

# Weighted a priori estimates for elliptic equations

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## Abstract

We give a simpler proof of the a priori estimates obtained in [DST1, DST2] for solutions of elliptic problems in weighted Sobolev norms when the weights belong to the Muckenhoupt class  $A_p$ . The argument is a generalization to bounded domains of the one used in  $\mathbb{R}^n$  to prove the continuity of singular integral operators in weighted norms.

In the case of singular integral operators it is known that the  $A_p$  condition is also necessary for the continuity. We do not know whether this is also true for the a priori estimates in bounded domains but we are able to prove a weaker result when the operator is the Laplacian or a power of it. We prove that a necessary condition is that the weight belongs to the local  $A_p$  class.

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# 1 Introduction

The goal of this paper is to prove weighted a priori estimates for solutions of linear elliptic problems with Dirichlet boundary conditions. More precisely, for a bounded smooth domain  $\Omega \subset \mathbb{R}^n$  we consider

$$\begin{cases} \mathcal{L}u = f \text{ in } \Omega \\ \mathcal{B}_j u = 0 \text{ on } \partial\Omega, \quad 1 \leq j \leq m-1 \end{cases}$$

where  $\mathcal{L}$  is an elliptic operator of order  $2m$  and  $\mathcal{B}_j$  differential operators of order  $m_j$  satisfying the properties introduced in the classic paper [ADN].

For  $1 < p < \infty$ , the a priori estimate

$$(1.1) \quad \|u\|_{W^{2m,p}(\Omega)} \leq C \|f\|_{L^p(\Omega)}.$$

is well-known. This result is usually referred as Agmon-Douglis-Nirenberg estimate because it is essentially contained in [ADN] although it is not explicitly written in this way in that paper. For completeness we give more details on this point in the next section.

The estimate (1.1) has been extended to weighted norms when  $\mathcal{L}$  is the Laplacian or a power of it in [DST1, DST2].

By a weight function we mean a locally integrable function  $w$  defined in  $\mathbb{R}^n$  and for  $1 \leq p < \infty$ , we define the Banach space  $L_w^p(\Omega)$  with norm given by

$$\|f\|_{L_w^p(\Omega)} = \left( \int_{\Omega} |f(x)|^p w(x) dx \right)^{1/p}.$$

and  $W_w^{2m,p}(\Omega)$  the Sobolev space of functions in  $L_w^p(\Omega)$  with derivatives up to order  $2m$  in  $L_w^p(\Omega)$  with the usual norm. For  $\Omega = \mathbb{R}^n$  we write simply  $L_w^p$  instead of  $L_w^p(\mathbb{R}^n)$ . With  $C$  we will denote a generic constant which can change its value even in the same line.

For  $1 < p < \infty$ , a weight is in the Muckenhoupt class  $A_p$  if

$$(1.2) \quad \left( \frac{1}{|Q|} \int_Q w \right) \left( \frac{1}{|Q|} \int_Q w^{-\frac{1}{p-1}} \right)^{p-1} \leq C$$

for all cubes  $Q$ . It is well known that the Hardy-Littlewood maximal operator is bounded in  $L_w^p$  if and only if  $w \in A_p$  (see for example [D]).

For  $\mathcal{L} = (-\Delta)^m$  it was proved in [DST2] (extending the results for  $m = 1$  given in [DST1]) that, if  $w$  is in  $A_p$ , then

$$(1.3) \quad \|u\|_{W_w^{2m,p}(\Omega)} \leq C \|f\|_{L_w^p(\Omega)},$$

In this paper we give a different proof of (1.3) generalizing to a bounded domain the classic arguments used to obtain the continuity of singular integral operators in  $L_w^p$ . The advantage of this proof is that it is simpler and it does not require estimates of derivatives of the Green function involving the distance to the boundary. On the other hand, our arguments are very general and apply to the class of operators considered in [ADN, K].

