



Normal Helly circular-arc graphs and its subclasses

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ABSTRACT

A Helly circular-arc model $\mathcal{M} = (C, \mathcal{A})$ is a circle C together with a Helly family \mathcal{A} of arcs of C . If no arc is contained in any other, then \mathcal{M} is a proper Helly circular-arc model, if every arc has the same length, then \mathcal{M} is a unit Helly circular-arc model, and if there are no two arcs covering the circle, then \mathcal{M} is a normal Helly circular-arc model. A Helly (resp. proper Helly, unit Helly, normal Helly) circular-arc graph is the intersection graph of the arcs of a Helly (resp. proper Helly, unit Helly, normal Helly) circular-arc model. In this article we study these subclasses of Helly circular-arc graphs. We show natural generalizations of several properties of (proper) interval graphs that hold for some of these Helly circular-arc subclasses. Next, we describe characterizations for the subclasses of Helly circular-arc graphs, including forbidden induced subgraphs characterizations. These characterizations lead to efficient algorithms for recognizing graphs within these classes. Finally, we show how these classes of graphs relate with straight and round digraphs.

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1. Introduction

A graph is an *interval graph* if its vertices can be arranged into a one-to-one mapping with a family of intervals of the real line in such a way that two vertices of the graph are adjacent if and only if their corresponding intervals have nonempty intersection. The family of intervals is called an *interval model* or *interval representation* of the graph. Similarly, a graph is a *circular-arc graph* if its vertices can be arranged into a one-to-one mapping with a family of arcs of some circle in such a way that two vertices of the graph are adjacent if and only if their corresponding arcs have nonempty intersection. The circle together with the family of arcs is called a *circular-arc model* or *circular-arc representation* of the graph.

Circular-arc graphs are a natural generalization of interval graphs; the intersection of segments of a line (i.e. intervals) is replaced with the intersection of segments of a circle (i.e. arcs). Despite the replacement of a line with a circle does not seem to be a significant modification, circular-arc graphs have a much more complex structure than interval graphs. One of the main reasons behind this complexity is that the circle introduces new ways for arcs to intersect. In an interval model, each family \mathcal{I} of pairwise intersecting intervals is associated with a unique nonempty segment (s, t) of the real line such that (s, t) is contained in every interval of \mathcal{I} , and (s, t) is inclusion-maximal. The situation for circular-arc models is quite different, since for some families of pairwise intersecting arcs there could be zero or more than one such segments. As a result of this generality, many properties that hold for interval graphs cannot be naturally generalized to circular-arc graphs. Let \mathcal{C} be the class of circular-arc models such that every family \mathcal{A} of pairwise intersecting arcs is associated with a unique nonempty segment (s, t) of the circle such that (s, t) is contained in every arc of \mathcal{A} , and (s, t) is inclusion-maximal. It is not hard to see that a circular-arc model \mathcal{M} belongs to \mathcal{C} if and only if \mathcal{M} has no two or three arcs that together cover its circle.

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The circular-arc models in \mathcal{C} are related with two important subclasses of circular-arc models, namely normal and Helly circular-arc models. A circular-arc model is *normal* when no two arcs cover the circle, while it is *Helly* when the arcs of every family of pairwise intersecting arcs share a common point of the circle. It is well known that if a circular-arc model is not Helly, then it contains two or three arcs that together cover the circle. Furthermore, if a normal circular-arc model has three arcs that cover the circle, then the model is not Helly. Consequently, \mathcal{C} is precisely the class of circular-arc models that are simultaneously normal and Helly. We define the *normal Helly* circular-arc graphs as those circular-arc graphs that admit a model in \mathcal{C} .

The purpose of this paper is to study the class of normal Helly circular-arc graphs, as well as its subclasses. Section 2 contains the basic definitions used throughout the paper, as well as the definitions of the many families of circular-arc models and graphs. Several reasons for studying these subclasses of circular-arc graphs are given in Section 3, where many properties that hold for interval and normal Helly circular-arc graphs, but do not hold for the general classes of normal circular-arc and Helly circular-arc graphs, are compiled. Of course, there is also a completely theoretical motivation for studying normal Helly circular-arc graphs, namely to study all the circular-arc subclasses that are obtained by the intersection of the most well known subclasses of circular-arc models. The forbidden induced subgraph characterizations of the normal Helly subclasses are described in Section 4, while the corresponding recognition problems are treated in Section 5. Finally, Section 6 provides some additional results on normal Helly circular-arc graphs that further relate this class of graphs with the class of interval graphs, and Section 7 discusses some possibilities for future research.

2. Preliminaries

For a graph G , we use $V(G)$ and $E(G)$ to denote the sets of vertices and edges of G , respectively, while we use n and m to denote $|V(G)|$ and $|E(G)|$, respectively. We will write uv to represent the edge of G between the pair of adjacent vertices u and v . For $v \in V(G)$, denote by $N_G(v)$ the set of vertices adjacent to v , and $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of v is $d_G(v) = |N_G(v)|$, and v is *universal* if $N_G[v] = V(G)$. Two vertices v and w are *twins* in G if $N_G[v] = N_G[w]$. We omit the subscript from N and d when there is no ambiguity about the referred graph.

The subgraph of G induced by $V \subseteq V(G)$, denoted by $G[V]$, is the graph that has V as vertex set and two vertices of $G[V]$ are adjacent if and only if they are adjacent in G . A *complete set* is a subset of pairwise adjacent vertices, while a *clique* is a maximal complete set. A *hole* is a chordless cycle with at least four vertices. A *bipartite* graph is a graph whose vertex set can be partitioned into two sets of pairwise non-adjacent vertices, where one of the partitions may be empty. A *co-bipartite* graph is the complement of a bipartite graph.

Similarly as for graphs, we use $V(D)$ and $E(D)$ to refer to the sets of vertices and edges of a digraph D , respectively, while we use n and m to denote $|V(D)|$ and $|E(D)|$, respectively. We write $v \rightarrow w$ to indicate that there is a directed edge from v to w in D , and $v \not\rightarrow w$ to indicate that there is no such directed edge. The *underlying graph* of D is the graph $G(D)$ with vertex set $V(D)$ such that v and w are adjacent in G if and only if v and w are adjacent in D . When $v \rightarrow w$, we say that v is an *in-neighbor* of w and that w is an *out-neighbor* of v . The *inset* of v is the set $N_D^-(v)$ of all the in-neighbors of v , and the *outset* of v is the set $N_D^+(v)$ of all the out-neighbors of v . The *closed inset* is $N_D^-[v] = N_D^-(v) \cup \{v\}$, and the *closed outset* is $N_D^+[v] = N_D^+(v) \cup \{v\}$. The cardinality of $N_D^-(v)$, denoted by $d_D^-(v)$, is the *indegree* of v and the cardinality of $N_D^+(v)$, denoted by $d_D^+(v)$, is the *outdegree* of v . As for graphs, we omit the subscripts in N and d when there is no ambiguity about the referred digraph.

The subdigraph of D induced by $V \subseteq V(D)$, denoted by $D[V]$, is the digraph that has V as vertex set and $v \rightarrow w$ in $D[V]$ if and only if $v \rightarrow w$ in D . A digraph D is an *oriented graph* when either $v \rightarrow w$ or $w \rightarrow v$, for every $v, w \in V(D)$. In other words, D is an oriented graph if it can be obtained from a graph G by choosing an orientation for each edge vw of G . In such case, we call D an *orientation* of G , i.e., D is an *orientation* of G if D is an oriented graph whose underlying graph is isomorphic to G .

In this paper we mainly deal with collections that are of a circular (cyclic) nature, such as circles (viewed as a collection of points), circular orderings, etc. Usually, the objects of a collection are labeled with some kind of index that identifies the position of the object inside the collection. Unless otherwise stated, we assume that all the operations on these indices are taken modulo the length of the collection. Furthermore, we may refer to negative indices and to indices greater than the length of the collection. In these cases, indices should also be understood modulo the length of the collection.

