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H^2 regularity for the $p(x)$-Laplacian in two-dimensional convex domains ☆

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ABSTRACT

In this paper we study the H^2 global regularity for solutions of the $p(x)$-Laplacian in two-dimensional convex domains with Dirichlet boundary conditions. Here $p : \Omega \to [p_1, \infty)$ with $p \in \mathrm{Lip}(\overline{\Omega})$ and $p_1 > 1$.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^2$ and let $p : \Omega \to (1, +\infty)$ be a measurable function. In this work, we study the H^2 global regularity of the weak solution of the following problem

$$\begin{cases} -\Delta_{p(x)} u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_{p(x)} u = \mathrm{div}(|\nabla u|^{p(x)-2} \nabla u)$ is the $p(x)$-Laplacian. The hypothesis over p, f and g will be specified later.

Note that, the $p(x)$-Laplacian extends the classical Laplacian ($p(x) \equiv 2$) and the p-Laplacian ($p(x) \equiv p$ with $1 < p < +\infty$). This operator has been recently used in image processing and in the modeling of electrorheological fluids, see [3,5,24].

Motivated by the applications to image processing problems, in [8], the authors study two numerical methods to approximate solutions of the type of (1.1). In Theorem 7.2, the authors prove the convergence in $W^{1,p(\cdot)}(\Omega)$ of the conformal Galerkin finite element method. It is of our interest to study, in a future work, the rate of this convergence. In general, all the error bounds depend on the global regularity of the second derivatives of the solutions, see for example [6,22]. However, there appear to be no existing regularity results in the literature that can be applied here, since all the results have either a first order or local character.

The H^2 global regularity for solutions of the p-Laplacian is studied in [22]. There the authors prove the following: Let $1 < p \leqslant 2$, $g \in H^2(\Omega)$, $f \in L^q(\Omega)$ ($q > 2$) and u be the unique weak solution of (1.1). Then:

- If $\partial\Omega \in C^2$ then $u \in H^2(\Omega)$;

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- If Ω is convex and $g = 0$ then $u \in H^2(\Omega)$;
- If Ω is convex with a polygonal boundary and $g \equiv 0$ then $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$.

Regarding the regularity of the weak solution of (1.1) when $f = 0$, in [1,7], the authors prove the $C^{1,\alpha}_{loc}$ regularity (in the scalar case and also in the vectorial case). Then, in the paper [15] the authors study the case where the functional has the so-called (p,q)-growth conditions. Following these ideas, in [17], the author proves that the solutions of (1.1) are in $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha > 0$ if Ω is a bounded domain in $\mathbb{R}^N$ $(N \geqslant 2)$ with $C^{1,\gamma}$ boundary, $p(x)$ is a Hölder function, $f \in L^\infty(\Omega)$ and $g \in C^{1,\gamma}(\overline{\Omega})$; while in [4], the authors prove that the solutions are in $H^2_{loc}(\{x \in \Omega\colon\ p(x) \leqslant 2\})$ if $p(x)$ is uniformly Lipschitz ($\mathrm{Lip}(\Omega)$) and $f \in W^{1,q(\cdot)}_{loc}(\Omega) \cap L^\infty(\Omega)$.

Our aim, it is to generalize the results of [22] in the case where $p(x)$ is a measurable function. To this end, we will need some hypothesis over the regularity of $p(x)$. Moreover, in all our result we can avoid the restriction $g = 0$, assuming some regularity of $g(x)$.

On the other hand, to prove our results, we can assume weaker conditions over the function f than the ones on [4]. Since, we only assume that $f \in L^{q(\cdot)}(\Omega)$, we do not have a priori that the solutions are in $C^{1,\alpha}(\Omega)$. Then we cannot use it to prove the H^2 global regularity. Nevertheless, we can prove that the solutions are in $C^{1,\alpha}(\overline{\Omega})$, after proving the H^2 global regularity.

The main results of this paper are:

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^2$ with C^2 boundary, $p \in \mathrm{Lip}(\overline{\Omega})$ with $p(x) \geqslant p_1 > 1$, $g \in H^2(\Omega)$ and u be the weak solution of (1.1). If*

(F1) *$f \in L^{q(\cdot)}(\Omega)$ with $q(x) \geqslant q_1 > 2$ in the set $\{x \in \Omega\colon\ p(x) \leqslant 2\}$;*
(F2) *$f \equiv 0$ in the set $\{x \in \Omega\colon\ p(x) > 2\}$,*

then $u \in H^2(\Omega)$.

Theorem 1.2. *Let Ω be a bounded domain in $\mathbb{R}^2$ with convex boundary, $p \in \mathrm{Lip}(\overline{\Omega})$ with $p(x) \geqslant p_1 > 1$, $g \in H^2(\Omega)$ and u be the weak solution of (1.1). If f satisfies (F1) and (F2) then $u \in H^2(\Omega)$.*

Using the above theorem we can prove the following:

Corollary 1.3. *Let Ω be a bounded convex domain in $\mathbb{R}^2$ with polygonal boundary, p and f as in the previous theorem, $g \in W^{2,q(\cdot)}(\Omega)$ and u be the weak solution of (1.1) then $u \in C^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$.*

Observe that this result extends the one in [17] in the case where Ω is a polygonal domain in $\mathbb{R}^2$.

Organization of the paper. The rest of the paper is organized as follows. After a short Section 2 where we collect some preliminary results, in Section 3, we study the H^2-regularity for the non-degenerated problem. In Section 4 we prove Theorem 1.1. Then, in Section 5, we study the regularity of the solution u of (1.1) if Ω is convex. In Section 6, we make some comments on the dependence of the H^2-norm of u on p_1. Lastly, in Appendices A and B we give some results related to elliptic linear equation with bounded coefficients and Lipschitz functions, respectively.

2. Preliminaries

We now introduce the spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ and state some of their properties.

Let Ω be a bounded open set of $\mathbb{R}^n$ and $p : \Omega \to [1,+\infty)$ be a measurable bounded function, called a variable exponent on Ω and denote $p_1 := \mathit{essinf}\, p(x)$ and $p_2 := \mathit{esssup}\, p(x)$.

We define the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ to consist of all measurable functions $u : \Omega \to \mathbb{R}$ for which the modular

$$\varrho_{p(\cdot)}(u) := \int_\Omega \left|u(x)\right|^{p(x)} dx$$

is finite. We define the Luxemburg norm on this space by

$$\|u\|_{L^{p(\cdot)}(\Omega)} := \inf\bigl\{k > 0\colon\ \varrho_{p(\cdot)}(u/k) \leqslant 1\bigr\}.$$

This norm makes $L^{p(\cdot)}(\Omega)$ a Banach space.

For the proofs of the following theorems, we refer the reader to [12].

Theorem 2.1 *(Hölder's inequality). Let $p, q, s : \Omega \to [1, +\infty]$ be measurable functions such that*

$$\frac{1}{p(x)} + \frac{1}{q(x)} = \frac{1}{s(x)} \quad \text{in } \Omega.$$

Then the inequality

$$\|fg\|_{L^{s(\cdot)}(\Omega)} \leqslant 2\|f\|_{L^{p(\cdot)}(\Omega)}\|g\|_{L^{q(\cdot)}(\Omega)}$$

holds for all $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{q(\cdot)}(\Omega)$.

Let $W^{1,p(\cdot)}(\Omega)$ denote the space of measurable functions u such that u and the distributional derivative ∇u are in $L^{p(\cdot)}(\Omega)$. The norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} := \|u\|_{p(\cdot)} + \big\||\nabla u|\big\|_{p(\cdot)}$$

makes $W^{1,p(\cdot)}(\Omega)$ a Banach space.

