Z}} \triangleq \bar{\mathcal{X}}_N \times \bar{\mathcal{U}}_N$, the set of admissible equilibrium points of the nominal system is given by

$$\bar{\mathcal{Z}}_s \triangleq \left\{(\bar{x}, \bar{u}) \in \rho\bar{\mathcal{Z}} \mid \bar{x} = A\bar{x} + B\bar{u}\right\},$$

where $\rho \in (0, 1)$ is a parameter (usually very close to 1) added to avoid those steady states and inputs that provide active constraints.

Now, let us define a measure of the economic performance,* that is, the economic objective of the plant as

$$\ell_{\text{eco}}(x, u, p), \tag{8}$$

where x and u are the state and the input of the system and p is a vector of bounded parameters, which takes into account prices, costs, production goals, etc. This parameter has to be considered as an input to the RTO layer, resulting from the economic scheduling and planning upper layer. We assume that the parameter p may change throughout the evolution of the plant because of market fluctuations or data reconciliation.[19]

The best admissible steady state is the optimal steady state provided by the RTO, and it can then be defined as follows.

Definition 3. The optimal steady state and input, $(x_s^{\text{eco}}, u_s^{\text{eco}})$, satisfies

$$\left(x_s^{\text{eco}}, u_s^{\text{eco}}\right) = \arg \min_{(x,u)\in\bar{\mathcal{Z}}_s} \ell_{\text{eco}}(x, u, p), \tag{9}$$

for a given p, and it is assumed to be unique.

Remark 2. Note that the optimal steady state and input depend on the value of p, that is, $(x_s^{\text{eco}}(p), u_s^{\text{eco}}(p))$. However, for the sake of clarity, in what follows, we will use the notation $(x_s^{\text{eco}}, u_s^{\text{eco}})$.

Assumption 2. The economic cost function $\ell_{\text{eco}}(x, u, p)$ is locally Lipschitz continuous in $(x_s^{\text{eco}}, u_s^{\text{eco}})$, that is, there exists a constant $\Gamma > 0$ such that

$$\left|\ell_{\text{eco}}(x, u, p) - \ell_{\text{eco}}\left(x_s^{\text{eco}}, u_s^{\text{eco}}, p\right)\right| \leq \Gamma \left|(x, u) - \left(x_s^{\text{eco}}, u_s^{\text{eco}}\right)\right|,$$

for all p and all $(x, u) \in \mathcal{Z}$, such that $|x - x_s^{\text{eco}}| \leq \varepsilon$ and $|u - u_s^{\text{eco}}| \leq \varepsilon$, with $\varepsilon > 0$.

3 | ROBUST FORMULATION OF THE ECONOMIC MPC

The main goal of this work is to design an economic MPC control law capable of maintaining feasibility for any change of the parameter p and ensuring robustness for any $w \in \mathcal{W}$.

As usual in economic MPC literature,[9] the idea is to use the economic cost function (8) as stage cost of the MPC controller. However, if the economic criterion changes, for instance, because of changes in the prices, expected demand, etc, the economically optimal admissible steady state $(x_s^{\text{eco}}, u_s^{\text{eco}})$, where the controller should steer the system,† may change, and the feasibility of the controller may be lost. In this work and following the idea presented in the work of Liu et al,[19] we use a slightly modified cost function, which considers artificial steady state variables and a relaxed terminal constraint, to ensure feasibility for any value of p. Moreover, with the purpose of guaranteeing robustness, the optimal control problem is formulated considering the nominal prediction model and the restricted constraints described in Section 2.1.

The proposed cost function reads

$$V_N(x, p; \mathbf{u}, x_a, u_a) = \sum_{j=0}^{N-1} \ell_{\text{eco}}\left(\bar{x}(j) - x_a + x_s^{\text{eco}}, \bar{u}(j) - u_a + u_s^{\text{eco}}, p\right) + V_f\left(\bar{x}(N), x_a\right) + V_O(x_a, u_a), \tag{10}$$

where (x_a, u_a), the artificial reference, are optimization variables that represent the best steady state that the controller can reach from x, in N steps, and are included to ensure feasibility under any change of the economic cost.[23,24] V_f is a penalty on the terminal state, and $V_O(x_a, u_a)$ is the so-called offset cost function devoted to penalize the lack of economic optimality of the artificial variables.

*Notice that no assumption is made on the positive definiteness of function (8).[3,9]

†In this work, we only consider the case of processes operated at steady state.

Assuming the prestabilizing gain K, of Section 2.1, we can write $V_N(x,p;\mathbf{u},x_a,u_a) = V_N(x,p;\mathbf{c},x_a,u_a)$, where each element of $\mathbf{c}$, $c(j;x)$, fulfills now $\bar{u}(j;x) = K(\bar{x}(j) - x_a) + u_a + c(j;x)$. This way, for any current state x, the optimization problem $P_N(x,p)$ to be solved at each time step is given by

Problem $P_N(x,p)$

$$\begin{aligned}
\min_{\mathbf{c},x_a,u_a} \quad & V_N(x,p;\mathbf{c},x_a,u_a) \\
\text{s.t.} \quad & \bar{x}(0) = x, \\
& \bar{x}(j+1) = A\bar{x}(j) + B\bar{u}(j), j \in \mathbb{I}_{[0:N-1]} \\
& \bar{u}(j) = K(\bar{x}(j) - x_a) + u_a + c(j), j \in \mathbb{I}_{[0:N-1]} \\
& \bar{x}(j) \in \bar{\mathcal{X}}_j, \ \ j \in \mathbb{I}_{[0:N-1]} \\
& \bar{u}(j) \in \bar{\mathcal{U}}_j, \ \ j \in \mathbb{I}_{[0:N-1]} \\
& (\bar{x}(N), x_a, u_a) \in \Omega_t,
\end{aligned}$$

where the last constraint is a terminal inequality constraint, necessary for stability reasons, and set $\Omega_t \subset \mathbb{R}^{2n+m}$ will be defined later on. In the latter optimization problem, x and p are the parameters, whereas the input sequence $\mathbf{c} = \{c(0), \dots, c(N-1)\}$ and the artificial variables x_a and u_a are the optimization variables. The optimal solution to Problem $P_N(x,p)$ and the optimal value of the cost function $V_N(x,p;\mathbf{c},x_a,u_a)$ are denoted respectively as $\mathbf{v}^0(x,p) = \{\mathbf{c}^0(x,p), x_a^0(x,p), u_a^0(x,p)\}$ and $V_N^0(x,p)$. The optimal control law, in the receding horizon fashion, is given by $\kappa_N(x,p) = u^0(0;x,p) = K(x - x_a^0(x,p)) + u_a^0(x,p) + c^0(0;x,p)$, where $c^0(0;x,p)$ is the first element of the solution sequence $\mathbf{c}^0(x,p)$.

The domain of attraction of the controller derived from Problem $P_N(x,p)$ is a compact set given by

$$\mathcal{X}_N = \left\{ x \in \mathcal{X} \, : \, \exists(\mathbf{c}, x_a, u_a) \text{ s.t. } \bar{x}(j) \in \bar{\mathcal{X}}_j, \bar{u}(j) \in \bar{\mathcal{U}}_j, j \in \mathbb{I}_{[0:N-1]}, (\bar{x}(N), x_a, u_a) \in \Omega_t \right\}.$$

To properly account for robust stability, some assumptions and definition regarding the terminal conditions of Problem $P_N(x,p)$ are made. First, the terminal set Ω_t ensures that, for all $(x,x_a,u_a) \in \Omega_t$, (x_a,u_a) is an admissible equilibrium point, the control input $u = K(x - x_a) + u_a$ is admissible, and the successor state remains in Ω_t for the same equilibrium point (x_a,u_a). The set Ω_t is characterized as follows.

Definition 4. (Robust invariant set for tracking[21])
Define the extended state $x_z = (x,x_a,u_a)$ and

$$A_z = \begin{bmatrix} A+BK & -BK & B \\ 0 & I_n & 0 \\ 0 & 0 & I_m \end{bmatrix}.$$

Define also the set

$$X_{z,i} = \{(x,x_a,u_a) : x \in \bar{\mathcal{X}}_i, K(x-x_a) + u_a \in \bar{\mathcal{U}}_i, (x_a,u_a) \in \bar{\mathcal{Z}}_s\}$$

and

$$\Sigma_t = \left\{ x_z : A_z^i x_z \in X_{z,i}, \ \text{for } i \geq 0 \right\},$$

which denotes the maximal robust admissible set for tracking. Then, the terminal robust invariant set for tracking, Ω_t, is given by

$$\Omega_t = \Sigma_t \ominus (\mathcal{R}_N \times \{0\} \times \{0\}).$$

Notice that the terminal constraint, $(\bar{x}(N),x_a,u_a) \in \Omega_t$, implies the following two conditions: (i) $\bar{x}(j) \in \bar{\mathcal{X}}_j$, $\bar{u}(j) \in \bar{\mathcal{U}}_j$, for $j > N$, with $\bar{x}(j+1) = A\bar{x}(j) + B\bar{u}(j)$ and $\bar{u}(j) = K(\bar{x}(j) - x_a) + u_a$; and (ii) $(x_a,u_a) \in \bar{\mathcal{Z}}_s$.

Assumption 3. (On the terminal cost, V_f)
Let $\Omega_t \subseteq \mathcal{X} \times \mathcal{X} \times \mathcal{U}$ be the terminal set presented above, and let $\kappa_f(x,x_a,u_a) = K(x - x_a) + u_a$ be a stabilizing local control law.[23] Moreover, let the terminal cost function $V_f(x,x_a)$ be continuous, with $V_f(x_a,x_a) = 0$ for all x_a, and such that for all $(x,x_a,u_a) \in \Omega_t$, it holds

$$V_f(x^+,x_a) \leq V_f(x,x_a) - \ell_{\text{eco}}\left(x - x_a + x_s^{\text{eco}}, \kappa_f(x,x_a,u_a) - u_a + u_s^{\text{eco}}, p\right) + \ell_{\text{eco}}\left(x_s^{\text{eco}}, u_s^{\text{eco}}, p\right). \tag{11}$$

Following the ideas presented in the work of Amrit et al,[14] we propose to use a terminal cost function given by

$$V_f(x,x_a) = \frac{1}{2}\|x - x_a\|_P^2 + g'(x - x_a), \tag{12}$$

where $P \in \mathbb{R}^{n\times n}$ is the solution of the Lyapunov equation $A_K' P A_K - P = -Q$, for a given matrix $Q \in \mathbb{R}^{n\times n}$, and $g' = t'(I - A_K)^{-1}$, for a given vector $t \in \mathbb{R}^n$. In Lemma 3 in the Appendix B, it is proved how a proper choice of Q and t allows Equation (12) to fulfill Assumption 3.

