The Star and Biclique Coloring and Choosability Problems

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Abstract

A biclique of a graph G is an induced complete bipartite graph. A
star of G is a biclique contained in the closed neighborhood of a vertex.
A star (biclique) k-coloring of G is a k-coloring of G that contains no
monochromatic maximal stars (bicliques). Similarly, for a list assignment
L of G, a star (biclique) L-coloring is an L-coloring of G in which no
maximal star (biclique) is monochromatic. If G admits a star (biclique) L-
coloring for every k-list assignment L, then G is said to be star (biclique)
k-choosable. In this article we study the computational complexity of
the star and biclique coloring and choosability problems. Specifically,
we prove that the star (biclique) k-coloring and k-choosability problems
are Σ_2^p-complete and Π_3^p-complete for $k > 2$, respectively, even when the
input graph contains no induced C_4 or K_{k+2}. Then, we study all these
problems in some related classes of graphs, including H-free graphs for
every H on three vertices, graphs with restricted diamonds, split graphs,
and threshold graphs.

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1   Introduction

Coloring problems are among the most studied problems in algorithmic graph theory. In its classical form, the k-coloring problem asks if there is an assignment of k colors to the vertices of a graph in such a way that no edge is monochromatic. Many generalizations and variations of the classical coloring problem have been defined over the years. One of such variations is the clique k-coloring problem, in which the vertices are colored so that no maximal clique is monochromatic. In this article we study the star and biclique coloring problems, which are variations of the coloring problem similar to clique colorings. A biclique is a set of vertices that induce a complete bipartite graph $K_{n,m}$, while a star is a biclique inducing the graph $K_{1,m}$. In the star (biclique) k-coloring problem, the goal is to color the vertices with k colors without generating monochromatic maximal stars (bicliques).

The clique coloring problem has been investigated for a long time, and it is still receiving a lot of attention. Recently, the clique k-coloring problem was proved to be Σ_2^p-complete [22] for every $k \geq 2$, and it remains Σ_2^p-complete for $k = 2$ even when the input is restricted to graphs with no odd holes [5]. The problem has been studied on many other classes of input graphs, for which it is was proved to be NP-complete or to require polynomial time (e.g. [2, 3, 4, 11, 16, 19]). Due to the close relation between cliques and bicliques, many problems on cliques have been translated in terms of bicliques (e.g. [1, 25, 27]). However, there are some classical problems on cliques whose biclique versions were not studied until recently [6, 12, 13, 14]. Clique colorings are examples of such problems; research on biclique colorings begun in 2010 in the Master Thesis of one of the authors [26] whose unpublished results are being extended in the present article. It is worth mentioning that, despite its youthfulness, at least two articles on biclique colorings were written: [19] develops a polynomial time algorithm for biclique coloring some unichord-free graphs, and [18] determines the minimum number of colors required by biclique colorings of powers of paths and cycles.

The list coloring problem is a generalization of the coloring problem in which every vertex v is associated with a list $L(v)$, and the goal is to color each vertex v with an element of $L(v)$ in such a way that no edge is monochromatic. Function L is called a list assignment, and it is a k-list assignment when $|L(v)| = k$ for every vertex v. A graph G is said to be k-choosable when it admits an L-coloring with no monochromatic edges, for every k-list assignment L. The choosability problem asks whether a graph is k-choosable. In the same way as the coloring problem is generalized to the clique (star, biclique) coloring problem, the choosability problem is generalized to the clique (star, biclique) choosability problem. That is, a graph G is clique (star, biclique) k-choosable when it admits an L-coloring generating no monochromatic maximal cliques (star, bicliques), for every k-list assignment L. The choosability problems seem harder than their coloring versions, because a universal quantifier on the list must be checked. This difficulty is reflected for the k-choosability and clique k-choosability problems in the facts that the former is Π_2^p-complete for every

$k \geq 3$ [15], whereas the latter is Π_3^p-complete for every $k \geq 2$ [22]. In [23] it is proven that every planar graph is clique 4-choosable. However, contrary to what happens with the clique coloring problem, there are not so many results regarding the complexity of the clique choosability problem for restricted classes of graphs.

In this paper we consider the star and biclique coloring and choosability problems, both for general graphs and for some restricted classes of graphs. The star and biclique coloring and choosability problems are defined in Section 2, where we introduce the terminology that will be used throughout the article. In Section 3, we prove that the star k-coloring problem is Σ_2^p-complete for $k \geq 2$, and that it remains Σ_2^p-complete even when its input is restricted to $\{K_{2,2}, K_{k+2}\}$-free graphs. Clearly, every maximal biclique of a $K_{2,2}$-free graph is a star. Thus, we obtain as a corollary that the biclique k-coloring problem on $\{K_{2,2}, K_{k+2}\}$-free graphs is Σ_2^p-complete as well. The completeness proof follows some of the ideas by Marx [22] for the clique coloring problem. In Section 4 we show that the star k-choosability problem is Π_3^p-complete for $k \geq 2$, and that it remains Π_3^p-complete for $\{K_{2,2}, K_{k+2}\}$-free graphs. Again, the Π_3^p-completeness of the biclique k-coloring problem on $\{K_{2,2}, K_{k+2}\}$-free is obtained as a corollary. As in [22], we require a structure to force a color on a vertex. The remaining sections of the article study the star and biclique coloring problems on graphs with restricted inputs. These graphs are related to the graph G that is generated to prove the Σ_2^p-completeness of the star coloring problem in Section 3. The aim is to understand what structural properties can help to make the problem simpler. In Section 5, we discuss the star and biclique coloring and choosability problems on K_3-free, P_3-free, $\overline{P_3}$-free, and $\overline{K_3}$-free graphs. For K_3, P_3 and $\overline{P_3}$, the star coloring and star choosability problems are almost trivial and can be solved in linear time. On the other hand, both problems are as hard as they can be for $\overline{K_3}$-free graphs, even when the input is a co-bipartite graph. In Section 6 we prove that the star coloring problem is NP-complete for diamond-free graphs and that the star choosability problem is Π_2^p-complete for a superclass of diamond-free graphs. If no induced $K_{i,i}$ is allowed for a fixed i, then the biclique coloring and the biclique choosability problems are also NP-complete and Π_2^p-complete. In Section 7, the star coloring and the star choosability problems on split graphs are proved to be NP-complete and Π_2^p-complete, respectively. Finally, Section 8 shows that the star coloring and the star choosability problems are equivalent for threshold graphs, and both can be solved in linear time. Table 1 sums up the results obtained in the article for the star coloring and star choosability problems.

2   Preliminaries

In this paper we work with simple graphs. The vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$, respectively. Write vw to denote the edge of G formed by vertices $v, w \in V(G)$. For the sake of simplicity, $E(G)$ is also considered as the family of subsets of $V(G)$ containing the set $\{v, w\}$ for each

Graph class	star k-coloring	star k-choosability
$\{K_{2,2}, K_{k+2}\}$-free	Σ_2^p-complete	Π_3^p-complete
K_3-free, P_3-free, $\overline{P_3}$-free	$O(n+m)$	$O(n+m)$
$\overline{K_3}$-free	NP-complete	Π_2^p-complete
co-bipartite		Π_2^p-complete
$\{W_4, \text{gem, dart}\}$-free	NP-complete	Π_2^p-complete
diamond-free	NP-complete	Π_2^p
$\overline{C_4}$-free	NP-complete	Π_2^p-complete
split	NP-complete	Π_2^p-complete
threshold	$O(n+m)$	$O(n+m)$

Table 1: Complexity results obtained in this article.

$vw \in E(G)$. For $v \in V(G)$, the *neighborhood* of v is the set $N_G(v)$ of vertices adjacent to v, while the *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$. A vertex w *dominates* v, and v is *dominated* by w, when $N_G[v] \subseteq N_G[w]$, while w *false dominates* v, and v is *false dominated* by w, when $N_G(v) \subseteq N_G(w)$. If $N_G[v] = N_G[w]$, then v and w are *twins*, and if $N_G(v) = N_G(w)$, then v and w are *false twins*. The *degree* of v is $d_G(v) = |N_G(v)|$. A vertex is an *isolated vertex*, a *leaf*, and a *universal vertex* when $d_G(v)$ equals 0, 1, and $|V(G)| - 1$, respectively. We omit the subscripts from N and d when no ambiguities arise.

The *complement* of G is the graph $\overline{G}$ where $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{vw \mid vw \notin E(G)\}$. For a graph H, the *union* of G and H is the graph $G \cup H$ where $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. Write $G = H$ to indicate that G and H are isomorphic. The *n-cycle* graph $(n \geq 3)$, denoted by C_n, is the connected graph that has n vertices of degree 2. The *n-path* graph, denoted by P_n, is the graph obtained from C_n by removing an edge. The *n-wheel* graph $(n \geq 3)$, denoted by W_n, is the graph obtained from C_n by inserting a universal vertex. The *diamond*, *gem*, and *dart* are the graphs shown in Figure 1. The *n-complete* graph, denoted by K_n, is the graph formed by n pairwise adjacent vertices. An *independent set* is a subset of $V(G)$ formed by pairwise non-adjacent vertices. Graph G is *bipartite* when $V(G)$ can be partitioned into two independent sets S and T. In this case, the unordered pair ST is called a *bipartition* of G. The *(n, m)-complete bipartite* graph $(n \geq 1, m \geq 1)$, denoted by $K_{n,m}$, is the graph isomorphic to $\overline{K_n \cup K_m}$. Note that $K_{n,m}$ is a bipartite graph. The graph $K_{1,n}$ is also called the *n-star* graph. The universal vertices of $K_{1,n}$ are referred to as the *centers* of $K_{1,n}$, while $K_{1,n}$ is said to be *centered* at v. Note that $K_{1,1}$ has two centers.

Let $W \subseteq V(G)$. We write $G[W]$ to denote the subgraph of G induced by W, and $G \setminus W$ to denote $G[V(G) \setminus W]$. Set W is said to be a *clique*, *biclique*, and *star* when $G[W]$ is isomorphic to a complete, bipartite complete, and star graph, respectively. For the sake of simplicity, we use the terms *clique* and *biclique* to refer to $G[W]$ as well. Moreover, we may refer to ST as a biclique or star when $G[S \cup T]$ is a bipartite complete or star graph with bipartition ST. The family of maximal cliques, maximal bicliques, and maximal stars are denoted by $\mathcal{C}(G)$,
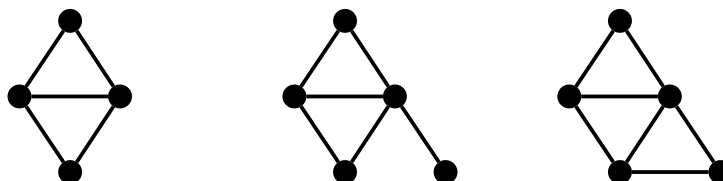
Figure 1: The diamond, dart, and gem graphs are shown from left to right.

$\mathcal{B}(G)$, and $\mathcal{S}(G)$, respectively.

A sequence of distinct vertices $P = v_1, \ldots, v_n$ is a *path* of *length* $n - 1$ when v_i is adjacent to v_{i+1}. If in addition v_n is adjacent to v_1, then P is a *cycle* of *length* n. A *tree* is a connected graph that contains no cycles. A *rooted tree* is a tree T with a fixed vertex $r \in V(T)$ called the *root* of T. The *parent* of v in T is the neighbor of v in its path to r. A path (resp. cycle) of G is *chordless* when $G[P] = P_n$ (resp. $G[P] = C_n$). A *hole* is a chordless cycle of length at least 4. A graph G is said to be *H-free*, for some graph H, when no induced subgraph of G is isomorphic to H. Similarly, G is $\mathcal{F}$-free, for any family of graph $\mathcal{F}$, when G is H-free for every $H \in \mathcal{F}$. A graph is *chordal* when it is $\{C_n\}_{n \geq 4}$-free, i.e., chordal graphs have no holes.

A *coloring* of G is a function ρ that maps each vertex $v \in V(G)$ to a color $\rho(v) \in \mathbb{N}$. When $\rho(v) \leq k$ for every $v \in V(G)$, ρ is called a *k-coloring*. We define $\rho(W) = \{\rho(v) \mid v \in W\}$ for any $W \subseteq V(G)$. Set W is said to be ρ-*monochromatic* when $|\rho(W)| = 1$. When there is no ambiguity, we say that W is *monochromatic* instead of ρ-monochromatic. For a family $\mathcal{F}$ of subsets of $V(G)$, we say that ρ is a *proper coloring* of $\mathcal{F}$ if no $W \in \mathcal{F}$ is monochromatic. Four kinds of families are considered in this article. A coloring ρ is a *vertex*, *clique*, *biclique*, and *star coloring* when ρ is a proper coloring of $E(G)$, $\mathcal{C}(G)$, $\mathcal{B}(G)$, and $\mathcal{S}(G)$, respectively. The problems of finding a proper coloring for these families are defined as follows.

VERTEX (RESP. CLIQUE, BICLIQUE, STAR) k-COLORING

INPUT: A connected graph G and a value $k \in \mathbb{N}$.
QUESTION: Is there a vertex (resp. clique, biclique, star) k-coloring of G?

List colorings are a generalization of colorings. A *list assignment* of G is a function that maps each vertex $v \in V(G)$ to a set $L(v) \subseteq \mathbb{N}$. When $|L(v)| = k$ for every $v \in V(G)$, L is called a *k-list assignment*. An *L-coloring* of G is a coloring ρ such that $\rho(v) \in L(v)$ for every $v \in V(G)$. Define $L(W) = \bigcup \{L(v) \mid v \in W\}$ for any $W \subseteq V(G)$. Given a family $\mathcal{F}$ of subset of $V(G)$ and a number $k \in \mathbb{N}$, graph G is said to be *k-choosable* with respect to $\mathcal{F}$ when there exists a proper

L-coloring of $\mathcal{F}$ for every k-list assignment L of G. Graph G is *vertex*, *clique*, *biclique*, and *star k-choosable* when G is k-choosable with respect to $E(G)$, $\mathcal{C}(G)$, $\mathcal{B}(G)$, and $\mathcal{S}(G)$, respectively. The problem of determining if G is k-choosable is defined as follows.

VERTEX (RESP. CLIQUE, BICLIQUE, STAR) k-CHOOSABILITY

INPUT: A connected graph G and a value $k \in \mathbb{N}$.
QUESTION: Is G vertex (resp. clique, biclique, star) k-choosable?

The vertex (resp. clique, biclique, star) *chromatic number*, denoted by $\chi(G)$ (resp. $\chi_C(G)$, $\chi_B(G)$, and $\chi_S(G)$), is the minimum $k \in \mathbb{N}$ such that G admits a vertex (resp. clique, biclique, star coloring) k-coloring. Similarly, the vertex (clique, biclique, star) *choice number*, denoted by $ch(G)$ (resp. $ch_C(G)$, $ch_B(G)$, $ch_S(G)$) is the minimum number $k \in \mathbb{N}$ such that G is vertex (resp. clique, biclique, star coloring) k-choosable. By definition, $\chi(G) \leq ch(G)$ and $\chi_*(G) \leq ch_*(G)$ for $* \in \{C, B, S\}$.

For a function f with domain D, the *restriction* of f to $D' \subseteq D$ is the function f' with domain D' where $f'(x) = f(x)$ for $x \in D'$. In such case, f is said to be an *extension* of f' to D. A *leafed vertex* is a vertex adjacent to a leaf. For the sake of simplicity, whenever we state that G *contains a leafed vertex* v, we mean that G contains v and a leaf adjacent to v. It is well known that $\chi(G) \geq \chi(H)$ for every induced subgraph H of G. Such a property is false for clique, biclique, and star colorings. In particular, for any graph H, it is possible to build a graph G such that $ch_*(G) = 2$ and G contains H as an induced subgraph. For $* = C$, graph G is built from H by iteratively inserting a twin of each vertex of H. Similarly, G is obtained by inserting false twins for $* = B$, while, by the next observation, G is obtained by inserting a leaf adjacent to each vertex for $* = S$.