Theorem 2.2. *Let $p'(x)$ be such that $1/p(x) + 1/p'(x) = 1$. Then $L^{p'(\cdot)}(\Omega)$ is the dual of $L^{p(\cdot)}(\Omega)$. Moreover, if $p_1 > 1$, $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ are reflexive.*

We define the space $W_0^{1,p(\cdot)}(\Omega)$ as the closure of the $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. Then we have the following version of Poincaré's inequity (see Theorem 3.10 in [21]).

Lemma 2.3 *(Poincaré's inequity). If $p : \Omega \to [1, +\infty)$ is continuous in $\overline{\Omega}$, there exists a constant C such that for every $u \in W_0^{1,p(\cdot)}(\Omega)$,*

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leqslant C\|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

In order to have better properties of these spaces, we need more hypotheses on the regularity of $p(x)$.

We say that p is log-*Hölder continuous* in Ω if there exists a constant C_{log} such that

$$\big|p(x) - p(y)\big| \leqslant \frac{C_{log}}{\log(e + \frac{1}{|x-y|})} \quad \forall x, y \in \Omega.$$

It was proved in [10, Theorem 3.7], that if one assumes that p is log-Hölder continuous then $C^\infty(\overline{\Omega})$ is dense in $W^{1,p(\cdot)}(\Omega)$ (see also [9,12,13,21,25]).

We now state the Sobolev embedding theorem (for the proofs see [12]). Let

$$p^*(x) := \begin{cases} \frac{p(x)N}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geqslant N \end{cases}$$

be the Sobolev critical exponent. Then we have the following:

Theorem 2.4. *Let Ω be a Lipschitz domain. Let $p : \Omega \to [1, \infty)$ and p be log-Hölder continuous. Then the imbedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^*(\cdot)}(\Omega)$ is continuous.*

3. H^2-regularity for the non-degenerated problem for any dimension

In this section we assume that Ω is a bounded domain in $\mathbb{R}^N$, with $N \geqslant 2$.

We want to study higher regularity of the weak solution of the regularized equation,

$$\begin{cases} -\operatorname{div}\big((\varepsilon + |\nabla u|^2)^{\frac{p(x)-2}{2}} \nabla u\big) = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

where $0 < \varepsilon \leqslant 1$, and $f \in \operatorname{Lip}(\Omega)$ and $g \in W^{1,p(\cdot)}(\Omega)$.

The existence of a weak solution of (3.2) holds by Theorem 13.3.3 in [12].

Remark 3.1. Given $\varepsilon \geqslant 0$, $p \in C^{\alpha_0}(\overline{\Omega})$ for some $\alpha_0 > 0$, and $g \in L^\infty(\Omega)$ we have the following results:

(1) Since $f, g \in L^\infty(\Omega)$, by Theorem 4.1 in [18], we have that $u \in L^\infty(\Omega)$.
(2) By Theorem 1.1 in [17], $u \in C_{loc}^{1,\alpha}(\Omega)$ for some α depending on p_1, p_2, $\|u\|_{L^\infty(\Omega)}$ and $\|f\|_{L^\infty(\Omega)}$. Moreover, given $\Omega_0 \subset\subset \Omega$, $\|u\|_{C^{1,\alpha}(\Omega_0)}$ depends on the same constants and $\operatorname{dist}(\Omega_0, \partial\Omega)$.

(3) Finally, by Theorem 1.2 in [17], if $\partial\Omega \in C^{1,\gamma}$ and $g \in C^{1,\gamma}(\partial\Omega)$ for some $\gamma > 0$ then $u \in C^{1,\alpha}(\overline{\Omega})$, where α and $\|u\|_{C^{1,\alpha}(\Omega)}$ depend on p_1, p_2, N, $\|u\|_{L^\infty(\Omega)}$, $\|p\|_{C^{\alpha_0}(\Omega)}$, α_0 and γ.

We will first prove the H^2-local regularity assuming only that $p(x)$ is Lipschitz. Then, we will prove the global regularity under the stronger condition that $\nabla p(x)$ is Hölder.

3.1. H^2-local regularity

While we were finishing this paper, we found the work [4], where the authors give a different proof of the H^2-local regularity of the solutions of (3.2). Anyhow, we leave the proof for the completeness of this paper.

Theorem 3.2. *Let $p, f \in \mathrm{Lip}(\Omega)$ with $p_1 > 1$ and u be a weak solution of (3.2), then $u \in H^2_{loc}(\Omega)$.*

Proof. First, let us define for any function F and $h > 0$,

$$\Delta^h F(x) = \frac{F(x + \mathbf{h}) - F(x)}{h},$$

where $\mathbf{h} = he_k$ and e_k is a vector of the canonical base of $\mathbb{R}^N$.

Let $\eta(x) = \xi(x)^2 \Delta^h u(x)$ where ξ is a regular function with compact support. Therefore, if we take $v_\varepsilon = (|\nabla u|^2 + \varepsilon)^{1/2}$ and $h < \mathrm{dist}(\mathrm{supp}(\xi), \partial\Omega)$, we have

$$\int_\Omega \bigl\langle v_\varepsilon(x)^{p(x)-2}\nabla u(x), \nabla\eta(x)\bigr\rangle\,dx = \int_\Omega f(x)\eta(x)\,dx,$$
$$\int_\Omega \bigl\langle v_\varepsilon(x+\mathbf{h})^{p(x+\mathbf{h})-2}\nabla u(x+\mathbf{h}), \nabla\eta(x)\bigr\rangle\,dx = \int_\Omega f(x+\mathbf{h})\eta(x)\,dx.$$

Subtracting, using that $\nabla\eta = 2\xi\nabla\xi\Delta^h u + \xi^2\Delta^h(\nabla u)$ and dividing by h we obtain

$$\begin{aligned}
I &= \int_\Omega \bigl\langle \Delta^h\bigl(v_\varepsilon(x)^{p(x)-2}\nabla u\bigr), \Delta^h(\nabla u)\bigr\rangle \xi^2\,dx \\
&= -2\int_\Omega \bigl\langle \Delta^h\bigl(v_\varepsilon(x)^{p(x)-2}\nabla u\bigr), \xi\nabla\xi\Delta^h u\bigr\rangle\,dx + \int_\Omega \xi^2\Delta^h f\,\Delta^h u\,dx \\
&= 2\int_\Omega \left(\int_0^1 v_\varepsilon(x+\mathbf{h}t)^{p(x+\mathbf{h}t)-2}\nabla u(x+\mathbf{h}t)\,dt\right)\frac{\partial}{\partial x_k}\bigl(\xi\nabla\xi\Delta^h u\bigr)\,dx \\
&\quad + \int_\Omega \xi^2\Delta^h f\,\Delta^h u\,dx \\
&= II + III.
\end{aligned}$$

Now, let us fix a ball B_R such that $B_{3R} \subset\subset \Omega$ and take $\xi \in C_0^\infty(\Omega)$ supported in B_{2R} such that $0 \leqslant \xi \leqslant 1$, $\xi = 1$ in B_R, $|\nabla\xi| \leqslant 1/R$ and $|D^2\xi| \leqslant CR^{-2}$.