We do not know whether the  $A_p$  condition is also necessary to have (1.3) but we prove a weaker result for the case of  $\mathcal{L} = (-\Delta)^m$ . Indeed, in order to have the a priori estimate (1.3) it is necessary that  $w \in A_p^{loc}(\Omega)$  (see below for the definition of this class).

## 2 Weighted a priori estimates

For a bounded domain  $\Omega \subset \mathbb{R}^n$  we consider the problem

$$(2.1) \quad \begin{aligned} \mathcal{L}u(x) &:= \sum_{0 \leq |\alpha| \leq 2m} a_\alpha(x) D^\alpha u(x) = f(x) && \text{in } \Omega \\ \mathcal{B}_j u(x) &:= \sum_{0 \leq |\alpha| \leq m_j} b_{j\alpha}(x) D^\alpha u(x) = 0 && \text{on } \partial\Omega \quad 1 \leq j \leq m-1 \end{aligned}$$

where  $\mathcal{L}$  is uniformly elliptic and  $\mathcal{B}_j$  satisfy the complementing conditions in the sense of [ADN].

Our arguments are based on the estimates for the Green function of (2.1) proved in [K], and therefore, we assume the hypotheses of that paper. Namely,  $m_j \leq 2m - 1$  and, for  $\ell_0 := \max_j(2m - m_j)$ , let  $\ell_1$  be an integer such that

$$(2.2) \quad \begin{aligned} \ell_1 &\geq \frac{3}{2}\ell_0, && \text{for } n \geq 3 \\ \ell_1 &\geq 2(\ell_0 + 1), && \text{for } n = 2. \end{aligned}$$

Then,

$$(2.3) \quad a_\alpha \in C^{\ell_1+1}(\Omega) \quad b_{j\alpha} \in C^{\ell_1+1}(\partial\Omega)$$

and

$$(2.4) \quad \partial\Omega \in C^{\ell_1+2m+1}$$

Under these assumptions, the existence of the Green function as well as estimates for it and its derivatives were proved in [K]. We state in the next theorem the estimates that we are going to use in our arguments.

**Theorem 2.1.** Under the hypotheses (2.2), (2.3) and (2.4) there exists the Green function  $G$  of (2.1), namely, the solution  $u$  is given by

$$(2.5) \quad u(x) = \int_{\Omega} G(x, y) f(y) dy.$$

Moreover, for  $0 < \alpha < 1$ , there exists a constant  $C$  depending on  $\mathcal{L}$ ,  $\mathcal{B}_j$ ,  $\Omega$  and  $n$  such that

$$(2.6) \quad |\gamma| \leq 2m \quad , \quad 2m - n - |\gamma| \neq 0 \quad \implies \quad |D_x^\gamma G(x, y)| \leq C \{ |x - y|^{2m - n - |\gamma|} + 1 \}$$

$$(2.7) \quad 2m - n - |\gamma| = 0 \quad \implies \quad |D_x^\gamma G(x, y)| \leq C \{ |\log |x - y|| + 1 \}$$

$$(2.8) \quad |\gamma| = 2m \quad \implies \quad |D_x^\gamma G(y, z) - D_x^\gamma G(x, z)| \leq C |y - x|^\alpha (|y - z|^{-n - \alpha} + |x - z|^{-n - \alpha}).$$

*Proof.* See Theorem 3.3 and its Corollary in [K, Page 965]. ■

Our proof of the weighted a priori estimates makes use of the classic un-weighted results. As we mentioned in the introduction, the a priori estimate (1.1) is essentially contained in [ADN] although not written explicitly there. Indeed, in the particular case of homogeneous boundary conditions that we are considering, Theorem 15.2 in page 704 of [ADN] says that

$$\|u\|_{W^{2m,p}(\Omega)} \leq C \{ \|f\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)} \}.$$

But, in view of the representation (2.5) and the bound for  $G$  given in (2.6), a standard application of the Young inequality yields  $\|u\|_{L^p(\Omega)} \leq C \|f\|_{L^p(\Omega)}$ , and therefore, (1.1) follows.