Assumption 4. (On the offset cost, V_O)
$V_O(x,u)$ is a positive definite strictly convex function such that the unique minimizer of

$$\min_{(x,u)\in\mathcal{Z}_s} V_O(x,u) \tag{13}$$

is (x_s^{eco}, u_s^{eco}). Furthermore, there exists a positive constant γ such that[19]

$$V_O(x,u) - V_O\left(x_s^{eco}, u_s^{eco}\right) \geq \gamma \left|x - x_s^{eco}\right|, \quad \forall\ (x,u) \in \bar{\mathcal{Z}}_s. \tag{14}$$

3.1 | Asymptotic stability and convergence

The usual way to prove asymptotic stability of a closed-loop system under an MPC controller is to use the optimal value of the cost function to be optimized as a Lyapunov function. This approach cannot be used in an economic MPC framework[3,9] since the optimal cost is not necessarily decreasing along the closed-loop trajectory, and in some cases, it may happen that $\ell_{eco}(x,u,p) < \ell_{eco}(x_s^{eco}, u_s^{eco}, p)$ in the transient regime. In this context, we need an extra assumption, which is dissipativity for the system with respect to the supply rate $s(x,u) = \ell_{eco}(x,u,p) - \ell_{eco}(x_s^{eco}, u_s^{eco}, p)$.

Assumption 5. (See the work of Angeli et al[3,9])
For each $(x_s^{eco}, u_s^{eco}) \in \bar{\mathcal{Z}}_s$ and a given p, there exists a continuous function $\lambda : \mathcal{X} \to \mathbb{R}$ (storage function) such that

$$\begin{aligned}&\min_{x,u} \ell_{eco}(x,u,p) + \lambda(x,p) - \lambda(x^+,p) \geq \ell_{eco}\left(x_s^{eco}, u_s^{eco}, p\right)\\ &\text{s.t. } x \in \mathcal{X}, u \in \mathcal{U},\end{aligned}$$

where $x^+ = Ax + Bu$. Moreover, defining the rotated stage cost function as

$$L(x,u,p) = \ell_{eco}(x,u,p) + \lambda(x,p) - \lambda(x^+,p) - \ell_{eco}\left(x_s^{eco}, u_s^{eco}, p\right),$$

there exists a $\mathcal{K}$-functions α such that $L(x,u,p) \geq \alpha(|x - x_s^{eco}|)$.

Remark 3. Note that the storage function depends on the value of p. However, for the sake of clarity, in what follows, we will use the notation $\lambda(x)$.

For the proposed controller, the rotated cost is $L(x - x_a + x_s^{eco}, u - u_a + u_s^{eco}, p)$. If we define $z = x - x_a$ and $v = u - u_a$, then we can rewrite the rotated cost in such a way that $L_r(z,v,p) = L(z + x_s^{eco}, v + u_s^{eco}, p)$. Such a cost enjoys the following property, which is a key point of the stability proof.[19]

Property 1.

1. If $(x_a, u_a) = (x_s^{eco}, u_s^{eco})$, then $L_r(x - x_s^{eco}, u - u_s^{eco}, p) = L(x,u,p)$, where $L(x,u,p)$ is the rotated cost function in the work of Angeli et al.[3,9]
2. If $(x,u) = (x_a, u_a)$, then $L_r(0,0,p) = L(x_s^{eco}, u_s^{eco}, p) = 0$.
3. $L_r(z,v,p) \geq \alpha_1(|z|) + \alpha_2(|v|)$ for certain $\mathcal{K}$-functions α_1 and α_2.

Next, we define the rotated terminal cost as

$$\tilde{V}_f(x,x_a) = V_f(x,x_a) + \lambda(x) - \lambda(x_a) - V_f(x_a,x_a).$$

The rotated terminal cost enjoys the following property.

Property 2. (Rotated terminal cost)
Let Assumption 3 hold, then the pair $(\tilde{V}_f(\cdot), L(\cdot))$ satisfies

$$\tilde{V}_f(x^+,x_a) \leq \tilde{V}_f(x,x_a) - L\left(x - x_a + x_s^{eco}, \kappa_f(x,x_a,u_a) - u_a + u_s^{eco}, p\right)$$

for states $x \in \mathcal{X}$, $x^+ = Ax + Bu$, $(x_a,u_a) \in \bar{\mathcal{Z}}_s$ and a given p.

Proof. This property is proved by adding $\lambda(x^+) - \lambda(x)$ in both sides of (11), that is,

$$\begin{aligned}V_f(x^+,x_a) - V_f(x,x_a) + \lambda(x^+) - \lambda(x) \leq &- \ell_{eco}\left(x - x_a + x_s^{eco}, \kappa_f(x,x_a,u_a) - u_a + u_s^{eco}, p\right)\\ &+ \ell_{eco}\left(x_s^{eco}, u_s^{eco}, p\right) + \lambda(x^+) - \lambda(x)\\ \tilde{V}_f(x^+,x_a) - \tilde{V}_f(x,x_a) \leq &\ L\left(x - x_a + x_s^{eco}, \kappa_f(x,x_a,u_a) - u_a + u_s^{eco}, p\right).\end{aligned}$$

□

On the basis of all the previous considerations, we can define an auxiliary cost function as follows:

$$\tilde{V}_N(x,p;\mathbf{u},x_a,u_a)=\sum_{j=0}^{N-1}L\left(\bar{x}(j)-x_a+x_s^{eco},\bar{u}(j)-u_a+u_s^{eco},p\right)+\tilde{V}_f\left(\bar{x}(N),x_a\right)+\tilde{V}_O(x_a,u_a),$$

where $\tilde{V}_O(x_a,u_a)$ is the rotated offset cost function, given by

$$\tilde{V}_O(x_a,u_a)=V_O(x_a,u_a)+\lambda(x_a)-\lambda\left(x_s^{eco}\right)-V_O\left(x_s^{eco},u_s^{eco}\right).$$

As for Equation (10), given the prestabilizing gain K, of Section 2.1, we can write $\tilde{V}_N(x,p;\mathbf{u},x_a,u_a=\tilde{V}_N(x,p;\mathbf{c},x_a,u_a)$, where each element of $\mathbf{c}$, $c(j;x)$, is such that $\bar{u}(j;x)=K(\bar{x}(j)-x_a)+u_a+c(j;x)$. Then, for any current state x, we can define the auxiliary optimization problem $\tilde{P}_N(x,p)$ given by

Problem $\tilde{P}_N(x,p)$

$$\begin{aligned}\min_{\mathbf{c},x_a,u_a}\ & \tilde{V}_N(x,p;\mathbf{c},x_a,u_a)\\ s.t.\ & \bar{x}(0)=x,\\ & \bar{x}(j+1)=A\bar{x}(j)+B\bar{u}(j),\quad j\in\mathbb{I}_{[0:N-1]}\\ & \bar{u}(j)=K(\bar{x}(j)-x_a)+u_a+c(j),\quad j\in\mathbb{I}_{[0:N-1]}\\ & \bar{x}(j)\in\bar{\mathcal{X}}_j,\quad j\in\mathbb{I}_{[0:N-1]}\\ & \bar{u}(j)\in\bar{\mathcal{U}}_j,\quad j\in\mathbb{I}_{[0:N-1]}\\ & (\bar{x}(N),x_a,u_a)\in\Omega_t.\end{aligned}$$

In Lemma 1 in the Appendix A, it is proved that the auxiliary optimization problem $\tilde{P}_N(x,p)$ delivers the same optimal solution as $P_N(x,p)$. Then, the optimal value of the cost function $\tilde{V}_N(x,p;\mathbf{u},x_a,u_a)$, denoted in what follows as $\tilde{V}_N^0(x,p)$, can be used to define a candidate Lyapunov function.

Theorem 1. *Consider that Assumptions 1 to 5 hold, and consider a given parameter p for the economic cost* $\ell_{eco}(x,u,p)$*. Then, for any initial state* $x\in\mathcal{X}_N$ *and for all* $\gamma>|g|$ *with g given in (12), the optimization problem* $\tilde{P}_N(x,p)$ *is recursively feasible, and the MPC control law drives the disturbed system (1) to* $x_s^{eco}\oplus\mathcal{R}_\infty$*, for any* $w\in\mathcal{W}$.

Proof.

i. *Recursive Feasibility*
   Recursive feasibility of the proposed Robust Economic MPC can be proved resorting to the work of Ferramosca et al,[21] lemmas 1 to 3.
ii. *Stability*
   Let us define the function $J(x,p)=\tilde{V}_N^0(x,p)-\tilde{V}_O(x_s^{eco},u_s^{eco})$ as a candidate Lyapunov function. Then, there exist $\mathcal{K}$-functions α_J, β_J, and σ such that

   i. $J(x,p)\geq\alpha_J(|x-x_s^{eco}|)$, for all $x\in\mathcal{X}_N$ (it follows from Property 1.3 and Lemma 4 in Appendix B[19]).
   ii. $J(x,p)\leq\beta_J(|x-x_s^{eco}|)$, for all $x\in\mathcal{X}_N$ (it follows from the fact that $\mathcal{X}_N$ is compact, $J(x_s^{eco},p)=0$, and $J(x,p)$ is continuous in $x=x_s^{eco}$ [2]).
   iii. $J(x^+,p)-J(x,p)\leq-\alpha_J(|x-x_s^{eco}|)+\sigma(|w|)$, for all $x\in\mathcal{X}_N$ and $w\in\mathcal{W}$, where σ is a $\mathcal{K}$-function (see Lemma 2 in Appendix A).

   Then, resorting to ISS arguments,[25] it can be proved that there exist a $\mathcal{KL}$ function β and a $\mathcal{K}$ function ϑ such that

$$\left|x(k)-x_s^{eco}\right|\leq\beta(\left|x(0)-x_s^{eco}\right|,k)+\vartheta(|w|)$$

for all initial state $x(0)\in\mathcal{X}_N$ and all disturbances $w\in\mathcal{W}$, which means that the closed-loop is ISS, and so, the disturbed system (1) is steered to $x_s^{eco}\oplus\mathcal{R}_\infty$. □

3.2 | Properties of the proposed controller

The proposed controller is a robust formulation of the economic MPC for changing economic criterion presented in the work of Ferramosca et al.[19] The main properties of the new formulation are listed below.