Observation 1 *Let G be a graph with a list assignment L, v be a leafed vertex of G, and l be a leaf adjacent to v. Then, any L-coloring of $G \setminus l$ can be extended into an L-coloring of G in such a way that there is no monochromatic maximal star with center in v.*

A *block* is a maximal set of twin vertices. If v and w are twin vertices, then $\{v\}\{w\}$ is both a maximal star and a maximal biclique, and thus v and w have different colors in any star or biclique L-coloring ρ. Consequently, $|\rho(W)| = W$ for any block W of G. We record this fact in the following observation.

Observation 2 *Let G be a graph with a list assignment L, and v, w be twin vertices. Then, $\rho(v) \neq \rho(w)$ for any star or biclique L-coloring ρ of G.*

3   Complexity of star and biclique coloring

In this section we establish the hardness of the star and biclique problems by showing that both problems are Σ_2^p-complete. The main result of this section is that STAR k-COLORING is Σ_2^p-complete for every $k \geq 2$, even when its input is restricted to $\{C_4, K_{k+2}\}$-free graphs. Since all the bicliques of a C_4-free graph are stars, this immediately implies that BICLIQUE k-COLORING is also Σ_2^p-complete for $\{C_4, K_{k+2}\}$-free graphs. The hardness results is obtained by reducing instances of the QSAT$_2$ problem. The QSAT$_h$ problem is known to be Σ_h^p-complete for every h [24], and is defined as follows.

Quantified 3-satisfiability with h alternations (QSAT$_h$)

INPUT: A formula $\phi(\vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2, \ldots, \vec{\mathbf{x}}_h)$ that is in 3-CNF if h is odd, while it is in 3-DNF if h is even.

QUESTION: Is $(\exists \vec{\mathbf{x}}_1)(\forall \vec{\mathbf{x}}_2)(\exists \vec{\mathbf{x}}_3) \ldots (Q_h \vec{\mathbf{x}}_h)\phi(\vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2, \vec{\mathbf{x}}_3, \ldots, \vec{\mathbf{x}}_h)$ (for $Q_h \in \{\exists, \forall\}$) true?

Recall that ϕ is in 3-CNF if it is a conjunction of clauses where each clause is a disjunction with three literals. Similarly, ϕ is in 3-DNF when it is a disjunction of clauses, each clause being a conjunction with three literals.

3.1   Keepers, switchers, and clusters

In this section we introduce the keeper, switcher, and cluster connections that are required for the reductions. The keeper connections are used to force the same color on a pair of vertices, in any star coloring. Conversely, the switcher connections force some vertices to have different colors. Finally, the cluster connections are used to represent the variables of a DNF formula. We begin defining the keeper connections.

Definition 1 [k-keeper] Let G be a graph and $v, w \in V(G)$. Say that $K \subset V(G)$ is a *k-keeper connecting v, w* ($k \geq 2$) when K can be partitioned into a clique $D = \{d_1, \ldots, d_{k-1}\}$ and $k - 1$ cliques $C_1, \ldots, C_{k-1}$ with k vertices each in such a way that $D \cup \{v, w\}$ and $C_i \cup \{d_i\}$ are cliques for $i \in \{1, \ldots, k-1\}$, and there are no more edges incident to vertices in K.

Figure 2 (a) shows a k-keeper connecting two vertices v and w. The main properties of k-keepers are summarized in the following lemma.

Lemma 1 *Let G be a graph and K be a k-keeper connecting $v, w \in V(G)$ ($k \geq 2$). Then,*

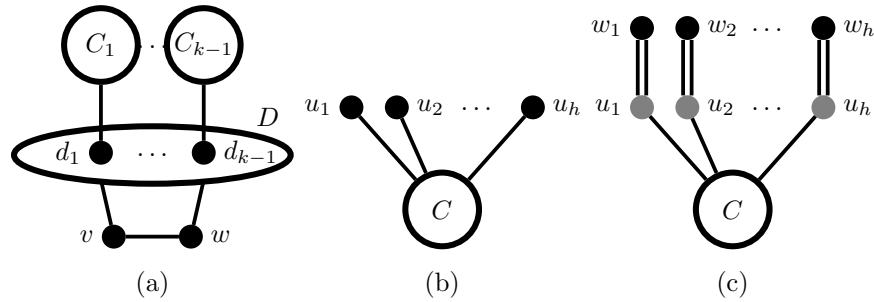 *(i) no induced hole or K_{k+2} contains a vertex in K,*

Figure 2:   (a) A k-keeper connecting v, w.   (b) A k-switcher connecting $\{u_1, \ldots, u_h\}$. (c) A long k-switcher connecting $\{w_1, \ldots, w_h\}$. In every figure, gray nodes represent leafed vertices, circular shapes represent cliques, and double lines represent a k-keeper connecting two vertices.

(ii) v and w have the same color in any star k-coloring of G, and

(iii) Any k-coloring ρ of $G \setminus K$ in which $\rho(v) = \rho(w)$ can be extended into a k-coloring of G in such a way that no monochromatic maximal star contains a vertex in K.

Proof: Let $C_1, \ldots, C_{k-1}$ and $D = \{d_1, \ldots, d_{k-1}\}$ be as in Definition 1. (i) is trivial.

(ii) Let ρ be a star k-coloring of G and fix $i, j \in \{1, \ldots, k-1\}$. Since C_i is a block of size k, it contains vertices c_i, c_j, c_v with colors $\rho(d_i), \rho(d_j)$, and $\rho(v)$, respectively, by Observation 2. Hence, $\rho(d_i) \neq \rho(v)$ since otherwise $\{d_i\}\{v, c_v\}$ would be a monochromatic maximal star, and, similarly, $\rho(d_i) \neq \rho(d_j)$ because $\{d_i\}\{d_j, c_j\}$ is not monochromatic. In other words, $|\rho(D)| = k-1$ and $\rho(v) \notin \rho(D)$. Replacing v and w in the reasoning above, we can conclude that $\rho(w) \notin \rho(D)$ as well. Therefore, $\rho(v) = \rho(w)$.

(iii) To extend ρ, set $\rho(D) = \{1, \ldots, k\} \setminus \{\rho(v)\}$, and $\rho(C_i) = \{1, \ldots, k\}$ for every $i \in \{1, \ldots, k-1\}$. It is not hard to see that no maximal star with a vertex in K is monochromatic.   □

The *switcher connection*, whose purpose is to force a set of vertices to have at least two colors, is now defined (see Figure 2 (b)).

Definition 2 [k-switcher] Let G be a graph and $U = \{u_1, \ldots, u_h\}$ be an independent set of G with $h \geq 2$. Say that $C \subset V(G)$ is a *k-switcher connecting U* ($k \geq 2$) when $|C| = k$, $C \cup \{u_i\}$ is a clique for $i \in \{1, \ldots, h\}$, and there are no more edges incident to vertices in C.

The following result is the analogous of Lemma 1 for switchers.

Lemma 2 *Let G be a graph and C be a k-switcher connecting $U \subset V(G)$ ($k \geq 2$). Then,*

(i) $|\rho(U)| \geq 2$ *for any star k-coloring ρ of G, and*

(ii) *Any k-coloring ρ of $G \setminus S$ in which $|\rho(U)| \geq 2$ can be extended into a k-coloring of G in such a way that no monochromatic maximal star has its center in C.*

As defined, switchers are not useful for proving the hardness of the star coloring problem for C_4-free graphs. The reason is that their connected vertices must have no common neighbors to avoid induced C_4's, and our proof requires vertices with different colors and common neighbors. To solve this problem we extend switchers into long switchers by combining them with keepers (see Figure 2 (c)). We emphasize that the set of vertices connected by a long switcher need not be an independent set.

Definition 3 [long k-switcher] Let G be a graph, and $W = \{w_1, \ldots, w_h\}$ be a set of vertices of G with $h \geq 2$. Say that $S \subset V(G)$ is a *long k-switcher connecting* W ($k \geq 2$) when S can be partitioned into an independent set of leafed vertices $U = \{u_1, \ldots, u_h\}$, a k-switcher C, and k-keepers $Q_1, \ldots, Q_h$ in such a way that C connects U, Q_i connects w_i, u_i for $i \in \{1, \ldots, h\}$, and there are no more edges adjacent to vertices in U.

The analogous of Lemma 2 follows; its proof is a direct consequence of Observation 1 and Lemmas 1 and 2.

Lemma 3 *Let G be a graph and S be a long k-switcher connecting $W \subset V(G)$ ($k \geq 2$). Then,*

(i) *no induced C_4 or K_{k+2} of G contains a vertex of S,*

(ii) $|\rho(W)| \geq 2$ *for any star k-coloring ρ of G, and*

(iii) *Any k-coloring ρ of $G \setminus S$ in which $|\rho(W)| \geq 2$ can be extended into a k-coloring of G in such a way that no monochromatic maximal star has its center in S.*

The last type of connection that we require is the cluster, which is used to represent variables of DNF formulas. As switchers, clusters are used to connect several vertices at the same time. Specifically, clusters require two sets X, $-X$, and a vertex s. The purpose of the connection is to encode all the valuations of the variables in a formula ϕ by using monochromatic stars with center in s. Thus, if a variable $\mathbf{x}$ is being represented by X and $-X$, then each monochromatic maximal star with center in s contains all the vertices in X and none of $-X$, when $\mathbf{x}$ is true, or it contains all the vertices of $-X$ and none of X, when $\mathbf{x}$ is false.

Definition 4 [ℓ-cluster] Let G be a graph with vertices s, $X = \{x_1, \ldots, x_\ell\}$, and $-X = \{-x_1, \ldots, -x_\ell\}$ ($\ell \geq 2$). Say that $K \subseteq V(G)$ is an *ℓ-cluster connecting* $\langle s, X, -X \rangle$ when K has ℓ leafed vertices $v_1, \ldots, v_\ell$, vertex s is adjacent to all the vertices in $X \cup -X \cup K$, sequence $x_1, -x_1, v_1, \ldots, x_\ell, -x_\ell, v_\ell$ is a hole, and there are no more edges incident to vertices in K. For $i \in \{1, \ldots, \ell\}$, we write $X[i]$ and $-X[i]$ to refer to x_i and $-x_i$, respectively.

Note that if K is an ℓ-cluster connecting $\langle s, X, -X \rangle$, then the subgraph induced by $K \cup X \cup -X \cup \{s\}$ is isomorphic to a 3ℓ-wheel that has s as its universal vertex. As mentioned, the main property of clusters is that its members can be colored in such a way that monochromatic stars represent valuations.

Lemma 4 *Let G be a graph and K be an ℓ-cluster connecting $\langle s, X, -X \rangle$ for $\{s\} \cup X \cup -X \subseteq V(G)$. Then,*

(i) *if $N(x) \cap N(x') \subseteq K \cup \{s\}$ for every $x, x' \in X \cup -X$, then no induced K_4 or C_4 of G contains a vertex in K,*

(ii) *if $\{s\}S$ is a maximal star of G with $S \cap K = \emptyset$, then $S \cap (X \cup -X)$ equals either X or $-X$, and*

(iii) *any k-coloring ρ of $G \setminus K$ ($k \geq 2$) can be extended into a k-coloring of G in such a way that $\rho(s) \notin \rho(K)$, and no monochromatic maximal star of G has its center in K.*

Proof: Let $K = \{v_1, \ldots, v_\ell\}$, $x_1, \ldots, x_\ell$, and $-x_1, \ldots, -x_\ell$ be as in Definition 4.

(i) For $i \in \{1, \ldots, \ell\}$, the non-leaf neighbors of v_i are $s, -x_i, x_{i+1}$. Since $-x_i$ and x_{i+1} are not adjacent and $N(-x_i) \cap N(x_{i+1}) = \{v_i, s\}$, it follows that v_i belongs to no induced K_4 or C_4 of G.

(ii) Suppose $S \cap K = \emptyset$. If $x_i \in S$ for some $i \in \{1, \ldots, \ell\}$, then $-x_i \notin S$ because $-x_i \in N(x_i)$. Then, since $v_i \notin S$, it follows that x_{i+1} belongs to S. Consequently, by induction on i, we obtain that $S \cap X$ equals either X or $\emptyset$. In the former case, $S \cap -X = \emptyset$ because every vertex in $-X$ has a neighbor in X. In the latter case, $S \cap -X = -X$ because $S \cap K = \emptyset$ and at least one of $\{x_{i+1}, -x_i, v_i\}$ belongs to S for every $i \in \{1, \ldots, \ell\}$.

(iii) Just extend ρ so that $\rho(s) \notin \rho(K)$, and color the leaves according to Observation 1.   □

3.2   Hardness of the star-coloring problem

It is well known that a problem P is Σ_2^p when the problem of authenticating a positive certificate of P is coNP [24]. For the star k-coloring problem, a k-coloring of G can be taken as a positive certificate. Since it is coNP to authenticate that a k-coloring of G is indeed a star coloring, we obtain that STAR k-COLORING is Σ_2^p. The following theorem states the Σ_2^p-hardness of the problem.

Theorem 1 STAR k-COLORING *is Σ_2^p-complete and it remains Σ_2^p-complete even when its input is restricted to $\{C_4, K_{k+2}\}$-free graphs.*

Proof: We already know that STAR k-COLORING belongs to Σ_2^p. To prove its hardness, we show a polynomial time reduction from QSAT$_2$. That is, for any 3-DNF formula $\phi(\vec{x}, \vec{y})$ with $\ell \geq 2$ clauses $\mathbf{P}_1, \ldots, \mathbf{P}_\ell$, and $n + m$ variables $\vec{x} = \mathbf{x}_1, \ldots, \mathbf{x}_n$, $\vec{y} = \mathbf{y}_1, \ldots, \mathbf{y}_m$, we build a graph G that admits a star k-coloring if and only if $(\exists \vec{x})(\forall \vec{y})\phi(\vec{x}, \vec{y})$ is true. For the sake of simplicity, in

this proof we use i, j, h, and q as indices that refer to values in $\{1, \ldots, n\}$, $\{1, \ldots, m\}$, $\{1, \ldots, \ell\}$, and $\{1, \ldots, k\}$.

Graph G can be divided into connection, inner, and leaf vertices. *Connection* vertices are in turn divided into a set of *clause* vertices $P = \{p_1, \ldots, p_\ell\}$, a set of *x-vertices* $X = \{x_1, \ldots, x_n\}$, two sets of *x-literal* vertices X_i and $-X_i$ with ℓ vertices each (for each i), two sets of *y-literal* vertices Y_j and $-Y_j$ with ℓ vertices each (for each j), a set of *color* vertices $C = \{c_1, \ldots, c_k\}$, and two special vertices s and t. Let $L_X = \bigcup_i (X_i \cup -X_i)$, and $L_Y = \bigcup_j^m (Y_j \cup -Y_j)$. *Inner* vertices are those vertices included in switchers, keepers, and clusters of G. Inner vertices and the edges between the connection vertices are given by the next rules.

Edges: s is adjacent to all the vertices in P, and if $\mathbf{x}_i$ (resp. $\overline{\mathbf{x}_i}$, $\mathbf{y}_j$, $\overline{\mathbf{y}_j}$) is a literal of $\mathbf{P}_h$, then p_h is adjacent to $-X_i[h]$ (resp. $X_i[h]$, $-Y_j[h]$, $Y_j[h]$).

Keepers: there is a k-keeper connecting s with t.

Long switchers: exist long k-switchers connecting $\{x_i, -X_i[h]\}$ and $\{X_i[h], -X_i[h]\}$ (for every i, h), $\{c_1, p_h\}$ (for every h), $\{c_2, y\}$ for every $y \in L_Y$, $\{c_q, w\}$ for every $q > 2$ and every connection vertex $w \neq s$, $\{c_1, c_2\}$, and $\{c_2, t\}$.

Variables: there are ℓ-clusters connecting $\langle s, X_i, -X_i \rangle$ and $\langle s, Y_j, -Y_j \rangle$.

Finally, each connection vertex other than s is leafed. This ends up the construction of G, which can be easily computed from $\phi(\vec{\mathbf{x}}, \vec{\mathbf{y}})$ in polynomial time. Figure 3 depicts a schema of the graph.