Let us remark that in the above mentioned Theorem 15.2 of [ADN] the authors assume that the norm on the left hand side of the estimate is finite, but this follows from their previous Theorem 7.3 (see also Remark 1 after that theorem) [ADN, Page 668] provided  $f$  is regular enough and using again the representation (2.5) to bound the norm of  $u$  appearing in the right hand side of that theorem. Then, in the general case one can proceed by a standard density argument.

Let us now recall the argument used in the case of singular integral operators that we are going to generalize. We will make use of the Hardy-Littlewood maximal operator

$$\mathcal{M}f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy$$

and of the sharp maximal operator

$$\mathcal{M}^\# f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where the supremums are taken over all cubes containing  $x$  and  $f_Q := \frac{1}{|Q|} \int_Q f$ .

If

$$Tf(x) = \lim_{\varepsilon \rightarrow 0} \int_{|x-y|>\varepsilon} K(x,y)f(y)dy$$

is a singular integral operator which is continuous in  $L^p$ , for  $1 < p < \infty$ , and  $K(x, y)$  satisfies

$$|K(x, z) - K(y, z)| \leq \frac{C|x - y|^\alpha}{|x - z|^{n+\alpha}}, \quad \text{for } |x - z| \geq 2|x - y|$$

with  $0 < \alpha < 1$ , then we have, for any  $s > 1$ ,

$$(2.9) \quad \mathcal{M}^\# Tf(x) \leq C(\mathcal{M}|f|^s(x))^{1/s}.$$

This estimate is well-known and its proof can be found in several books, although the hypotheses on the operator are not stated usually as we are doing here. Indeed, the proof given in [D, Lemma 7.9] only uses the hypotheses given above.

In the next lemma we prove a version of (2.9) in a bounded domain. With this goal we introduce the local sharp maximal operator

$$\mathcal{M}_\Omega^\# f(x) = \sup_{\Omega \supset Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy.$$

**Lemma 2.2.** Let  $u$  be the solution of (2.1) and assume that the hypotheses of Theorem 2.1 are satisfied. If  $|\gamma| = 2m$  we have, for any  $s > 1$  and any  $x \in \Omega$ ,

$$\mathcal{M}_\Omega^\#(D^\gamma u)(x) \leq C(\mathcal{M}|f|^s)^{1/s}(x).$$

*Proof.* We extend  $f$  by zero outside of  $\Omega$ . Let  $Q \subset \Omega$  be a cube such that  $x \in Q$  and  $Q^*$  an expanded cube of  $Q$  by a factor 2. We decompose  $f = f_1 + f_2$  with  $f_1 = f\chi_{Q^*}$  and call  $u_1$  and  $u_2$  the solutions of (2.1) with  $f_1$  and  $f_2$  as right hand sides respectively.

It is known that we can replace  $f_Q$  by any constant. We choose  $a = D^\gamma u_2(x)$ . Then,

$$\frac{1}{|Q|} \int_Q |D^\gamma u(y) - a| dy \leq \frac{1}{|Q|} \int_Q |D^\gamma u_1(y)| dy + \frac{1}{|Q|} \int_Q |D^\gamma u_2(y) - D^\gamma u_2(x)| dy = (i)+(ii).$$

Now, given  $s > 1$ , using the Hölder inequality and (1.1), we have

$$\begin{aligned} (i) &\leq \left( \frac{1}{|Q|} \int_Q |D^\gamma u_1(y)|^s dy \right)^{1/s} \leq C \left( \frac{1}{|Q|} \int_\Omega |f_1(y)|^s dy \right)^{1/s} \\ &= C \left( \frac{1}{|Q|} \int_{Q^*} |f(y)|^s dy \right)^{1/s} \leq C(\mathcal{M}|f|^s)^{1/s}(x). \end{aligned}$$

On the other hand, if  $x \notin \text{supp } f_2$  we can take the derivative inside the integral in the expression for  $u_2$  given by (2.5). Then, since  $\text{supp } f_2 \subset (Q^*)^c$ , for  $x \in Q$  we have

$$(ii) \leq \frac{1}{|Q|} \int_Q \int_{(Q^*)^c} |D_x^\gamma G(y, z) - D_x^\gamma G(x, z)| |f_2(z)| dz dy.$$