- **Feasibility and domain of attraction.** Since the domain of attraction $\mathcal{X}_N$ does not depend on the optimal steady state, then for all initial condition $x\in\mathcal{X}_N$, every admissible steady state is reachable. Moreover, since the trajectory remains in $\mathcal{X}_N$, if the economic criterion changes, problem $P_N(x, p)$ does not lose feasibility. However, in practice, it may be difficult to know beforehand if an initial state is in $\mathcal{X}_N$ in order to close the loop with the controller. It is industrial practice to manually operate the plant to an admissible equilibrium point before closing the control loop. Then, the initial state is an equilibrium point, that is, $(x, u) \in \bar{\mathcal{Z}}_s$. In this practical case, feasibility of the proposed controller is ensured since the initial state would be in the domain of attraction of the controller even for $N = 1$.

- **Local economic optimality.** The proposed controller may be suboptimal while $x_a \not\approx x_s^{\text{eco}}$, that is, its performance may be different from standard economic MPC controllers.[9] This suboptimality is a consequence of the particular cost to minimize and the relaxed terminal constraint, and in some sense, it is the price one has to pay for always ensuring feasibility. However, it is proved in the aforementioned work[19] that under some mild assumption on the cost $V_O(\cdot, \cdot)$, the controller ensures the economic optimality at least locally, that is, in a region $\mathcal{X}_N^e \subseteq \mathcal{X}_N$, where $\mathcal{X}_N^e$ is the domain of attraction of the standard economic MPC.[9] Note that, since it is the stage cost that provides the economic optimality, one is free to choose the offset cost function $V_O(\cdot, \cdot)$ in any form such that it satisfies Assumption 4.
- **Robust Strategy.** The proposed controller ensures robustness for any additive bounded disturbances $w \in \mathcal{W}$ and maintains the closed-loop system in a region of the state space around the economically optimal operation point, given by the minimal RPI set, that is, the system (1) is steered to $x_s^{\text{eco}} \oplus \mathcal{R}_\infty$, for all $w \in \mathcal{W}$.

4 | ILLUSTRATIVE EXAMPLE

In this section, in order to demonstrate the properties and the benefits of the proposed robust economic MPC strategy, some simulations results will be presented. First, a brief description of the considered system is shown. Then, the results of dynamic simulations are presented.

4.1 | System description

The system that we consider is the four-tank plant.[26] This is composed of four diagonally interconnected tanks. The controlled flow pumps A and B extract water from the lower tank by pouring it into tanks 1 and 4, pump A, and tanks 2 and 3 pump B. All tanks are discharged by gravity as shown in Figure 1.

Basically, the purpose of the plant is to control the levels of tanks 1 and 2, which represent the two outputs of the system. These outputs are strongly coupled since if it is desired to increase the level of the reservoir 1, increasing the flow rate of the pump A also increases the level of the reservoir 4, which, when discharged thereon, will increase the level thereof.

This way, the water levels in the four tanks (h_1, h_2, h_3, h_4) are the state of the system, whereas the inlet flows (q_a, q_b) are the manipulated variables.

The system is interesting because of the following. (i) All state variables are accessible since the heights of the liquids can be measured. (ii) The system model is nonlinear. (iii) The variables to be controlled are strongly coupled.

A state-space continuous-time nonlinear model of the quadruple-tank process system is given by Johansson[26]

$$\frac{\mathrm{d}h_1}{\mathrm{d}t} = -\frac{a_1}{S}\sqrt{2gh_1} + \frac{a_3}{S}\sqrt{2gh_3} + \frac{\gamma_a}{S}\frac{q_a}{3600} \tag{15a}$$

$$\frac{\mathrm{d}h_2}{\mathrm{d}t} = -\frac{a_2}{S}\sqrt{2gh_2} + \frac{a_4}{S}\sqrt{2gh_4} + \frac{\gamma_b}{S}\frac{q_b}{3600} \tag{15b}$$

$$\frac{\mathrm{d}h_3}{\mathrm{d}t} = -\frac{a_3}{S}\sqrt{2gh_3} + \frac{(1-\gamma_b)}{S}\frac{q_b}{3600} \tag{15c}$$

$$\frac{\mathrm{d}h_4}{\mathrm{d}t} = -\frac{a_4}{S}\sqrt{2gh_4} + \frac{(1-\gamma_a)}{S}\frac{q_a}{3600}, \tag{15d}$$

where, the parameters of the plant are

- S: Cross-section of the tanks ($[m^2]$).
- a_i: Discharge constant of the tank i ($[m^2]$).
- h_i: Water level of the tank i (state of the system) ($[m]$).
- q_a, q_b: Flow produced by the pumps A and B ($[m^3/h]$).
- g: The acceleration of gravity ($[m/s^2]$).
- γ_a, γ_b: Ratio of the three-way valves.

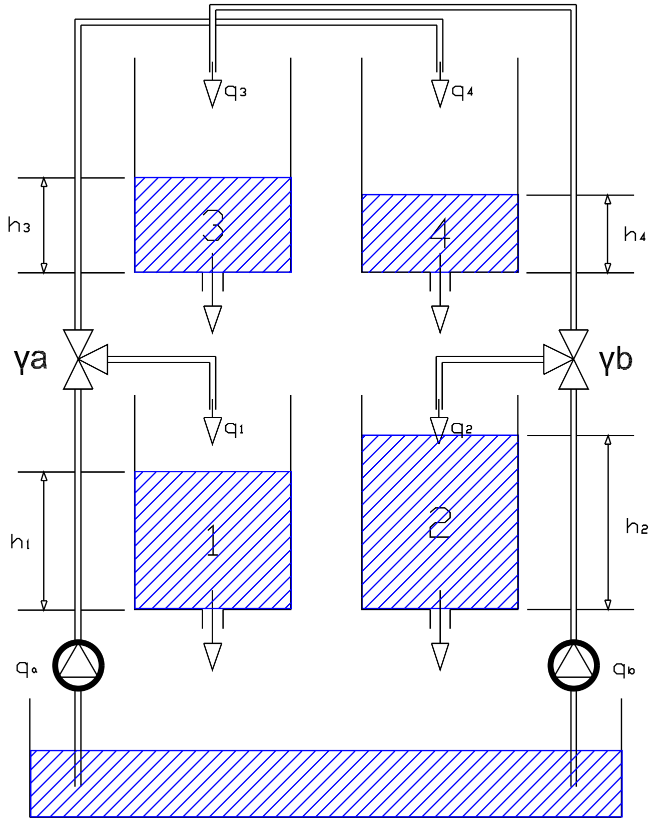

FIGURE 1 The quadruple tank process [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 1 Parameters of the four-tank plant

	Value	Unit	Description
a_1	1.310e-4	m^2	Discharge constant of tank 1
a_2	1.507e-4	m^2	Discharge constant of tank 2
a_3	9.267e-5	m^2	Discharge constant of tank 3
a_4	8.816e-5	m^2	Discharge constant of tank 4
S	0.06	m^2	Cross-section of all tanks
γ_a	0.3		Parameter of the 3-way valve
γ_b	0.4		Parameter of the 3-way valve

The value of these parameters are shown in the Table 1. These values have been estimated on the experimental plant located at the Control Laboratory of the University of Seville (Spain). For a detailed description of this experimental plant, please refer to the work of Alvarado et al.[27]

Linearizing the model at an operating point given by $h^o = (0.6487, 0.6639, 0.6498, 0.6592)$, $q^o = (1.63, 2)$, and defining $x_i = h_i - h_i^o$, $u_j = q_j - q_j^o$, where $i = 1, \dots, 4$ and $j = a, b$, we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \begin{bmatrix} \frac{-1}{\tau_1} & 0 & \frac{1}{\tau_3} & 0 \\ 0 & \frac{-1}{\tau_2} & 0 & \frac{1}{\tau_4} \\ 0 & 0 & \frac{-1}{\tau_3} & 0 \\ 0 & 0 & 0 & \frac{-1}{\tau_4} \end{bmatrix} x + \begin{bmatrix} \frac{\gamma_a}{\chi} & 0 \\ 0 & \frac{\gamma_b}{\chi} \\ 0 & \frac{(1-\gamma_b)}{\chi} \\ \frac{(1-\gamma_a)}{\chi} & 0 \end{bmatrix} u + w, \qquad y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x,$$

where $\chi = 3600 * S$, and $\tau_i = \frac{S}{a_i}\sqrt{\frac{2h_i^o}{g}} \geq 0, i = 1, \dots, 4$, are the time constants of each tank.

The system has been discretized using the zero-order hold method with a sample time of $T_s = 5[s]$.

The set $\mathcal{W}$ of possible disturbances realization is given by $\mathcal{W} = \{w \in \mathbb{R}^4 : \|w\|_\infty \leq 5 \times 10^{-3}\}$, and it was selected to account for plant-model mismatches in the usual operating points.[28]

The system must fulfill the following constraints: $\mathcal{X} = \{x \in \mathbb{R}^4 : 0.2 \leq x_{1,2} \leq 1.36; 0.2 \leq x_{3,4} \leq 1.30\}$ and $\mathcal{U} = \{u \in \mathbb{R}^2 : [0,0] \leq u \leq [3.26,4]\}$. In what follows, set $\mathcal{Y}$ will denote the set of admissible outputs given by $\mathcal{Y} = \text{Proj}_y(\mathcal{X})$. Matrix K has been chosen as the linear quadratic regulator gain, for $Q = I_4$ and $R = 0.001 I_2$, and it is given by

$$K = \begin{bmatrix} -6.5049 & -4.9114 & 3.0196 & -22.5094 \\ -6.1002 & -9.5974 & -21.7712 & 2.3534 \end{bmatrix}.$$

The prediction horizon has been chosen as $N = 5$.

The economic objective is to minimize the plant energetic consumption,[29] by minimizing the voltage of the two pumps, and at the same time to maximize the volume of water in the tanks 1 and 2. Then, the economic cost function are given by

$$f_{\text{eco}}(y,u,p) = \left(q_a^2 + \psi q_b^2\right) + \theta \frac{V_{\min}}{S(h_1 + h_2)},$$

where$V_{\min} = 0.012[m^3]$is the minimum volume of water to be accumulated in the tanks. $y = (h_1, h_2)$, $u = (q_a, q_b)$, and $p = (\psi, \theta)$ are the prices on the cost function. Note that this function is strictly convex in $(x, u)$and twice differentiable.