2.1. Circular-arc models

A *circular-arc model* \mathcal{M} is a pair $(C(\mathcal{M}), \mathcal{A}(\mathcal{M}))$ where $C(\mathcal{M})$ is a circle and $\mathcal{A}(\mathcal{M})$ is a finite family of open arcs of $C(\mathcal{M})$. The arcs in $\mathcal{A}(\mathcal{M})$ are said to be *arcs* of \mathcal{M} , and $C(\mathcal{M})$ is said to be the *circle* of \mathcal{M} . Unless explicitly stated, we always choose the clockwise direction for traversing $C(\mathcal{M})$. For $s, t \in C(\mathcal{M})$, write (s, t) to mean the open arc of $C(\mathcal{M})$ defined by traversing the circle from s to t . Call s, t the *extremes* of (s, t) , while s is the *beginning point* and t is the *ending point* of the arc. For each $A \in \mathcal{A}(\mathcal{M})$, represent by $s(A)$ the beginning point of A and by $t(A)$ the ending point of A . The *extremes* of \mathcal{M} are those of all the arcs $A \in \mathcal{A}(\mathcal{M})$.

Throughout this paper, we assume that no pair of extremes of a circular-arc model coincide and that no single arc entirely covers the circle of the model (see [15]). A *segment* of \mathcal{M} is an arc of $C(\mathcal{M})$ formed by two consecutive extremes of \mathcal{M} . Say that $\epsilon > 0$ is *small enough* if $\epsilon < \ell$, where ℓ is the minimum among the lengths of all the segments of \mathcal{M} . *Duplicating the*

Table 1

Subclasses of circular-arc models. The five columns on the right show the properties that each class of model satisfies.

Name of the model	Acronym	I	U	P	H	N
Circular-arc	CA	0	0	0	0	0
Normal circular-arc	NCA	0	0	0	0	1
Helly circular-arc	HCA	0	0	0	1	0
Normal Helly circular-arc	NHCA	0	0	0	1	1
Proper circular-arc	PCA	0	0	1	0	0
Normal proper circular-arc	PNCA	0	0	1	0	1
Proper Helly circular-arc	PHCA	0	0	1	1	0
Normal proper Helly circular-arc	NPHCA	0	0	1	1	1
Unit circular-arc	UCA	0	1	1	0	0
Normal unit circular-arc	NUCA	0	1	1	0	1
Unit Helly circular-arc	UHCA	0	1	1	1	0
Normal unit Helly circular-arc	NUHCA	0	1	1	1	1
Interval	IG	1	0	0	1	1
Proper interval	PIG	1	0	1	1	1
Unit interval	UIG	1	1	1	1	1

arc A means inserting the arc $(s(A) + \epsilon, t(A) + \epsilon)$ into \mathcal{M} , for some small enough ϵ . An s -sequence is a maximal sequence of consecutive beginning points. Similarly, a t -sequence is a maximal sequence of consecutive ending points. In general, an *extreme sequence* means either an s -sequence or a t -sequence. The *complement* of $A \in \mathcal{A}(\mathcal{M})$ is the arc $\bar{A} = (t(A), s(A))$, and the *complement* of \mathcal{M} is the circular-arc model $\bar{\mathcal{M}} = (C(\mathcal{M}), \{\bar{A} \mid A \in \mathcal{A}(\mathcal{M})\})$.

There are five elementary classes of circular-arc models that are of special interest [35]. Let \mathcal{M} be a circular-arc model. If the arcs of every family of pairwise intersecting arcs share a common point, then \mathcal{M} is a *Helly circular-arc (HCA)* model. If \mathcal{M} has no pair of arcs that together cover the circle, then \mathcal{M} is a *normal circular-arc (NCA)* model. When no arc is contained in any other arc, we say that \mathcal{M} is a *proper circular-arc (PCA)* model. If, in addition, all the arcs have the same length, then \mathcal{M} is a *unit circular-arc (UCA)* model. Finally, \mathcal{M} is an *interval circular-arc (ICA)* model when some point of $C(\mathcal{M})$ is covered by no arcs.

An *interval model* is a finite family \mathcal{I} of open intervals on the real line. It is easy to see that every interval circular-arc model \mathcal{M} is in a one-to-one correspondence with an interval model \mathcal{I} . For this reason we will use interval models in replacement of ICA models, and every definition on circular-arc models is translated to interval models with such correspondence.

The five elementary classes of circular-arc models can be combined so as to generate a total of 32 classes of circular-arc models, as follows. Let $X \subseteq \{N, P, U, H, I\}$. Say that \mathcal{M} is an X CA model if \mathcal{M} is an x CA model, for every $x \in X$. For instance, \mathcal{M} is an $\{N,H\}$ CA model if \mathcal{M} is both an NCA and an HCA model. Clearly, if \mathcal{M} is an X CA model, then \mathcal{M} is also a Y CA for every $Y \subseteq X$. Not all the 32 classes of circular-arc models are different, because some of the properties are implied by others. For example, every UCA model is proper, thus every UCA model is a $\{U, P\}$ CA model as well. Similarly, every interval model is Helly and normal. This leaves us with 15 different classes of circular-arc models that are listed in Table 1. To avoid the set notation to name a class of circular-arc models, we choose a better acronym for each class of circular-arc models.

2.2. Circular-arc graphs

A graph G is a *circular-arc graph* if there is a one-to-one correspondence between the vertices of G and a family \mathcal{A} of arcs of a circle C such that two vertices are adjacent if and only if their corresponding arcs have nonempty intersection. The circular-arc model (C, \mathcal{A}) is called a *model* or *representation* of G , and G is said to *admit* the model (C, \mathcal{A}) . In other words, G is a circular-arc graph if it is isomorphic to the intersection graph of \mathcal{A} . Say that two circular-arc models are *equivalent* when their intersection graphs are isomorphic.

By restricting the attention to a subclass of circular-arc models, we obtain a special class of circular-arc graphs, formed by those graphs that admit a model of the subclass. For $X \subseteq \{N, P, U, H, I\}$, say that G is an X CA graph when G admits an X CA model. As before, these 32 classes of circular-arc graphs are not all different. Not even the fifteen classes defined by the special models in Table 1 are all different, because graphs may admit many circular-arc models.

Theorem 1 ([44], See Theorem 8). *Every PIG model is equivalent to a UIG model.*

Theorem 2 ([48], See Theorem 4). *Every PCA model is equivalent to an NPCA model.*

In total, ten different subclasses of circular-arc graphs are obtained by combining the normal, proper, unit, Helly, and interval properties of a circular-arc model. These classes are enumerated in Table 2. As before, we avoid the set notation by defining appropriate acronyms for each subclass. For historic reasons, proper interval graphs are also called unit interval graphs and the acronym UIG is also used for PIG graphs.

In this paper we will show characterizations, by means of forbidden induced subgraphs, for the three subclasses of NHCA graphs. For PHCA and UHCA graphs we are able to describe the family of forbidden induced subgraphs, while for NHCA graphs we only show those circular-arc graphs that are not NHCA. The characterizations for the three NHCA subclasses yield

Table 2
Subclasses of circular-arc graphs, according to the class of models that they admit.

Name of the class	Acronyms	Admitted models
Circular-arc	CA	CA
Normal circular-arc	NCA	NCA
Helly circular-arc	HCA	HCA
Normal Helly circular-arc	NHCA	NHCA
Proper circular-arc	PCA	PCA and NPCA
Proper Helly circular-arc	PHCA	PHCA and NPHCA
Unit circular-arc	UCA	UCA and NUCA
Unit Helly circular-arc	UHCA	UHCA and NUHCA
Interval	IG	Interval
Proper interval (unit interval)	PIG (UIG)	UIG and PIG

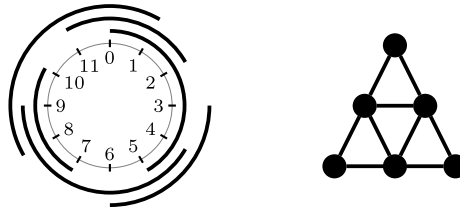


Fig. 1. The circular-arc model $CI(3, 1)$ and its intersection graph.