Therefore, using (2.8) and that  $|y - z| \sim |x - z|$  and  $|x - z| \geq \frac{\ell(Q)}{2}$  we obtain

$$\begin{aligned} (ii) &\leq C \frac{l(Q)^\alpha}{|Q|} \int_Q \int_{(Q^*)^c} \frac{|f(z)|}{|x - z|^{n+\alpha}} dz dy \leq Cl(Q)^\alpha \int_{(Q^*)^c} \frac{|f(z)|}{|x - z|^{n+\alpha}} dz \\ &\leq Cl(Q)^\alpha \sum_{k=0}^{\infty} \int_{2^{k-1}l(Q) < |z-x| \leq 2^k l(Q)} \frac{|f(z)|}{|x - z|^{n+\alpha}} dz \leq C\mathcal{M}f(x) \end{aligned}$$

where the last inequality follows by a standard argument (see [D, Lemma 7.9] for details).  $\blacksquare$

The following Lemma is a slightly modified version of [DRS, Theorem 5.23] because we are using a different definition of the sharp maximal operator. The reader can easily check that the proof given in that paper applies to our case.

**Lemma 2.3.** For  $f \in L^1_{loc}(\Omega)$  and  $w \in A_p$ , if  $f_\Omega = \frac{1}{|\Omega|} \int_\Omega |f|$ , then

$$(2.10) \quad \|f - f_\Omega\|_{L^p_w(\Omega)} \leq C \|\mathcal{M}_\Omega^\# f\|_{L^p_w(\Omega)}.$$

Now we can state and prove the main result of this section.

**Theorem 2.4.** Let  $u$  be the solution of (2.1) and assume that the hypotheses of Theorem 2.1 are satisfied. Then, for  $1 < p < \infty$ ,  $w \in A_p$  and  $f \in L^p_w(\Omega)$ , there exists a constant  $C$  depending on  $\mathcal{L}$ ,  $\mathcal{B}_j$ ,  $\Omega$ ,  $n$  and  $w$  such that

$$\|u\|_{W_w^{2m,p}(\Omega)} \leq C \|f\|_{L^p_w(\Omega)}.$$

*Proof.* If  $2m - n < |\gamma| < 2m$ , using (2.5) and (2.6) we obtain

$$\begin{aligned} |D^\gamma u(x)| &= \left| \int_{\Omega} D_x^\gamma G(x, y) f(y) dy \right| \leq C \int_{\Omega} \frac{|f(y)|}{|x - y|^{|\gamma| + n - 2m}} dy \\ &\leq C \sum_{k=0}^{\infty} \int_{2^{-(k+1)d} < |x-y| \leq 2^{-k}d} \frac{|f(y)|}{|x - y|^{|\gamma| + n - 2m}} dy \leq C \mathcal{M}f(x) \end{aligned}$$

where  $d$  denotes the diameter of  $\Omega$ . For  $|\gamma| \leq 2m - n$  we obtain the same estimate using now that, in view(2.6) and (2.7), for any  $\varepsilon > 0$ ,  $|G(x, y)| \leq C|x - y|^{-\varepsilon}$ .

Consequently, it follows from the boundedness of the maximal operator for  $A_p$  weights that

$$\|u\|_{W_w^{2m-1,p}(\Omega)} \leq C \|f\|_{L_w^p(\Omega)}.$$

It rests to estimate  $D^\gamma u$  for  $|\gamma| = 2m$  which is the most difficult part. From Lemmas 2.3 and 2.2 we have

$$\begin{aligned} (2.11) \quad \int_{\Omega} |D^\gamma u(x) - (D^\gamma u)_\Omega|^p w(x) dx &\leq C \int_{\Omega} \left| \mathcal{M}_\Omega^\#(D^\gamma u)(x) \right|^p w(x) dx \\ &\leq C \int_{\Omega} (\mathcal{M}|f|^s(x))^{p/s} w(x) dx. \end{aligned}$$