Before dealing with the star k-coloring problem on G, we show that G is $\{C_4, K_{k+2}\}$-free. By statement (i) of Lemmas 1, 3 and 4, it suffices to prove that the subgraph H induced by the connection vertices is $\{K_{k+2}, C_4\}$-free. For this, observe that any induced C_4 or K_{k+2} must contain a vertex of $L_X \cup L_Y$ because $\{s\}P$ is an induced star of H and x_i, c_q, and t have degree at most 1 in H. Now, $X_i[h]$ has at most three neighbors in H, namely $-X_i[h]$, s, and maybe p_h. Hence, since $N(-X_i[h]) = \{s, X_i[h]\}$ when $X_i[h]$ is adjacent to p_h, we obtain that $X_i[h]$ belongs to no induced K_{k+2} nor C_4. A similar analysis is enough to conclude no vertex of $L_X \cup L_Y$ belongs to an induced K_{k+2} nor C_4, thus H is $\{C_4, K_{k+2}\}$-free.

Now we complete the proof by showing that $(\exists \vec{\mathbf{x}})(\forall \vec{\mathbf{y}})\phi(\vec{\mathbf{x}}, \vec{\mathbf{y}})$ is true if and only if G admits a star k-coloring. Suppose first that $(\forall \vec{\mathbf{y}})\phi(\vec{\mathbf{x}}, \vec{\mathbf{y}})$ is true for some valuation $\nu : \vec{\mathbf{x}} \to \{0, 1\}$, and define ρ as the k-coloring of G that is obtained by the following two steps. First, set $\rho(c_q) = q$, $\rho(X_i) = \rho(x_i) = 2 - \nu(\mathbf{x}_i)$, $\rho(-X_i) = 1 + \nu(\mathbf{x}_i)$, $\rho(p_k) = 2$, and $\rho(L_Y) = \rho(s) = \rho(t) = 1$. Next, iteratively set ρ for the leaves and inner vertices according to Observation 1, and statement (iii) of Lemmas 1, 3, and 4. Observe that the second step is well defined because every pair of vertices connected by k-keepers have the same color, while every pair of vertices connected by long k-switchers have different colors.
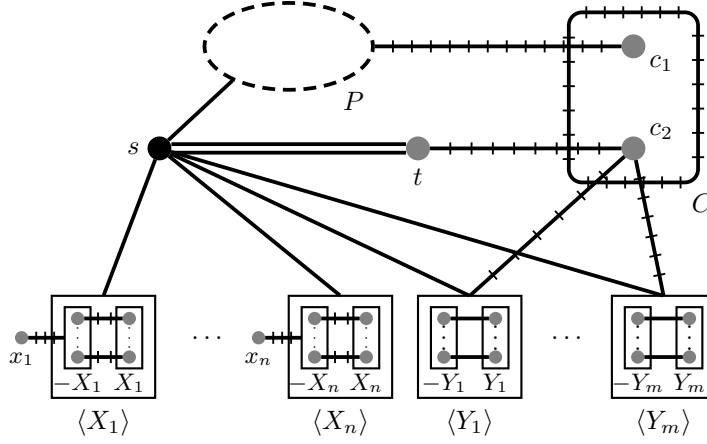
Figure 3: Schema of the graph obtained from ϕ in Theorem 1. For the sake of simplicity, we omit $c_3, \ldots, c_k$ and the edges from P to $L_X \cup L_Y$. Circular shapes drawn with dashes represent independent sets; marked edges between two vertices represent k-switchers connecting them; circular shapes with marks represent sets of vertices pairwise connected by k-switchers; and squares represent sets of vertices.

We claim that ρ is a star k-coloring of G. To see why, consider a maximal star $\{w\}S$ of G and observe that w cannot be a leaf. If $w \neq s$, then $\{w\}S$ is not monochromatic by Observation 1 and Lemmas 1, 3 and 4. Suppose, then, that $w = s$ and, moreover, that $S \setminus P$ is monochromatic. Then, no ℓ-cluster intersects S by statement (iii) of Lemma 4. Consequently, by statement (ii) of Lemma 4, $S \cap (X_i \cup -X_i)$ equals either X_i or $-X_i$, while $S \cap (Y_j \cup -Y_j)$ equals either Y_j or $-Y_j$ for every i and every j. Extend ν to include $\vec{\mathbf{y}}$ in its domain, so that $\nu(\mathbf{y}_j) = 1$ if and only if $Y_j \subseteq S$. By hypothesis, $\nu(\phi(\vec{\mathbf{x}}, \vec{\mathbf{y}})) = 1$, thus there is some clause $\mathbf{P}_h$ whose literals are all true according to ν. If p_h has some neighbor in $-X_i$, then $\nu(\mathbf{x}_i) = 1$, thus $-X_i \not\subset S$ because $\rho(-X_i) = 2$. Similarly, if p_h has some neighbor in X_i (resp. $-Y_j$, Y_j), then $X_i \not\subset S$ (resp. $-Y_j \not\subset S$, $Y_j \not\subset S$). Therefore, since P is an independent set and $S \subset \{t\} \cup L_X \cup L_Y \cup P$, we obtain that $p_k \in S$, thus $\{s\}S$ is not monochromatic.

For the converse, let ρ be a star k-coloring of G. Since there is are long k-switchers connecting $\{c_1, c_2\}$ and $\{c_q, w\}$ for every $q > 2$ and every connection vertex $w \neq s$, we obtain that $|\rho(C)| = k$ and $\rho(w) \in \{\rho(c_1), \rho(c_2)\}$ by statement (ii) of Lemma 3. Define $\nu \colon \vec{\mathbf{x}} + \vec{\mathbf{y}} \to \{0, 1\}$ as any valuation in which $\nu(\mathbf{x}_i) = 1$ if and only if $\rho(x_i) = \rho(c_1)$. Since $\nu(\mathbf{y}_j)$ can take any value from $\{0, 1\}$, it is enough to prove that $\nu(\phi(\vec{\mathbf{x}}, \vec{\mathbf{y}})) = 1$. Let $V_X = \bigcup_i((X_i \mid \nu(\mathbf{x}_i) = 1) \cup (-X_i \mid \nu(\mathbf{x}_i) = 0))$ and $V_Y = \bigcup_j((Y_j \mid \nu(\mathbf{y}_j) = 1) \cup (-Y_j \mid \nu(\mathbf{y}_j) = 0))$.

Let $S = V_X \cup V_Y \cup \{t\}$. By construction, S is an independent set, thus $\{s\}S$ is a star of G. Recall that there are long k-switchers connecting $\{x_i, -X_i[h]\}$ and

$\{X_i[h], -X_i[h]\}$. Hence $\rho(x_i) = \rho(X_i)$ and $\rho(-X_i) \neq \rho(x_i)$ by statement (ii) of Lemma 3. This implies that $\rho(V_X) = \{1\}$. Similarly, there are long k-switchers connecting $\{c_2, y\}$ for each $y \in L_Y$, thus $\rho(V_Y) \subset \rho(L_Y) = 1$ as well. Finally, using Lemma 1, we obtain that $\rho(s) = \rho(t) = 1$ because there is a long k-switcher connecting $\{c_2, t\}$ and a k-keeper connecting t, s. So, by hypothesis, $\{s\}S$ is not a maximal star, which implies that $S \cup \{w\}$ is also an independent set for some $w \in N(s)$.

Since t and s have the same neighbors in the k-keeper K connecting them, it follows that $w \notin K$. Similarly, all the vertices in a cluster are adjacent to at least one vertex of $V_X \cup V_Y$. Finally, each vertex of $L_X \cup L_Y$ either belongs or has some neighbor in $V_X \cup V_Y$. Consequently, $w = p_h$, i.e., w represents some clause $\mathbf{P}_h$. If $\mathbf{x}_i$ is a literal of $\mathbf{P}_h$, then p_h is adjacent to $-X_i[h]$. Hence, since $V_X \cup V_Y \cup \{p_h\}$ is an independent set, it follows that $-X_i[h] \notin V_X$. By the way V_X is defined, this means that $\nu(\mathbf{x}_i) = 1$. Similar arguments can be used to conclude that if $\mathbf{l}$ is a literal of $\mathbf{P}_h$, then $\nu(\mathbf{l}) = 1$. That is, $\mathbf{P}_h$ is satisfied by ν, thus $(\exists \vec{\mathbf{x}})(\forall \vec{\mathbf{y}})\phi(\vec{\mathbf{x}}, \vec{\mathbf{y}})$ is true. $\qquad \square$

3.3 Graphs with no short holes and small forbidden subgraphs

Note that every hole H of the graph G defined in Theorem 1 either 1. contains an edge xy for vertices x, y connected by a k-keeper, or 2. contains a path $x, v, -x$ for a vertex v in a cluster K connecting $\langle s, X, -X \rangle$ with $x \in X$ and $-x \in -X$. A slight modification of G can be used in the proof of Theorem 1 so as to enlarge the hole H. In case 1., xy can be subdivided by inserting a vertex z in such a way that x, z and z, y are connected by k-keepers. Similarly, in case 2., dummy vertices not adjacent to any p_h can be inserted into X and $-X$ so as to increase the distance between x and $-x$ in H. Neither of these modifications generates a new hole in G. Thus, in Theorem 1 we can use a iterative modification of G whose induced holes have length at least h. The following corollary is then obtained.

Corollary 1 *For every $h \in O(1)$,* STAR k-COLORING *is Σ_2^p-complete when the input is restricted to K_{k+2}-free graphs whose induced holes have length at least h.*

An interesting open question is what happens when h grows to infinity, i.e., what is the complexity of star k-coloring a chordal graph or a chordal K_{k+2}-free graph. In following sections we consider the star coloring and star choosability problems in two subclasses of chordal graphs, namely split and threshold graphs.

Theorem 1 also shows that the star 2-coloring problem is hard for $\{K_4, C_4\}$-free graphs, which is a class of graphs defined by forbidding two small subgraphs. Thus, another interesting question posed by Theorem 1 is what happens when other small graphs are forbidden, so as to understand what structural properties can simplify the problem. Following sections discuss the coloring problems from this perspective as well. In particular, we study the problem for: every H-free graphs where H has three vertices; a superclass of diamond-free graphs;

and split and threshold graphs. Before dealing with this restricted versions, we establish the complexity of the star k-choosability problem for $\{C_4, K_{k+2}\}$-free graphs.

4   Complexity of the choosability problems

In this section we deal with the list version of the star and biclique-coloring problems. The goal is to show that STAR k-CHOOSABILITY and BICLIQUE k-CHOOSABILITY are Π_3^p-complete problems even when their inputs are restricted to $\{C_4, K_{k+2}\}$-free graphs. Again, only one proof is required because the star and biclique-choosability problems coincide for C_4-free graphs. In This opportunity, however, the proof is by induction on k. That is, we first conclude that the STAR 2-CHOOSABILITY problem is Π_3^p-complete with a polynomial-time reduction from the QSAT$_3$, and next we show that STAR k-CHOOSABILITY can be reduced in polynomial time to the STAR $(k+1)$-CHOOSABILITY. The proof for $k = 2$ is similar to the proof of Theorem 1; however, we did not find an easy way to generalize it for $k > 2$ because long switchers generate graphs that are not star k-choosable.

4.1   Keepers, switchers, clusters and forcers

For the case $k = 2$ we require the keeper and cluster connections once again, and a new version of the long switcher. We begin reviewing the main properties of keepers, switchers, and clusters with respect to the star choosability problem.

Lemma 5 *Let G be a graph, L be a k-list assignment of G, and K be a k-keeper connecting $v, w \in V(G)$ ($k \geq 2$). Then, any L-coloring ρ of $G \backslash (K \cup \{w\})$ can be extended into an L-coloring of G in such a way that no monochromatic maximal star contains a vertex in K.*

Proof: Let $C_1, \ldots, C_{k-1}$ and $D = \{d_1, \ldots, d_{k-1}\}$ be the vertices of K as in Definition 1. Extend ρ into an L-coloring of G such that $|\rho(D) \backslash \{\rho(v)\}| = k-1$, $|\rho(C_i)| = k$ for every $i \in \{1, \ldots, k-1\}$, and $\rho(w) \notin \rho(D)$. Since L is a k-list assignment, such an extension can always be obtained. Furthermore, no monochromatic maximal star has a vertex in K. $\square$

Lemma 6 *Let G be a graph, L be a k-list assignment of G, and C be a k-switcher connecting $U \subset V(G)$ ($k \geq 2$). Then, any L-coloring ρ of $G \setminus C$ in which $|\rho(U)| \geq 2$ can be extended into an L-coloring of G in such a way that no monochromatic maximal star has its center in C.*

By the previous lemma, if C is a k-switcher connecting U, then $G[U \cup C]$ is star k-choosable. This property does not hold for long switchers because it is no longer true that a k-keeper connects vertices of the same color. So, to avoid induced C_4's, we need a new version of the long switcher. This new switcher is defined only for 2-colorings, and it is star 2-choosable as desired. We refer

to this switcher as the *list switcher*. In short, the difference between the long 2-switcher and the list switcher is that the latter has no leafed vertices and 2-keepers are replaced by edges (see Figure 4 (a)). Its definition is as follows.

Definition 5 [list switcher] Let G be a graph, and $W = \{w_1, \ldots, w_h\}$ be a set of vertices of G with $h \geq 2$. Say that $S \subset V(G)$ is a *list switcher connecting* W when S can be partitioned into an independent set $U = \{u_1, \ldots, u_h\}$ and a 2-switcher C in such a way that C connects U, $w_i u_i \in E(G)$ for $i \in \{1, \ldots, h\}$, and there are no more edges adjacent to vertices in U.

The following lemma is equivalent to Lemma 3 for list switchers.

Lemma 7 *Let G be a graph and S be a list switcher connecting $W \subset V(G)$. Then,*

(i) *no induced C_4 or K_4 of G contains a vertex of S,*

(ii) *$|\rho(W)| \geq 2$ for any star k-coloring ρ of G, and*

(iii) *for any 2-list assignment L of G, every L-coloring ρ of $G \setminus S$ in which $|\rho(W)| \geq 2$ can be extended into an L-coloring of G in such a way that no monochromatic maximal star contains a vertex in S.*

Proof: We only prove (iii). Let $w_1, \ldots, w_h$, and $u_1, \ldots, u_h$ be as in Definition 5, and $\{x, y\}$ be the 2-switcher connecting $\{u_1, \ldots, u_h\}$. Suppose, without loss of generality, that $\rho(w_1) \neq \rho(w_2)$, and observe that either $\rho(w_1) \notin L(x)$ or $\rho(w_2) \notin L(y)$ or $L(x) = L(y) = \{\rho(w_1), \rho(w_2)\}$. In this setting, extend ρ to include x and y in such a way that $\rho(x) \neq \rho(y)$, $\rho(x) \neq \rho(w_1)$, and $\rho(y) \neq \rho(w_2)$. Following, extend ρ into an L-coloring of G such that $\rho(u_1) \neq \rho(y)$, $\rho(u_2) \neq \rho(x)$, and $\rho(u_i) \neq \rho(w_i)$ for $i \in \{3, \ldots, h\}$. It is not hard to see that no monochromatic maximal star contains a vertex in S. $\square$

Finally, the proof of statement (iii) of Lemma 4 implies the following lemma.

Lemma 8 *Let G be a graph, L be a 2-list assignment of G, and K be an ℓ-cluster connecting $\langle s, X, -X \rangle$ for $\{s\} \cup X \cup -X \subseteq V(G)$. Then, any L-coloring ρ of $G \setminus K$ can be extended into an L-coloring of G in such a way that $\rho(s) \notin \rho(K)$, and no monochromatic maximal star of G has its center in K.*

Besides the 2-keepers, list switchers, and clusters, we use forth kind of connection that can be used to force the color of a given vertex when an appropriate list assignment is chosen. This connection is called the *forcer* and, contrary to the other connections, it connects only one vertex.