By Remark 3.1, there exists a constant $C_1 > 0$ such that $|\nabla u| \leqslant C_1$ in B_{3R}, therefore we get

$$\begin{aligned}
II &\leqslant 2\int_{B_{2R}} \frac{C}{R}\bigl|\Delta^h u_{x_k}\bigr|\xi\,dx + 2\int_{B_{2R}} \frac{C}{R^2}\bigl|\Delta^h u\bigr|\,dx \\
&\leqslant \frac{C}{R}\int_{B_{2R}} \bigl|\Delta^h(\nabla u)\bigr|\xi\,dx + CR^{N-2}.
\end{aligned}$$

On the other hand, since f is Lipschitz we have that

$$\bigl|f(x+\mathbf{h}) - f(x)\bigr| \leqslant C_2 h$$

for some constant $C_2 > 0$. This implies that

$$III \leqslant C_2 R^N.$$

Therefore, summing II and III, and using Young's inequality, we have that for any $\delta > 0$

$$I \leqslant \delta \int_{B_{2R}} \left|\Delta^h(\nabla u)\right|^2 \xi^2\, dx + C, \tag{3.3}$$

for some constant C depending on R and δ.

On the other hand observe that $I = I_1 + I_2$ where

$$I_1 = \frac{1}{h} \int_{B_{2R}} \left\langle \left(v_\varepsilon(x+\mathbf{h})^{p(x+\mathbf{h})-2} \nabla u(x+\mathbf{h}) - v_\varepsilon(x)^{p(x+\mathbf{h})-2} \nabla u(x)\right), \Delta^h(\nabla u)\right\rangle \xi^2\, dx,$$

and

$$I_2 = \frac{1}{h} \int_{B_{2R}} \left\langle \left(v_\varepsilon(x)^{p(x+\mathbf{h})} - v_\varepsilon(x)^{p(x)}\right) \frac{\nabla u(x)}{v_\varepsilon(x)^2}, \Delta^h(\nabla u)\right\rangle \xi^2\, dx.$$

Using that $p(x)$ is Lipschitz and the fact that $|\nabla u(x)| \leqslant C_1$ we have that, for some b between $p(x+h)$ and $p(x)$,

$$\frac{1}{h}\left|v_\varepsilon(x)^{p(x+\mathbf{h})} - v_\varepsilon(x)^{p(x)}\right| = \left|v_\varepsilon(x)^b \log\left(v_\varepsilon(x)\right) \frac{p(x+\mathbf{h}) - p(x)}{h}\right| \leqslant C,$$

for some constant $C > 0$ depending on $p_1, p_2, \varepsilon, C_1$ and the Lipschitz constant of $p(x)$.

Therefore, we have that

$$-I_2 \leqslant C C_1 \varepsilon^{-1} \int_{B_{2R}} \left|\Delta^h(\nabla u)\right| \xi^2\, dx.$$

By (3.3), the last inequality and using again Young's inequality we have that, for any $\delta > 0$,

$$I_1 \leqslant \delta \int_{B_{2R}} \left|\Delta^h(\nabla u)\right|^2 \xi^2\, dx + C, \tag{3.4}$$

for some constant $C > 0$ depending on p_1, p_2, ε, C_1 and the Lipschitz constant of $p(x)$.

To finish the proof, we have to find a lower bound for I_1. By the well-known inequality, we have that

$$\left\langle \left(v_\varepsilon(x+\mathbf{h})^{p(x+h)-2} \nabla u(x+\mathbf{h}) - v_\varepsilon(x)^{p(x+\mathbf{h})-2} \nabla u(x)\right), \left(\nabla u(x+\mathbf{h}) - \nabla u(x)\right)\right\rangle \geqslant C_\varepsilon \left|\nabla u(x+\mathbf{h}) - \nabla u(x)\right|^2,$$

where

$$C_\varepsilon = \begin{cases} \varepsilon^{(p(x+\mathbf{h})-2)/2} & \text{if } p(x+\mathbf{h}) \geqslant 2, \\ (p(x+\mathbf{h}) - 1)\varepsilon^{(p(x+\mathbf{h})-2)/2} & \text{if } p(x+\mathbf{h}) \leqslant 2. \end{cases}$$

Therefore, using that $p_1 > 1$, we arrive at

$$I_1 \geqslant \int_{B_{2R}} C h^{-2} \left|\nabla u(x+\mathbf{h}) - \nabla u(x)\right|^2 \xi^2\, dx = C \int_{B_{2R}} \left|\Delta^h\left(\nabla u(x)\right)\right|^2 \xi^2\, dx.$$

Finally combining the last inequality with (3.4) we have that

$$\int_{B_R} \left|\Delta^h\left(\nabla u(x)\right)\right|^2 dx \leqslant C(N, p, f, \varepsilon).$$

This proves that $u \in H^2_{loc}(\Omega)$. □

3.2. H^2-global regularity

Now we want to prove that if $f \in \mathrm{Lip}(\Omega)$ and $g \in C^{1,\beta}(\partial\Omega)$, the regularized equation (3.2) has a weak solution $u \in C^2(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ for an $\alpha \in (0,1)$. We already know, by Remark 3.1, that $u \in C^{1,\alpha}(\overline{\Omega})$. Then, we only need to prove that $u \in C^2(\Omega)$.

Lemma 3.3. *Let Ω be a bounded domain in $\mathbb{R}^N$ with $\partial\Omega \in C^{1,\gamma}$, $p \in C^{1,\beta}(\Omega) \cap C^{\alpha_0}(\overline{\Omega})$, $f \in \mathrm{Lip}(\Omega)$ and $g \in C^{1,\beta}(\partial\Omega)$. Then, the Dirichlet Problem (3.2) has a solution $u \in C^2(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$.*

Proof. Observe that by Theorem 3.2, we know that the solution is in $H^2_{loc}(\Omega)$. Then for any $\Omega' \subset\subset \Omega$ we can derive the equation and look at the solution of (3.2) as the solution of the following equation,

$$\begin{cases} L_\varepsilon u = a(x) & \text{in } \Omega', \\ u = u & \text{on } \partial\Omega'. \end{cases} \tag{3.5}$$

Here,

$$L_\varepsilon u = a^\varepsilon_{ij}(x) u_{x_i x_j}$$

with

$$a^\varepsilon_{ij}(x) = \delta_{ij} + \big(p(x) - 2\big)\frac{u_{x_i} u_{x_j}}{v_\varepsilon^2}, \qquad v_\varepsilon = \big(\varepsilon + |\nabla u|^2\big)^{\frac{1}{2}} \quad \text{and}$$

$$a_\varepsilon(x) = \ln(v_\varepsilon)\langle \nabla u, \nabla p\rangle + f v_\varepsilon^{2-p}. \tag{3.6}$$

The operator L_ε is uniformly elliptic in Ω, since for any $\xi \in \mathbb{R}^N$

$$\min\big\{(p_1 - 1), 1\big\}|\xi|^2 \leqslant a^\varepsilon_{ij}\xi_i\xi_j \leqslant \max\big\{(p_2 - 1), 1\big\}|\xi|^2. \tag{3.7}$$

On the other hand, by Remark 3.1, $u \in C^{1,\alpha}(\overline{\Omega})$. Then, $a^\varepsilon_{ij} \in C^\alpha(\overline{\Omega})$, since $\varepsilon > 0$. Using that $f \in \mathrm{Lip}(\Omega)$, we have that $a \in C^\rho(\Omega)$ where $\rho = \min(\alpha, \beta)$. If $\partial\Omega' \in C^2$, as u is the unique solution of (3.5), by Theorem 6.13 in [19], we have that $u \in C^{2,\rho}(\Omega')$. This ends the proof. □

Remark 3.4. By the H^2 global estimate for linear elliptic equations with $L^\infty(\Omega)$ coefficients in two variables (see Lemma A.1 and (3.7)) we have that

$$\|u\|_{H^2(\Omega)} \leqslant C\big(\|a_\varepsilon\|_{L^2(\Omega)} + \|g\|_{H^2(\Omega)}\big)$$

where u is the solution of (3.2) and C is a constant independents of ε.

4. Proof of Theorem 1.1

Before proving the theorem, we will need a global bound for the derivatives of the solutions of (3.2).