But it is known that there exists  $s$  depending only on  $w$  such that  $1 < s < p$  and  $w \in A_{p/s}$  (see for example [D]), and using the boundedness of  $\mathcal{M}$  in  $L_w^{p/s}$  we obtain from (2.11),

$$\int_{\Omega} |D^\gamma u(x) - (D^\gamma u)_\Omega|^p w(x) dx \leq C \int_{\Omega} |f(x)|^p w(x) dx.$$

Then,

$$\int_{\Omega} |D^\gamma u(x)|^p w(x) dx \leq C \left( \int_{\Omega} |f(x)|^p w(x) dx + \int_{\Omega} |(D^\gamma u)_\Omega|^p w(x) dx \right)$$

and so it only remains to estimate the last term. But, since  $w$  is integrable in  $\Omega$ , it is enough to show that

$$(2.12) \quad |(D^\gamma u)_\Omega| \leq C \|f\|_{L_w^p(\Omega)}.$$

Taking  $1 < s < p$  and using the a priori estimate (1.1) for  $s$  we have

$$\begin{aligned} |(D^\gamma u)_\Omega| &\leq \left( \frac{1}{|\Omega|} \int_{\Omega} |(D^\gamma u)_\Omega|^s dx \right)^{\frac{1}{s}} \leq C \left( \frac{1}{|\Omega|} \int_{\Omega} |f(x)|^s dx \right)^{\frac{1}{s}} \\ &\leq C \left( \frac{1}{|\Omega|} \int_{\Omega} |f(x)|^p w(x) dx \right)^{\frac{1}{p}} \left( \frac{1}{|\Omega|} \int_{\Omega} w(x)^{-\frac{s}{p-s}} dx \right)^{\frac{p-s}{sp}} \end{aligned}$$

where we have used the Hölder inequality first with  $s$  and then with  $p/s$ . Then, choosing  $s$  such that  $w \in A_{p/s}$  the last term on the right hand side is finite, and therefore, (2.12) is proved. ■

### 3 Necessary condition

It is known that the  $A_p$  condition is also necessary for the continuity of singular integral operators [S]. Then, it is natural to ask whether the same is true for the weighted a priori estimates. We do not know the answer but we prove in this section a weaker result, namely, a necessary condition to have the weighted a priori estimates for  $\mathcal{L} = (-\Delta)^m$  is that the weight belong to the  $A_p^{loc}(\Omega)$  class extensively studied in [HSV].

In this section it is more convenient to work with balls instead of cubes. To recall the definition of the  $A_p^{loc}$  class, first we consider  $0 < \beta < 1$  and define the  $A_p^\beta(\Omega)$  class as follows. A weight  $w$  belongs to this class if

$$\left( \frac{1}{|B|} \int_B w \right) \left( \frac{1}{|B|} \int_B w^{-\frac{1}{p-1}} \right)^{p-1} \leq C$$

for all balls  $B \subset \Omega$  such that  $\text{diam}(B) < \beta \text{dist}(B, \partial\Omega)$ . It was proved in [HSV] that the classes  $A_p^\beta(\Omega)$  are independent of  $\beta$ , namely, if  $0 < \beta < 1$  and  $0 < \gamma < 1$ , we have that  $w \in A_p^\beta(\Omega)$  if and only if  $w \in A_p^\gamma(\Omega)$ . In view of this fact, we say that  $w \in A_p^{loc}(\Omega)$  if  $w \in A_p^\beta(\Omega)$  for some  $0 < \beta < 1$ . We will call a ball satisfying this condition for some  $\beta$ , admissible. To simplify notation, we will use the usual notation  $w(S) = \int_S w(x) dx$ .

**Proposition 3.1.** Let  $w$  be a weight. Then,  $w \in A_p^{loc}(\Omega)$  if and only if

$$(3.1) \quad (f_B)^p \leq \frac{C}{w(B)} \int_B f^p w dx$$

for all  $f$  nonnegative and for all admissible ball  $B$ .