Definition 6 [k-forcer] Let G be a graph and $v \in V(G)$. Say that $F \subset V(G)$ is a *k-forcer connecting* v ($k \geq 2$) when F can be partitioned into sets of leafed vertices A and B, and cliques $C(a, b)$ for $a \in A \cup \{v\}$ and $b \in B$ in such a way that $|A| = k - 1$, $|B| = k^k - 1$, $C(a, b)$ is a k-switcher connecting $\{a, b\}$, and there are no more edges incident to vertices in $A \cup B$.
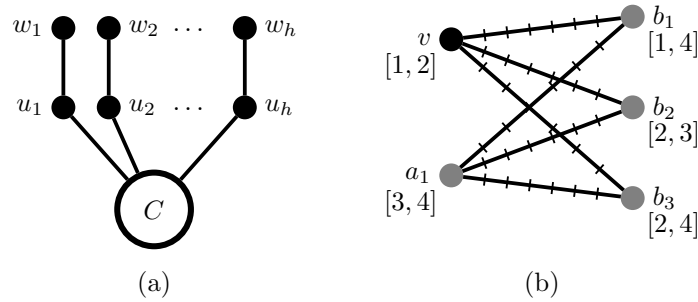
Figure 4: (a) A list switcher connecting $\{u_1, \ldots, u_h\}$. (b) A 2-forcer connecting v with a 2-list assignment L; note that 1 is the unique L-admissible color for v.

Let L be a k-list assignment of G and F be a k-forcer connecting $v \in V(G)$. We say that $c \in L(v)$ is *L-admissible for v* when there is an L-coloring ρ of G in which $\rho(v) = c$ and no monochromatic maximal star has its center in F. Clearly, if c is L-admissible for v, then any L-coloring ρ of $G \setminus F$ in which $\rho(v) = c$ can be extended into an L-coloring of G in such a way that no monochromatic maximal star has its center in F. The main properties of forcers are summed up in the following lemma (see Figure 4 (b)).

Lemma 9 *Let G be a graph and F be a k-forcer connecting $v \in V(G)$ $(k \geq 2)$. Then,*

  (i) *no induced K_{k+2} or C_4 of G contains a vertex in F,*

  (ii) *for every k-list assignment L of G there is an L-admissible color for v, and*

 (iii) *every k-list assignment L of $G \setminus F$ can be extended into a k-list assignment of G in which v has a unique L-admissible color.*

Proof: Let A, B, and $C(a, b)$ be as in Definition 6, and define $A^* = A \cup \{v\}$. Statement (i) follows from the fact that no pair of vertices in $A^* \cup B$ have a common neighbor.

(ii) Let H be the complete bipartite graph with bipartition $\{h_a \mid a \in A^*\}\{h_b \mid b \in B\}$, and M be a k-list assignment of H where $M(h_a) = L(a)$ for every $a \in A^* \cup B$. In [22] it is proven that H admits a vertex M-coloring γ. Define ρ as any L-coloring of G in which $\rho(a) = \gamma(h_a)$ for $a \in A^* \cup B$, where k-switchers and leaves are colored according to Lemma 6 and Observation 1, respectively. The coloring of the k-switchers is possible because γ is a vertex M-coloring. By Observation 1, no vertex in $A \cup B$ is the center of a maximal star, while by Lemma 6, no vertex in $C(a, b)$ is the center of a maximal monochromatic star for $a \in A^*$ and $b \in B$. That is, $\rho(v)$ is L-admissible for v.

(iii) Extend L into a k-list assignment of G such that 1. $L(a) \cap L(a') = \emptyset$ for every pair of vertices $a, a' \in A^*$, 2. $\mathcal{L}(B) = \{L(b) \mid b \in B\}$ is a family of different subsets included in $L(A^*)$ such that $|L(b) \cap L(a)| = 1$ for every $a \in A^*$

and $b \in B$, and 3. $L(C(a,b)) = L(b)$ for every $a \in A^*$, and $b \in B$. Define H and M as in statement (ii). By statement (ii), there is an L-coloring ρ of G that contains no monochromatic maximal star with center in F. Since $C(a,b)$ is a block of G, it follows that $\rho(C(a,b)) = L(b)$ by Observation 2. Then, since no maximal star with center in $C(a,b)$ is monochromatic, it follows that $\rho(a) \neq \rho(b)$ for every $a \in A^*$, $b \in B$. Thus, if γ is the coloring such that $\gamma(h_a) = \rho(a)$ for every $a \in A^* \cup B$, then γ is a vertex M-coloring of H. Consequently, as proven in [22], $\gamma(h_v) = \rho(v)$ is the unique color of $L(v)$ that belongs to the subset of $L(A^*) \notin \mathcal{L}(B)$. $\qquad\square$

4.2 Hardness of the star choosability problem

A problem P is Π_3^p when the problem of authenticating a negative certificate of P is Σ_2^p [24]. For the star k-choosability problem, a k-list assignment of G can be taken as the negative certificate. Using arguments similar to those in Section 3.2 for star k-colorings, it is not hard to see that it is a Σ_2^p problem to authenticate whether a graph G admits no L-colorings for a given k-list assignment L. Therefore, STAR k-CHOOSABILITY is Π_3^p. In this section we establish the hardness of STAR k-CHOOSABILITY. For $k = 2$ we reduce the complement of an instance of QSAT$_3$ into an instance of STAR 2-CHOOSABILITY. Then, we proceed by induction showing how to reduce an instance of STAR k-CHOOSABILITY into an instance of STAR $(k+1)$-CHOOSABILITY for every $k \geq 2$.

The proof for the case $k = 2$ is, in some sense, an extension of Theorem 1. The goal is to force the true literals of z variables to have the same color as s, so that a monochromatic maximal star centered at s appears when the formula is false.

Theorem 2 STAR 2-CHOOSABILITY *is* Π_3^p-*hard, and it remains* Π_3^p-*hard even when its input is restricted to* $\{C_4, K_4\}$-*free graphs.*

Proof: The hardness of STAR 2-CHOOSABILITY is obtained by reducing the complement of QSAT$_3$. That is, given a 3-DNF formula $\phi(\vec{z}, \vec{x}, \vec{y})$ with ℓ clauses $\mathbf{P}_1, \ldots, \mathbf{P}_\ell$, and $n + m + o$ variables $\vec{x} = \mathbf{x}_1, \ldots, \mathbf{x}_n$, $\vec{y} = \mathbf{y}_1, \ldots, \mathbf{y}_m$, $\vec{z} = \mathbf{z}_1, \ldots, \mathbf{z}_o$, we build a graph G that is 2-list-choosable if and only if $(\forall \vec{z})(\exists \vec{x})(\forall \vec{y})$ $\phi(\vec{z}, \vec{x}, \vec{y})$ is true. For the sake of simplicity, in this proof we use i, j, h, and f to refer to values in $\{1, \ldots, n\}$, $\{1, \ldots, m\}$, $\{1, \ldots, \ell\}$, and $\{1, \ldots, o\}$, respectively.

Graph G is similar to the graph in Theorem 1. Its vertex set is again divided into connection, inner, and leaf vertices. In turn, *connection* vertices are divided into a set $P = \{p_1, \ldots, p_\ell\}$, a set $X = \{x_1, \ldots, x_n\}$, sets $X_i, -X_i, Y_j, -Y_j, Z_f$, and $-Z_f$ with ℓ vertices each, and two vertices s, t. Let $L_X = \bigcup_i (X_i \cup (-X_i))$, $L_Y = \bigcup_j (Y_j \cup (-Y_j))$, and $L_Z = \bigcup_f (Z_f \cup (-Z_f))$.

Inner vertices are those vertices included in 2-keepers, list switcher, clusters and 2-forcers. The following rules define inner vertices and the edges between connection vertices. **Edges:** s is adjacent to all the vertices in P, and if $\mathbf{x}_i$ (resp. $\overline{\mathbf{x}_i}$, $\mathbf{y}_j$, $\overline{\mathbf{y}_j}$, $\mathbf{z}_f$, $\overline{\mathbf{z}_f}$) is a literal of $\mathbf{P}_h$, then p_h is adjacent to $-X_i[h]$ (resp. $X_i[h]$, $-Y_j[h]$, $Y_j[h]$, $Z_f[h]$, $-Z_f[h]$). **Keepers:** s and t are connected by a
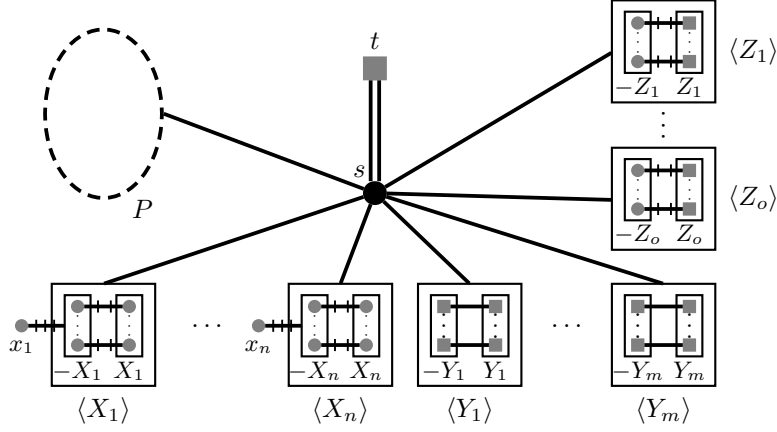
Figure 5: Schema of the graph obtained from ϕ in Theorem 2; for the sake of simplicity, we omit the edges from P to $L_X \cup L_Y \cup L_Z$. Square vertices represent vertices connected to a 2-forcer.

2-keeper. **List switchers:** there are list switchers connecting $\{x_i, -X_i[h]\}$ and $\{X_i[h], -X_i[h]\}$ (for every i, h), $\{Z_f[h], -Z_f[h]\}$ (for every f, h), and $\{p_h, t\}$ (for every h). **Clusters:** there are ℓ-clusters connecting $\langle s, X_i, -X_i \rangle$, $\langle s, Y_j, -Y_j \rangle$, and $\langle s, Z_f, -Z_f \rangle$. **Forcers:** there are 2-forcers connecting each vertex of Z_f (for every f) and each vertex of L_Y.

Finally, every connection vertex other than s is leafed. This ends up the construction of G (see Figure 5), which can be easily computed from $\phi(\vec{\mathbf{z}}, \vec{\mathbf{x}}, \vec{\mathbf{y}})$ in polynomial time. Arguments similar to those in Theorem 1 are enough to conclude that G is $\{C_4, K_4\}$-free.

We now show that $(\forall \vec{\mathbf{z}})(\exists \vec{\mathbf{x}})(\forall \vec{\mathbf{y}})\phi(\vec{\mathbf{z}}, \vec{\mathbf{x}}, \vec{\mathbf{y}})$ is true if and only if G is 2-choosable. We first show that if $(\forall \vec{\mathbf{z}})(\exists \vec{\mathbf{x}})(\forall \vec{\mathbf{y}})\phi(\vec{\mathbf{z}}, \vec{\mathbf{x}}, \vec{\mathbf{y}})$ is true, then G admits a star L-coloring ρ for any 2-list assignment L. The L-coloring ρ is obtained by executing the following algorithm.

Step 1: For every w connected to a forcer, let $\rho(w)$ be L-admissible for w. Such a color always exists by statement (ii) of Lemma 9. Suppose, w.l.o.g., that $\rho(t) = 1$ and let $\nu(\vec{\mathbf{z}})$ be a valuation of $\vec{\mathbf{z}}$ such that $\nu(\mathbf{z}_f) = 1$ if and only if $\rho(Z_f) = \{1\}$.

Step 2: By hypothesis, ν can be extended to include $\vec{\mathbf{x}}$ so that $(\forall \mathbf{y})\nu(\phi(\vec{\mathbf{z}}, \vec{\mathbf{x}}, \vec{\mathbf{y}}))$ is true. If $L(x_i) \neq L(-X_i[h])$ or $L(X_i[h]) \neq L(-X_i[h])$ for some h, then:

Step 2.1: Let $\rho(X_i[h]) \neq \rho(-X_i[h])$ in such a way that $\rho(X_i[h]) = 1$ if and only if $1 \in L(X_i[h])$ and $\nu(\mathbf{x}_i) = 1$, while $\rho(-X_i[h]) = 1$ if and only if $1 \in L(-X_i[h])$ and $\nu(\mathbf{x}_i) = 0$.

Step 2.2: Let $\rho(x_i) \neq \rho(-X_i[h])$.

Step 2.3: Let $\rho(-X_i[k]) \neq \rho(x_i)$ and $\rho(X_i[k]) \neq \rho(-X_i[k])$ for every $k \neq h$.

If $L(x_i) = L(-X_i) = L(X_i)$, then:

Step 2.4: Let $\rho(x_i) = \rho(X_i) = 1$ if and only if $1 \in L(x_i)$ and $\nu(\mathbf{x}_i) = 1$, and $\rho(-X_i) \neq \rho(x_i)$.

Note that, whichever case gets executed, $\rho(x_i) \notin \rho(-X_i)$ and $\rho(-X_i[h]) \neq \rho(X_i[h])$ for every h.

Step 3: Let $\rho(-Z_f[h]) \neq \rho(Z_f[h])$ and $\rho(p_h) \neq 1$.

Step 4: Let ρ for s, leaves, and inner vertices be as in Observation 1, and Lemmas 5, 7, 8 and 9. Observe that this is always possible. In particular, observe that every pair of vertices connected by a list switcher have different colors, while every vertex connected to a forcer has an L-admissible color.

We claim that ρ is a star L-coloring of G. Let $\{w\}S$ be any maximal star of G. By Observation 1 and Lemmas 5, 7, 8 and 9, $\{w\}S$ is not monochromatic when $w \neq s$. Suppose, for the rest of the proof, that $w = s$ and $\{s\}(S \setminus P)$ is monochromatic. By Lemma 5, this implies that $t \in S$, thus $\rho(s) = \rho(t) = 1$. Also, by Lemma 8, S intersects no k-cluster, thus $S \subseteq \{t\} \cup L_X \cup L_Y \cup L_Z \cup P$. Moreover, by statement (ii) of Lemma 4, $S \cap (X_i \cup -X_i)$ equals either X_i or $-X_i$, $S \cap (Y_j \cup -Y_j)$ equals either Y_j or $-Y_j$, and $S \cap (Z_f \cup -Z_f)$ is either Z_f or $-Z_f$, for every i, j, and f. Extend ν to $\vec{y}$ so that $\nu(\mathbf{y}_j) = 1$ if and only if $Y_j \subset S$. By hypothesis, $\nu(\phi(\vec{z}, \vec{x}, \vec{y})) = 1$, thus there is some clause $\mathbf{P}_h$ whose literals are all true according to ν. If p_h has some neighbor in $-Y_j$, then $\nu(\mathbf{y}_j) = 1$, thus $Y_j \subset S$ and $-Y_j \cap S = \emptyset$. If p_h has some neighbor $-z_h \in -Z_f$, then $\nu(\mathbf{z}_f) = 1$ which means, by the way ν is defined for $\vec{z}$ in Step 1, that $\rho(Z_f) = \{1\}$. Consequently, by Step 3, $\rho(-z_h) \neq 1$, i.e., $-z_h \notin S$. Similarly, if p_h has some neighbor in Z_f, then $\nu(\mathbf{z}_f) = 0$ which means that $\rho(Z_f) \neq \{1\}$. Thus, there must exist at least one vertex $z_f \in Z_f$ with $\rho(z_f) \neq 1$. Then, since $\rho(S) = 1$, it follows that $Z_f \not\subset S$. Finally, if p_h has a neighbor in $-X_i$, then $\nu(\mathbf{x}_i) = 1$, thus $\rho(-X_i[h]) \neq 1$ for some h by either Step 2.1 or Step 2.4. Hence, $\rho(-X_i) \neq \{1\}$, thus $-X_i \not\subset S$. Analogously, p_h has no neighbors in $X_i \cap S$. Summing up, since $P \cup \{t\}$ is an independent set, it follows that p_h has no neighbors in S, thus $p_h \in S$ and $\{s\}S$ is not monochromatic by Step 3.