Lemma 4.1. *Let $f \in L^{q(\cdot)}(\Omega)$ with $q'(x) \leqslant p^*(x)$, $g \in W^{1,p(\cdot)}(\Omega)$, $\varepsilon > 0$ and u_ε be the weak solution of (3.2) then*

$$\|\nabla u_\varepsilon\|_{L^{p(\cdot)}(\Omega)} \leqslant C$$

where C is a constant depending on $\|f\|_{L^{q(\cdot)}(\Omega)}$, $\|g\|_{W^{1,p(\cdot)}(\Omega)}$ but not on ε.

Proof. Let

$$J(v) := \int_\Omega \frac{1}{p(x)}\big(|\nabla v|^2 + \varepsilon\big)^{p(x)/2}\,dx.$$

By the convexity of J and using (3.2) we have that

$$\begin{aligned} J(u_\varepsilon) &\leqslant J(g) - \int_\Omega \big(|\nabla u_\varepsilon|^2 + \varepsilon\big)^{(p-2)/2}\nabla u_\varepsilon(\nabla g - \nabla u_\varepsilon)\,dx \\ &\leqslant C\left(1 + \int_\Omega f(u_\varepsilon - g)\,dx\right) \\ &\leqslant C\big(1 + \|f\|_{L^{q(\cdot)}(\Omega)}\|u_\varepsilon - g\|_{L^{q'(\cdot)}(\Omega)}\big) \\ &\leqslant C\big(1 + \|f\|_{L^{q(\cdot)}(\Omega)}\|\nabla u_\varepsilon - \nabla g\|_{L^{p(\cdot)}(\Omega)}\big), \end{aligned}$$

where in the last inequality we are using that $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^*(\cdot)}(\Omega)$ continuously and Poincaré's inequality.

Thus we have that there exists a constant independent of ε such that

$$\int_\Omega |\nabla u_\varepsilon|^{p(x)}\,dx \leqslant C\big(1 + \|\nabla u_\varepsilon\|_{L^{p(\cdot)}(\Omega)}\big),$$

and using the properties of the $L^{p(\cdot)}(\Omega)$-norms this means that

$$\|\nabla u_\varepsilon\|^m_{L^{p(\cdot)}(\Omega)} \leqslant C\bigl(1+\|\nabla u_\varepsilon\|_{L^{p(\cdot)}(\Omega)}\bigr),$$

for some $m>1$. Therefore $\|\nabla u_\varepsilon\|_{L^{p(\cdot)}(\Omega)}$ is bounded independent of ε. $\square$

To prove Theorem 1.1, we will use the results of Section 3. Therefore, we will first need to assume that $p \in C^{1,\beta}(\Omega)\cap C(\overline{\Omega})$.

Theorem 4.2. *Let Ω be a bounded domain in $\mathbb{R}^2$ with C^2 boundary, $p\in C^{1,\beta}(\Omega)\cap C^{\alpha_0}(\overline{\Omega})$ with $p(x)\geqslant p_1>1$, $g\in H^2(\Omega)$ and u be the weak solution of (1.1). If f satisfies* (F1) *and* (F2) *then $u\in H^2(\Omega)$.*

Proof. Let $f_\varepsilon \in \mathrm{Lip}(\Omega)$ and $g_\varepsilon\in C^{2,\alpha}(\overline{\Omega})$ such that

$$f_\varepsilon \to f \quad \text{strongly in } L^{q(\cdot)}(\Omega),$$
$$g_\varepsilon \to g \quad \text{strongly in } H^2(\Omega),$$

as $\varepsilon\to 0$. Observe that, since $f(x)=0$ if $p(x)>2$, we can take $f_\varepsilon\equiv 0$ in $\{x\in\Omega\colon\ p(x)>2\}$.

Now, let us consider the solution of (3.2) as the solution of

$$\begin{cases} a^\varepsilon_{11}(x)\dfrac{\partial^2 u_\varepsilon}{\partial x_1^2}+2a^\varepsilon_{12}(x)\dfrac{\partial^2 u_\varepsilon}{\partial x_1\partial x_2}+a^\varepsilon_{22}(x)\dfrac{\partial^2 u_\varepsilon}{\partial x_2^2}=a_\varepsilon(x) & \text{in } \Omega,\\ u_\varepsilon = g_\varepsilon & \text{on } \partial\Omega, \end{cases}$$

where a^ε_{11}, a^ε_{22}, a^ε_{12}, a_ε are defined as in Lemma 3.3, substituting f and g by f_ε and g_ε respectively. By Lemma 3.3 we know that $u_\varepsilon\in C^2(\Omega)\cap C^{1,\alpha}(\overline{\Omega})$.

First we will prove the $\{u_\varepsilon\}_{\varepsilon\in(0,1]}$ is bounded in $H^2(\Omega)$. By Remark 3.4, we have that

$$\begin{aligned}\|u_\varepsilon\|_{H^2(\Omega)} &\leqslant C\bigl(\bigl\|a_\varepsilon(x)\bigr\|_{L^2(\Omega)}+\|g_\varepsilon\|_{H^2(\Omega)}\bigr)\\ &\leqslant C\bigl(\bigl\|\ln(v_\varepsilon)\nabla u_\varepsilon\nabla p\bigr\|_{L^2(\Omega)}+\bigl\|f_\varepsilon v^{2-p}\bigr\|_{L^2(\Omega)}+\|g_\varepsilon\|_{H^2(\Omega)}\bigr).\end{aligned} \tag{4.8}$$

Taking $\Omega_1=\{x\in\Omega\colon\ |\nabla u_\varepsilon(x)|>1\}$, using that $p(x)$ is Lipschitz and Hölder's inequality, we have

$$\bigl\|\ln(v_\varepsilon)\nabla u_\varepsilon\nabla p\bigr\|_{L^2(\Omega)} \leqslant C\bigl\|\ln^2(v_\varepsilon)\nabla u_\varepsilon\bigr\|^{1/2}_{L^{p'(\cdot)}(\Omega_1)}\|\nabla u_\varepsilon\|^{1/2}_{L^{p(\cdot)}(\Omega_1)}+C. \tag{4.9}$$

On the other hand, since $q(x)\geqslant q_1>2$, we have that $q'(x)\leqslant p^*(x)$. Then, as $\|f_\varepsilon\|_{L^{q(\cdot)}(\Omega)}$ and $\|g_\varepsilon\|_{H^2(\Omega)}$ are bounded independent of ε, using Lemma 4.1 we conclude that $\|\nabla u_\varepsilon\|_{L^{p(\cdot)}(\Omega)}$ is uniformly bounded.

Observe that, for all $s>0$ there exists a constant $C>0$ such that

$$\ln(v_\varepsilon)\leqslant Cv_\varepsilon^{s/2} < C|\nabla u_\varepsilon|^{s/2} \quad \text{in } \Omega_1,$$

thus

$$\begin{aligned}\bigl\|\ln^2(v_\varepsilon)|\nabla u_\varepsilon|\bigr\|_{L^{p'(\cdot)}(\Omega_1)} &\leqslant C\bigl\||\nabla u_\varepsilon|^{1+s}\bigr\|_{L^{p'(\cdot)}(\Omega_1)}\\ &\leqslant C\|\nabla u_\varepsilon\|^{(1+s)}_{L^{p'(\cdot)(1+s)}(\Omega_1)}\\ &\leqslant C\|u_\varepsilon\|^{(1+s)}_{H^2(\Omega_1)}.\end{aligned}$$

In the last line, we are using that $2^*=\infty$, since $N=2$.