*Proof.* The proof follows as in the case of the  $A_p$  weights given in [S, Chapter V, Section 1.4] and [D, Theorem 7.1] using the results in [HSV] and considering admissible balls. ■

Let  $f \in L_w^p(\Omega)$  and consider the homogeneous boundary value problem

$$(3.2) \quad \begin{cases} (-\Delta)^m u = f & \text{in } \Omega \\ \left( \frac{\partial}{\partial \nu} \right)^j u = 0 & \text{on } \partial\Omega, \quad 1 \leq j \leq m-1 \end{cases}$$



then

$$(3.3) \quad u(x) = \int_{\Omega} G_m(x, y) f(y) dy$$

where  $G_m(x, y)$  is the Green function of the operator  $(-\Delta)^m$  in  $\Omega$ . It is well known that

$$G_m(x, y) = \Gamma(x - y) + h(x, y)$$

where  $\Gamma$  is the fundamental solution given by

$$\begin{cases} c_{m,n}|x|^{2m-n} & n \text{ odd, or } n \text{ even and } n > 2m \\ c_{m,n}|x|^{2m-n} \log |x| & n \text{ even and } n \leq 2m \end{cases}$$

and, for each  $y \in \Omega$ ,  $h(x, y)$  satisfies

$$\begin{cases} (-\Delta_x)^m h(x, y) = 0 & x \in \Omega \\ \left( \frac{\partial}{\partial \nu} \right)^j h(x, y) = - \left( \frac{\partial}{\partial \nu} \right)^j \Gamma(x - y) & x \in \partial\Omega, 0 \leq j \leq m - 1. \end{cases}$$

Let us recall that, for  $|\gamma| = 2m$ , a standard argument yields

$$(3.4) \quad D_x^\gamma \int_{\Omega} \Gamma(x - y) f(y) dy = \lim_{\varepsilon \rightarrow 0} \int_{|x-y|>\varepsilon} D^\gamma \Gamma(x - y) f(y) dy + c(x) f(x)$$

where  $c$  is a bounded function.

We will use the ideas given in [S, Chapter V, Section 4.6].

**Lemma 3.2.** For  $|\gamma| = 2m$  we have

1. There exists  $u_0 \in \mathbb{R}^n$  with  $|u_0| = 1$  and a constant  $C_0$  such that, for all positive numbers  $t$ ,

$$|D^\gamma \Gamma(tu_0)| \geq C_0 t^{-n}.$$

2. There exists  $t_0$  such that if  $u = t_0 u_0$  and  $|v| \leq 2$  then for all  $0 \neq r \in \mathbb{R}$ ,

$$|D^\gamma \Gamma(r(u + v)) - D^\gamma \Gamma(ru)| \leq \frac{1}{2} |D^\gamma \Gamma(ru)|.$$

*Proof.* One can check that  $D^\gamma \Gamma$  is homogeneous of degree  $-n$  and not identically zero. Then, there exists  $u_0 \in \mathbb{R}^n$  with  $|u_0| = 1$  such that  $|D^\gamma \Gamma(u_0)| =: C_0 > 0$ , and therefore, (1) follows from the homogeneity.

To prove (2) we observe first that, by homogeneity, it is enough to show that the statement holds for  $r = 1$ .

Take  $v \in \mathbb{R}^n$  satisfying  $|v| \leq k|u|$  with  $k \leq \frac{1}{2}$  to be chosen below. Then, for some  $\xi$  in the segment joining  $u$  and  $u + v$ ,

$$|D^\gamma \Gamma(u + v) - D^\gamma \Gamma(u)| \leq |\nabla D^\gamma \Gamma(\xi)| |v| \leq C |\xi|^{-n-1} |v|$$

and using  $|u| \leq 2|\xi|$ ,  $|v| \leq k|u|$  and (1) we obtain,

$$|D^\gamma \Gamma(u + v) - D^\gamma \Gamma(u)| \leq C_1 k |u|^{-n} \leq \frac{C_1 k}{C_0} |D^\gamma \Gamma(u)|.$$

Consequently it is enough to choose  $k$  such that  $\frac{C_1}{C_0} k \leq \frac{1}{2}$ . Now, since  $|v| \leq 2$ , if we choose  $t_0 = \frac{2}{k}$  our hypothesis  $|v| \leq k|u|$  is verified and the proof is concluded.  $\blacksquare$

The following result is proved in ([DST2, Proposition 3.3]).