$O(n + m)$ time algorithms for the related recognition problems. Such characterizations and algorithms are known for almost all the classes of circular-arc graphs listed in Table 2. For the classes of interval, PIG, UCA, and PCA graphs the families of forbidden induced subgraphs are known (see [29] for interval graphs, [44] for PIG graphs, and [48] for UCA and PCA graphs), while, for HCA graphs, only the list of forbidden circular-arc graphs is known [24]. With respect to the recognition problem, there are $O(n + m)$ time recognition algorithms for the classes of CA, PCA, UCA, HCA, interval, and PIG graphs (see, e.g., [25,39] for CA, [7,26] for PCA, [26,36] for UCA, [24] for HCA, [3] for interval graphs, and [4,7] for PIG), but there is no known polynomial time recognition algorithm for NCA graphs. There is, however, an $O(n^5 m^6 \log m)$ time algorithm to recognize co-bipartite NCA graphs [42,18].

Let ϵ be a small enough value for a circular-arc model \mathcal{M} . If we replace an arc $A \in \mathcal{A}(\mathcal{M})$ with the arc $(s(A) + \epsilon, t(A) + \epsilon)$, we obtain a model \mathcal{M}' equivalent to \mathcal{M} . This implies that any circular-arc graph admits an infinite number of equivalent models. However, \mathcal{M} and \mathcal{M}' are essentially the same model, since all we are interested is in how the arcs of \mathcal{M} intersect. Say that two circular-arc models \mathcal{M} and \mathcal{M}' have *equal extremes*, or simply that \mathcal{M} and \mathcal{M}' are *equal*, if their extremes appear in the same order. With this definition, every circular-arc graph admits a finite number of non-equal models. When we informally say that a graph G admits k models, we mean that G admits k non-equal models.

We will use the same terminology employed for graphs and vertices when talking about circular-arc models and arcs. For example, we say that an arc is *universal* to mean that its corresponding vertex is universal in the intersection graph. Similarly, we call a model *connected* when its intersection graph is connected. For $k \in \mathbb{N}_0$, define $U_k(\mathcal{M})$ to be the model obtained from \mathcal{M} by removing all but k of its universal arcs, if they exist. Clearly, $\mathcal{M} = U_k(\mathcal{M})$ if and only if \mathcal{M} has at most k universal arcs.

In a circular-arc model, all the arcs that cover some point p of the circle form a complete set. If this complete set is also a clique \mathcal{Q} , then we call p a *clique point* and we say that \mathcal{Q} is *represented* by p . Recall that, by definition, a circular-arc model is Helly precisely when the arcs of every family of pairwise intersecting arcs share a common point. In other words, an HCA model is a circular-arc model in which every clique is represented by a clique point.

Tucker [48] defined the class of $CI(n, k)$ graphs so as to characterize those PCA graphs that are not UCA. For relative prime values n and k such that $n > 2k$, define $CI(n, k)$ as the circular-arc model that is built as follows (see also Fig. 1). Let C be a circle of length $4n$. Draw n arcs A_0, \dots, A_{n-1} of length $4k + 1$ such that each A_i begins at $4ki$ and ends at $4k(i + 1) + 1$. Afterwards, draw n arcs B_0, \dots, B_{n-1} of length $4k - 1$ such that each B_i begins at $4ki + 2k + 1$ and ends at $4k(i + 1) + 2k$. The intersection graph of the model $CI(n, k)$ is called the $CI(n, k)$ graph. Note that every $CI(n, k)$ model is an NPCA model by definition.

Theorem 3 ([48]). *Let G be a PCA graph. Then G is a UCA graph if and only if it contains no $CI(n, k)$ as an induced subgraph, with n and k relative primes and $n > 2k$.*

As we already mentioned, Tucker [48] proved that every PCA graph admits an NPCA model (see also [15]). A simple proof was recently obtained by Kaplan and Nussbaum [26] using the following lemma.

Lemma 1 ([26]). *Let \mathcal{M} be a PCA model of a graph G . If A_1, A_2 are two arcs of \mathcal{M} that together cover the circle, then both A_1 and A_2 are universal.*

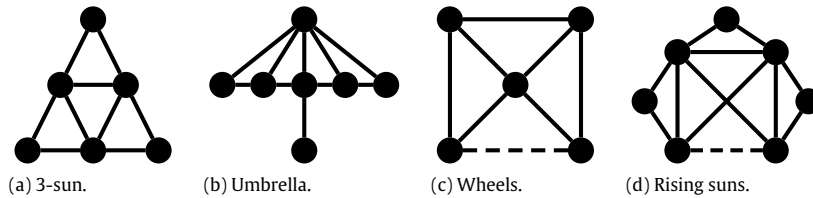


Fig. 2. HCA graphs that are not NHCA. Wheels have at least 5 vertices and rising suns have at least 7 vertices.

Tucker’s theorem is obtained as a corollary.

Theorem 4 ([26,48]). *Every PCA model \mathcal{M} is equivalent to an NPCA model. Furthermore, if \mathcal{M} has k universal arcs, then an NPCA model equivalent to \mathcal{M} can be obtained by duplicating $k - 1$ times the universal arc of $U_1(\mathcal{M})$, if it exists.*

The furthermore part of Theorem 4 yields an $O(n)$ time algorithm for transforming a PCA model \mathcal{M} of a graph G into an NPCA model of G , in $O(n)$ time. First compute $U_1(\mathcal{M})$ and then duplicate the universal arc $k - 1$ times. To remove the universal arcs of \mathcal{M} so as to obtain $U_1(\mathcal{M})$, the following corollary of Lemma 1 is used.

Lemma 2 ([26]). *An arc A of a PCA model \mathcal{M} is universal if and only if A contains at least $n - 1$ extremes of \mathcal{M} .*

3. Why study subclasses of NHCA graphs?

One of the most important algorithmic properties about interval models is that the arcs of every family of pairwise intersecting intervals share a common point. The first linear-time recognition algorithm for interval graphs is based on this fact [3], as are as well some of the newer recognition algorithms (e.g. [16]). When interval models are generalized to circular-arc models, the Helly property is lost. The class of Helly circular-arc graphs lies between the CA and IG classes and, for this reason, HCA graphs preserve many nice properties of interval graphs that are lost even for UCA graphs. Usually, these properties involve the cliques of the graphs. Just to give one of the many examples, consider the two theorems below.

Theorem 5 ([14,11]). *Let G be a graph. Then G is an interval graph if and only if there is an ordering of the cliques of G such that, for every vertex v , the cliques containing v form a linear range of the ordering.*

Theorem 6 ([13]). *Let G be a graph. Then G is an HCA graph if and only if there is an ordering of the cliques of G such that, for every vertex v , the cliques containing v form a range of the ordering.*

It is not hard to find a counterexample to Theorem 6 when HCA graphs are replaced by circular-arc graphs. Of course, there are also some properties that hold for interval graphs but do not hold for HCA graphs, because in an HCA model there could be two arcs covering the circle. In a similar way, we can expect the gap from UIG graphs to UCA graphs to be as big as the gap from interval graphs to circular-arc graphs, as well as that from UIG graphs to PCA graphs.

In this section we motivate the study of the NHCA subclasses by showing some of the properties that are lost because of these gaps. That is, we show several properties that hold in the NHCA subclasses and are not easy to extend to the more general classes of circular-arc graphs. These properties are generalizations or slight modifications of properties that hold for the interval graph subclasses, in the same sense as Theorems 5 and 6.

The first property that we describe is what we call the “local interval” property. Clearly, in an interval model \mathcal{I} , the submodel of \mathcal{I} induced by the closed neighborhood of an interval I is an interval model, for every $I \in \mathcal{I}$. This property is not true for general HCA models and for general NCA models. Even more, there are some graphs, such as the 4-wheel (see Fig. 2), for which no circular-arc model satisfies the local interval property. On the other hand, if we restrict our attention to an NHCA model \mathcal{M} , we can see that the submodel of \mathcal{M} induced by the closed neighborhood of an arc A is an interval model, for every $A \in \mathcal{A}(\mathcal{M})$. Hence, NHCA models will retain many properties of interval models that deal with the interval nature of the neighborhood of an arc. The local interval property of NHCA models follows from Theorem 7 below which is observed by McKee in [40]. For the sake of simplicity, we take Theorem 7 as an alternative definition of NHCA, PHCA, and UHCA graphs. Hence, sometimes we omit the reference to this theorem.