Then, by the last inequality, (4.8) and (4.9), we get

$$\|u_\varepsilon\|_{H^2(\Omega)}\leqslant C\bigl(\|u_\varepsilon\|^{(1+s)/2}_{H^2(\Omega)}+\bigl\|f_\varepsilon v_\varepsilon^{2-p}\bigr\|_{L^2(\Omega)}+1\bigr). \tag{4.10}$$

Taking

$$A_1=\bigl\{x\in\Omega\colon\ p(x)=2\bigr\} \quad\text{and}\quad A_2=\bigl\{x\in\Omega\colon\ p(x)<2\bigr\}$$

and using that $f_\varepsilon\equiv 0$ in $\{x\in\Omega\colon\ p(x)>2\}$, we have that

$$\bigl\|f_\varepsilon v_\varepsilon^{2-p}\bigr\|_{L^2(\Omega)}\leqslant\|f_\varepsilon\|_{L^2(A_1)}+\bigl\|f_\varepsilon v_\varepsilon^{2-p}\bigr\|_{L^2(A_2)}.$$

Since $\|f_\varepsilon\|_{L^2(A_1)}$ is bounded, to prove that $\{u_\varepsilon\}_{\varepsilon\in(0,1]}$ is bounded in $H^2(\Omega)$, we only have to find a bound of $\|f_\varepsilon v_\varepsilon^{2-p}\|_{L^2(A_2)}$.

Let us define in A_2 the function

$$\tilde{q}(x) = \begin{cases} \frac{1}{2p(x)-3} + 1 & \text{if } \frac{1}{q(x)} + \frac{3}{2} \leqslant p(x) < 2, \\ \frac{q(x)}{2} + 1 & \text{if } p(x) < \frac{1}{q(x)} + \frac{3}{2}. \end{cases}$$

It is easy to see that $2 < \tilde{q}(x) \leqslant q(x)$ for any $x \in A_2$.

On the other hand, let us denote $\mu(x) = \frac{2\tilde{q}(x)}{\tilde{q}(x)-2}$ and $\gamma(x) = \mu(x)(2 - p(x))$ then

$$1 < 1 + \frac{2}{q_2} \leqslant \gamma(x) \leqslant \max\left\{2, 2 + \frac{8}{q_1 - 2}\right\} \quad \forall x \in A_2.$$

Now, using Hölder's inequality with exponent $\tilde{q}(x)/2$, we have

$$\left\| f_\varepsilon v_\varepsilon^{2-p} \right\|_{L^2(A_2)} \leqslant C \| f_\varepsilon \|_{L^{\tilde{q}(\cdot)}(A_2)} \left\| v_\varepsilon^{2-p} \right\|_{L^{\mu(\cdot)}(A_2)}. \tag{4.11}$$

Then, if $\|v_\varepsilon\|_{L^{\gamma(\cdot)}(A_2)} \leqslant 1$ we have $\|v_\varepsilon^{2-p}\|_{L^{\mu(\cdot)}(A_2)} \leqslant 1$ and since $\tilde{q}(x) \leqslant q(x)$ we get

$$\left\| f_\varepsilon v_\varepsilon^{2-p} \right\|_{L^2(A_2)} \leqslant C.$$

If $\|v\|_{L^{\gamma(\cdot)}(A_2)} \geqslant 1$, we have

$$\left\| v_\varepsilon^{2-p} \right\|_{L^{\mu(\cdot)}(A_2)} \leqslant \|v_\varepsilon\|_{L^{\gamma(\cdot)}(A_2)}^{2-p_1} \leqslant C\left(1 + \|\nabla u_\varepsilon\|_{L^{\gamma(\cdot)}(A_2)}^{2-p_1}\right), \tag{4.12}$$

where in the last inequality we are using that $\varepsilon \leqslant 1$.

Since $2^* = \infty$ and $1 < \gamma_1 \leqslant \gamma(x) \leqslant \gamma_2 < \infty$, by the Sobolev embedding inequality, we have that

$$\|\nabla u_\varepsilon\|_{L^{\gamma(\cdot)}(A_2)}^{2-p_1} \leqslant C\|u_\varepsilon\|_{H^2(A_2)}^{2-p_1} \leqslant C\|u_\varepsilon\|_{H^2(\Omega)}^{2-p_1}.$$

Combining this last inequality with inequalities (4.12), (4.11), (4.10) and the fact that $\tilde{q}(x) \leqslant q(x)$, we get

$$\|u_\varepsilon\|_{H^2(\Omega)} \leqslant C\left(\|u_\varepsilon\|_{H^2(\Omega)}^{(1+s)/2} + \|u_\varepsilon\|_{H^2(\Omega)}^{2-p_1} + 1\right).$$

Finally, we get that for any $0 < s < 1$ there exists a constant $C = C(p, g, f, s)$ such that

$$\|u_\varepsilon\|_{H^2(\Omega)} \leqslant C.$$

Then, there exists a subsequence still denoted $\{u_\varepsilon\}_{\varepsilon \in (0,1]}$ and $u \in H^1(\Omega)$ such that

$$u_\varepsilon \to u \quad \text{strongly in } H^1(\Omega),$$
$$u_\varepsilon \rightharpoonup u \quad \text{weakly in } H^2(\Omega).$$

It is clear that u satisfies the boundary condition.

Lastly, by Proposition 3.2 in [2], there exists a constant $M > 0$ independent of ε such that

$$\left|\left(\varepsilon + |\nabla u_\varepsilon|^2\right)^{\frac{p(x)-2}{2}} \nabla u_\varepsilon - \left(\varepsilon + |\nabla u|^2\right)^{\frac{p(x)-2}{2}} \nabla u\right| \leqslant M\left|\nabla(u_\varepsilon - u)\right|^{p(x)-1} \tag{4.13}$$

for all $x \in \Omega$. Then, passing to the limit in the weak formulation of (3.2) and using the above inequality, we have that

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla \varphi \, dx = \int_\Omega f \varphi \, dx$$

for any $\varphi \in C_0^\infty(\Omega)$. Therefore $u \in H^2(\Omega)$ and solves (1.1). □

Now, we are able to prove the theorem.

Proof of Theorem 1.1. First, we consider the case $p \in C^1(\overline{\Omega})$. Let $p_\varepsilon \in C^\infty(\overline{\Omega})$ such that $p_\varepsilon \to p$ in $C^1(\Omega)$. Now, we define

$$f_\varepsilon(x) = \begin{cases} f(x) & \text{if } p_\varepsilon(x) \leqslant 2, \\ 0 & \text{if } p_\varepsilon(x) > 2. \end{cases} \tag{4.14}$$

Observe that $f_\varepsilon \to f$ in $L^{q(\cdot)}(\Omega)$ as $\varepsilon \to 0$.