**Lemma 3.3.** There exists a constant  $C$  such that, for  $|x - y| \leq d(x)$ ,

$$|D_x^\gamma h(x, y)| \leq C d(x)^{-n}.$$

Now, using the representation (3.3) and (3.4), we have

$$(3.5) \quad D^\gamma u(x) = T_\gamma f(x) + c(x)u(x)$$

where  $T_\gamma$  is defined by

$$T_\gamma f(x) = \lim_{\varepsilon \rightarrow 0} \int_{|x-y|>\varepsilon} D^\gamma \Gamma(x-y) f(y) dy + \int_\Omega D_x^\gamma h(x, y) f(y) dy.$$

We can now prove the main result of this section.

**Theorem 3.4.** Let  $u$  be the solution of (3.2). If  $w$  is a weight such that the following a priori estimate

$$\|u\|_{W_w^{2m,p}(\Omega)} \leq C \|f\|_{L_w^p(\Omega)}$$

holds, then  $w \in A_p^{loc}(\Omega)$ .

*Proof.* In view of (3.5), since  $c$  is a bounded function, it is enough to prove that, if for any  $\gamma$  such that  $|\gamma| = 2m$ ,

$$\|T_\gamma f\|_{L_w^p(\Omega)} \leq C \|f\|_{L_w^p(\Omega)}$$

then  $w \in A_p^{loc}(\Omega)$ .

We write  $T_\gamma = T_1 + T_2$  where

$$T_1 f(x) = \lim_{\varepsilon \rightarrow 0} \int_{|x-y|>\varepsilon} D^\gamma \Gamma(x-y) f(y) dy$$

and

$$T_2 f(x) = \int_{\Omega} D_x^\gamma h(x, y) f(y) dy.$$

Consider an admissible ball  $B = B(\bar{x}, r)$ , i.e.,  $2r < \beta \text{dist}(B, \partial\Omega)$ , with  $\beta$  to be determined later and  $B' := B(\bar{x} + ru, r)$ , with  $u = t_0 u_0$  as in Lemma 3.2. We will see that  $\beta$  can be taken so that  $B'$  is also admissible. Let  $x \in B'$  and  $z \in \partial\Omega$  then  $x = \bar{x} + ru + rx'$  with  $|x'| \leq 1$ ,

$$\begin{aligned} |x - z| &\geq |\bar{x} - z| - r|u + x'| \\ &\geq \text{dist}(B, \partial\Omega) - r|u + x'| \\ &\geq \frac{2r}{\beta} - r|u + x'| \\ &\geq 2r \left( \frac{1}{\beta} - \frac{t_0 + 1}{2} \right), \end{aligned}$$

so, taking  $\beta$  satisfying  $\frac{1}{\beta} - \frac{(t_0+1)}{2} > 1$ , i.e.  $\beta < \frac{2}{t_0+3}$  and  $\frac{1}{\alpha} = \frac{1}{\beta} - \frac{(t_0+1)}{2}$ , we have  $2r < \alpha \text{dist}(B', \partial\Omega)$ .