Theorem 7 ([40]). *An x CA model is x HCA if and only if no two or three arcs cover its circle, for $x \in \{N, P, U\}$.*

Corollary 1 (Local Interval Property for Models). *A circular-arc model \mathcal{M} is an NHCA model if and only if, for every $A \in \mathcal{A}(\mathcal{M})$, the submodel of \mathcal{M} induced by $N[A]$ is an interval model.*

The local interval property of NHCA graphs implies that many simple algorithms that work for interval graphs also work for NHCA graphs. Consider, for example, the problem of finding every clique of an interval graph. A *clique segment* in an interval model \mathcal{M} is a segment (s, t) where s is a beginning point and t is an ending point. In \mathcal{M} , every clique point belongs to a clique segment, and all the points in a clique segment are clique points. Thus, the set of intervals that contain a given

clique segment induce a clique and each clique is represented by exactly one clique segment. It is trivial to find every clique of an interval graph by computing the clique segments in one of its interval models. For HCA graphs the situation is quite different. First, not every segment of the form (s, t) is formed by clique points (with s and t being two consecutive beginning and ending points, respectively). Second, not every clique of the model is represented by exactly one clique segment, i.e., there are several clique segments whose containing arcs induce the same clique. Thus, the $O(n)$ time algorithm to find every clique point is not so trivial (see [30,47]). However, NHCA graphs share the same property as interval graphs: every segment (s, t) represents a clique, and every clique is represented by exactly one of such clique segments. Thus, the same algorithm for interval graphs works as well for NHCA graphs. Another example in where the local interval property has some advantages is in the dynamic recognition problem. It is somewhat easy to insert a new edge in $O(1)$ time into a PHCA graph, but it is not so easy for PCA graphs (see [47]). Section 6.3 further develops these interval concepts for NHCA and PHCA graphs.

Another similarity between interval and NHCA graphs has to do with the forbidden induced subgraph structure. For proper interval graphs the structure is described by the following theorem by Roberts. Here $K_{1,3}$ is the graph with four vertices, where one of these vertices is universal and the other three vertices are pairwise non-adjacent.

Theorem 8 ([44]). *Let G be an interval graph. Then the following are equivalent:*

- (i) G does not contain $K_{1,3}$ as an induced subgraph.
- (ii) G is a proper interval graph.
- (iii) G is a unit interval graph.

For NHCA graphs we obtain a similar result.

Theorem 9. *Let G be an NHCA graph. Then G is a PHCA graph if and only if G contains no $K_{1,3}$ as an induced subgraph.*

Proof. Clearly, the $K_{1,3}$ graph is not PCA, so it is not PHCA. For the converse, suppose that G is an NHCA graph with an NHCA model \mathcal{M} . Sort every extreme sequence of \mathcal{M} in such a way that no arc with an extreme in the sequence is properly contained in another arc with an extreme in the sequence (such sorting is always possible, see e.g. [43] or Algorithm 4). This sorting does not change the intersections between the arcs of \mathcal{M} , thus the sorted model \mathcal{M}' is also an NHCA model of G . Now, if some arc $A_1 \in \mathcal{A}(\mathcal{M}')$ is contained in some other arc $A_2 \in \mathcal{A}(\mathcal{M}')$, it is because there is some ending point between $s(A_1)$ and $s(A_2)$, and there is some beginning point between $t(A_2)$ and $t(A_1)$. In other words, there are two arcs L and R of \mathcal{M}' such that $s(A_1), t(L), s(A_2), t(A_2), s(R)$ and $t(A_1)$ appear in this order in \mathcal{M}' . Since \mathcal{M}' is NHCA then $L \neq R$ and $L \cap R = \emptyset$, i.e., the intersection graph of A_1, L, A_2, R is isomorphic to $K_{1,3}$. \square

Implication (ii) \implies (iii) of Theorem 8 is lost, because the graph $CI(n, k)$ is PHCA but not UHCA for every $n > 3k$. Indeed, by definition, every $CI(n, k)$ graph admits a PCA model in which no family with at most $\lceil n/k \rceil > 3$ arcs cover the circle. To retain the “proper=unit” property with a similar definition as the one in Theorem 7, we should ask that no set of arcs cover the circle. This is because the unit length property of UIG graphs is global, and it does not depend only on the neighborhood of each vertex.

Next, we consider clique graphs of HCA graphs and NHCA graphs. The *clique graph* $K(G)$ of a graph G is the intersection graph of the cliques of G , and for a class \mathcal{C} of graphs we denote $K(\mathcal{C}) = \{K(G) \mid G \in \mathcal{C}\}$. Clique graphs of interval graphs were studied by Hedman [17], who proved that the PIG and $K(\text{IG})$ classes are equal. On the other hand, clique graphs of HCA graphs were first studied by Durán and Lin [9], who proved that clique graphs of HCA graphs are both PCA and HCA. The question that motivated us to study PHCA graphs first, and NHCA graphs later, was whether or not the class of PHCA graphs is equal to the $K(\text{HCA})$ class. The answer is no, but almost, as it is shown in the next theorem.

Theorem 10 ([31]). *Let H be a graph and U the set of universal vertices of H . Then H is the clique graph of some HCA graph G if and only if:*

- (i) H is a PHCA graph or
- (ii) $H \setminus U$ is a co-bipartite PHCA graph and $|U| \geq 2$.

However, the analog of Hedman’s result can be obtained for NHCA graphs [31]. That is, the $K(\text{NHCA})$ and the PHCA classes of graphs are equal. Hedman [17] proved also that for every PIG graph G there is a PIG graph H such that $K(H)$ is isomorphic to G . That is, the PIG and the $K(\text{PIG})$ classes are also equal. The same result holds for PHCA graphs, i.e., for every PHCA graph G there is a PHCA graph H such that $K(H)$ and G are isomorphic [31].

McKee [40] found many other similarities between the $K(\text{NHCA})$ and $K(\text{IG})$ classes that cannot be easily generalized to $K(\text{HCA})$. A *clique path* of a graph G is a spanning path P of $K(G)$ such that, for every $v \in V(G)$, the cliques containing v are consecutive in P . Similarly, a *clique cycle* of G is a spanning cycle C of $K(G)$ such that, for $v \in V(G)$, the cliques containing v are consecutive in C . A vertex is *simplicial* when it belongs to a unique clique. By Theorem 5, every connected interval graph admits a clique path, while by Theorem 6 every non-interval HCA graph admits a clique cycle. A major difference between interval graphs and HCA graphs is that the former admit a unique clique path when at most two simplicial vertices are allowed, while the latter may admit many clique cycles even when they have no simplicial vertices. Part of the reason why HCA graphs with no simplicial vertices admit many clique cycles is due to the fact that some cliques can be represented by many clique segments. This is not the case with NHCA graphs (cf. above) and, moreover, non-interval NHCA graphs with

no simplicial vertices admit a unique clique cycle as proved by McKee [40]. Clique paths have other properties that can be generalized to clique trees but that do not generalize well to clique cycles; McKee shows that some of these properties are satisfied by clique cycles of non-interval NHCA graphs.

Now we consider a property of PIG graphs that can be preserved by restricting the attention to PHCA graphs. A classic characterization of interval graphs is that interval graphs are those graphs whose clique matrix has the consecutive-ones property for columns [14]. Similarly, a graph is HCA if and only if its clique matrix has the circular-ones property for columns [13]. The definitions of clique matrix, consecutive and circular-ones properties are given in Section 6. We can think that the consecutive and circular-ones properties for columns are due to the Helly property of the interval and HCA graphs, respectively. On the other hand, it is well known that a graph is a proper interval graph if and only if its clique matrix has the consecutive-ones property for both its rows and its columns (see e.g. [8, 10, 12]). The analogous theorem for the circular-ones property can be proved for PHCA graphs (see Section 6). That is, the clique matrix of a graph has the circular-ones properties for both rows and columns if and only if the graph is a PHCA graph.