Then, by Theorem 4.2, the solution u_ε of (1.1) (with p_ε and f_ε instead of p and f) is bounded in $H^2(\Omega)$ by a constant independent of ε. Therefore, there exists a subsequence still denoted $\{u_\varepsilon\}_{\varepsilon\in(0,1]}$ and $u \in H^2(\Omega)$ such that

$$\begin{aligned} &u_\varepsilon \to u \quad \text{in } H^1(\Omega),\\ &u_\varepsilon \rightharpoonup u \quad \text{weakly in } H^2(\Omega). \end{aligned} \tag{4.15}$$

It remains to prove that u is a solution of (1.1). Let $\varphi \in C_0^\infty(\Omega)$, then

$$\begin{aligned} \int_\Omega f_\varepsilon \varphi\,dx &= \int_\Omega |\nabla u_\varepsilon|^{p_\varepsilon(x)-2}\nabla u_\varepsilon \nabla\varphi\,dx\\ &= \int_\Omega |\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon \nabla\varphi\,dx + \int_\Omega \big(|\nabla u_\varepsilon|^{p_\varepsilon(x)-2} - |\nabla u_\varepsilon|^{p(x)-2}\big)\nabla u_\varepsilon \nabla\varphi\,dx. \end{aligned} \tag{4.16}$$

Therefore, using that $H^2(\Omega) \hookrightarrow W^{1,p(\cdot)}(\Omega)$ compactly, we have that

$$\int_\Omega |\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon \nabla\varphi\,dx \to \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla\varphi\,dx. \tag{4.17}$$

On the other hand, we have

$$\big|\nabla u_\varepsilon(x)\big|^{p_\varepsilon(x)-1} - \big|\nabla u_\varepsilon(x)\big|^{p(x)-1} = \big|\nabla u_\varepsilon(x)\big|^{b_\varepsilon(x)} \log\big(\big|\nabla u_\varepsilon(x)\big|\big)\big(p_\varepsilon(x) - p(x)\big),$$

where $b_\varepsilon(x) = p_\varepsilon(x)\theta + (1-\theta)p(x) - 1$ for some $0 < \theta < 1$. Therefore, using that $2^* = \infty$ and that $p_\varepsilon \to p$ uniformly, we obtain

$$\int_\Omega \big(|\nabla u_\varepsilon|^{p_\varepsilon(x)-2} - |\nabla u_\varepsilon|^{p(x)-2}\big)\nabla u_\varepsilon \nabla\varphi\,dx \to 0. \tag{4.18}$$

Then, using that $f_\varepsilon \to f$ in $L^{q(\cdot)}(\Omega)$, (4.16), (4.17) and (4.18), we conclude that u is a solution of (1.1).

Now, we consider the case $p \in \mathrm{Lip}(\overline{\Omega})$. By Lemmas B.1 and B.2 there exists $p_\varepsilon \in C^1(\overline{\Omega})$ such that $|\Omega \setminus \Omega_0| < \varepsilon$ where

$$\Omega_0 = \big\{x \in \Omega\colon\ p_\varepsilon(x) = p(x) \text{ and } \nabla p_\varepsilon(x) = \nabla p(x)\big\}.$$

We define f_ε as in (4.14). Then, the solution u_ε of (1.1) with p_ε and f_ε instead of p and f is bounded in $H^2(\Omega)$ by a constant independent of ε. Therefore there exists a subsequence still denoted $\{u_\varepsilon\}_{\varepsilon\in(0,1]}$ and $u \in H^2(\Omega)$ satisfying (4.15).

Lastly, we prove that u is a solution of (1.1). Let $\varphi \in C_0^\infty(\Omega)$. By Hölder's inequality, since $2^* = \infty$ and by (3) of Lemma B.2 we have

$$\begin{aligned} &\int_{\Omega\setminus\Omega_0} \big(|\nabla u_\varepsilon|^{p_\varepsilon(x)-2} - |\nabla u_\varepsilon|^{p(x)-2}\big)\nabla u_\varepsilon \nabla\varphi\,dx\\ &\quad\leqslant C\big(\|\nabla u_\varepsilon\|_{L^{p_\varepsilon}(\Omega)}\|1\|_{L^{p_\varepsilon}(\Omega\setminus\Omega_0)} + \|\nabla u_\varepsilon\|_{L^{p}(\Omega)}\|1\|_{L^{p}(\Omega\setminus\Omega_0)}\big)\\ &\quad\leqslant C\|u_\varepsilon\|_{H^2(\Omega)}\big(\|1\|_{L^{p_\varepsilon}(\Omega\setminus\Omega_0)} + \|1\|_{L^{p}(\Omega\setminus\Omega_0)}\big). \end{aligned}$$

Then, since $\|u_\varepsilon\|_{H^2(\Omega)}$ is bounded independent of ε and $|\Omega \setminus \Omega_0| < \varepsilon$ we obtain that

$$\int_{\Omega\setminus\Omega_0} \big(|\nabla u_\varepsilon|^{p_\varepsilon(x)-2} - |\nabla u_\varepsilon|^{p(x)-2}\big)\nabla u_\varepsilon \nabla\varphi\,dx \to 0.$$

Therefore, since (4.16), (4.17) again hold, using that $f_\varepsilon \to f$ in $L^{q(\cdot)}(\Omega)$, and the above equation, we conclude that u is a solution of (1.1). □

5. The convex case

Lastly, we want to prove that the solution is in $H^2(\Omega)$ if we only assume that $\partial\Omega$ is convex. We want to remark here that this result generalizes the one in Theorem 2.2 in [22] in two ways. In that paper the authors consider the case $p = constant$ and $g = 0$. Instead, we are allowed to cover the case where g is any function in $H^2(\Omega)$ and $p(x) \in \mathrm{Lip}(\overline{\Omega})$.

Remark 5.1. Let Ω be a convex set and $p : \Omega \to [1,\infty)$ be log-continuous in $\overline{\Omega}$. Then, there exists a sequence $\{\Omega_m\}_{m\in\mathbb{N}}$ of convex subset of Ω with C^2 boundary such that $\Omega_m \subset \Omega_{m+1}$ for any $m \in \mathbb{N}$ and $|\Omega \setminus \Omega_m| \to 0$.

(1) Then, there exists a constant C depending on $p(x)$, $|\Omega|$ such that

$$\|v\|_{L^{p(\cdot)}(\Omega_m)} \leqslant C\|\nabla v\|_{L^{p(\cdot)}(\Omega_m)} \quad \forall v \in W_0^{1,p(\cdot)}(\Omega_m),$$

for any $m \in \mathbb{N}$. This follows by Theorem 3.3 in [21], using that $\Omega_m \subset \Omega_{m+1}$ for any $m \in \mathbb{N}$.

(2) The Lipschitz constants of Ω_m ($m \in \mathbb{N}$) are uniformly bounded (see Remark 2.3 in [22]). Therefore, the extension operators

$$E_{1,m} : W^{1,p(\cdot)}(\Omega_m) \to W^{1,p(\cdot)}(\Omega) \quad \text{and} \quad E_{2,m} : H^2(\Omega_m) \to H^2(\Omega)$$

define as Theorem 4.2 in [11] satisfy that $\|E_{1,m}\|$ and $\|E_{2,m}\|$ are uniformly bounded.

(3) By (2) and Corollary 8.3.2 in [12], there exists a constant C independent of m such that

$$\|v\|_{L^{p^*(\cdot)}(\Omega_m)} \leqslant C\|v\|_{W^{1,p(\cdot)}(\Omega_m)} \quad \forall v \in W^{1,p(\cdot)}(\Omega_m),$$

for any $m \in \mathbb{N}$.

We want to remark that all the constants of the above inequalities are independent of p_1 (see Section 6 for the applications).