Now, we will show that for  $x \in B'$  and  $y \in B$ ,  $D^\gamma \Gamma(x - y)$  has constant sign. Indeed, writing  $x = \bar{x} + ru + rx'$  and  $y = \bar{x} + ry'$  with  $|x'|, |y'| \leq 1$ , we have  $x - y = ru + rv$  with  $|v| = |x' - y'| \leq 2$ . Then, by (2) from Lemma 3.2 we obtain, for  $D^\gamma \Gamma(ru) > 0$ ,

$$(3.6) \quad \frac{1}{2} D^\gamma \Gamma(ru) \leq D^\gamma \Gamma(x - y) \leq \frac{3}{2} D^\gamma \Gamma(ru),$$

while for  $D^\gamma \Gamma(ru) < 0$

$$(3.7) \quad \frac{3}{2} D^\gamma \Gamma(ru) \leq D^\gamma \Gamma(x - y) \leq \frac{1}{2} D^\gamma \Gamma(ru).$$

Consequently, taking  $f \in C_0^\infty(B)$  positive, we have

$$|T_1 f(x)| = \int_B |D^\gamma \Gamma(x - y)| f(y) dy.$$

and moreover, using (3.6), (3.7) and property (1) of Lemma 3.2,

$$|T_1 f(x)| \geq \frac{1}{2} \int_B |D^\gamma \Gamma(ru)| f(y) dy \geq C_0 (rt_0)^{-n} \int_B f(y) dy = C_1 f_B.$$

with a constant  $C_1$  depending only on  $t_0$ ,  $C_0$  and  $n$ .

On the other hand, in order to bound  $|T_2 f(x)|$  we use Lemma 3.3. We require  $|x - y| \leq d(x)$ , but,

$$|x - y| = |ru + r(x' - y')| \leq r(|u| + |x' - y'|) \leq r(t_0 + 2) < \frac{\alpha}{2} d(x)(t_0 + 2)$$

and then, we need  $\frac{\alpha}{2}(t_0 + 2) < 1$  or equivalently  $\beta < \frac{2}{2t_0+3}$ . Now, we have that

$$\begin{aligned} |T_2 f(x)| &\leq \int_B |D_x^\alpha h(x, y)| f(y) dy \leq C \int_B d(x)^{-n} f(y) dy \leq C(2r)^{-n} \alpha^n \int_B f(y) dy \\ &= C_2 \alpha^n f_B. \end{aligned}$$

where  $C$  is the constant appearing in Lemma 3.3 and  $C_2$  depends on  $n$  and  $C$ .

Summing up we have

$$|T_\gamma f(x)| \geq (C_1 - C_2 \alpha^n) f_B.$$

In order to have  $C_1 - C_2 \alpha^n > 0$  it is enough that  $\beta < \frac{1}{\frac{t_0+1}{2} + (\frac{C_2}{C_1})^{\frac{1}{n}}}$ .

Taking into account the other conditions for  $\beta$  we choose  $\beta < \min \left\{ \frac{2}{2t_0+3}, \frac{1}{\frac{t_0+1}{2} + (\frac{C_2}{C_1})^{\frac{1}{n}}} \right\}$ , and then, we obtain  $f_B \leq C |T_\gamma f(x)|$ , for any  $x \in B'$ . Therefore,

$$\int_{B'} f_B^p w(x) dx \leq C \int_{B'} |T_\gamma f(x)|^p w(x) dx \leq C \int_\Omega |T_\gamma f(x)|^p w(x) dx$$

and consequently, applying the continuity of  $T_\gamma$  in  $L_w^p$ , we obtain

$$(3.8) \quad (f_B)^p w(B') \leq C \int_B f(x)^p w(x) dx.$$

Analogously, changing roles of  $B$  and  $B'$  and taking  $f$  with support in  $B'$ , it follows that

$$(3.9) \quad (f_{B'})^p w(B) \leq C \int_{B'} f(x)^p w(x) dx.$$

provided that  $C_1 - C_2 \beta^n > 0$ . But this inequality holds for our previous election of  $\beta$  because  $\beta < \alpha$ .

By a passage to the limit, the inequalities (3.8) and (3.9) extend to any non-negative function  $f$  supported in  $B$  or  $B'$  respectively. If we consider  $f = \chi_{B'}$  in (3.9) we obtain

$$w(B) \leq C w(B').$$

Using this in (3.8) we get

$$(f_B)^p w(B) \leq C \int_B f(x)^p w(x) dx$$

but, by Proposition 3.1, this means that  $w \in A_p^{loc}(\Omega)$ . ■

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