4.2 | Dynamic simulations

In order to observe the dynamic performance of the system, it is considered a starting point that is the linearisation point of the nominal model h_i^o.

On the other hand, three economic costs have been considered on the basis of the following values of prices: $p_1 = (5, 10)$ and $p_2 = (0.5, 100)$ and $p_3 = (5, 100)$. For these cases, the economically optimal steady conditions and optimal costs are, respectively,

$$\begin{aligned}
(y_{s1}^{\text{eco}}, u_{s1}^{\text{eco}}) &= (0.2951, 0.3997, 1.4623, 1.3196), \quad & f_{\text{eco}}(y_{s1}^{\text{eco}}, u_{s1}^{\text{eco}}, p_1) &= 13.7234,\\
(y_{s2}^{\text{eco}}, u_{s2}^{\text{eco}}) &= (1.1118, 0.9473, 1.6652, 2.9832), \quad & f_{\text{eco}}(y_{s2}^{\text{eco}}, u_{s2}^{\text{eco}}, p_2) &= 16.9358,\\
(y_{s3}^{\text{eco}}, u_{s3}^{\text{eco}}) &= (0.5153, 0.8517, 2.4144, 1.3196), \quad & f_{\text{eco}}(y_{s3}^{\text{eco}}, u_{s3}^{\text{eco}}, p_3) &= 29.1662.
\end{aligned}$$

The chosen terminal cost function $V_f(x, x_a)$ presented in Section 3 satisfies Assumption 3 and has the next values for matrices P and g for each values of prices p_1, p_2, and p_3

$$P_1 = 10^4 * \begin{bmatrix} 9.0172 & -1.5784 & 1.2552 & -3.5418 \\ -1.5784 & 7.3651 & -4.3798 & 0.8197 \\ 1.2552 & -4.3798 & 3.9311 & -0.6338 \\ -3.5418 & 0.8197 & -0.6338 & 2.4651 \end{bmatrix}, \qquad g_1 = \begin{bmatrix} -224.0255 \\ -253.8485 \\ -753.2375 \\ -89.7624 \end{bmatrix}$$

$$P_2 = 10^4 * \begin{bmatrix} 2.4811 & -0.4343 & 0.3454 & -0.9745 \\ -0.4343 & 2.0265 & -1.2051 & 0.2255 \\ 0.3454 & -1.2051 & 1.0816 & -0.1744 \\ -0.9745 & 0.2255 & -0.1744 & 0.6783 \end{bmatrix}, \qquad g_2 = \begin{bmatrix} -155.5548 \\ -134.4142 \\ -128.1216 \\ -139.6735 \end{bmatrix}$$

$$P_3 = 10^4 * \begin{bmatrix} 9.4163 & -1.6482 & 1.3108 & -3.6985 \\ -1.6482 & 7.6911 & -4.5737 & 0.8560 \\ 1.3108 & -4.5737 & 4.1051 & -0.6618 \\ -3.6985 & 0.8560 & -0.6618 & 2.5742 \end{bmatrix}, \qquad g_3 = \begin{bmatrix} -382.3292 \\ -373.7694 \\ -688.4208 \\ -144.6703 \end{bmatrix}.$$

The results of the simulation are presented in Figures 2 to 4. In particular, Figures 2 and 3 show the evolution of the controlled outputs h_1 and h_2 and the evolution of the control inputs q_a and q_b, respectively. The economic steady output, the artificial references, and the real output are depicted, respectively, in blue dash-dotted, red dashed, and black solid lines. It can be observed how the inputs and outputs of the system are affected throughout the time by the additive disturbances w, highlighting the robust behavior of the controlled system. This behavior is also visible in Figure 4, where the output space evolution of this simulation is shown. The green shaded sets represent the projections onto the output space of the minimal RPIs centered in the economically optimal steady states, that is, $y_{si}^{\text{eco}} \oplus C\mathcal{R}_\infty$. It can be noticed that the controller drives the controlled system toward the economically optimal operation point. The deviation is due to the presence of the disturbance, which is considered, in this case, having a truncated Gaussian distribution. It can be observed that the system is asymptotically steered to the robust invariant region around the optimal steady state and then maintained there, ie, $y \to y_s^{\text{eco}} \oplus C\mathcal{R}_\infty$ as $\bar{y} \to y_s^{\text{eco}}$, which means that the proposed controller satisfies the economic objective even in the presence of disturbances.

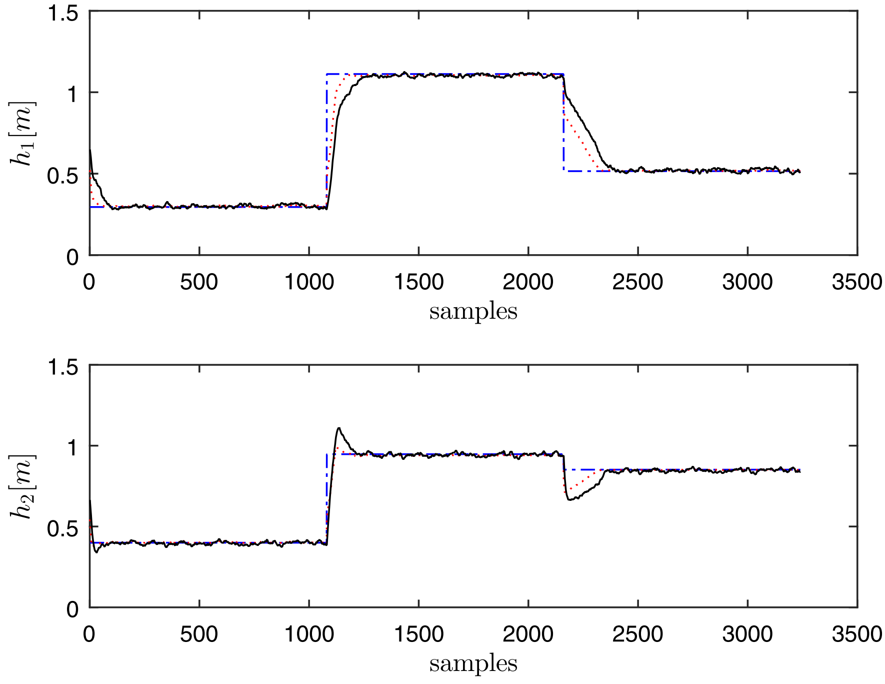

FIGURE 2 Evolution of the outputs h_1 and h_2: system output in black solid line, artificial reference in red dotted line, economic optimum in blue dash-dotted line [Colour figure can be viewed at wileyonlinelibrary.com]

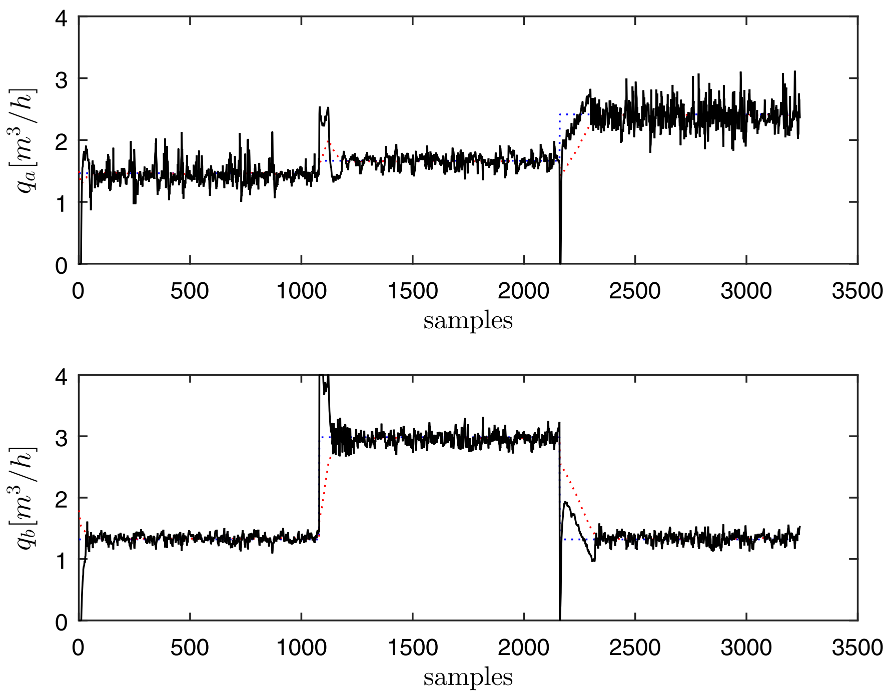

FIGURE 3 Evolution of the control inputs q_a and q_b: system output in black solid line, artificial reference in red dotted line [Colour figure can be viewed at wileyonlinelibrary.com]

Moreover, there is no loss of feasibility when changing the economic parameter p, and therefore, the controller is able to drive the system to a new economic steady state. This is thanks to the role played by the artificial variables in the design of the controller.

Figure 5 shows the economic cost for the different values of p_1, p_2, and p_3. We can observe that the economic cost function robustly converges to the optimal value of f_{eco} for all changes of p.

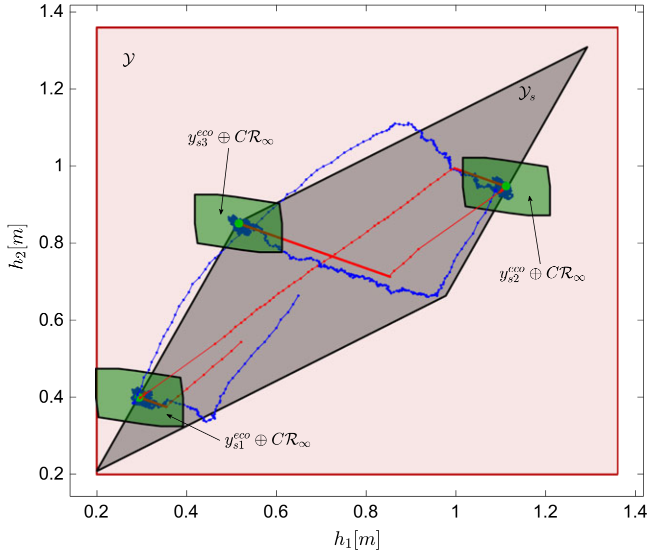

FIGURE 4 Output space evolution of the closed-loop system in case of a truncated Gaussian distribution [Colour figure can be viewed at wileyonlinelibrary.com]

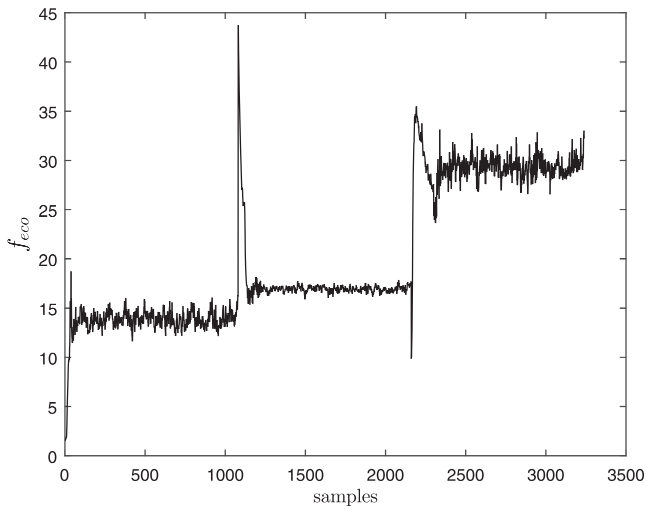

FIGURE 5 Evolution of the economic cost f_{eco}, for different values of p

Summarizing, this illustrative example shows the following.