Proof of Theorem 1.2. We begin taking $\{\Omega_m\}_{m\in\mathbb{N}}$ as in Remark 5.1 and u_m the solution of

$$\begin{cases} -\Delta_{p(x)} u_m = f & \text{in } \Omega_m, \\ u_m = g & \text{on } \partial\Omega_m. \end{cases}$$

By Theorem 1.1, $u_m \in H^2(\Omega_m)$ for any $m \in \mathbb{N}$. Moreover, u_m solves

$$\begin{cases} L^m u_m = a_{ij}^m(x) u_{m,x_i x_j} = a^m(x) & \text{in } \Omega_m, \\ u_m = g & \text{on } \partial\Omega_m, \end{cases}$$

with

$$a_{ij}^m(x) = \delta_{ij} + \big(p(x) - 2\big)\frac{u_{m,x_i}(x) u_{m,x_j}(x)}{|\nabla u_m(x)|^2},$$

$$a^m(x) = \ln\big(\big|\nabla u_m(x)\big|\big)\big\langle \nabla u_m(x), \nabla p(x)\big\rangle + f(x)\big|\nabla u_m(x)\big|^{2-p(x)}.$$

Then $v_m = u_m - g$ solves

$$\begin{cases} L^m v_m = -L^m g + a^m(x) & \text{in } \Omega_m, \\ v_m = 0 & \text{on } \partial\Omega_m. \end{cases}$$

Thus, using that $v_m \in H^2(\Omega_m) \cap H_0^1(\Omega_m)$ and since the coefficients $a_{ij}^m(x)$ are bounded independent of m, we can argue as in Theorem 2.2 in [22] and obtain

$$\begin{aligned} \|v_m\|_{H^2(\Omega_m)} &\leqslant C\big\|-L^m g + f|\nabla u_m|^{2-p(\cdot)} + \ln\big(|\nabla u_m|\big)|\nabla u_m|\big\|_{L^2(\Omega_m)} \\ &\leqslant C\big(\big\||\nabla u_m|^{2-p(\cdot)}\big\|_{L^2(\Omega_m)} + \big\|\ln\big(|\nabla u_m|\big)|\nabla u_m|\big\|_{L^2(\Omega_m)} + 1\big) \end{aligned} \tag{5.19}$$

where the constant C is independent of m.

As in Lemma 4.1 we can prove, using Remark 5.1(1) and (3), that the norms $\|\nabla u_m\|_{L^{p(\cdot)}(\Omega_m)}$ are uniformly bounded. Therefore, proceeding as in Theorem 4.2, we obtain

$$\big\|\ln\big(|\nabla u_m|\big)|\nabla u_m|\big\|_{L^2(\Omega_m)} + \big\|f|\nabla u_m|^{2-p}\big\|_{L^2(\Omega_m)} \leqslant C\big(\|\nabla u_m\|_{L^{p'(\cdot)(1+s)}(\Omega_{1,m})}^{(1+s)/2} + \|\nabla u_m\|_{L^{\gamma(\cdot)}(A_{2,m})}^{2-p_1} + 1\big), \tag{5.20}$$

with C independent of m, where

$$\Omega_{1,m} = \big\{x \in \Omega_m\colon \big|\nabla u_m(x)\big| > 1\big\} \quad \text{and} \quad A_{2,m} = \big\{x \in \Omega_m\colon p(x) < 2\big\}.$$

Now, using Remark 5.1(3) and (2), we have that for any $r > 1$

$$\begin{aligned} \|v_m\|_{W^{1,r}(\Omega_m)} &\leqslant \|E_{2,m} v_m\|_{W^{1,r}(\Omega)} \\ &\leqslant C\|E_{2,m} v_m\|_{H^2(\Omega)} \\ &\leqslant C\|v_m\|_{H^2(\Omega_m)} \end{aligned} \tag{5.21}$$

where C is independent of m.

Therefore, using (5.19), (5.20) and (5.21), we get

$$\begin{aligned}\|v_m\|_{H^2(\Omega_m)} &\leqslant C\big(\|v_m\|_{H^2(\Omega_m)}^{(1+s)/2} + \|v_m\|_{H^2(\Omega_m)}^{2-p_1} + \|g\|_{H^2(\Omega_m)}^{(1+s)/2} + \|g\|_{H^2(\Omega_m)}^{2-p_1} + 1\big)\\ &\leqslant C\big(\|v_m\|_{H^2(\Omega_m)}^{(1+s)/2} + \|v_m\|_{H^2(\Omega_m)}^{2-p_1} + 1\big),\end{aligned}$$

where the constant C is independent of m. This proves that $\{\|v_m\|_{H^2(\Omega_m)}\}_{m\in\mathbb{N}}$ is bounded.

Now we have, as in the proof of Theorem 2.2 in [22], that there exist a subsequence still denote $\{v_m\}_{m\in\mathbb{N}}$ and a function $v \in H^2(\Omega)\cap H_0^1(\Omega)$ such that

$$v_m \to v \quad \text{strongly in } H^1(\Omega')$$

for any $\Omega' \subset\subset \Omega$. Then $u = v + g \in H^2(\Omega)$ and

$$u_m \to u \quad \text{strongly in } H^1(\Omega')$$

for any $\Omega' \subset\subset \Omega$. Thus, using (4.13), we have

$$|\nabla u_m|^{p(x)-2}\nabla u_m \to |\nabla u|^{p(x)-2}\nabla u \quad \text{strongly in } L^{p'(\cdot)}(\Omega') \tag{5.22}$$

for any $\Omega' \subset\subset \Omega$.

On the other hand, for any $\varphi \in C_0^\infty(\Omega)$ there exists m_0 such that for all $m \geqslant m_0$

$$\int_{\Omega_m} |\nabla u_m|^{p(x)-2}\nabla u_m \nabla\varphi\, dx = \int_{\Omega_m} f\varphi\, dx.$$

Therefore, using (5.22) we have that u is a weak solution of (1.1). □

Proof of Corollary 1.3. By the previous theorem we have that $u \in H^2(\Omega)$, then we can derive Eq. (1.1) and obtain

$$\begin{cases} -a_{ij}(x)u_{x_ix_j} = a(x) & \text{in } \Omega,\\ u = g & \text{on } \partial\Omega,\end{cases}$$

where

$$\begin{aligned} a_{ij}(x) &= \delta_{ij} + \big(p(x)-2\big)\frac{u_{x_i}(x)u_{x_j}(x)}{|\nabla u(x)|^2},\\ a(x) &= \ln\big(\big|\nabla u(x)\big|\big)\big\langle\nabla u(x), \nabla p(x)\big\rangle + f(x)\big|\nabla u(x)\big|^{2-p(x)}.\end{aligned}$$

Using that $f \in L^{q(\cdot)}(\Omega)$ with $q(x) \geqslant q_1 > 2$ and following the lines in the proof of Theorem 4.2, we have that $a(x) \in L^s(\Omega)$ with $s > 2$. Therefore, by Remark A.3, we have that $u \in C^{1,\alpha}(\overline{\Omega})$. □

6. Comments

In the image processing problem it is of interest the case where p_1 is close to 1. By this reason, we are also interested in the dependence of the H^2-norm on p_1.

If $N = 2$, $g \in H^2(\Omega)$ and u_ε is the solution of (3.2), we have by Lemma A.1, (3.6) and (3.7), that there exists a constant C independent of p_1 and ε such that

$$\|u_\varepsilon\|_{H^2(\Omega)} \leqslant \frac{C}{(p_1-1)^\kappa}\big(\|a_\varepsilon\|_{L^2(\Omega)} + \|g\|_{H^2(\Omega)}\big),$$

where $\kappa = 1$ if Ω is convex and $\kappa = 2$ if $\partial\Omega \in C^2$. Therefore, using that the Poincaré inequality and the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^*(\cdot)}(\Omega)$ hold in the case $p_1 = 1$ and following the lines of Theorem 1.1 and Theorem 1.2 we have that

$$\|u\|_{H^2(\Omega)} \leqslant \frac{C}{(p_1-1)^\kappa},$$

where the constant C is independent of p_1.

Appendix A. Regularity results for elliptic linear equations with coefficients in L^∞

Let Ω be a bounded open subset of $\mathbb{R}^2$ and

$$\mathcal{M}u = a_{ij}(x)u_{x_i x_j},$$

such that $a_{ij} = a_{ji}$ and for any $\xi \in \mathbb{R}^N$

$$\lambda|\xi|^2 \leqslant a_{ij}(x)\xi_i\xi_j \leqslant \Lambda|\xi|^2, \tag{A.1}$$

and

$$M_1 \leqslant a_{11}(x) + a_{22}(x) \leqslant M_2 \quad \text{in } \Omega \tag{A.2}$$

where λ, Λ, M_1 and M_2 are positive constant.