For the converse, suppose G is star 2-choosable, and consider any valuation ν of $\vec{z}$. Define L to be a 2-list assignment of G so that $\nu(\mathbf{z}_f)$ is the unique color admissible for all the vertices in Z_f, 1 is the unique color admissible for all the vertices in $L_Y \cup \{t\}$, and $L(w) = \{0, 1\}$ for every vertex not connected to a 2-forcer. By statement (iii) of Lemma 9, such list assignment L always exists. Let ρ be a star L-coloring of G and extend ν to include $\vec{x} + \vec{y}$ in its domain so that $\nu(\mathbf{x}_i) = \rho(v_i)$. Note that $\nu(\mathbf{y}_j)$ can take any value from $\{0, 1\}$, so it is enough to prove that $\nu(\phi(\vec{x}, \vec{y}, \vec{z})) = 1$. Define $V_X = \bigcup_i((X_i \mid \nu(\mathbf{x}_i) = 1) \cup (-X_i \mid \nu(\mathbf{x}_i) = 0))$, $V_Y = \bigcup_j((Y_j \mid \nu(\mathbf{y}_j) = 1) \cup (-Y_j \mid \nu(\mathbf{y}_j) = 0))$, and $V_Z = \bigcup_f((Z_f \mid$

$\nu(\mathbf{z}_f) = 1) \cup (-Z_f \mid \nu(\mathbf{z}_f) = 0))$, and let $S = \{s\}(\{t\} \cup V_X \cup V_Y \cup V_Z)$. As in Theorem 1, it can be observed that (i) S is a monochromatic star and (ii) every vertex in $N(s) \setminus P$ is either adjacent or equal to a vertex in S. Thus, since ρ is a star L-coloring of G, there must be some vertex p_h adjacent to no vertex in $V_X \cup V_Y \cup V_Z$. Moreover, such vertex p_h corresponds to some clause $\mathbf{P}_h$ whose literals are all true by the way ν is defined. $\square$

The proof for $k > 2$ is by induction, i.e., we reduce STAR k-CHOOSABILITY into STAR $(k + 1)$-CHOOSABILITY for every $k \geq 2$. Roughly speaking, the idea of the reduction is to insert a vertex z that forbids every vertex of the reduced graph to have the same color as z.

Theorem 3 STAR k-CHOOSABILITY *is* Π_3^p-*complete for every* $k \geq 2$, *and it remains* Π_3^p-*complete when the input is restricted to* $\{C_4, K_{k+2}\}$-*free graphs.*

Proof: The proof is by induction on k. The base case $k = 2$ corresponds to Theorem 2. For the inductive step, we show how to transform a $\{C_4, K_{k+2}\}$-free graph G_k into a $\{C_4, K_{k+3}\}$-free graph G_{k+1} so that G_k is star k-choosable if and only if G_{k+1} is star $(k + 1)$-choosable.

The vertices of G_{k+1} are divided into connection and inner vertices. Connection vertices comprise a set W inducing G_k and a vertex z. Inner vertices are included in $(k + 1)$-forcers or $(k + 1)$-switchers connecting connection vertices. There is a $(k + 1)$-forcer connecting z, and a $(k + 1)$-switcher connecting $\{z, w\}$ for every $w \in W$. Let $C(w)$ be the $(k + 1)$-switcher connecting $\{w, z\}$, i.e., $C(w) \cup \{w\}$ and $C(w) \cup \{z\}$ are cliques of G_{k+1}. By statement (i) of Lemmas 2 and 9, G_{k+1} is $\{K_{k+3}, C_4\}$-free.

Suppose G_k is star k-choosable. Let L_{k+1} be a $(k + 1)$-list assignment of G_{k+1}, and $c(z) \in L(z)$ be L-admissible for z. Recall that $c(z)$ always exists by statement (ii) of Lemma 9. Define L_k as a k-list assignment of $G_{k+1}[W]$ such that $L_k(w) \subseteq L_{k+1}(w) \setminus \{c(z)\}$ for $w \in W$. By hypothesis, there is a star L_k-coloring ρ of $G_{k+1}[W]$. Define σ to be the L_{k+1}-coloring of G_{k+1} such that $\sigma(w) = \rho(w)$ for $w \in W$ and $\sigma(z) = c(z)$. Inner vertices are colored according to Lemma 6 and statement (ii) of Lemma 9. Clearly, if $\{w\}S$ is a maximal star of G_{k+1} and w is a connection vertex, then either $w = z$ or $\{w\}S$ includes a maximal star of $G_k[W]$. Whichever the case, $\{w\}S$ is not monochromatic, i.e., σ is a star coloring of G_{k+1}.

For the converse, let L_k be an k-list assignment of $G_{k+1}[W]$ and take a color $c \notin L(V(G_{k+1}))$. Define L_{k+1} as any $(k + 1)$-list assignment of G_{k+1} such that c is the unique L_{k+1}-admissible color for z, and $L_{k+1}(w) = L_{k+1}(C(w)) = L_k(w) \cup \{c\}$ for every $w \in W$. Such list assignment always exists by statement (iii) of Lemma 9. Let σ be a star L_{k+1}-coloring of G_{k+1}. By construction, $\sigma(z) = \{c\}$, and by Lemma 2, $c \notin \sigma(W)$. Hence, the restriction ρ of σ to W is an L_k-coloring of $G_{k+1}[W]$. Moreover, if $\{w\}S$ is a maximal star of $G_{k+1}[W]$, then $\{w\}(S \cup \{x\})$ is a maximal star of G_{k+1}, for every $x \in C(w)$. Since $C(w)$ is a block of G_{k+1} and $L(C(w)) = L(w)$, it follows that $\rho(w) = \sigma(w) \in \sigma(C(w))$. Hence, $\{w\}S$ is not monochromatic. $\square$

5   Forbidding graphs of order 3

The previous sections dealt with time complexity of the star and biclique coloring and choosability problems. The remaining of the article is devoted to these problems in other restricted classes of graphs. As discussed in Section 3.3, we are interested in classes of graphs that are related to chordal graphs or can be defined by forbidding small induced subgraphs. In this section, we study the classes of H-free graphs, for every graph H on three vertices.

There are four graphs with exactly three vertices, namely K_3, P_3, $\overline{P_3}$, and $\overline{K_3}$. The following theorem shows that K_3-free graphs are star 2-choosable.

Theorem 4 *Every K_3-free graph is star 2-choosable. Furthermore, for any 2-list assignment, a star L-coloring can be obtained in linear time.*

Proof: Let L be a 2-list assignment of a K_3-free graph G, T be a rooted tree subgraph of G with $V(T) = V(G)$, r be the root of T, and $p(v)$ be the parent of v in T for each $v \in V(G) \setminus \{r\}$. Define ρ to be an L-coloring of G where $\rho(r) \in L(r)$ and $\rho(v) \in L(v) \setminus \{\rho(p(v))\}$ for every $v \in V(G) \setminus \{r\}$. Since G is K_3-free, $\{v\}S$ is a maximal star of G for $v \in V(G)$ only if $S = N(v)$, hence $\{v\}S$ is not monochromatic. Observe that a BFS traversal of G is enough to compute ρ, thus ρ is computed in linear time from G.   □

As a corollary, we obtain that $\{C_4, K_3\}$-free graphs are biclique 2-choosable also. However, this corollary can be easily strengthened so as to include those K_3-free graphs that are *biclique-dominated*. A graph G is *biclique-dominated* when every maximal biclique is either a star or has a false dominated vertex. Some interesting classes of graphs are K_3-free and biclique-dominated, including hereditary biclique-Helly graphs [6].

Theorem 5 *Every K_3-free graph that is biclique-dominated is biclique 2-choosable. Furthermore, for any 2-list assignment, a biclique L-coloring can be computed in polynomial time.*

Proof: Let L be a 2-list assignment of a K_3-free graph G that is biclique-dominated. The algorithm for biclique L-coloring G has two steps. First, apply Theorem 4 on G so as to obtain a star L-coloring ρ of G. Second, traverse each vertex w and, for each v that is false dominated by w, change $\rho(v)$ with any color in $L(v) \setminus \rho(w)$. (It is not important if $\rho(v)$ or $\rho(w)$ are later changed when other vertices are examined.) The coloring thus generated is a biclique L-coloring. Indeed, if a maximal biclique contains a false dominated vertex v, then it also contains the vertex w such that $\rho(v)$ was last changed in the second step while traversing w. Since false domination is a transitive relation, it follows that $\rho(v) \neq \rho(w)$ when the second step is completed. On the other hand, if S is a maximal biclique with no false dominated vertices, then S is a star. Since the colors of the vertices of S are not affected by the second step, we obtain that S is not monochromatic. It is not hard to see that the algorithm requires polynomial time.   □

Coloring a connected P_3-free graph is trivial because the unique connected P_3-free graph G with n vertices is K_n. Thus, $ch_B(G) = \chi_B(G) = \chi_S(G) = ch_S(G) = n$.

Theorem 6 *If G is a connected P_3-free graph with n vertices, then $ch_B(G) = \chi_B(G) = \chi_S(G) = ch_S(G) = n$.*

The case of $\overline{P_3}$-free graphs, examined in the next theorem, is not much harder.

Theorem 7 *If G is a $\overline{P_3}$-free graph with k universal vertices, then $ch_B(G) = \chi_B(G) = \chi_S(G) = ch_S(G) = \max\{2, k\}$.*

Proof: Let K be the set of universal vertices of G. Clearly, K is a block of G, thus $ch_B(G) \geq k$ and $ch_S(G) \geq k$ by Observation 2. For the other bound, let L be a k-list assignment of G, and $B_1, \ldots, B_n$ be the sets of vertices that induce components of $\overline{G} \setminus K$. Define ρ as an L-coloring of G such that $|\rho(K)| = k$ and $|\rho(B_i)| = 2$ for $i \in \{1, \ldots, n\}$. Note that B_i is a set of false twin vertices $(i \in \{1, \ldots, j\})$ because B_i is a clique of $\overline{G}$. Thus, every maximal star or biclique S is formed by two vertices of K or it contains a set B_i for some $i \in \{1, \ldots, n\}$. Whichever the case, S is not monochromatic, thus ρ is a star and biclique L-coloring. □

The remaining class is the class of $\overline{K_3}$-free graphs. By definition, if G is $\overline{K_3}$-free, then every maximal star and every maximal biclique of G has $O(1)$ vertices. Thus, it takes polynomial time to determine if an L-coloring of G is a star or biclique coloring, for any k-list assignment L. Hence, when restricted to $\overline{K_3}$-free graphs, the star and biclique k-coloring problems belong to NP, while the star and biclique k-choosability problems belong to Π_2^p. The next theorem shows that, when $k \geq 3$, the choosability problems are Π_2^p-complete even when the input is further restricted to co-bipartite graphs.

Theorem 8 STAR k-CHOOSABILITY *and* BICLIQUE k-CHOOSABILITY *are* Π_2^p-*complete for every $k \geq 3$ when the input is restricted to co-bipartite graphs.*

Proof: The proof is obtained by reducing the problem of determining if a connected bipartite graph with no false twins is vertex k-choosable, which is known to be Π_2^p-complete [15]. Let G be a connected bipartite graph with no false twins, XY be a bipartition of G, and $k \in \mathbb{N}$. Define H to be the bipartite graph obtained from G by inserting, for every $vw \in E(G)$, the stars $\{a_i(vw)\}A_i(vw)$ $(i \in \{1, \ldots, 4\})$ with $|A_i(vw)| = k - 1$ and the edges $va_1(vw)$, $va_3(vw)$, $wa_2(vw)$, $wa_4(vw)$, $a_1(vw)a_2(vw)$, and $a_3(vw)a_4(vw)$ (see Figure 6). We claim that G is vertex k-choosable if and only if $\overline{H}$ is star (resp. biclique) k-choosable.

Suppose first that G is vertex k-choosable, and let L be a k-list assignment of H and M be the restriction of L to $V(G)$. By hypothesis, G admits a vertex M-coloring ρ. Define σ to be any vertex L-coloring of H so that $\sigma(v) = \rho(v)$
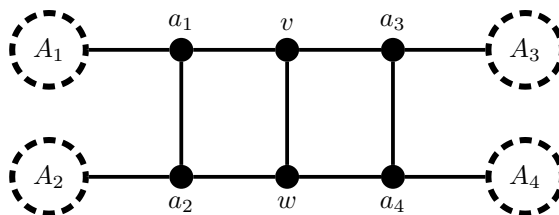
Figure 6: Transformation applied to vw in Theorem 8; each independent set has $k-1$ vertices.

for $v \in V(G)$, and $|\sigma(A_i(vw) \cup \{a_i(vw)\})| = k$ for every $vw \in E(G)$ and every $i \in \{1, \ldots, 4\}$. It is not hard to see that such a coloring always exists. Clearly, every maximal star (resp. biclique) S of $\overline{H}$ is formed by two twins of $\overline{H}$ or it contains two vertices that are adjacent in H. In the latter case S is not σ-monochromatic because σ is a vertex coloring of H, while in the former case S is not σ-monochromatic because both of its vertices must belong to $A_i(vw)$, as G has no false twins, for some $vw \in E(G)$ and some $i \in \{1, \ldots, 4\}$.

For the converse, suppose $\overline{H}$ is star (resp. biclique) k-choosable, and let M be a k-list assignment of G. Define σ to be a star (resp. biclique) L-coloring of $\overline{H}$, for the k-list assignment L of H where $L(a) = L(v) = M(v)$ for every $vw \in E(G)$ with $v \in X$, and every $a \in A_i(vw) \cup \{a_i(vw)\}$ with $i \in \{1, \ldots, 4\}$. Suppose, to obtain a contradiction, that $\sigma(v) = \sigma(w)$ for some $vw \in E(G)$ with $v \in X$ and $w \in Y$. Then, for every $a \in A_i(vw)$ ($i \in \{1, \ldots, 4\}$), we obtain that $\sigma(a) \neq \sigma(v)$ because $\{a\}\{v, w\}$ is a maximal star (resp. biclique) of $\overline{H}$. Hence, since $A_i(vw)$ is a block of $\overline{H}$, we obtain by Observation 2 that $\sigma(A_i(vw)) = L(v) \setminus \{\sigma(v)\}$ for every $i \in \{1, \ldots, 4\}$. Consequently, since $\{b\}\{a, a_1(vw)\}$ is a maximal star (resp. biclique) for every $b \in A_2(vw)$ and every $a \in A_1(vw)$, it follows that $\sigma(a_1(vw)) = \sigma(v)$. Analogously, $\sigma(a_i(vw)) = \sigma(v)$ for every $i \in \{1, \ldots, 4\}$. But then, $\{a_i(vw) \mid 1 \leq i \leq 4\}$ is a monochromatic maximal biclique that contains a maximal star, a contradiction. Therefore, $\sigma(v) \neq \sigma(w)$ for every edge vw of $V(G)$, which implies that the restriction of σ to $V(G)$ is a vertex M-coloring of G. □

Let G be a K_3-free graph with no false twins, and define H as the K_3-free graph that is obtained from G as in Theorem 8. By fixing the list assignment that maps each vertex to $\{1, \ldots, k\}$ in the proof of Theorem 8, it can be observed that G admits a vertex k-coloring if and only if $\overline{H}$ admits a star (resp. biclique) k-coloring, for every $k \geq 3$. The problem of determining if a connected K_3-free graph with no false twins admits a vertex k-coloring is known to be NP-complete [17, 20]. Hence, the star and biclique k-coloring problems are NP-complete when restricted to $\overline{K_3}$-free graphs, for every $k \geq 3$.

Theorem 9 STAR k-COLORING *and* BICLIQUE k-COLORING *are* NP-*complete for every $k \geq 3$ when the input is restricted to $\overline{K_3}$-free graphs.*

6   Graphs with restricted diamonds

The graph G defined in Theorem 1 contains a large number of induced diamonds. For instance, to force different colors on a pair of vertices v and w, a k-switcher C connecting $\{v, w\}$ is used. Such switcher contains $O(k^2)$ diamonds, one for each edge of C. An interesting question is, then, whether induced diamonds can be excluded from Theorem 1. The answer is no, as we prove in this section that the star coloring problem is NP-complete for diamond-free graphs. By taking a deeper look at G, it can be noted that every diamond of $G \setminus \{s\}$ has a pair of twin vertices. In order to prove that the star coloring problem is NP-complete for diamond-free graphs, we show that the problem is NP even for the larger class of graphs in which every diamond has two twin vertices. This class corresponds to the class of $\{W_4, \text{dart}, \text{gem}\}$-free graphs (cf. below), and is worth to note that its graphs may admit an exponential number of maximal stars. We also study the biclique coloring problem on this class, for which we prove that the problem is NP when there are no induced $K_{i,i}$ for $i \in O(1)$. At the end of the section, we study the star and biclique choosability problems, which turn to be Π_2^p-hard for $\{C_4, \text{dart}, \text{gem}\}$-free graph.