- Despite the presence of the nonvanishing additive disturbance w, the controller is able to steer the system toward the economically optimal operation point, maintaining it in a region around the steady state given by the RPI. In fact, once the output evolution enters the set $y_s^{\text{eco}} \oplus C\mathcal{R}_\infty$, it never leaves it unless a change in the economic criterion, that is, a change in the parameter p, occurs.
- The artificial reference allows the controller to maintain the feasibility when the economically optimal steady state changes. That is, if the economic goal changes, the controller is able to bring the system to a new point of feasible operation, keeping the stability and convergence properties.

Remark 4. Note that the online computational demand of the proposed robust controller is not greater than the one of a nominal one since every polytopic set involved in the online MPC optimization problem is computed offline.

5 | CONCLUSIONS

In this work, a new robust economic MPC for a changing economic criterion has been presented for the case of linear systems with additive bounded disturbances.

The overall idea is to robustly consider the economic optimization cost into the controller formulation. In this way, the proposed controller is able to bring the system to an economic optimal operation, ensuring economic optimality and maintaining the closed-loop system in a region of the state space around the economically optimal operation point, given by the minimal RPI Set. By means of the economic MPC formulation and the use of artificial variables, stability and convergence to the optimum are ensured. Robustness is achieved by means of a nominal prediction model and the usage of restricted constraints to properly accounts for the disturbance effects.

The main benefits of the proposed controller are as follows.

- The closed loop robustly converges to the optimal economic point that minimizes ℓ_{eco}.
- The controller remains feasible under any change of the economic objective and any disturbance realization.
- Economic optimality is maintained under some mild assumptions on the offset cost function.

A simulation example of a four-tank system showed that the strategy could be useful from an application point of view.

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APPENDIX A

PROOF OF THEOREM

Lemma 1. *Consider that Assumptions 1 to 5 hold, and consider a given parameter p for the economic cost* $\ell_{\text{eco}}(x, u, p)$*. Then, for any initial state* $x \in \mathcal{X}_N$*, problem* $P_N(x, p)$ *and the auxiliary problem* $\tilde{P}_N(x, p)$ *deliver the same optimal solution.*

Proof. From the definition of rotated cost in Equation 15, we have that

$$\begin{aligned}
\tilde{V}_N(x, p; \mathbf{u}, x_a, u_a) &= \sum_{j=0}^{N-1} L\big(\bar{x}(j) - x_a + x_s^{\text{eco}}, \bar{u}(j) - u_a + u_s^{\text{eco}}, p\big) + \tilde{V}_f\left(\bar{x}(N), x_a\right) + \tilde{V}_O(x_a, u_a) \\
&= \sum_{j=0}^{N-1} \left[\ell_{\text{eco}}\left(\bar{x}(j) - x_a + x_s^{\text{eco}}, \bar{u}(j) - u_a + u_s^{\text{eco}}, p\right) + \lambda(\bar{x}(j)) - \lambda(\bar{x}(j+1)) - \ell_{\text{eco}}\left(x_s^{\text{eco}}, u_s^{\text{eco}}, p\right)\right] \\
&\quad + V_f\left(\bar{x}(N), x_a\right) + \lambda(\bar{x}(N)) - \lambda(x_a) - V_f(x_a, x_a) + V_O(x_a, u_a) + \lambda(x_a) - \lambda\big(x_s^{\text{eco}}\big) - V_O\big(x_s^{\text{eco}}, u_s^{\text{eco}}\big).
\end{aligned}$$

Notice that $V_f(x_a, x_a) = 0$ and $\sum_{j=0}^{N-1} \lambda(\bar{x}(j)) - \lambda(\bar{x}(j+1)) = \lambda(\bar{x}) - \lambda(\bar{x}(N))$. Hence,

$$\begin{aligned}
\tilde{V}_N(x, p; \mathbf{u}, x_a, u_a) &= \sum_{j=0}^{N-1} \left[\ell_{\text{eco}}(\bar{x}(j) - x_a + x_s^{\text{eco}}, \bar{u}(j) - u_a + u_s^{\text{eco}}, p)\right] + \lambda(\bar{x}) - \lambda(\bar{x}(N)) - N\ell_{\text{eco}}\left(x_s^{\text{eco}}, u_s^{\text{eco}}, p\right) \\
&\quad + V_f\left(\bar{x}(N), x_a\right) + \lambda\left(\bar{x}(N)\right) + V_O(x_a, u_a) - \lambda\left(x_s^{\text{eco}}\right) - V_O\left(x_s^{\text{eco}}, u_s^{\text{eco}}\right) \\
&= \sum_{j=0}^{N-1} \ell_{\text{eco}}\left(\bar{x}(j) - x_a + x_s^{\text{eco}}, \bar{u}(j) - u_a + u_s^{\text{eco}}, p\right) + V_f\left(\bar{x}(N), x_a\right) + V_O(x_a, u_a) \\
&\quad - N\ell_{\text{eco}}\left(x_s^{\text{eco}}, u_s^{\text{eco}}, p\right) - V_O\left(x_s^{\text{eco}}, u_s^{\text{eco}}\right) + \lambda(x) - \lambda\left(x_s^{\text{eco}}\right),
\end{aligned}$$

where the first term is $V_N(x, p; \mathbf{u}, x_a, u_a)$, Equation (10), and the second does not depend on the decision variable vector $(\mathbf{u}, x_a, u_a)$. Thus, $\tilde{V}_N(\cdot)$ and $V_N(\cdot)$ differ by terms that are constant for a given initial state x.

Therefore, since problem $P_N(x, p)$ and problem $\tilde{P}_N(x, p)$ have the same set of constraint and their objective function only differs by constant terms, we conclude that the solutions of $P_N(x, p)$ and $\tilde{P}_N(x, p)$ are equal. □

Lemma 2. *Consider that Assumptions 1 to 5 hold, and consider a given parameter p for the economic cost* $\ell_{\text{eco}}(x, u, p)$*. For any initial state* $x(k) \in \mathcal{X}_N$*, consider the optimal solution to problem* $\tilde{P}_N(x(k), p)$*,* $\tilde{V}_N^0(x(k), p)$ *and define the function* $J(x(k), p) = \tilde{V}_N^0(x(k), p) - \tilde{V}_O(x_s^{\text{eco}}, u_s^{\text{eco}})$*. Then,*

$$J\left(x(k+1), p\right) - J\left(x(k), p\right) \leq -\alpha_J\left(\left|x(k) - x_s^{\text{eco}}\right|\right) + \sigma\left(|w(k)|\right)$$

for all $x(k) \in \mathcal{X}_N$ *and* $w(k) \in \mathcal{W}$*, where* σ *and* α_J *are* $\mathcal{K}$*-function.*

Proof. To prove this lemma, as standard in MPC, let us compare two costs, the cost at time $k+1$ and at time k. Since the measured state $x(k+1)$ is uncertain, we consider the case that $w(k) = 0$ and $w(k) \neq 0$. In what follows, we will use the notation x for $x(k)$, x^+ for $x(k+1)$, and w for $w(k)$.

First of all, notice that, if $w = 0$, following same arguments as in the work of Ferramosca et al,[19] it can be shown that

$$J(x^+, p) - J(x, p) \leq -\alpha_J\left(\left|x - x_s^{\text{eco}}\right|\right).$$

Consider now that $w \neq 0$. We proceed now to compare the optimal cost $\tilde{V}_N^0(x, p)$ at time k, with $\tilde{V}_N(x^+, p; \tilde{\mathbf{u}}, x_a, \tilde{u}_a)$, that is, the cost at time $k+1$ given by the feasible candidate sequence

$$\begin{aligned}
\tilde{\mathbf{u}} &= \{\tilde{u}(0), \tilde{u}(1), \ldots, \tilde{u}(N-1)\} \\
&= \left\{u^0(1;x), u^0(2;x), \ldots, u^0(N-1,x), K\left(x(N) - x_a^0\right) + u_a^0\right\} \\
\tilde{x}_a &= x_a^0(x) \\
\tilde{u}_a &= u_a^0(x).
\end{aligned}$$

Notice that it is not possible to compare $\tilde{V}_N(x^+, p; \tilde{\mathbf{u}}, \tilde{x}_a, \tilde{u}_a)$ and $\tilde{V}_N^0(x, p)$ using standard nominal MPC arguments[2] since $x^+ \neq Ax + Bu^0(0;x)$ because of the presence of the disturbance. However, taking into account the nominal prediction model and Lemma 6, we can state that $x^+ = \bar{x}(1;x) + w$, where $\bar{x}(1;x) = Ax + Bu^0(0;x)$. Therefore,

$$\begin{aligned}
\Delta V_N &= \tilde{V}_N(x^+, p; \tilde{\mathbf{u}}, \tilde{x}_a, \tilde{\mathbf{u}}) - \tilde{V}_N^0(x, p) \\
&= \tilde{V}_N(x^+, p; \tilde{\mathbf{u}}, \tilde{x}_a, \tilde{u}_a) - \tilde{V}_N\left(\bar{x}(1;x), p; \tilde{\mathbf{u}}, \tilde{x}_a, \tilde{u}_a\right) + \tilde{V}_N\left(\bar{x}(1;x), p; \tilde{\mathbf{u}}, \tilde{x}_a, \tilde{u}_a\right) - \tilde{V}_N^0(x, p) \\
&= \Delta V_N^w + \Delta V_N^{\text{nom}},
\end{aligned}$$

where

$$\begin{aligned}
\Delta V_N^w &= \tilde{V}_N(x^+, p; \tilde{\mathbf{u}}, \tilde{x}_a, \tilde{u}_a) - \tilde{V}_N(\bar{x}(1;x), p; \tilde{\mathbf{u}}, \tilde{x}_a, \tilde{u}_a) \\
\Delta V_N^{\text{nom}} &= \tilde{V}_N\left(\bar{x}(1;x), p; \tilde{\mathbf{u}}, \tilde{x}_a, \tilde{u}_a\right) - \tilde{V}_N^0(x, p).
\end{aligned}$$