In the next lemma, we will give an H^2-bound for solutions of

$$\begin{cases} \mathcal{M}u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{A.3}$$

In fact, the following result is proved in Theorem 37, III in [23], but the dependence of the bounds on the ellipticity and the L^∞-norm of $(a_{ij}(x))$ are not explicit. Then, following the proof of the mentioned theorem we can prove

Lemma A.1. *Let Ω be a bounded domain in $\mathbb{R}^2$, $f \in L^2(\Omega)$ and $g \in H^2(\Omega)$. Then, if u is a solution of (A.3) and $u \in H^2(\Omega)$ we have that*

$$\|u\|_{H^2(\Omega)} \leqslant \frac{C}{\lambda^\kappa}\big(\|f\|_{L^2(\Omega)} + \|g\|_{H^2(\Omega)}\big),$$

where $\kappa = 1$ if Ω is convex and $\kappa = 2$ if $\partial\Omega \in C^2$ and C is a constant independent of λ.

Proof. In this proof, we denote $u_{ij} = u_{x_i x_j}$ for all $i, j = 1, 2$ and C is a constant independent of λ.

First, we consider the case $g \equiv 0$. Using (A.1), we have that

$$\big(a_{11}(x) + a_{22}(x)\big)\big(u_{12}^2 - u_{11}u_{22}\big) = \sum_{i,j,k=1}^{2} a_{ij}u_{ki}u_{kj} - \Delta u \sum_{ij=1}^{2} a_{ij}u_{ij} \geqslant \lambda \sum_{ik=1}^{2} u_{ki}^2 - \Delta u f(x).$$

Then, using Young's inequality, we get

$$\frac{\lambda}{2(a_{11}(x) + a_{22}(x))} \sum_{ik=1}^{2} u_{ki}^2 \leqslant \frac{4}{\lambda(a_{11}(x) + a_{22}(x))} f(x)^2 + u_{12}^2 - u_{11}u_{22},$$

and by (A.2), we have that

$$\sum_{ik=1}^{2} u_{ki}^2 \leqslant \frac{C}{\lambda^2} f(x)^2 + \frac{C}{\lambda}\big(u_{12}^2 - u_{11}u_{22}\big). \tag{A.4}$$

Now, using (37.4) and (37.6) in [23], we have that for any $u \in H^2(\Omega)$

$$\int_\Omega \big(u_{12}^2 - u_{11}u_{22}\big)\,dx = -\int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu}\right)^2 \frac{H}{2}\,ds \tag{A.5}$$

where H is the curvature of $\partial\Omega$. If Ω is convex, then $H \geqslant 0$ and therefore, using (A.4) and (A.5), we have that

$$\big\|D^2 u\big\|_{L^2(\Omega)} \leqslant \frac{C}{\lambda}\|f\|_{L^2(\Omega)}. \tag{A.6}$$

In the general case, we can use the following inequality

$$\int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu}\right)^2 ds \leqslant C\left(\big(1 + \delta^{-1}\big)\int_\Omega |\nabla u|^2\,dx + \delta \int_\Omega \sum_{ik=1}^{2} u_{ki}^2\,dx\right) \tag{A.7}$$

for any $\delta > 0$. See Eq. (37.6) of [23].

Then, by (A.4), (A.5), using that H is bounded and (A.7) (choosing δ properly) we arrive at

$$\int_\Omega \sum_{ik=1}^{2} u_{ki}^2\,dx \leqslant \frac{C}{\lambda^2}\left(\int_\Omega f(x)^2\,dx + \int_\Omega |\nabla u|^2\,dx\right). \tag{A.8}$$

On the other hand, using that $Lu = f$ in Ω, (A.1) and the Poincaré inequality, we have

$$\|\nabla u\|_{L^2(\Omega)} \leqslant \frac{C}{\lambda}\|f\|_{L^2(\Omega)}. \tag{A.9}$$

Therefore, by (A.8) and (A.9), we get

$$\left\|D^2u\right\|_{L^2(\Omega)} \leqslant \frac{C}{\lambda^2}\|f\|_{L^2(\Omega)}.$$

Thus, by the last inequality, (A.9) and (A.6) the lemma is proved in the case $g = 0$.
When g is any function in $H^2(\Omega)$ the lemma follows taking $v = u - g$. □

The following theorem is proved in Corollary 8.1.6 in [20].

Theorem A.2. *Let Ω be a convex polygonal domain in $\mathbb{R}^2$, $\mathcal{M}$ satisfying (A.1) and $u \in H^2(\Omega) \cap H_0^1(\Omega)$ be a solution of (A.3) with $g = 0$ and $f \in L^p(\Omega)$ with $p > 2$. Then $\nabla u \in C^\mu(\overline{\Omega})$ for some $0 < \mu < 1$.*

Remark A.3. Observe that the above theorem holds also if we consider any $g \in W^{2,p}(\Omega)$, since we can take $v = u - g$ in (A.3) and use that $W^{2,p}(\Omega) \hookrightarrow C^{1,1-2/p}(\overline{\Omega})$.

Appendix B. Lipschitz functions

Using the linear extension operator defined in [14], we have the following lemma.

Lemma B.1. *Let Ω be a bounded open domain with Lipschitz boundary and $f \in \mathrm{Lip}(\overline{\Omega})$. Then, there exists a function $\overline{f} : \mathbb{R}^N \to \mathbb{R}$ such that $\overline{f}$ is a Lipschitz function, $\sup_{\mathbb{R}^N} \overline{f} = \inf_{\overline{\Omega}} f$ and $\inf_{\mathbb{R}^N} \overline{f} = \max_{\overline{\Omega}} f$.*

Lemma B.2. *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a Lipschitz function. Then for each $\varepsilon > 0$, there exists a C^1 function $f_\varepsilon : \mathbb{R}^N \to \mathbb{R}$ such that*

(1) $|\{x \in \mathbb{R}^N\colon\ f_\varepsilon(x) \neq f(x) \text{ or } Df_\varepsilon(x) \neq Df(x)\}| \leqslant \varepsilon$.
(2) *There exists a constant C depending only on N such that*

$$\|Df_\varepsilon\|_{L^\infty(\mathbb{R}^N)} \leqslant C\,\mathrm{Lip}(f).$$

(3) *If $1 < f_1 \leqslant f(x) \leqslant f_2$ in $\mathbb{R}^N$, we have*

$$1 < f_\varepsilon(x) \leqslant f_2 + C\varepsilon^{\frac{1}{N}} \quad \text{in } \mathbb{R}^N$$

with C a constant depending only on N.

Proof. Items (1) and (2) follow by Theorem 1, p. 251 in [16].
To prove (3), let us define

$$\Omega_0 = \left\{x \in \mathbb{R}^N\colon\ f_\varepsilon(x) = f(x) \text{ and } Df_\varepsilon(x) = Df(x)\right\}$$

and let us suppose that there exists $x \in \mathbb{R}^N \setminus \Omega_0$ such that $f_\varepsilon(x) = f_2 + \delta$ with $\delta > 0$. If $x_0 \in \Omega_0$, by (2), we have

$$C\,\mathrm{Lip}(f)|x - x_0| \geqslant f_\varepsilon(x) - f_\varepsilon(x_0) = f_2 + \delta - f(x_0) \geqslant \delta.$$

Then $B_\rho(x) \subset \mathbb{R}^N \setminus \Omega_0$ where $\rho = \delta(C\,\mathrm{Lip}(f))^{-1}$ and using (1) we get $\delta \leqslant C\varepsilon^{1/N}$, for some constant C independent of ε.
Analogously we can prove the other inequality. □

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