Finally, the boxicity of NHCA graphs was studied by Bhowmick and Chandran [2]. Given two graphs G_1 and G_2 with the same vertex set V , define the *intersection* of G_1 and G_2 as the graph $G_1 \cap G_2$ with vertex set V such that vw is an edge of $G_1 \cap G_2$ if and only if vw is an edge of both G_1 and G_2 . The *boxicity* of a graph G is the minimum number of interval graphs whose intersection is isomorphic to G . Clearly, interval graphs are precisely the graphs with boxicity 1. On the other hand, the boxicity of a general circular-arc graph can be as large as $n/2$ [45]. Bhowmick and Chandran proved that NHCA graphs have boxicity at most 3. If, furthermore, no four arcs of a circular-arc model cover the circle, then the boxicity is at most 2.

4. The structure of the NHCA subclasses

In this section we present characterizations by forbidden induced subgraphs for the classes of NHCA, PHCA, and UHCA graphs. These characterizations follow the same spirit as Theorem 9, in the sense that they show when a graph of some of these classes belongs to a subclass of it. The characterizations shown in this section immediately yield $O(n + m)$ time recognition algorithms for all the classes. In the next section we further discuss the algorithmic implications of these characterizations.

The following proposition is used several times throughout this section.

Proposition 1. *Let \mathcal{M} be a circular-arc model and B_1, \dots, B_k be a hole in \mathcal{M} . If $A \in \mathcal{A}(\mathcal{M})$ is an arc which is not contained in any other arc, then either:*

- (i) A and B_i cover the circle for some $1 \leq i \leq k$,
- (ii) A, B_i , and B_{i+1} cover the circle for some $1 \leq i \leq k$,
- (iii) $A \subset (B_i \cup B_{i+1}) \setminus (B_{i+2} \cup \dots \cup B_{i-1})$ for some $1 \leq i \leq k$, or
- (iv) A, B_i, \dots, B_j is an induced hole of \mathcal{M} , for some $1 \leq i, j \leq k$.

Proof. If $A = B_i$ for some $1 \leq i \leq n$, then (iv) follows. Suppose then that A is not an arc of the hole. Traverse $C(\mathcal{M})$ from $t(A)$ and let B_i be the arc whose beginning point appears first. If $s(B_i) \in A$ then A and B_{i-1} must cover the circle. Otherwise, if $t(B_i) \in A$ then A, B_{i-1} , and B_i cover the circle. Finally, suppose that neither $s(B_i)$ nor $t(B_i)$ are points of A , and let B_j be the arc whose ending point appears first in a counterclockwise traversal of $C(\mathcal{M})$ from $s(A)$. If $i - 1 = j + 1$, then it follows that $A \subset B_{i-1}$, which is a contradiction to the fact that A is not properly contained in any other arc. Otherwise, $A, B_{i-1}, \dots, B_{j+1}$ induce a hole whenever $i - 1 \neq j + 2$ or $A \subset (B_{i-2} \cup B_{i-1}) \setminus (B_i \cup \dots \cup B_{i-3})$ whenever $i - 1 = j + 2$. \square

4.1. Normal Helly circular-arc graphs

We begin with the problem of determining when an HCA graph admits an NHCA model. Wheels, 3-suns, rising suns, and umbrellas are the forbidden subgraphs involved in the characterization. The 3-sun and the umbrella are the graphs depicted in Fig. 2(a) and (b), respectively. The n -wheel, for $n \geq 4$, is the graph obtained by inserting one universal vertex into a hole of length n (see Fig. 2(c)). Finally, the n -rising sun, for $n \geq 4$, is the graph that is obtained from a path v_2, \dots, v_{n-1} by first adding two universal vertices v_1 and v_n , and then inserting three vertices w_1, w_{n-1}, w_n such that w_i is adjacent only to v_i and to v_{i+1} , for $i \in \{1, n - 1, n\}$ (see Fig. 2(d)). The 3-sun graph is denoted by S_3 , the umbrella is denoted by U , the n -wheel is denoted by W_n , and the n -rising sun is denoted by R_n .

Interval graphs were characterized by families of forbidden induced subgraphs by Lekkerkerker and Boland [29]. When restricted to HCA graphs, the characterization reads as follows.

Theorem 11 ([29]). *An HCA graph is an interval graph if and only if it does not contain holes, 3-suns, rising suns, or umbrellas as induced subgraphs.*

McKee [40] provides a characterization of NHCA graphs in terms of its cycles, in which he observes that wheels are not NHCA graphs. It turns out the characterization by Lekkerkerker and Boland can be generalized to NHCA graphs by replacing the forbidden holes by wheels. In order to prove the generalization, we analyze first how do HCA models of NHCA graphs look.

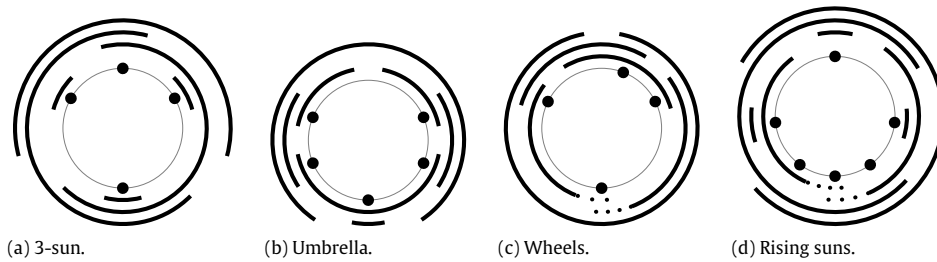


Fig. 3. HCA models of the graphs in Fig. 2. Points in the circles are used to mark the clique points.

Theorem 12. Let \mathcal{M} be an HCA model of a graph G . Then, \mathcal{M} is equivalent to an NHCA model if and only if \mathcal{M} contains no wheels, 3-suns, rising suns, or umbrellas as induced submodels.

Proof. Let \mathcal{M}' be any HCA model of a wheel W_k , for $k \geq 4$. Such a model exists as it is depicted in Fig. 3(c). Model \mathcal{M}' has at least one clique point for each clique of W_k ; the universal arc covers all of these clique points, while each of the other arcs of the submodel covers exactly two of them. Consequently, there are two arcs covering $C(\mathcal{M}')$, i.e., \mathcal{M}' is not normal. Thus, whenever \mathcal{M} is equivalent to an NHCA model, \mathcal{M} does not contain an induced submodel of W_k . The proofs for the 3-sun, the umbrella and the rising suns follow analogously.

For the converse, suppose that \mathcal{M} contains none of the forbidden submodels and yet \mathcal{M} has two arcs A_1 and A_2 that cover the circle which, w.l.o.g., are not contained in other arcs of \mathcal{M} . Suppose also, to obtain a contradiction, that \mathcal{M} has k arcs B_1, \dots, B_k that induce a hole in that order. If $A_1 = B_i$ for some $1 \leq i \leq k$, then A_2 is adjacent to all the arcs of the hole, i.e., A_2, B_1, \dots, B_k induce a wheel. Otherwise, we need to consider four cases by Proposition 1:

1. A_1 and B_i cover the circle for some $1 \leq i \leq k$. In this case, A_1 intersects all the arcs of the hole, i.e., A_1, B_1, \dots, B_k induce a wheel. Therefore, this case cannot happen.
2. A_1, B_i , and B_{i+1} cover the circle for some $1 \leq i \leq k$. Then 1. holds because \mathcal{M} is Helly.
3. A_1 is contained in $B_i \cup B_{i+1}$ for some $1 \leq i \leq k$. In this case, A_2 intersects all the arcs of B_1, \dots, B_k , thus A_2, B_1, \dots, B_k induce a wheel. This case is also impossible.
4. A_1, B_i, \dots, B_j is a hole of \mathcal{M} for some $1 \leq i, j \leq k$. As in Case 2, A_2 intersects all the arcs of this new hole. Thus, $A_2, A_1, B_i, \dots, B_j$ induce a wheel, again a contradiction.