Note that, following same argument as in the works of Amrit et al an Ferramosca et al,[14,19] it can be shown that

$$\begin{aligned}
\Delta V_N^{\text{nom}} &= \tilde{V}_N\left(\bar{x}(1;x), p; \tilde{\mathbf{u}}, \tilde{x}_a, \tilde{u}_a\right) - \tilde{V}_N^0(x, p) \\
&= -L\left(x - x_a^0 + x_s^{\text{eco}}, u^0(0;x) - u_a^0 + u_s^{\text{eco}}, p\right).
\end{aligned}$$

Let us now analyze ΔV_N^w. First of all, let us define the sequence of nominal predictions corresponding to the feasible candidate sequence $\tilde{\mathbf{u}}$, starting from x^+ at time $k+1$

$$\begin{aligned}
\tilde{x}(j;x^+) &= A^j x^+ + \sum_{i=0}^{j-i} A^i B\tilde{u}(j-i-1) \\
&= \bar{x}(j+1;x) + A_K^j w,
\end{aligned}$$

where the last equality comes from Lemma 6 (Appendix B).

Then,

$$
\begin{aligned}
\Delta V_N^w =& \tilde{V}_N(x^+,p;\tilde{\mathbf{u}},\tilde{x}_a,\tilde{u}_a) - \tilde{V}_N(\bar{x}(1;x),p;\tilde{\mathbf{u}},\tilde{x}_a,\tilde{u}_a) \\
=& \sum_{j=0}^{N-1} L\left(\tilde{x}(j;x^+)-\tilde{x}_a+x_s^{\mathrm{eco}},\tilde{u}(j)-\tilde{u}_a+u_s^{\mathrm{eco}},p\right) + \tilde{V}_f\big(\tilde{x}(N;x^+),\tilde{x}_a\big) + \tilde{V}_O(\tilde{x}_a,\tilde{u}_a) \\
&- \sum_{j=0}^{N-1} L\left(\bar{x}(j+1;x)-\tilde{x}_a+x_s^{\mathrm{eco}},\tilde{u}(j)-\tilde{u}_a+u_s^{\mathrm{eco}},p\right) - \tilde{V}_f(\bar{x}(N+1;x),\tilde{x}_a) - \tilde{V}_O(\tilde{x}_a,\tilde{u}_a) \\
=& \sum_{j=0}^{N-1} \ell_{\mathrm{eco}}\left(\tilde{x}(j;x^+)-x_a^0+x_s^{\mathrm{eco}},u^0(j+1;x)-\tilde{u}_a+u_s^{\mathrm{eco}},p\right) + V_f\big(\tilde{x}(N;x^+),x_a^0\big) + V_O\big(x_a^0,u_a^0\big) \\
&+ \lambda(x^+) - \lambda\big(x_s^{\mathrm{eco}}\big) - N\ell_{\mathrm{eco}}\big(x_s^{\mathrm{eco}},u_s^{\mathrm{eco}},p\big) + \lambda\big(\tilde{x}(N;x^+)\big) - \lambda\big(x_a^0\big) - V_f\big(x_a^0,x_a^0\big) - V_O\big(x_s^{\mathrm{eco}},u_s^{\mathrm{eco}}\big) \\
&- \sum_{j=0}^{N-1} \ell_{\mathrm{eco}}\big(\bar{x}(j+1;x)-x_a^0+x_s^{\mathrm{eco}},u^0(j+1;x)-u_a^0+u_s^{\mathrm{eco}},p\big) - V_f\big(\bar{x}(N+1;x),x_a^0\big) - V_O\big(x_a^0,u_a^0\big) \\
&- \lambda(\bar{x}(1;x)) + \lambda\big(x_s^{\mathrm{eco}}\big) + N\ell_{\mathrm{eco}}\big(x_s^{\mathrm{eco}},u_s^{\mathrm{eco}},p\big) - \lambda(\bar{x}(N+1;x)) + \lambda\big(x_a^0\big) + V_f\big(x_a^0,x_a^0\big) + V_O\big(x_s^{\mathrm{eco}},u_s^{\mathrm{eco}}\big) \\
=& \sum_{j=0}^{N-1} \left(\ell_{\mathrm{eco}}(\tilde{x}(j;x^+)-x_a^0+x_s^{\mathrm{eco}},u^0(j+1;x)-\tilde{u}_a+u_s^{\mathrm{eco}},p) - \ell_{\mathrm{eco}}(\bar{x}(j+1;x)-x_a^0+x_s^{\mathrm{eco}},u^0(j+1;x)-u_a^0+u_s^{\mathrm{eco}},p)\right) \\
&+ V_f\big(\tilde{x}(N;x^+),x_a^0\big) - V_f\big(\bar{x}(N+1;x),x_a^0\big) + \lambda(x^+) - \lambda(\bar{x}(1;x)) + \lambda\big(\tilde{x}(N;x^+)\big) - \lambda(\bar{x}(N+1;x)),
\end{aligned}
$$

where the second to last equality are similar to the ones developed in Lemma 1. Notice that, since $\lambda(\cdot)$ is continuous in the compact set $\mathcal{X}$, then it is uniformly continuous on $\mathcal{X}$, and hence,

$$
\begin{aligned}
\lambda(x^+) - \lambda(\bar{x}(1;x)) &\leq \left|\lambda(x^+) - \lambda(\bar{x}(1;x))\right| \\
&\leq \sigma_{\lambda_1}\left(\left|x^+ - \bar{x}(1;x)\right|\right) = \sigma_{\lambda_1}(|w|) \\
\lambda\big(\tilde{x}(N;x^+)\big) - \lambda(\bar{x}(N+1;x)) &\leq \left|\lambda\big(\tilde{x}(N;x^+)\big) - \lambda(\bar{x}(N+1;x))\right| \\
&\leq \bar{\sigma}_{\lambda_2}\left(\left|\tilde{x}(N;x^+) - \bar{x}(N+1;x)\right|\right) = \bar{\sigma}_{\lambda_2}\left(\left|A_k^N w\right|\right) \leq \sigma_{\lambda_2}(|w|),
\end{aligned}
$$

where $\sigma_{\lambda_1}, \quad \sigma_{\lambda_2}$, and $\bar{\sigma}_{\lambda_2}$ are $\mathcal{K}$-function.

Furthermore, taking into account that $\tilde{x}(N;x^+) = \bar{x}(N+1;x) + A_k^N w$, the following bound can be obtained:

$$
\begin{aligned}
V_f\big(\tilde{x}(N;x^+)\big) - V_f(\bar{x}(N+1;x)) &\leq \sigma_{V_f}\left(\left|\tilde{x}(N;x^+) - \bar{x}(N+1;x)\right|\right) \\
&\leq \sigma_{V_f}(\left|A_K^N\right|\,|w|) \\
&\leq \sigma_{V_f}(|w|),
\end{aligned}
$$

where the first inequality comes from uniform continuity of $V_f(x)$[‡] and σ_{V_f} is a $\mathcal{K}$-function. Moreover,

$$
\begin{aligned}
&\sum_{j=0}^{N-1} \ell_{\mathrm{eco}}\left(\tilde{x}(j;x^+)-x_a^0+x_s^{\mathrm{eco}},u^0(j+1;x)-\tilde{u}_a+u_s^{\mathrm{eco}},p\right) - \ell_{\mathrm{eco}}\left(\bar{x}(j+1;x)-x_a^0+x_s^{\mathrm{eco}},u^0(j+1;x)-u_a^0+u_s^{\mathrm{eco}},p\right) \\
\leq\ & \sum_{j=0}^{N-1} \left|\ell_{\mathrm{eco}}\left(\tilde{x}(j;x^+)-x_a^0+x_s^{\mathrm{eco}},u^0(j+1;x)-\tilde{u}_a+u_s^{\mathrm{eco}},p\right) - \ell_{\mathrm{eco}}\left(\bar{x}(j+1;x)-x_a^0+x_s^{\mathrm{eco}},u^0(j+1;x)-u_a^0+u_s^{\mathrm{eco}},p\right)\right| \\
\leq\ & \Gamma \sum_{j=0}^{N-1} \left|\left(\tilde{x}(j;x^+)-\bar{x}(j+1;x)\right)\right| \\
\leq\ & \Gamma \sum_{j=0}^{N-1} \left|A_K^j\right|\,|w| \\
\leq\ & \Gamma\,|w| \sum_{j=0}^{N-1} \left|A_K^j\right| = \Gamma \frac{1-|A_K|^N}{1-|A_K|}\,|w| \\
\leq\ & \sigma_L(|w|),
\end{aligned}
$$

[‡] V_f is uniformly continuos because it is continuous in a compact.

where the second inequality comes from Lipschitz continuity of $\ell_{\text{eco}}(x,u,p)$, Γ is the Lipschitz constant, and σ_L is a $\mathcal{K}$-function.

Therefore,

$$\begin{aligned}\Delta V_N^w &\leq \sigma_{\lambda_1}(|w|)+\sigma_{\lambda_2}(|w|)+\sigma_{V_f}(|w|)+\sigma_L(|w|)\\ &\leq \sigma(|w|),\end{aligned}$$

being σ is a $\mathcal{K}$-function. Because of the previous inequality, we can conclude that

$$\begin{aligned}\Delta V_N &= \Delta V_N^{\text{nom}}+\Delta V_N^w\\ &\leq -L\left(x-x_a^0+x_s^{\text{eco}},u^0(0;x)-u_a^0+u_s^{\text{eco}},p\right)+\sigma(|w|).\end{aligned}$$

By optimality of the solution at time $k+1$, we can state that $\tilde{V}_N^0(x^+,p)\leq \tilde{V}_N(x^+,p;\tilde{\mathbf{u}},\tilde{x}_a,\tilde{u}_a)$, and then,

$$\begin{aligned}\Delta V_N^0 &= \tilde{V}_N^0(x^+,p)-\tilde{V}_N^0(x,p)\\ &\leq \Delta V_N\\ &\leq -L\left(x-x_a^0+x_s^{\text{eco}},u^0(0;x)-u_a^0+u_s^{\text{eco}},p\right)+\sigma(|w|)\\ &\leq -\alpha\left(\left|x-x_a^0\right|\right)+\sigma(|w|)\\ &\leq -\alpha_J\left(\left|x-x_s^{\text{eco}}\right|\right)+\sigma(|w|),\end{aligned}$$

where the last inequality comes from Lemma 4, which conlcudes the proof.