Let G be a graph. Say that $v \in V(G)$ is *block separable* if every pair of adjacent vertices $w, z \in N(v)$ not dominating v are twins in $G[N(v)]$. The following lemma shows that $\{W_4, \text{dart}, \text{gem}\}$-free graphs are precisely those graphs in which every induced diamond has twin vertices, and they also correspond to those graphs is which every vertex is block separable. This last condition is crucial in the NP coloring algorithms.

Theorem 10 *The following statements are equivalent for a graph G.*

  (i) *G is $\{W_4, \text{dart}, \text{gem}\}$-free.*

 (ii) *Every induced diamond of G contains a pair of twin vertices.*

(iii) *Every $v \in V(G)$ is block separable.*

Proof: (i) $\Longrightarrow$ (ii) If $D \subseteq V(G)$ induces a diamond with universal vertices v, w and there exists $x \in N(v) \setminus N(w)$, then $D \cup \{x\}$ induces a W_4, a dart, or a gem in G depending on the remaining adjacencies between x and the vertices of D.

(ii) $\Longrightarrow$ (iii) Suppose $v \in V(G)$ is not block separable, thus $N[v]$ contains two adjacent vertices w and z not dominating v that are not twins in $H = G[N[v]]$; say $d_H(z) \geq d_H(w)$. Then, v and z are the universal vertices of a diamond containing w and a vertex in $N_H(z) \setminus N_H(w)$, i.e., G contains an induced diamond with no twin vertices.

(iii) $\Longrightarrow$ (i) The W_4, dart, and gem graphs have a vertex of degree 4 that is not block separable.   □

Note that if v is block separable, then $N[v]$ can be partitioned into sets $B_0, \ldots, B_\ell$ where $v \in B_0$ and each B_i is a block of $G[N[v]]$. Moreover, no vertex in B_i is adjacent to a vertex in B_j, for $1 \leq i < j \leq \ell$. We refer to $B_0, \ldots, B_\ell$ as the *block separation* of v. By definition, $\{v\}S$ is a maximal star of G with

$|S| > 1$ if and only if $\ell > 1$, $|S \cap B_0| = 0$ and $|S \cap B_i| = 1$ for $i \in \{1, \dots, \ell\}$. By Theorem 10, every vertex of a $\{W_4$, dart, gem$\}$-free graph admits a block separation, hence the next result follows.

Lemma 10 *Let G be a $\{W_4$, dart, gem$\}$-free graph with a coloring ρ. Then, ρ is a star coloring of G if and only if*

- $|\rho(B)| = |B|$ *for every block B of G, and*

- *for every $v \in V(G)$ with block separation $B_0, \dots, B_\ell$, there exists B_i such that $\rho(v) \notin \rho(B_i)$.*

It is well known that the blocks of a graph G can be computed in $O(n + m)$ time. Hence, it takes $O(d(v)^2)$ time obtain the block separation of a block separable vertex v, and, consequently, the star k-coloring and the star k-choosability problems are in NP and Π_2^p for $\{W_4$, dart, gem$\}$-free graphs, respectively.

Theorem 11 STAR k-COLORING *is* NP *when the input is restricted to $\{W_4$, dart, gem$\}$-free graphs.*

Theorem 12 STAR k-CHOOSABILITY *is* Π_2^p *when the input is restricted to $\{W_4$, dart, gem$\}$-free graphs.*

We now consider the biclique coloring problem. The algorithm for determining if a coloring ρ is a biclique coloring of G is divided in two steps. First, it checks that no monochromatic maximal star is a maximal biclique. Then, it checks that G contains no monochromatic maximal biclique $K_{i,j}$ with $2 \le i \le j$.

For the first step, suppose ρ is a coloring of G where $\rho(B) = |B|$ for every block B of G. Let v be a vertex with a block separation $B_0, \dots, B_\ell$. As discussed above, $\{v\}S$ is a maximal star if and only if $\ell > 1$, $|S \cap B_0| = 0$, and $|S \cap B_i| = 1$ for every $i \in \{1, \dots, \ell\}$. If $\{v\}S$ is not a maximal biclique, then there exists $w \in V(G) \setminus N[v]$ adjacent to all the vertices in S. Observe that w has at most one neighbor in B_i with color c, for each color c. Otherwise, taking into account that twin vertices have different colors, v, w, y, z would induce a diamond with no twin vertices, for $y, z \in N(v) \cap N(w)$. Therefore, at most one monochromatic maximal star with center v is included in a biclique containing w, for each $w \in V(G) \setminus N(v)$. Thus, to check if there is a monochromatic maximal biclique containing v we first check whether $\prod_{i=1}^{\ell} |\{z \in B_i \mid \rho(z) = \rho(v)\}| < n$. If negative, then ρ is not a biclique coloring of G. Otherwise, all the monochromatic maximal stars with center in v are generated in polynomial time, and for each such star $\{v\}S$ it is tested if there exists $w \in V(G) \setminus N[v]$ adjacent to all the vertices in S.

Lemma 11 *If a $\{W_4$, dart, gem$\}$-free graph G and a coloring ρ are given as input, then it takes polynomial time to determine if there exists a monochromatic maximal biclique $\{v\}S$ with $v \in V(G)$.*

For the second step, suppose S is an independent set with at least two vertices, and let $I = \bigcap_{v \in S} N(v)$. Note that if $w, z \in I$ are adjacent, then they are twins in G because w, z are the universal vertices of any induced diamond formed by taking a pair of vertices in S. Hence, I can be partitioned into a collection $B_1, \ldots, B_\ell$ of blocks of G where no vertex in B_i is adjacent to a vertex in B_j, for $1 \leq i < j \leq \ell$. Thus, ST is a maximal biclique of G if and only if no vertex of $V(G) \setminus S$ is complete to I and $|T \cap B_i| = 1$ for every $i \in \{1, \ldots, \ell\}$. That is, G has a monochromatic maximal biclique ST if and only if $|\rho(S)| = 1$, each block of I has a vertex of color $\rho(S)$, and $\bigcap_{w \in I} N(w) = S$.

Lemma 12 *Let G be a $\{W_4, dart, gem\}$-free graph. If an independent set S and a coloring ρ of G are given as input, then it takes polynomial time to determine if G has a monochromatic maximal biclique ST with $T \subseteq V(G)$.*

If G is $K_{i,i}$-free for some constant i, then every biclique ST of G with $|S| \leq |T|$ has $|S| < i$. Thus, to determine if ρ is a biclique coloring of G, it is sufficient to traverse every independent set S of G with $O(i)$ vertices and to check that there exists no $T \subset V(G)$ such that ST is a monochromatic maximal biclique. By Lemmas 11 and 12, it takes polynomial time to determine if there exists T such that ST is a monochromatic maximal star. Since there are $n^{O(1)}$ independent sets with at most i vertices, the algorithm requires polynomial time. We thus conclude that BICLIQUE k-COLORING and BICLIQUE k-CHOOSABILITY are respectively NPand Π_2^p when the input is restricted to $\{K_{i,i}, W_4, dart, gem\}$-free graphs.

Theorem 13 BICLIQUE k-COLORING *is* NP*when the input is restricted to $\{K_{i,i}, W_4, dart, gem\}$-free graphs, for $i \in O(1)$.*

Theorem 14 BICLIQUE k-CHOOSABILITY *is* Π_2^p *when the input is restricted to $\{K_{i,i}, W_4, dart, gem\}$-free graphs, for $i \in O(1)$.*

In the rest of this section, we discuss the completeness of the star and biclique coloring and choosability problems. As in Sections 3 and 4, only one proof is used for each problem because C_4-free graphs are considered. For the reductions, two restricted satisfiability problems are required, namely NAE-SAT and NAE∀∃SAT. A valuation ν of a CNF formula ϕ is a *nae-valuation* when all the clauses of ϕ have a true and a false literal. The formula $(\exists \vec{x})\phi(\vec{x})$ is *nae-true* when $\phi(\vec{x})$ admits a nae-valuation, while $(\forall \vec{x})\phi(\vec{x})$ is *nae-true* when every valuation of $\phi(\vec{x})$ is a *nae-valuation*. NAE-SAT is the NP-complete problem (see [9]) in which a CNF formula ϕ is given, and the goal is to determine if ϕ admits a nae-valuation. Analogously, NAE∀∃SAT is the Π_2^p-complete problem (see [7]) in which a CNF formula $\phi(\vec{x}, \vec{y})$ is given, and the purpose is to determine if $(\forall \vec{x})(\exists \vec{y})\phi(\vec{x}, \vec{y})$ is nae-true. We begin discussing the completeness of the star coloring problem. In order to avoid induced diamonds, we define a replacement of long switchers.

Definition 7 [diamond k-switcher] Let G be a graph and $U = \{u_1, \ldots, u_h\}$ be an independent set of G with $h \geq 2$. Say that $S \subset V(G)$ is a *diamond k-switcher*

connecting U $(k \geq 2)$ when S can be partitioned into a vertex w_1, a set of leafed vertices $\{w_2, \ldots, w_h\}$, a family $Q_1, \ldots, Q_h$ of k-keepers, and a clique C with k vertices in such a way that $C \cup \{w_1\}$ is a clique, $\{w_1\}\{w_2, \ldots, w_h\}$ is a star, Q_i connects u_i, w_i for $i \in \{1, \ldots, h\}$, and there are no more edges adjacent to vertices in S.

A diamond k-switcher is depicted in Figure 7. The main properties of diamond switchers are given in the next lemma.
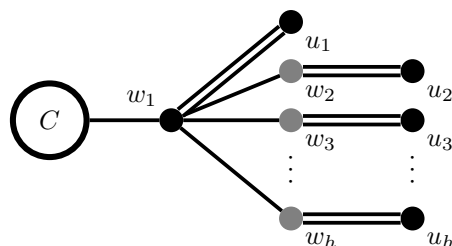

Figure 7: A diamond k-switcher connecting $\{u_1, \ldots, u_h\}$.

Lemma 13 *Let G be a graph and S be a diamond k-switcher connecting $U \subset V(G)$ $(k \geq 2)$. Then,*

(i) no induced C_4, diamond, or K_{k+2} of G contains a vertex of S,

(ii) $|\rho(U)| \geq 2$ for any star k-coloring ρ of G, and

(iii) Any k-coloring ρ of $G \setminus S$ in which $|\rho(U)| \geq 2$ can be extended into a k-coloring of G in such a way that no monochromatic maximal star has its center in S.

Proof: Let $U = \{u_1, \ldots, u_h\}$, $w_1, \ldots, w_h$, Q_1, $\ldots$, Q_h, and C be as in Definition 7. Statement (i) follows by statement (i) of Lemma 1, observing that U is an independent set and that no induced diamond can contain a vertex in a k-keeper.

(ii) Let ρ be a star k-coloring of G. Since C is a block of size k, it contains a vertex c with color $\rho(w_1)$ by Observation 2. Then, taking into account that $\{w_1\}\{c, u_1, w_2, \ldots, w_h\}$ is a maximal star and $\rho(u_i) = \rho(w_i)$ by statement (ii) of Lemma 1 for $i \in \{1, \ldots, h\}$, it follows that $\rho(u_i) \neq \rho(u_1)$ for some $i \in \{2, \ldots, h\}$.

(iii) To extend ρ, first set $\rho(C) = \{1, \ldots, k\}$ and $\rho(w_i) = \rho(u_i)$ (for $i \in \{1, \ldots, h\}$), and then iteratively extend ρ to color the leaves and the k-keepers according to Observation 1 and statement (iii) of Lemma 1. □

We are now ready to prove the NP-completeness of the star-coloring problem.

Theorem 15 STAR k-COLORING *is* NP-*complete when the input is restricted to* $\{C_4, diamond, K_{k+2}\}$-*free graphs for every $k \geq 2$.*

Proof: By Theorem 11, STAR k-COLORING is NPfor $\{C_4, \text{diamond}, K_{k+2}\}$-free graphs. For the hardness part, we show a polynomial time reduction from NAE-SAT. That is, given a CNF formula ϕ with ℓ clauses $\mathbf{P}_1, \ldots, \mathbf{P}_\ell$ and n variables $\mathbf{x}_1, \ldots, \mathbf{x}_n$, we define a $\{C_4, \text{diamond}, K_{k+2}\}$-free graph G such that ϕ admits a nae-valuation if and only if G admits a star k-coloring.

The vertices of G are divided into *connection* and *inner* vertices. For each $i \in \{1, \ldots, n\}$ there are two connection vertices $x_i, -x_i$ *representing* the literals $\mathbf{x}_i$ and $\overline{\mathbf{x}_i}$, respectively. Also, there are $k - 2$ connection vertices $y_3, \ldots, y_k$. Let $X = \{x_1, \ldots, x_n, -x_1, \ldots, -x_n\}$, $Y = \{y_3, \ldots, y_k\}$, and $P_h = \{x \in X \mid x \text{ represents a literal in } \mathbf{P}_h\}$ for $h \in \{1, \ldots, \ell\}$. Inner vertices are the vertices included in diamond k-switchers connecting connection vertices. For each $v \in X \cup Y$ and each $y \in Y$ there is a *color* diamond k-switcher connecting $\{v, y\}$. Also, for each $i \in \{1, \ldots, n\}$ there is a *valuation* diamond k-switcher connecting $\{x_i, -x_i\}$. Finally, there is a *clause* diamond k-switcher connecting P_h for every $h \in \{1, \ldots, \ell\}$. Observe that $X \cup Y$ is an independent set of G. Thus, by statement (i) of Lemma 13, G is $\{C_4, \text{diamond}, K_{k+2}\}$-free.

Suppose ϕ has a nae-valuation $\nu : \vec{\mathbf{x}} \to \{0, 1\}$, and let ρ be a k-coloring of the connection vertices such that $\rho(x_i) = 2 - \nu(\mathbf{x}_i)$ and $\rho(y_j) = j$ for $i \in \{1, \ldots, n\}$ and $j \in \{3, \ldots, k\}$. Clearly, every color or valuation k-switcher connects a pair of vertices that have different colors. Also, since ν is a nae-valuation, every set P_h ($h \in \{1, \ldots, \ell\}$) has two vertices representing literals $\mathbf{l}_1$ and $\mathbf{l}_2$ of $\mathbf{P}_h$ with $\nu(l_1) \neq \nu(l_2)$. Hence, P_h is not monochromatic, thus every clause diamond k-switcher connects a non-monochromatic set of vertices. Therefore, by statement (iii) of Lemma 13, ρ can be iteratively extended into a star k-coloring of G.

For the converse, suppose G admits a star k-coloring ρ. By applying statement (ii) of Lemma 13 while considering the different kinds of diamond k-switchers, we observe the following facts. First, according to the color diamond k-switchers, $|\rho(Y)| = k - 2$ and $\rho(X) \cap \rho(Y) = \emptyset$. Then, we can assume that $\rho(X) \subseteq \{1, 2\}$ and $\rho(Y) = \{3, \ldots, k\}$. Hence, by the valuation diamond k-switcher connections, we obtain that $\rho(x_i) \neq \rho(-x_i)$ for every $i \in \{1, \ldots, n\}$. Thus, the mapping $\nu : \vec{\mathbf{x}} \to \{0, 1\}$ such that $\nu(\mathbf{x}_i) = 2 - \rho(x_i)$ is a valuation. Moreover, by the clause diamond k-switcher connections, P_h is not monochromatic for $h \in \{1, \ldots, \ell\}$. Consequently, ν is a nae-valuation of ϕ. $\square$

Observe that the graph G defined in Theorem 15 is not chordal. However, as discussed in Section 3.3, every edge xy such that x, y are connected by a k-keeper (inside the diamond k-switchers) can be subdivided so as to eliminate all the induced holes of length at most i, for every $i \in O(1)$.

We now deal with the star choosability problem. Recall that long switchers are not well suited for the star choosability problem because they contain keepers, and vertices connected by keepers need not have the same colors in every list coloring. Keepers are also present inside diamond switchers, thus it is not a surprise that diamond k-switchers are not star k-choosable. For this reason, as in Section 4, the proof is by induction, using list switchers for $k = 2$. Since list switchers contain induced diamonds, the Σ_2^p-hardness will be obtained for

$\{W_4$, gem, dart$\}$-free graphs, and not for diamond-free graphs. Unfortunately, we did not find a way to avoid these diamonds. Moreover, some kind of forcers are required as well; our forcers have induced diamonds that we were not able to remove either. The hardness proof for $k = 2$ is, in some sense, a combination of the proofs of Theorems 2 and 15. Roughly speaking, the idea is to force the colors of the universal variables of the input formula as in Theorem 2, while a nae-valuation is encoded with colors as in Theorem 15.