Since none of the above cases can occur, it follows that \mathcal{M} has no induced holes. Thus, G is hole-free and it contains no rising suns, 3-suns, or umbrellas, which implies that G is an interval graph by Theorem 11. Consequently, \mathcal{M} is equivalent to some interval model of G . \square

The characterization by minimal forbidden induced subgraphs then follows easily.

Corollary 2. An HCA graph is NHCA if and only if it contains no wheels, 3-suns, rising suns, or umbrellas as induced subgraphs.

There is also a strong consequence for circular-arc models that can be used for negative certification and which is also useful for the characterization of NHCA graphs in terms of NCA graphs.

Corollary 3. Every circular-arc model of a non-interval NHCA graph is NHCA.

Proof. Let \mathcal{M} be any circular-arc model of a non-interval NHCA graph G . Since G is NHCA then we can apply the algorithm in [37] with input \mathcal{M} , to obtain an HCA model \mathcal{M}' of G . By definition, \mathcal{M}' is equivalent to some NHCA model, because G is NHCA. If \mathcal{M}' has two arcs that cover the circle, then we can use the same arguments as in Theorem 12 to prove that G is an interval graph, a contradiction. Otherwise, \mathcal{M}' is NHCA, i.e., there are not two or three arcs that cover $C(\mathcal{M}')$.

The algorithm in [37] works in such a way that every arc of \mathcal{M} is included in some arc of \mathcal{M}' . Thus, in \mathcal{M} there could not be two or three arcs covering the circle, i.e., \mathcal{M} is NHCA. \square

Now we proceed with the characterization of NHCA graphs in terms of NCA graphs. In this case, the forbidden induced subgraphs are the wheels, the 3-sun, the rising suns, the umbrella, and the tent graph \bar{S}_3 (see Fig. 4). Again, we require the characterization of interval graphs by Lekkerkerker and Boland which reads as follows when restricted to NCA graphs.

Theorem 13 ([29]). An NCA graph is an interval graph if and only if it does not contain holes, 3-suns, rising suns, umbrellas, or tents as induced subgraphs.

The partial characterization of NHCA graphs in terms of NCA graphs follows the same scheme as before, first we analyze the NCA models of NHCA graphs and then the characterization is obtained as a corollary.

Theorem 14. Let \mathcal{M} be an NCA model of a graph G . Then, \mathcal{M} is equivalent to an NHCA model if and only if \mathcal{M} contains no wheels, 3-suns, rising suns, umbrellas, or tents, as induced submodels.



Fig. 4. The tent graph and one of its NCA models.

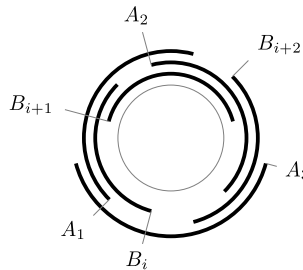


Fig. 5. Claim 2 of Theorem 14, $A_1, B_{i+1}, A_2, A_3, B_i$ induce a wheel.

Proof. Wheels, 3-suns, rising suns, umbrellas, and tents admit circular-arc models that are not NHCA (see Figs. 3 and 4). Hence they are not NHCA, by Corollary 3.

The converse is somewhat similar to the converse of Theorem 12, but it needs a few tweaks. Suppose that \mathcal{M} contains none of the forbidden submodels and, yet, \mathcal{M} has three arcs A_1, A_2 and A_3 that cover the circle. We may assume that none of these three arcs is contained in any other arc because \mathcal{M} has no two arcs that cover the circle. To obtain a contradiction, suppose that \mathcal{M} has k arcs B_1, \dots, B_k that induce a hole \mathcal{H} in this order.

Claim 1: A_j, B_i, B_{i+1} do not cover the circle, for $1 \leq i \leq k$ and $1 \leq j \leq 3$. Otherwise, A_j intersects all the arcs of \mathcal{H} , i.e., $\mathcal{H} \cup \{A_j\}$ induces a wheel.

Claim 2: There is a hole $\mathcal{H}' \subset \mathcal{H} \cup \{A_1, A_2\}$ that contains at least one of A_1 and A_2 . Suppose, to obtain a contradiction, that this is not the case. Then, by Proposition 1 and Claim 1, it follows that $A_1 \subset (B_i \cup B_{i+1}) \setminus (B_{i+2}, \dots, B_{i-1})$ and that $A_2 \subset (B_j \cup B_{j+1}) \setminus (B_{j+2}, \dots, B_{j-1})$, for $1 \leq i, j \leq n$. By Claim 1, A_3 does not cover the circle with B_i and B_{i+1} , thus $i \neq j$. So, either $t(A_1) \in B_{i+1} \cap A_2$ and $j = i + 1$ or $t(A_2) \in B_i \cap A_1$ and $i = j + 1$. Assume the former w.l.o.g., thus $A_2 \subset (B_{i+1} \cup B_{i+2}) \setminus (B_{i+2}, \dots, B_i)$ (see Fig. 5). Now, consider the position of arc A_3 . Since A_1, A_2, A_3 cover the circle, then A_3 crosses both $t(B_{i+2})$ and $s(B_i)$. By Claim 1, A_3 crosses neither $s(B_{i+1})$ nor $t(B_{i+1})$, thus $A_1, B_{i+1}, A_2, A_3, B_i$ induce a wheel where A_1 is the universal arc.

By Claim 2, there is a hole $\mathcal{H}' \subset \mathcal{H} \cup \{A_1, A_2\}$ that contains at least one of A_1, A_2 , say A_1 . If we exchange \mathcal{H} with \mathcal{H}' and A_1 with A_3 in Claim 2, we obtain that there is a hole $\mathcal{H}'' \subset \mathcal{H}' \cup \{A_2, A_3\}$ that, w.l.o.g., contains A_2 as well. If A_1 is not an arc of \mathcal{H}'' , it is because A_1 is covered by two arcs of the hole, one of which is A_2 . In this case, call B to the arc of \mathcal{H}'' that together with A_2 covers A_1 . Otherwise, call $B = A_1$. Summing up, $\mathcal{H}'' = B, A_2, B_i, \dots, B_j$, for some pair $1 \leq i, j \leq k$. Since A_1, A_2, A_3 cover the circle, then B, A_2, A_3 cover the circle, thus A_3 must intersect all the arcs of \mathcal{H}'' , a contradiction. This contradiction appears because we assume that there is a hole in \mathcal{M} . Therefore, G contains no holes, 3-suns, umbrellas, rising suns, or tents as induced subgraphs. This implies that G admits an interval model equivalent to \mathcal{M} by Theorem 13. \square

Corollary 4. An NCA graph is NHCA if and only if it contains no wheels, 3-suns, rising suns, umbrellas, or tents as induced subgraphs.

The proofs of Theorems 12 and 14 can be combined so as to obtain a characterization of when a circular-arc graph admits an NHCA model. We are not going to prove the theorem to avoid repetitions, instead we just give a sketch of the proof.

Theorem 15. A circular-arc graph is NHCA if and only if it contains no wheels, 3-suns, rising suns, umbrellas, or tents as induced subgraphs.

Proof. Wheels, 3-suns, rising suns, umbrellas and tents are not NHCA by Theorem 14.