Thus, we can conclude that

$$J(x^+,p)-J(x,p)\leq -\alpha_J(\left|x-x_s^{\text{eco}}\right|)+\sigma(|w|),$$

for all $x\in\mathcal{X}_N$ and $w\in\mathcal{W}$. □

APPENDIX B

LEMMATA

Lemma 3. *Let Assumptions 1 and 2 hold. Let the terminal cost function be given by* $V_f(x,x_a)=\frac{1}{2}\|x-x_a\|_P^2+g'(x-x_a)$, *where* $P\in\mathbb{R}^{n\times n}$ *is the solution of the Lyapunov equation* $A_K'PA_K-P=-Q$, *for a given* $Q\in\mathbb{R}^{n\times n}$, *and* $g'=t'(I-A_K)^{-1}$, *for a given* $t\in\mathbb{R}^n$. *Then, there exists a proper choice of Q and t such that* $V_f(x,x_a)$ *fulfills Assumption 3.*

Proof. First of all, let us assume, without loss of generality that $(x_s^{\text{eco}},u_s^{\text{eco}})=(0,0)$. Then, by the mean value theorem, it there exists an x in the interval (x_a,x_s^{eco}) such that

$$\begin{aligned}\ell_{\text{eco}}(x-x_a+x_s^{\text{eco}},\kappa_f(x,x_a,u_a)-u_a+u_s^{\text{eco}},p) =&\ell_{\text{eco}}(x-x_a,K(x-x_a),p)\\ =&\ell_{\text{eco}}(0,0)+\nabla\ell_{\text{eco}(0,0)}(x-x_a)\\ &+\frac{1}{2}(x-x_a)'H_{\ell_{\text{eco}}}(x)(x-x_a),\end{aligned}$$

where $\nabla\ell_{\text{eco}}(0,0)$ is the gradient of $\ell_{\text{eco}}(x-x_a,K(x-x_a),p)$ w.r.t. x evaluated in $(0,0)$, and $H_{\ell_{\text{eco}}}(x)$ is the Hessian matrix of $\ell_{\text{eco}}(x-x_a,K(x-x_a),p)$ w.r.t. x.

Now, let us define $t=\nabla\ell_{\text{eco}}(0,0)$. Moreover, let

$$\lambda^*=\max_{x\in\text{Proj}_x(\Omega_t)}\left\{\lambda_M\left(H_{\ell_{\text{eco}}}(x)\right)\right\},$$

where $\lambda_M(H_{\ell_{\text{eco}}}(x))$ is the maximum eigenvalue of $H_{\ell_{\text{eco}}}(x)$ and $\text{Proj}_x(\Omega_t)$ is the projection of Ω_t onto x. Then, $\lambda^*(x-x_a)'(x-x_a)\geq(x-x_a)'H_{\ell_{\text{eco}}}(x)(x-x_a)$ for all $(x,x_a,u_a)\in\Omega_t$.

We can now define matrix Q as $Q=\lambda^*I+\alpha I$, with $\alpha>-\lambda^*$. Then,

$$(x-x_a)'Q(x-x_a)\geq(x-x_a)'H_{\ell_{\text{eco}}}(x)(x-x_a)$$

for all $(x,x_a,u_a)\in\Omega_t$.

Therefore,

$$\begin{aligned}\ell_{\text{eco}}(x-x_a,K(x-x_a),p)-\ell_{\text{eco}}(0,0,p)&=\nabla\ell_{\text{eco}(0,0)}(x-x_a)+\frac{1}{2}(x-x_a)'H_{\ell_{\text{eco}}}(x)(x-x_a)\\ &\leq t'(x-x_a)+\frac{1}{2}\|x-x_a\|_Q^2,\end{aligned}$$

or in general,

$$\ell_{\text{eco}}(x - x_a + x_s^{\text{eco}}, \kappa_f(x, x_a, u_a) - u_a + u_s^{\text{eco}}, p) - \ell_{\text{eco}}(x_s^{\text{eco}}, u_s^{\text{eco}}, p) \le t'(x - x_a) + \frac{1}{2}\|x - x_a\|_Q^2 .$$

Taking now the increment of the terminal cost under the control law $\kappa_f(x, x_a, u_a) = K(x - x_a) + u_a$, we have

$$\begin{aligned}
V_f(x^+, x_a) - V_f(x, x_a) &= \frac{1}{2}\|x^+ - x_a\|_P^2 + g'(x^+ - x_a) - \frac{1}{2}\|x - x_a\|_P^2 - g'(x - x_a) \\
&= \frac{1}{2}\|A_K(x - x_a)\|_P^2 + g'(A_K(x - x_a)) - \frac{1}{2}\|x - x_a\|_P^2 - g'(x - x_a) \\
&= \frac{1}{2}\|x - x_a\|_{A_K' P A_K - P}^2 - g'(I - A_K)(x - x_a) \\
&= -\frac{1}{2}\|x - x_a\|_Q^2 - t'(I - A_K)^{-1}(I - A_K)(x - x_a) \\
&= -\frac{1}{2}\|x - x_a\|_Q^2 - t'(x - x_a) \\
&\le -\ell_{\text{eco}}\left(x - x_a + x_s^{\text{eco}}, \kappa_f(x, x_a, u_a) - u_a + u_s^{\text{eco}}, p\right) + \ell_{\text{eco}}\left(x_s^{\text{eco}}, u_s^{\text{eco}}, p\right).
\end{aligned}$$

Moreover, notice that $V_f(x_a, x_a) = 0$. Then, by choosing $t = \nabla\ell_{\text{eco}(0,0)}$ and $Q = \lambda^* I + \alpha I$, with $\alpha > -\lambda^*$, the terminal cost function (12) satisfies Assumption 3. □

In what follows, we introduce some lemmas necessary for the proof of Theorem 1.

Lemma 4. *Consider that Assumptions 1 to 5 hold and $w = 0$. For a given p, let x_s^{eco} be the optimal steady state. For all $x \in \mathcal{X}_N$ and $x_a^0 \in \mathcal{X}_s = Proj_x(\bar{\mathcal{Z}}_s)$, define the function $e(x) = x - x_a^0$. Then, there exists a $\mathcal{K}$-function α_e such that*

$$|e(x)| \ge \alpha_e\left(\left|x - x_s^{\text{eco}}\right|\right). \tag{B1}$$

Proof. Because of the convexity of the sets $\mathcal{X}_N$ and $\mathcal{X}_s$, $e(x)$ is a continuous function.[2] Moreover, let us consider these two cases:

1. $|e(x)| = 0$ iff $x = x_s^{\text{eco}}$. In fact, (i) if $e(x) = 0$, then $x = x_a^0$, and from Lemma 5, this implies that $x_a^0 = x_s^{\text{eco}}$; (ii) if $x = x_s^{\text{eco}}$, then by the economic optimality of x_s^{eco}, it is $x_a^0 = x_s^{\text{eco}}$, and then, $x = x_a^0$.
2. $|e(x)| > 0$ for all $|x - x_s^{\text{eco}}| > 0$. In fact, for any $x \ne x_s^{\text{eco}}$, $|e(x)| \ne 0$, and moreover, $|x - x_s^{\text{eco}}| > 0$. Then, $|e(x)| > 0$.

Then, since $\mathcal{X}_N$ is compact, in virtue of chapter 5, lemma 6 in the work of Khalil,[30] there exists a $\mathcal{K}$-function α_e such that $|e(x)| \ge \alpha_e(|x - x_s^{\text{eco}}|)$. □

Lemma 5. *Consider that Assumptions 1 to 5 hold and $w = 0$. Take $\gamma > |g|$, where g is defined in (12). Let the optimal solution to Problem $\tilde{P}_N(x, p)$, at time k and for a given p, be such that $x(k) = x_a^0(x(k))$ and $\bar{u}(k) = u_a^0(x(k))$. Then, $x(k) = x_s^{\text{eco}}$, and $\bar{u}(k) = u_s^{\text{eco}}$.*

Proof. In this proof, the time dependence will be omitted for the sake of clarity. Since the optimal solution to problem $\tilde{P}_N(x, p)$ is such that $x = x_a^0$ and $\bar{u} = u_a^0$, then because of the terminal constraint, $(x, \bar{u}) \in \bar{\mathcal{Z}}_s$, and the optimal cost function is given by $\tilde{V}_N^0(x_a^0, p) = \sum_{j=0}^{N-1} L(x_s^{\text{eco}}, u_s^{\text{eco}}, p) + \tilde{V}_O(x_a^0, u_a^0) = \tilde{V}_O(x_a^0, u_a^0)$. The latter cost form comes from the fact that the optimal predicted input and state trajectories correspond to the action of keeping the system at the equilibrium (x_a, u_a), for the N steps of the horizon.

The lemma will be proved by contradiction. Assume that $(x, \bar{u}) = (x_a^0, u_a^0) \ne (x_s^{\text{eco}}, u_s^{\text{eco}})$. Hence, given that both, (x_a^0, u_a^0) and $(x_s^{\text{eco}}, u_s^{\text{eco}})$ belong to the convex set $\bar{\mathcal{Z}}_s$ (ie, they are feasible equilibrium pairs), there exists a $\hat{\beta} \in (0, 1)$ such that for any $\beta \in [\hat{\beta}, 1)$

1. $(\hat{x}_s, \hat{u}_s) = \beta(x_a^0, u_a^0) + (1 - \beta)(x_s^{\text{eco}}, u_s^{\text{eco}})$.
2. $(\hat{x}_s, \hat{u}_s) \in \bar{\mathcal{Z}}_s$.
3. The control law $u = K(x - \hat{x}_s) + \hat{u}_s$ drives the system from x_a^0 to $\hat{x}_s$ in an admissible way.