Theorem 16 STAR 2-CHOOSABILITY *is Π_2^p-hard when its input is restricted to* $\{C_4$, *dart, gem,* $K_4\}$-*free graphs.*

Proof: The hardness of STAR 2-CHOOSABILITY is obtained by a reduction from NAE$\forall\exists$SAT. That is, given a CNF formula $\phi(\vec{z}, \vec{x})$ with ℓ clauses $\mathbf{P}_1, \ldots, \mathbf{P}_\ell$, and $m+n$ variables $\vec{x} = \mathbf{x}_1, \ldots, \mathbf{x}_n$, $\vec{z} = \mathbf{z}_1, \ldots, \mathbf{z}_m$, we build a $\{C_4$, dart, gem, $K_4\}$-free graph G that is star 2-choosable if and only if $(\forall\vec{z})(\exists\vec{x})\phi(\vec{z}, \vec{x})$ is nae-true. For the sake of simplicity, in this proof we use i, f, and h as indices that refer to values in $\{1, \ldots, n\}$, $\{1, \ldots, m\}$, and $\{1, \ldots, \ell\}$.

Graph G is an extension of the graph in Theorem 15 for $k = 2$ (replacing diamond switcher with list switchers). It has a *connection* vertex x_i (resp. $-x_i$, z_f, $-z_f$) *representing* $\mathbf{x}_i$ (resp. $\overline{\mathbf{x}_i}$, $\mathbf{z}_f$, $\overline{\mathbf{z}_f}$) for each i (and each f), and two *connection* vertices t, $-t$. Let $L_X = \{x_i, -x_i \mid 1 \le i \le n\}$, $Z = \{z_1, \ldots, z_m\}$, $-Z = \{-z_1, \ldots, -z_m\}$, and $P_h = \{x \mid x \in X \cup Z \cup -Z \text{ represents a literal in } \mathbf{P}_h\}$. Graph G also has *inner vertices* which are the vertices in list switchers and 2-forcers connecting connection vertices. There are 2-forcers connecting each vertex of $Z \cup \{t, -t\}$, and list switchers connecting: $\{t, x_i, -x_i\}$ and $\{-t, x_i, -x_i\}$ for each i; $\{z_f, -z_f\}$ for each f; and $P_h \cup \{t\}$ and $P_h \cup \{-t\}$ for each h.

Let L be a 2-list assignment of G, and suppose $(\forall\vec{z})(\exists\vec{x})\phi(\vec{z}, \vec{x})$ is nae-true. Define ρ as an L-coloring of the connection vertices satisfying the following conditions.

(i) $\rho(v)$ is any color L-admissible for $v \in Z \cup \{t, -t\}$. Such a color always exists by statement (ii) of Lemma 9. Suppose, w.l.o.g., that $\rho(t) = 1$ and $\rho(-t) \in \{0, 1\}$, and define $\nu(\vec{z})$ as a valuation of $\vec{z}$ such that $\nu(\mathbf{z}_f) = 1$ if and only if $\rho(z_f) = 1$.

(ii) By hypothesis, ν can be extended into a nae-valuation of $\phi(\vec{z}, \vec{x})$. If $\nu(\mathbf{x}_i) \in L(x_i)$, then $\rho(x_i) = \nu(\mathbf{x}_i)$. Otherwise, $\rho(x_i) \in L(x_i) \backslash \{1 - \nu(\mathbf{x}_i)\}$. Similarly, $\rho(-x_i) = 1 - \nu(\mathbf{x}_i)$ if $1 - \nu(\mathbf{x}_i) \in L(-x_i)$, while $\rho(-x_i) \in L(-x_i) \backslash \{\nu(\mathbf{x}_i)\}$ otherwise.

(iii) $\rho(-z_f) \in L(z_f) \backslash \{\rho(z_f)\}$.

It is not hard to see that ρ can always be obtained. Observe that $\rho(-z_f) \ne \rho(z_f)$, $\rho(x_i) = \rho(-x_i)$ only if $\rho(x_i), \rho(-x_i) \notin \{\rho(t), \rho(-t)\}$, and $\rho(P_h)$ is monochromatic only if $\rho(P_h) \not\subseteq \{\rho(t), \rho(-t)\}$. Therefore, ρ can be extended into an L-coloring of G by statement (iii) of Lemma 7 and statement (ii) of Lemma 9.

For the converse, suppose G is star 2-choosable, and consider any valuation ν of $\vec{z}$. Define L to be a 2-list assignment of G such that 1 is the unique color

admissible for t, 0 is the unique color admissible for $-t$, $\nu(\mathbf{z}_f)$ is the unique color admissible for z_f, and $L(v) = \{0,1\}$ for $v \in V(G) \setminus (Z \cup \{t,-t\})$. By statement (iii) of Lemma 9, such a list assignment always exists. Let ρ be a star L-coloring of G. By repeatedly applying statement (ii) of Lemma 7, it can be observed that none of $\{t, x_i, -x_i\}$, $\{-t, x_i, -x_i\}$, $\{t\} \cup P_h$, and $\{-t\}$ are monochromatic. Therefore, ν is a nae-valuation of $\phi(\vec{\mathbf{z}}, \vec{\mathbf{x}})$. $\square$

Note that if C is a k-switcher, then every induced diamond that contains a vertex $u \in C$ also contains a twin of u. Hence, by Theorem 10, if F is a k-forcer connecting v, then no vertex in F belongs to an induced dart or gem. Consequently, if G_k is a $\{C_4, \text{dart, gem}, K_{k+2}\}$-free graph, then the graph G_{k+1} defined in the proof of Theorem 3 is $\{C_4, \text{dart, gem}, K_{k+3}\}$-free. That is, a verbatim copy of the proof of Theorem 3 can be used to conclude the following.

Theorem 17 STAR k-CHOOSABILITY *is* Π_2^p*-complete when its input is restricted to* $\{C_4, \text{dart, gem}, K_{k+2}\}$*-free graphs for every* $k \geq 2$.

7   Split graphs

In this section we consider the star coloring and star choosability problems restricted to split graphs. The reason for studying split graphs is that they form an important subclass of chordal graphs, and also correspond to the class of $\{2K_2, C_4, C_5\}$-free graphs [10]. A graph G is *split* when its vertex set can be partitioned into an independent set $S(G)$ and a clique $Q(G)$. There are $O(d(v))$ maximal stars centered at $v \in V(G)$, namely $\{v\}(N(v) \cap S(G))$ and $\{v\}((N(v) \cup \{w\}) \setminus N(w))$ for $w \in Q(G)$. Thus, the star coloring and star choosability problems on split graphs are NP and Π_2^p, respectively. In this section we prove the completeness of both problems.

We begin observing that G admits a star coloring with $\beta + 1$ colors, where β is the size of the maximum block. Indeed, each block B of $Q(G)$ is colored with colors $\{1, \ldots, |B|\}$, while each vertex of $S(G)$ is colored with color $\beta + 1$. We record this fact in the following observation.

Observation 3 *If G is a split graph whose blocks have size at most β, then G admits a star coloring using $\beta + 1$ colors. Furthermore, such a coloring can be obtained in linear time.*

Computing a star coloring of a split graph using $\beta + 1$ colors is easy, but determining if β colors suffice is an NP-complete problem. The proof of hardness is almost identical to the one in Section 6 for $\{C_4, \text{diamond}, K_{k+2}\}$-free graphs. That is, given a CNF formula ϕ we build a graph G using *split* switchers in such a way that ϕ admits a nae-valuation if and only if G admits a star k-coloring. As switchers, *split* switchers force a set of vertices to have at least two colors in a star k-coloring. The difference is that split switchers do so in a split graph (see Figure 8).
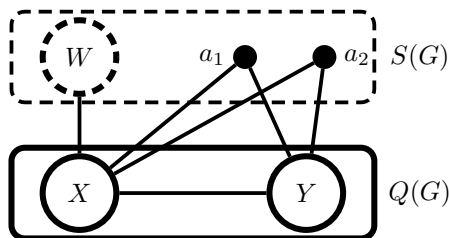
Figure 8: A split k-switcher connecting a set W.

Definition 8 [split k-switcher] Let G be a split graph and $W \subseteq S(G)$ with $|W| \geq 2$. Say that $S \subset V(G) \setminus W$ is a *split k-switcher connecting W* when S can be partitioned into two sets $X, Y \subseteq Q(G)$ and two vertices $a_1, a_2 \in S(G)$ in such a way that $|X| = |Y| = k$, $X \cup \{v\}$ is a clique for every $v \in W \cup \{a_1, a_2\}$, $Y \cup \{a_1\}$ and $Y \cup \{a_2\}$ are cliques, there are no more edges between vertices in $X \cup Y$ and vertices in $S(G)$, and there are no more edges incident to a_1 and a_2.

The properties of split k-switchers are summarized in the following lemma.

Lemma 14 *Let G be a split graph and S be a split k-switcher connecting $W \subset S(G)$. Then,*

(i) *$|\rho(W)| > 1$ for every star k-coloring ρ of G, and*

(ii) *For every k-list assignment of G, any L-coloring ρ of $G \setminus S$ in which $|\rho(W)| > 1$ can be extended into an L-coloring of G in such a way that no monochromatic maximal star has its center in S.*

Proof: Let X, Y, a_1, and a_2 be as in Definition 8.

(i) Let $\rho(G)$ be a star k-coloring of G. By definition, X and Y are blocks of G, thus $|\rho(X)| = |\rho(Y)| = k$ by Observation 2. Let x_c and y_c be the vertices with color c in X and Y, respectively. Since the maximal star $\{x_c\}(W \cup \{y_c\})$ is not monochromatic, it follows that $\rho(W) \neq \{c\}$. Therefore, $|\rho(W)| > 1$.

(ii) To extend ρ to S, define $|\rho(X)| = |\rho(Y)| = k$, and $\rho(a_1) \neq \rho(a_2)$. Let $\{v\}V$ be a maximal star with $v \in S$. If either $v \in Y$ or $v \in X$ and $V \cap Y = \emptyset$, then $\{a_1, a_2\} \subseteq V$, thus $\{v\}V$ is not monochromatic. Otherwise, if $v \in X$ and $V \cap Y \neq \emptyset$, then $W \subseteq V$, thus $\{v\}V$ is not monochromatic as well. $\square$

By replacing diamond k-switchers with split k-switchers in the proof of Theorem 15, the NP-completeness of the star k-coloring problem for split graphs is obtained. For the sake of completeness, we sketch the proof, showing how to build the graph G from the CNF formula.

Theorem 18 STAR k-COLORING *restricted to split graphs is* NP-*complete for every $k \geq 2$.*

Proof: Split graphs have $O(n + m)$ maximal stars, hence STAR k-COLORING is NP for split graphs. For the hardness part, let ϕ be a CNF formula with m clauses $\mathbf{P}_1, \ldots, \mathbf{P}_m$ and n variables $\mathbf{x}_1, \ldots, \mathbf{x}_n$. Define G as the split graphs with *connection* and *inner* vertices, as follows. For each $i \in \{1, \ldots, n\}$ there are two connection vertices $x_i, -x_i$ *representing* the literals $\mathbf{x}_i$ and $\overline{\mathbf{x}_i}$, respectively. Also, there are $k - 2$ connection vertices $y_3, \ldots, y_k$. Let $X = \{x_1, \ldots, x_n, -x_1, \ldots, -x_n\}$, $Y = \{y_3, \ldots, y_k\}$, and $P_h = \{x \in X \mid x$ represents a literal in $\mathbf{P}_h\}$ for $h \in \{1, \ldots, \ell\}$. Inner vertices form the split k-switchers connecting connection vertices. For each $v \in X \cup Y$ and each $y \in Y$ there is a *color* split k-switcher connecting $\{v, y\}$. Also, for each $i \in \{1, \ldots, n\}$ there is a *valuation* split k-switcher connecting $\{x_i, -x_i\}$. Finally, there is a *clause* split k-switcher connecting P_h for every $h \in \{1, \ldots, \ell\}$. Clearly, G is a split graph, and, as in the proof of Theorem 15, G admits star k-star coloring if and only if ϕ admits a nae-valuation. $\qquad\square$

For the star choosability problem of split graphs, the idea is to adapt the proof of Theorem 16, providing a new kind of forcer. This new forcer is called the *split k-forcer* and it is just a k-forcer where its k-switchers form a clique. For the sake of completeness, we include its definition.

Definition 9 [split k-forcer] Let G be a split graph and $v \in S(G)$. Say that $F \subseteq V(G)$ is a *split k-forcer connecting* v $(k \geq 2)$ when F can be partitioned into sets $A, B \subseteq S(G)$ and $C(a, b) \subseteq Q(G)$ for $a \in A \cup \{v\}$ and $b \in B$ in such a way that $|A| = k - 1$, $|B| = k^k - 1$, $C(a, b) \cup \{a\}$ and $C(a, b) \cup \{b\}$ are cliques, there are no more edges between vertices in $F \cap Q(G)$ and vertices in $S(G)$, and there are no more edges incident to vertices in $A \cup B$.

Let L be a k-list assignment of a split graph G and F be a split k-forcer connecting $v \in V(G)$. As in Section 4, we say that $c \in L(v)$ is *L-admissible for* v when there is an L-coloring ρ of G such that $\rho(v) = c$ and no monochromatic maximal star has its center in F. The following lemma resembles Lemma 9.

Lemma 15 *Let G be a split graph and F be a split k-forcer connecting $v \in S(G)$. Then,*

   (i) for every k-list assignment L of G there is an L-admissible color for v, and

   (ii) every k-list assignment L of $G \setminus F$ can be extended into a k-list assignment of G in which v has a unique L-admissible color.

Proof: The lemma can be proven with a verbatim copy of the proof of Lemma 9. In particular, observe that a split k-forcer can be obtained from a k-forcer by inserting the edges between $C(a, b)$ and $C(a', b')$, for every $a, a' \in A \cup \{v\}$ and $b, b' \in B$. $\qquad\square$

The hardness of the star choosability problem is also obtained by adapting the proof of Theorem 16. We remark, however, that in this case no induction is required, because split k-switchers are k-choosable by statement (ii) of Lemma 14. The proof is sketched in the following theorem.

Theorem 19 STAR k-CHOOSABILITY *restricted to split graphs is* Π_2^p-*complete for every* $k \geq 2$.

Proof: STAR k-CHOOSABILITY is Π_2^p for split graphs because split graphs have a polynomial amount of maximal stars. Let $\phi(\vec{\mathbf{z}}, \vec{\mathbf{x}})$ be a CNF formula with ℓ clauses $\mathbf{P}_1, \ldots, \mathbf{P}_\ell$, and $m + n$ variables $\vec{\mathbf{x}} = \mathbf{x}_1, \ldots, \mathbf{x}_n$, $\vec{\mathbf{z}} = \mathbf{z}_1, \ldots, \mathbf{z}_m$. Use i, f, h, and q to denote indices in $\{1, \ldots, n\}$, $\{1, \ldots, m\}$, $\{1, \ldots, \ell\}$, and $\{3, \ldots, k\}$, respectively. Define G as the split graph that has a *connection* vertex $x_i \in S(G)$ (resp. $-x_i$, z_f, $-z_f$) *representing* $\mathbf{x}_i$ (resp. $\overline{\mathbf{x}_i}$, $\mathbf{z}_f$, $\overline{\mathbf{z}_f}$), and k *connection* vertices t, $-t$, $y_3, \ldots, y_k$. Let $X = \{x_i, -x_i \mid 1 \leq i \leq n\}$, $Y = \{y_3, \ldots, y_k\}$, $Z = \{z_1, \ldots, z_m\}$, $-Z = \{-z_1, \ldots, -z_m\}$, and $P_h = \{x \mid x \in X \cup Z \cup -Z$ represents a literal in $\mathbf{P}_h\}$. Graph G also has *inner vertices* that are the vertices of split k-switchers and split k-forcers connecting connection vertices. There are split k-forcers connecting z_f, t, $-t$, and split k-switchers connecting: $\{t, x_i, -x_i\}$, $\{-t, x_i, -x_i\}$, $\{z_f, -z_f\}$, $P_k \cup \{t\}$, $P_k \cup \{-t\}$, and $\{v, y_q\}$ for every $v \in X \cup Y \cup -Z \cup \{t, -t\}$.