For the converse, suppose that there are two arcs A_1, A_2 that cover a circular-arc model \mathcal{M} , and that \mathcal{M} has an induced hole B_1, \dots, B_k . Then, by Proposition 1, we can either find a wheel in \mathcal{M} as in Theorem 12 or A_1, B_i, B_{i+1} cover the circle for some $1 \leq i \leq k$. In this last case, A_1 is universal to all the arcs of the hole which also implies that \mathcal{M} contains an induced wheel. Then, as before, \mathcal{M} is NCA and the result follows from Theorem 14. \square

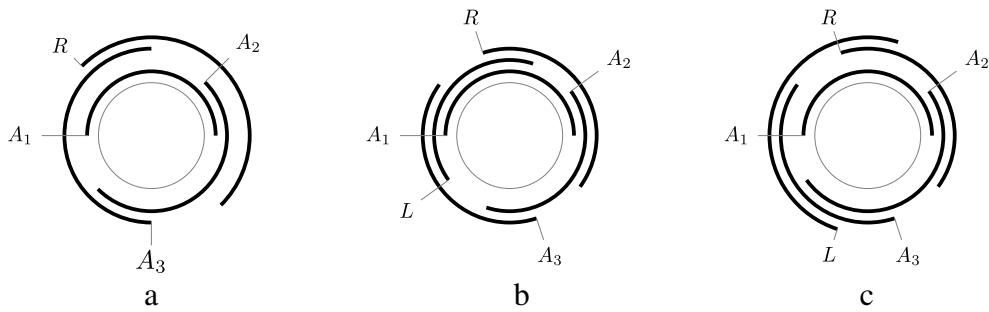


Fig. 6. Proof of Theorem 16. If A_1 is completely covered, then G is not a PHCA graph.

4.2. Proper Helly circular-arc graphs

Up to this point we have characterized which circular-arc (resp. HCA, NCA) graphs admit an NHCA model and which NHCA graphs admit a PHCA model. In this section we characterize which PCA graphs are also PHCA. For the proof we may use the same arguments of Theorem 12 to show that every PCA model of a non-interval PHCA graph is in fact a PHCA model. This would yield an elegant short proof. But instead, we prefer to do a constructive proof that shows how a PCA model of an interval graph can be transformed into a proper interval model. As we will see in Section 5.2, this proof yields an $O(n)$ time algorithm to transform a PCA model into a PHCA model. We remark that this proof is the same that appeared in [32], with some minor corrections.

Theorem 16. Let \mathcal{M} be a PCA model of a graph G . Then the following are equivalent:

- (i) \mathcal{M} is equivalent to a PHCA model.
- (ii) \mathcal{M} contains no induced submodel of W_4 or S_3 .
- (iii) $U_1(\mathcal{M})$ is HCA or $U_0(\mathcal{M})$ is a PIG model.

Proof. (i) \implies (ii): neither W_4 nor S_3 are NHCA graphs by Theorem 12, thus \mathcal{M} cannot have induced submodels of them.

(ii) \implies (iii): let \mathcal{M} be a PCA model, containing no induced submodels of W_4 and S_3 . By Lemma 1, $\mathcal{M}_1 = U_1(\mathcal{M})$ is an NPCA model. If \mathcal{M}_1 is not an HCA model, then \mathcal{M}_1 contains three arcs A_1, A_2, A_3 covering $C(\mathcal{M}_1)$ (see [24]). No two arcs cover $C(\mathcal{M}_1)$, thus we may assume that $s(A_1), t(A_3), s(A_2), t(A_1), s(A_3), t(A_2)$ appear in this order in a traversal of $C(\mathcal{M}_1)$. First, we prove that one of the above three arcs must be universal. Suppose the contrary. Then, there exist arcs B_1, B_2 and B_3 such that B_i does not intersect A_i , for $i \in \{1, 2, 3\}$. However, since \mathcal{M}_1 is a proper model, it follows that B_i intersects A_j, A_k for $\{j, k\} = \{1, 2, 3\} \setminus \{i\}$. The latter leads to a contradiction because the intersection graph of $\{A_i, B_i\}_{i \in \{1, 2, 3\}}$ is isomorphic to S_3 , when B_1, B_2, B_3 are pairwise disjoint, or it contains an induced W_4 . Consequently, one of A_1, A_2, A_3 , say A_1 , is the unique universal arc of \mathcal{M}_1 .

Next, we examine the arc A_1 in \mathcal{M}_1 . We will prove that there is no pair of arcs L, R such that $t(L) \in A_1, s(R) \in A_1$, and $s(R)$ precedes $t(L)$ in a traversal of $C(\mathcal{M})$ from $s(A_1)$. To obtain a contradiction for this fact, assume the contrary and discuss the following alternatives.

1. $L = A_3$. In this situation, R, A_2, A_3 are three arcs covering C . Because A_1 is the unique universal arc of \mathcal{M}_1 , we know that R, A_2, A_3 are not universal. Consequently, as above, \mathcal{M}_1 contains a submodel of W_4 or S_3 , a contradiction (Fig. 6(a)).
2. $R = A_2$. Similar to alternative 1.
3. $L \neq A_3$ and $R \neq A_2$. By Cases 1 and 2, above, it suffices to examine the situation where $s(R), t(L) \in (t(A_3), s(A_2))$. Suppose $L \cap A_2 = R \cap A_3 = \emptyset$. In this case, the arcs A_1, A_2, A_3, L, R form a forbidden W_4 , impossible (Fig. 6(b)). Alternatively, let $L \cap A_2 \neq \emptyset$. Then the arcs A_2, L, R cover the circle and none of them is the universal arc A_1 , an impossibility (Fig. 6(c)). The situation $R \cap A_3 \neq \emptyset$ is similar.

By the above cases, we conclude that all the ending points must precede the beginning points in $(s(A_1), t(A_1))$. Let t and s be the last ending point and the first beginning point inside $(s(A_1), t(A_1))$, respectively. Taking into account that A_1 is the only universal arc, we conclude that no point of the segment $(t, s) \subset A_1$ of $C(\mathcal{M})$ can be contained in any arc of \mathcal{M} except A_1 . Hence $\mathcal{M}_0 = \mathcal{M}_1 \setminus \{A_1\}$ is a PIG model.

(iii) \implies (i): suppose first that $\mathcal{M}_1 = U_1(\mathcal{M})$ is an HCA model. Duplicating the universal arc as many times as required, we can include in the model all the universal arcs that have been possibly removed from it, obtaining a model equivalent to \mathcal{M} . Clearly, the obtained model is both PCA and HCA.

Next, suppose that $\mathcal{M}_0 = U_0(\mathcal{M})$ is a PIG model and \mathcal{M}_1 is not an HCA model, thus $\mathcal{M}_0 \neq \mathcal{M}_1$. Since $\mathcal{M}_0 \neq \mathcal{M}_1$, it follows that \mathcal{M}_1 has some universal arc A . We prove that $\mathcal{M}' := \mathcal{M}_0 \cup \{\bar{A}\}$ is a PIG model equivalent to \mathcal{M}_1 . By Lemma 1, \mathcal{M}_1 is an NPCA model, thus there is exactly one extreme of each non-universal arc inside A . Hence, \bar{A} contains exactly one extreme of each non-universal arc. This means that \bar{A} is a universal arc of \mathcal{M}' , and that \mathcal{M}' is an NPCA model equivalent to \mathcal{M}_1 . On the other hand, since \mathcal{M}_0 is an interval model, it follows that there is some point $p \in A$ which is crossed only by A in \mathcal{M}_1 .

Therefore, p is not crossed by any arc of \mathcal{M}' , which implies that \mathcal{M}' is an interval model as well. Summing up, \mathcal{M}' is a PIG model equivalent to \mathcal{M}_1 . Duplicating the universal arc of \mathcal{M}' we can include all the universal arcs that were removed from \mathcal{M}_1 , to obtain a PIG model of G . \square

The characterization in terms of the forbidden subgraphs is depicted in the corollary below.

Corollary 5. *A PCA graph is PHCA if and only if it contains no induced W_4 and no induced S_3 .*

In implication (ii) \implies (iii), the non-Helly PCA model $\mathcal{M}_1 = U_1(\mathcal{M})$ has a universal arc A whose removal yields the interval model $U_0(\mathcal{M})$. This arc A can be replaced by \bar{A} , and then all the arcs of $\mathcal{M} \setminus \mathcal{M}_1$ can be inserted once again into \mathcal{M}_0 by duplicating \bar{A} as in (iii) \implies (i). The model so obtained is a PIG model equivalent to \mathcal{M} . Thus, if some PHCA graph admits a non-Helly PCA model, then the graph is in fact a PIG graph. The following corollary, which is the analog of Corollary 3, reflects this fact.