Therefore, defining as $\hat{\mathbf{u}}$ the sequence of control actions derived from the control law $\hat{u}(j) = K(\hat{x}(j) - \hat{x}_s) + \hat{u}_s$, with $\hat{x}(0) = x_a^0$, it is inferred that $\hat{\mathbf{u}}$ is a feasible solution to problem $\tilde{P}_N(x, p)$. Then, by the optimality of the assumed solution, we have that $\tilde{V}_N^0(x_a^0, p) = \tilde{V}_N(x_a^0, p; \mathbf{u}^0, x_a^0, u_a^0) \le \tilde{V}_N(x_a^0, p; \hat{\mathbf{u}}, \hat{x}_s, \hat{u}_s)$. Then, taking into account that $\tilde{V}_N^0(x_a^0, p) = \tilde{V}_O(x_a^0, u_a^0)$ and developing the second term of the latter inequality, it follows that

$$\begin{aligned}
\tilde{V}_O(x_a^0, u_a^0) &\le \sum_{j=0}^{N-1} L\left(\hat{x}(j) - \hat{x}_s + x_s^{\text{eco}}, \hat{u}(j) - \hat{u}_s + u_s^{\text{eco}}, p\right) + \tilde{V}_f(\hat{x}(N), \hat{x}_s) + \tilde{V}_O(\hat{x}_s, \hat{u}_s) \\
&\le \tilde{V}_f(\hat{x}(0), \hat{x}_s) + \tilde{V}_O(\hat{x}_s, \hat{u}_s) \\
&= \tilde{V}_f\left(x_a^0, \hat{x}_s\right) + \tilde{V}_O(\hat{x}_s, \hat{u}_s) \\
&= V_f\left(x_a^0, \hat{x}_s\right) + \lambda\left(x_a^0\right) - \lambda(\hat{x}_s) - V_f(\hat{x}_s, \hat{x}_s) + V_O(\hat{x}_s, \hat{u}_s) + \lambda(\hat{x}_s) - \lambda\left(x_s^{\text{eco}}\right) - V_O\left(x_s^{\text{eco}}, u_s^{\text{eco}}\right) \\
&= V_f\left(x_a^0, \hat{x}_s\right) + V_O(\hat{x}_s, \hat{u}_s) + d,
\end{aligned} \tag{B2}$$

where $d = \lambda(x_a^0) - \lambda(x_s^{\text{eco}}) - V_O(x_s^{\text{eco}}, u_s^{\text{eco}})$.

The second inequality comes from Property 2 since

$$\tilde{V}_f(\hat{x}(j+1), \hat{x}_s) \le \tilde{V}_f(\hat{x}(j), \hat{x}_s) - L\left(\hat{x}(j) - \hat{x}_s + x_s^{\text{eco}}, \kappa(\hat{x}(j), \hat{x}_s, \hat{u}_s) - \hat{u}_s + u_s^{\text{eco}}, p\right),$$

where $\kappa(\hat{x}(j), \hat{x}_s, \hat{u}_s) = \hat{u}(j) = K(\hat{x}(j) - \hat{x}_s) + \hat{u}_s$, with $\hat{x}(0) = x_a^0$, and $\hat{x}(j+1) = A\hat{x}(j) + B\hat{u}(j)$.

Now, taking $j = 0$, we have that

$$\tilde{V}_f(\hat{x}(1), \hat{x}_s) \le \tilde{V}_f(\hat{x}(0), \hat{x}_s) - L(\hat{x}(0) - \hat{x}_s + x_s^{\text{eco}}, \hat{u}(0) - \hat{u}_s + u_s^{\text{eco}}, p),$$

whereas taking $j = 1$, we have that

$$\begin{aligned}
\tilde{V}_f(\hat{x}(2), \hat{x}_s) &\le \tilde{V}_f(\hat{x}(1), \hat{x}_s) - L\left(\hat{x}(1) - \hat{x}_s + x_s^{\text{eco}}, \hat{u}(1) - \hat{u}_s + u_s^{\text{eco}}, p\right) \\
&\le \tilde{V}_f(\hat{x}(0), \hat{x}_s) - \sum_{j=0}^{1} L\left(\hat{x}(j) - \hat{x}_s + x_s^{\text{eco}}, \hat{u}(j) - \hat{u}_s + u_s^{\text{eco}}, p\right).
\end{aligned}$$

Therefore, by induction, for $j = N - 1$, we have

$$\begin{aligned}
\tilde{V}_f(\hat{x}(N), \hat{x}_s) &\le \tilde{V}_f(\hat{x}(N-1), \hat{x}_s) - L\left(\hat{x}(N-1) - \hat{x}_s + x_s^{\text{eco}}, \hat{u}(N-1) - \hat{u}_s + u_s^{\text{eco}}, p\right) \\
&\le \tilde{V}_f(\hat{x}(0), \hat{x}_s) - \sum_{j=0}^{N-1} L\left(\hat{x}(j) - \hat{x}_s + x_s^{\text{eco}}, \hat{u}(j) - \hat{u}_s + u_s^{\text{eco}}, p\right),
\end{aligned}$$

which can be written as

$$\tilde{V}_f(\hat{x}(0), \hat{x}_s) \ge \sum_{j=0}^{N-1} L(\hat{x}(j) - \hat{x}_s + x_s^{\text{eco}}, \hat{u}(j) - \hat{u}_s + u_s^{\text{eco}}, p) + \tilde{V}_f(\hat{x}(N), \hat{x}_s).$$

Now, we resume Equation (B2). Given that $(\hat{x}_s, \hat{u}_s)$ is a convex combination of (x_a^0, u_a^0) and $(x_s^{\text{eco}}, u_s^{\text{eco}})$, with $\beta \in (0, 1)$ as the convex parameter, the idea is to compute the cost as a function of β. Define the function $W(\beta)$, which is given by

$$\begin{aligned}
W(\beta) &= V_f\left(x_a^0, \hat{x}_s\right) + V_O(\hat{x}_s, \hat{u}_s) + d \\
&= \frac{1}{2}\left\|x_a^0 - \hat{x}_s\right\|_P^2 + g'\left(x_a^0 - \hat{x}_s\right) + V_O(\hat{x}_s, \hat{u}_s) + d \\
&= (1-\beta)^2 \frac{1}{2}\left\|x_a^0 - x_s^{\text{eco}}\right\|_P^2 + (1-\beta) g'\left(x_a^0 - x_s^{\text{eco}}\right) + V_O(\hat{x}_s, \hat{u}_s) + d.
\end{aligned}$$

Notice that $W(1) = V_O(x_a^0, u_a^0) + d = \tilde{V}_O(x_a^0, u_a^0) = \tilde{V}_N^0(x_a^0, p)$.

Taking the partial of W about β, we have that

$$\frac{\partial W}{\partial \beta} = -(1-\beta)\left\|x_a^0 - x_s^{\text{eco}}\right\|_P^2 - g'\left(x_a^0 - x_s^{\text{eco}}\right) + q'\left(x_a^0 - x_s^{\text{eco}}, u_a^0 - u_s^{\text{eco}}\right),$$

where $q' \in \partial V_O(\hat{x}_s, \hat{u}_s)$, defining $\partial V_O(\hat{x}_s, \hat{u}_s)$ as the subdifferential of $V_O(\hat{x}_s, \hat{u}_s)$. Evaluating this partial for $\beta = 1$, we obtain

$$\left.\frac{\partial W}{\partial \beta}\right|_{\beta=1} = -g'\left(x_a^0 - x_s^{\text{eco}}\right) + \bar{q}'\left(x_a^0 - x_s^{\text{eco}}, u_a^0 - u_s^{\text{eco}}\right),$$

where $\bar{q}' \in \partial V_O(x_a^0, u_a^0)$, defining $\partial V_O(x_a^0, u_a^0)$ as the subdifferential of $V_O(x_a^0, u_a^0)$.

Then, from Assumption 4, we can state that, for every x_a^0 and x_s^{eco},

$$\bar{q}'\left(x_a^0 - x_s^{eco}, u_a^0 - u_s^{eco}\right) \geq V_O\left(x_a^0, u_a^0\right) - V_O\left(x_s^{eco}, u_s^{eco}\right) \geq \gamma\left|x_a^0 - x_s^{eco}\right|.$$

Therefore,

$$\left.\frac{\partial W}{\partial \beta}\right|_{\beta=1} \geq -g'\left(x_a^0 - x_s^{eco}\right) + \gamma\left|x_a^0 - x_s^{eco}\right| \geq -|g|\left|x_a^0 - x_s^{eco}\right| + \gamma\left|x_a^0 - x_s^{eco}\right|$$
$$= (\gamma - |g|)\left|x_a^0 - x_s^{eco}\right|.$$

Since $\gamma > |g|$ and $|x_a^0 - x_s^{eco}| > 0$, hence $\frac{\partial W}{\partial \beta}|_{\beta=1} > 0$.

This means that there exists a $\beta \in [\hat{\beta}, 1)$ such that the cost to move the system from x_a^0 to $\hat{x}_s$, $W(\beta)$, is smaller than the cost to remain in x_a^0, which is given by $W(1) = \tilde{V}_O(x_a^0, u_a^0) = \tilde{V}_N^0(x_a^0, p)$.

This contradicts the optimality of the solution to problem $\tilde{P}_N(x, p)$, and hence, $x = x_a^0 = x_s^{eco}$, and $\bar{u} = u_a^0 = u_s^{eco}$, which proves the lemma. □

Remark 5. Summarizing, this lemma proves that if the system converges to an equilibrium point x_a^0, then this point is the economically optimal steady state, x_s^{eco}.

The following lemmas will be only stated here, but the interested reader may refer to the work of Ferramosca et al[21] for detailed proofs.

Lemma 6. (See lemma 1 in the work of Ferramosca et al[21])
For all $j = 0, \ldots, N-1$ *and for all* $k \geq 0$, $\tilde{x}(j; x(k+1)) - \bar{x}(j+1; x(k)) = A_K^j w(k)$.

Lemma 7. (See lemma 2 in the work of Ferramosca et al[21])
If $x^0(j; x(k)) \in \mathcal{X}_j$ *and* $u^0(j; x(k)) \in \mathcal{U}_j$*, then*

$$\tilde{x}(j-1; x(k+1)) \in \mathcal{X}_{j-1}, \qquad \forall\, j = 0, \ldots, N,$$
$$\tilde{u}(j-1; x(k+1)) \in \mathcal{U}_{j-1}, \qquad \forall\, j = 1, \ldots, N-1.$$

Lemma 8. (Recursive feasibility of the terminal constraint. See lemma 4 in the work of Ferramosca et al[21])
For all $k \geq 0$,

$$\left(\bar{x}^0(N; x(k)), x_a^0(k), u_a^0(k)\right) \in \Omega_t.$$