Following the proof of Theorem 16, it can be observed that, for $k = 2$, G admits a star L-coloring, for a k-list assignment L of G, if and only if $(\forall \vec{\mathbf{x}})(\exists \vec{\mathbf{y}})\phi(\vec{\mathbf{x}}, \vec{\mathbf{y}})$ is nae-true. For $k > 2$, observe that if $(\forall \vec{\mathbf{x}})(\exists \vec{\mathbf{y}})\phi(\vec{\mathbf{x}}, \vec{\mathbf{y}})$ is nae-true, then a star L-coloring ρ is obtained if $\rho(Y)$ is taken so that $|\rho(Y)| = k - 2$ and $\rho(t), \rho(-t) \notin \rho(Y)$. Conversely, if G admits a star L-coloring for the k-list assignment in which $L(v) = \{0, \ldots, k-1\}$ for every connecting vertex $v \notin Z \cup \{t, -t\}$, then $\rho(X) = \{0, 1\}$, thus a nae-valuation of $\phi(\vec{\mathbf{x}}, \vec{\mathbf{y}})$ is obtained. $\square$

To end this section, consider the more general class of $\overline{C_4}$-free graphs. By definition, $\{v\}S$ is a star of a graph G if and only if S is a maximal independent set of $G[N(v)]$. In [8], it is proved that $G[N(v)]$ has $O(d(v)^2)$ maximal independent sets when G is $\overline{C_4}$-free. Thus, $\overline{C_4}$-free graphs have $O(nm)$ maximal stars, which implies that the star coloring and star choosability problems on this class are NP and Π_2^p, respectively.

Theorem 20 STAR k-COLORING *and* STAR k-CHOOSABILITY *are respectively* NP-*complete and* Π_2^p-*complete for every* $k \geq 2$ *when the input is restricted to* $\overline{C_4}$-*free graphs.*

8   Threshold graphs

Threshold graphs form a well studied class of graphs which possesses many definitions and characterizations [10, 21]. The reason for studying them in this article is that threshold graphs are those split graphs with no induced P_4's. Equivalently, a graph is a threshold graph if and only if it is $\{2K_2, P_4, C_4\}$-free.

In this section we develop a linear time algorithm for deciding if a threshold graph G admits a star k-coloring. If affirmative, then a star k-coloring of G can be obtained in linear time. If negative, then a certificate indicating why G admits no coloring is obtained. We prove also that G is star k-choosable if

and only if G admits a star k-coloring. Thus, deciding whether G is star k-choosable takes linear time as well. It is worth noting that threshold graphs can be encoded with $O(n)$ bits using two sequences of natural numbers (cf. below). We begin this section with some definitions on such sequences.

Let $S = s_1, \ldots, s_r$ be a sequence of natural numbers. Each $i \in \{1, \ldots, r\}$ is called an *index* of S. For $k \in \mathbb{N}$, we write $S = [k]$ and $S \leq [k]$ to respectively indicate that $s_i = k$ and $s_i \leq k$ for every index i. Similarly, we write $S > [k]$ when $S \not\leq [k]$, i.e., when $s_i > k$ for some index i. Note that S could be empty; in such case, $S = [k]$ and $S \leq [k]$ for every $k \in \mathbb{N}$. For indices i, j, we use $S[i, j]$ to denote the sequence $s_i, \ldots, s_j$. If $i > j$, then $S[i, j] = \emptyset$. Similarly, we define $S(i, j] = S[i + 1, j]$, $S[i, j) = S[i, j - 1]$, and $S(i, j) = S[i + 1, j - 1]$.

A *threshold representation* is a pair (Q, S) of sequences of natural numbers such that $|Q| = |S| + 1$. Let $Q = q_1, \ldots, q_{r+1}$ and $S = s_1, \ldots, s_r$. Each threshold representation defines a graph $G(Q, S)$ whose vertex set can be partitioned into $r + 1$ blocks $Q_1, \ldots, Q_{r+1}$ with $|Q_1| = q_1, \ldots, |Q_{r+1}| = q_{r+1}$ and r independent sets $S_1, \ldots, S_r$ with $|S_1| = s_1, \ldots, |S_r| = s_r$ such that, for $1 \leq i \leq j \leq r$, the vertices in Q_i are adjacent to all the vertices in $S_j \cup Q_{j+1}$. It is well known that G is a connected threshold graph if and only if it is isomorphic to $G(Q, S)$ for some threshold representation (Q, S) [10, 21]. The following observation describes all the maximal stars of $G(Q, S)$.

Observation 4 *Let (Q, S) be a threshold representation and v be a vertex of $G(Q, S)$. Then, $\{v\}W$ is a maximal star of $G(Q, S)$ if and only if there are indices $i \leq j$ of Q such that $v \in Q_i$, and $W = \{w\} \cup \bigcup_{h=i}^{j-1} S_h$ for some vertex $w \in Q_j$.*

For $k \in \mathbb{N}$, we say that index i of Q is *k-forbidden* for the threshold representation (Q, S) when either $q_i > k$ or $q_i = k$ and there exists some index $j > i$ such that $q_j = k$, $Q(i, j) = [k - 1]$ and $S[i, j) = [1]$. The next theorem shows how to obtain a star k-coloring of G when a threshold representation is provided.

Theorem 21 *The following statements are equivalent for a threshold representation (Q, S).*

1. *$G(Q, S)$ is star k-choosable.*

2. *$G(Q, S)$ admits a star k-coloring.*

3. *No index of Q is not k-forbidden for (Q, S).*

Proof: (i) $\Longrightarrow$ (ii) is trivial.

(ii) $\Longrightarrow$ (iii). Suppose $G(Q, S)$ admits a star k-coloring ρ and yet Q contains some k-forbidden index i. Since Q_i is a block of $G(Q, S)$, then $q_i \leq k$ by Observation 2. Hence, $q_i = k$ and there exists an index $j > i$ such that $q_j = k$, $Q(i, j) = [k - 1]$, and $S[i, j) = [1]$. Let w_h be the unique vertex in S_h for $h \in \{i, \ldots, j - 1\}$. By Observation 2, both Q_i and Q_j have at least one vertex of each color $c \in \{1, \ldots, k\}$, while for each index $h \in \{i + 1, \ldots, j - 1\}$ there exists

a color c_h such that $\rho(Q_h) = \{1, \ldots, k\} \setminus \{c_h\}$. Consequently, there are indices $a < b$ in $\{i, \ldots, j\}$ such that $\rho(w_a) \in \rho(Q_a) \cap \rho(Q_b)$ and $c_h = \rho(w_h) = \rho(w_a)$ for every index $h \in \{a+1, \ldots, b-1\}$. Indeed, it is enough to take $a \in \{i, \ldots, j-1\}$ as the maximum index with $\rho(w_a) \in \rho(Q_a)$ and $b \in \{a, \ldots, j\}$ as the minimum index such that $\rho(w_a) \notin \rho(Q_{a+1}) \cup \ldots \cup \rho(Q_{b-1})$. Therefore, if v_a and v_b are the vertices of Q_a and Q_b with color c, respectively, then $\{v_a\}\{v_b, w_{a+1}, \ldots, w_{b-1}\}$ is a monochromatic maximal star by Observation 4, a contradiction.

(iii) $\Longrightarrow$ (i). Let L be a k-list assignment of $G(Q, S)$ and define w_i as any vertex of S_i for each index i of I. For each index i of Q, define $p(i) \in \{1, \ldots, i\}$ as the minimum index such that $Q(p(i), i] < [k]$. Let ρ be an L-coloring of $G(Q, S)$ that satisfies all the following conditions for every index i of Q:

(1) $|\rho(Q_i)| = q_i$,

(2) if $q_i < k$ and $i \neq r+1$, then $\rho(w_{p(i)})$ and $\rho(w_i)$ do not belong to $\rho(Q_i)$,

(3) if $q_i < k-1$ and $1 < i < r+1$, then $\rho(w_i) \neq \rho(w_{p(i)})$, and

(4) if $s_i > 1$ and $i \neq r+1$, then $|\rho(S_i)| \geq 2$.

A coloring satisfying all the above conditions can obtained iteratively, by coloring the vertices in $Q_i \cup S_i$ before coloring the vertices in $Q_j \cup S_j$ for every pair of indices $i < j$. We claim that ρ is a star L-coloring of $G(Q, S)$. To see why, let $\{v\}W$ be a maximal star of $G(Q, S)$. By Observation 4, there are two indices $i \leq j$ of Q such that $v \in Q_i$ and $W = \{w\} \cup S_i \cup \ldots \cup S_{j-1}$ for some $w \in Q_j$. If $i = j$, then $\rho(v) \neq \rho(w)$ by (1). If $S[i, j] > [1]$, then S_h is not monochromatic by (4). If $q_i < k$, then $\{v\} \cup S_i$ is not monochromatic by (2). If $q_i = k$ and $q_j < k$, then $p(i) \in \{i, \ldots, j-1\}$, thus $\{v\} \cup S_{p(j)} \subset W$ is not monochromatic by (2). Finally, if $q_i = q_j = k$ and $S[i, j] = [1]$, then there exists index h such that $q_h < k-1$; otherwise i would be a k-forbidden index of (Q, S). Then, by (3), $\rho(w_h) \neq \rho(w_{p(h)})$ which implies that W is not monochromatic. Summing up, we conclude that $G(Q, S)$ has no monochromatic maximal star. $\square$

Theorem 21 has several algorithmic consequences for a threshold representation (Q, S) of a graph G. As mentioned, G is a split graph where $Q(G) = \bigcup Q_i$ and $S(G) = \bigcup S_i$, thus $\chi_S(G)$ is either k or $k+1$, for $k = \max(Q)$. While deciding if k colors suffice for a general split graph is an NP-complete problem, only $O(|Q|)$ time is needed to decide whether $\chi_S(G) = k$ when (Q, S) is given as input; it is enough to find a k-forbidden index of Q. Furthermore, if $k < \chi_S(G)$, then a k-forbidden index can be obtained in $O(|Q|)$ time as well. Also, if $\chi_S(G) = k$, then a star k-coloring ρ of G can be obtained in linear time by observing rules (1)–(4) of implication (iii) $\Longrightarrow$ (i). To obtain ρ, begin traversing Q to find $p(i)$ for every index i of Q. Then, color the vertices of each block Q_i with colors $1, \ldots, q_i$. Following, color the vertices $w_1, \ldots, w_r$ in that order, taking the value of q_i into account for each index i of S. If $q_i \geq k-1$, then $\rho(w_i) = k$; otherwise, $\rho(w_i)$ is any value in $\{q_i + 1, \ldots, k\} \setminus \rho(w_{p(i)})$. Finally, color the vertices in $S_i \setminus \{w_i\}$ with color 1 for each index i of S. To encode ρ only

two values are required for each index i of Q, namely, q_i and $\rho(w_i)$. Thus, ρ can be obtained in $O(|Q|)$ time as well. Finally, (Q, S) can be obtained in $O(n+m)$ time from G when G is encoded with adjacency lists [10, 21]. Thus, all the above algorithms take $O(n+m)$ when applied to the adjacency list representation of G. We record all these observation in the theorem below.

Theorem 22 *Let (Q, S) be a threshold representation of a graph G. The problems of computing $\chi_S(G)$, $ch_S(G)$, a k-forbidden index of Q for $k < \chi_S(G)$, and a $\chi_S(G)$-coloring of G can be solved in $O(|Q|)$ time when (Q, S) is given as input. In turn, all these problems take $O(n+m)$ time when an adjacency list representation of G is given.*

9   Further remarks and open problems

In this paper we investigated the time complexity of the star and biclique coloring and choosability problems. In this section we discuss some open problems that follow from our research.

Theorem 1 states that the star k-coloring problem is Σ_2^p-complete even when the input is restricted to $\{C_4,\ K_{k+2}\}$-free graphs. In Section 3.3 we discussed how to generalize this theorem to include $\{C_i,\ K_{k+2}\}$-free graphs, for every $i \in O(1)$. An interesting question is what happens when i grows to infinity, i.e., what happens when chordal graphs are considered. By Theorem 18, we know that the star k-coloring problem is at least NP-hard on chordal graphs. Is it NP-complete or not? Similarly, by Theorem 19, the star k-choosability problem on chordal graphs is at least Π_2^p-hard; is it Π_2^p-complete?

Problem 1 *Determine the time complexity of the star k-coloring (k-choosability) problem on chordal graphs and chordal K_{k+2}-free graphs.*

To prove the Σ_2^p-completeness of the star k-coloring problem we showed how to transform a formula $\phi(\vec{\mathbf{x}}, \vec{\mathbf{y}})$ into a graph G. Graph G has many connection vertices that are joined together by k-keepers and k-switchers. The purpose of the k-keepers is to force two vertices to have the same colors, while k-switchers are used to force different colors on a pair of vertices. Both k-keepers and k-switchers contain blocks of size k. By taking a close examination at Lemmas 1 and 2, it can be seen that these blocks play an important role when colors need to be forced. An interesting question is whether these blocks can be avoided.

Problem 2 *Determine the time complexity of the star k-coloring (k-choosability) problem on graph where every block has size at most j, for $j \leq k$.*

Keepers and switchers not only have blocks of size k; when a k-keeper or a k-switcher connects two vertices v and w, a clique of size $k+1$ containing v and w is generated. We know that the star k-coloring problem is Σ_2^p-complete for K_{k+2}-free graphs, but what happens when K_{k+1}-free graphs are considered? The answer for the case $k = 2$ is given by Theorem 4, i.e., the star 2-coloring problem is easy on K_3-free graphs. And for larger values of k?

Problem 3 *Determine the time complexity of the star k-coloring (k-choosability) problem on K_j-free graphs, for $k > 2$ and $4 \leq j \leq k + 1$.*

In Section 4, the Π_3^p-completeness of the star k-choosability problem on C_4-free graphs is proved by induction. For the case $k = 2$, a graph G_2 is built from a DNF formula using the same ideas that we used to prove the hardness of the star 2-coloring problem. Then, for the case $k > 2$, the graph G_{k-1} is transformed into a graph G_k. The reason for splitting the proof in two cases is that long k-switchers can no longer be included into G_k. So, to avoid the inclusion of induced C_4's, list switchers are used to build G_2, while k-switchers are used to transform G_{k-1} into G_k for $k > 2$. This way, each generated graph G_k is C_4-free. We did not find a way to extend the holes in G_k as it is done in Section 3.3 for the star k-coloring problem. Thus, G_k contains C_5 as an induced subgraph for every $k > 2$, while G_2 contains C_7 as an induced subgraph. Is the problem simpler when such holes are avoided?

Problem 4 *Determine the time complexity of the star k-choosability problem when all the holes in the input graph have length at least i, for $i \in O(1)$.*

The star coloring problem is easier when any of the graphs on three vertices does not appear as an induced subgraph, as discussed in Section 5. Similarly, the biclique coloring problem is simpler when it contains no induced P_3, $\overline{P_3}$, or $\overline{K_3}$, or when it is K_3-free and biclique-dominated. Also, by Theorem 13, the biclique coloring problem on K_3-free graphs is "only" NPwhen it contains no induced $K_{i,i}$ for $i \in O(1)$.

Problem 5 *Determine the time complexity of the biclique k-coloring (k-choosability) problem on K_3-free graphs.*

The star and biclique coloring problems are also simplified when diamonds are forbidden, as seen in Section 6. By Theorem 11, the star k-coloring problem is NPfor diamond-free graphs, while the biclique k-coloring problem is NPfor {diamond, $K_{i,i}$}-free graphs ($i \in O(1)$) by Theorem 13. Additionally, if holes and nets are forbidden, then the star k-coloring problem can be solved easily [26]. This leaves at least two interesting questions.

Problem 6 *Determine the time complexity of the biclique k-coloring problems on diamond-free graphs.*

Problem 7 *Determine the time complexity of the star k-choosability problems on diamond-free graphs.*

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