Corollary 6. *Every PCA model of a non-interval PHCA graph is PHCA.*

Proof. By Corollary 3, every PCA model of a non-interval PHCA graph is also NHCA. Thus, the model is both PCA and HCA. \square

4.3. Unit Helly circular-arc graphs

Theorem 16 describes when a PCA model can be transformed into an equivalent PHCA model. An almost verbatim copy of its proof can be used to characterize those UCA models which are equivalent to some UHCA model. However, this characterization can be done easily once Corollary 6 is known.

Theorem 17. *A graph is UHCA if and only if it is PHCA and UCA. Moreover, every UCA model of a non-interval UHCA graph is UHCA.*

Proof. Clearly, every UHCA model is both PHCA and UCA. For the converse, let G be a PHCA and UCA graph and observe that G contains no induced $K_{1,3}$. If G is also an interval graph, then it is a UIG graph by Theorem 8. If G is not an interval graph then, by Corollary 6, every UCA model of G is also HCA. \square

The partial forbidden induced subgraph characterizations of UHCA graphs are shown below.

Corollary 7. *A UCA graph is UHCA if and only if it contains no induced W_4 .*

Proof. It follows from Theorems 17 and 3, Corollary 5, and the fact that $S_3 = CI(3, 1)$. \square

Corollary 8. *A PHCA graph is UHCA if and only if it contains no induced $CI(n, k)$ graph with $n > 3k$.*

Proof. By Theorem 3, UHCA graphs contain no induced $CI(n, k)$ for $n > 3k$. For the converse, let G be a PHCA graph having no induced $CI(n, k)$ with $n > 3k$. By definition, every $CI(n, k)$ model with $2 < n < 3k$ has three arcs that together cover the circle. Then, since $CI(n, k)$ graphs are not interval graphs, it follows by Corollary 6 that $CI(n, k)$ graphs with $2 < n < 3k$ are not PHCA. So, in G there is no induced $CI(n, k)$ for $n > 2k$. Thus, Theorem 3 and Corollary 7 imply that G is a UHCA graph. \square

The whole picture of the CA class hierarchy is depicted in Fig. 7. Each box of the picture represents a subclass of circular-arc graphs. An upright edge from the box corresponding to the class \mathcal{C}_1 to the box corresponding to the class \mathcal{C}_2 means that \mathcal{C}_1 is properly contained in \mathcal{C}_2 . Graphs that belong to \mathcal{C}_2 but not to \mathcal{C}_1 appear beside the edge corresponding to the inclusion of \mathcal{C}_1 in \mathcal{C}_2 , except for the edge between the CA and the NCA classes since this family is unknown. Finally, the label \emptyset beside the edge between the CA and HCA classes represents the family of obstacles that were defined in [37].

5. The algorithms

The characterizations described in the last section lead directly to $O(n + m)$ time algorithms for recognizing graphs in each of the NHCA subclasses. This complexity is linear when the input is the graph represented by the adjacency lists. So, the recognition problem for NHCA, PHCA, and UHCA graphs is well solved. As for the model construction, first we refer to the more general class of circular-arc graphs. Considering that the corresponding models of this class can be represented just by $O(n)$ elements, there is a motivation for trying to find $O(n)$ time algorithms that solve the recognition and model construction problems for each of the subclasses. In this case, the input is a general circular-arc model of a graph G and the question is deciding whether G belongs to a restricted class of circular-arc graphs and, whenever affirmative, constructing the corresponding restricted circular-arc model. For example, given an arbitrary circular-arc model of a graph G , algorithms running in $O(n)$ time have been recently described to construct, whenever possible, a proper circular-arc model of G [43], a unit circular-arc model of G [36], or a Helly circular-arc model of G [24].

In the present section we consider the representation problems for the classes of NHCA, PHCA, and UHCA graphs. We propose new algorithms along the above lines and we discuss the related certification and authentication problems.

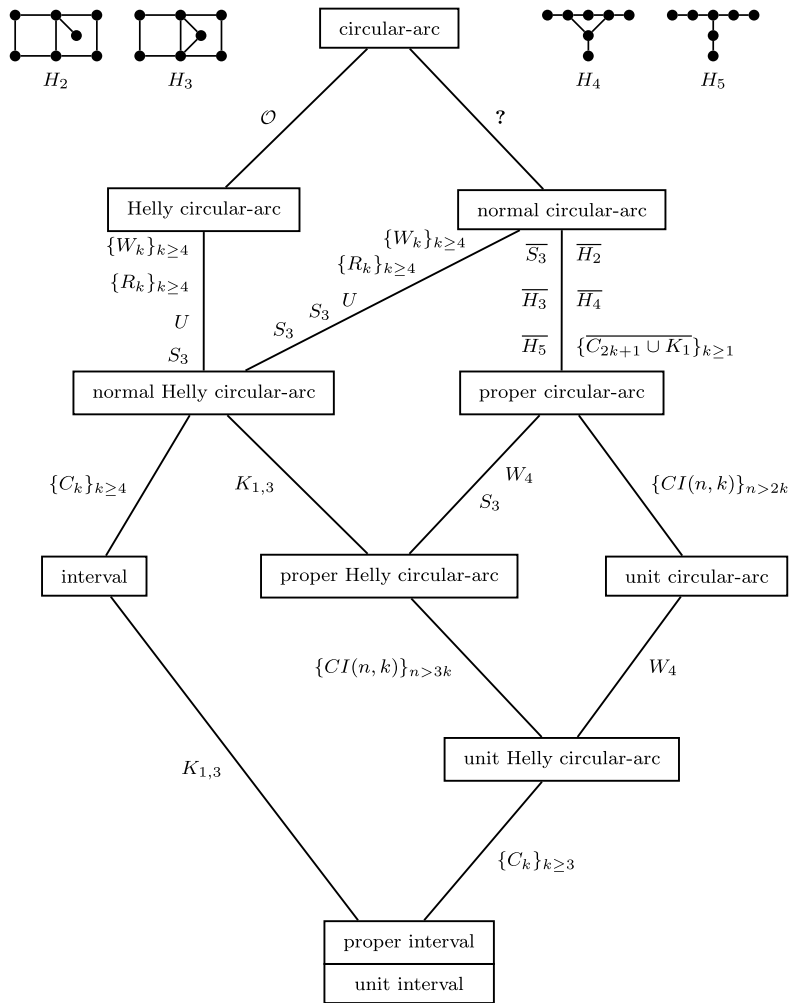


Fig. 7. The class hierarchy of circular-arc graphs.

5.1. Recognition of NHCA graphs

In this section we develop an algorithm to test whether a circular-arc graph is NHCA, when the input is a circular-arc model. The algorithm follows directly from Corollary 3 and its time complexity is $O(n + m)$, improving over the previous $O(nm)$ time algorithm by McKee [40].

Let \mathcal{M} be a circular-arc model of a graph G . If there are no two or three arcs covering $C(\mathcal{M})$, then G is an NHCA graph and \mathcal{M} is an NHCA model of G , so there is nothing to be done for this case. Otherwise, it is enough to check if \mathcal{M} is equivalent to an interval model since, by Corollary 3, those NHCA graphs that admit a non-NHCA model are interval graphs. If \mathcal{M} is equivalent to an interval model, then G is an NHCA graph and any interval model of G is an NHCA model of G .

We use Algorithm 1 to test whether there are two or three arcs covering the circle. There, $NEXT(A)$ represents the arc crossing $t(A)$ whose ending point reaches farthest, whereas $NEXT^2(A)$ represents $NEXT(NEXT(A))$. In the second traversal of the first loop, N is the arc reaching farthest among those crossing e , when e is reached. Thus, $NEXT$ is correctly computed in the first loop. Now, observe that if A_1 and A_2 cover $C(\mathcal{M})$, then A_1 and $NEXT(A_1)$ also cover $C(\mathcal{M})$. Similarly, if A_1, A_2, A_3 cover $C(\mathcal{M})$, then $A_1, NEXT(A_1)$ and $NEXT^2(A_1)$ also cover $C(\mathcal{M})$. Thus, \mathcal{M} is an NHCA model if and only if neither $NEXT(A)$ nor $NEXT^2(A)$ crosses $s(A)$, for every $A \in \mathcal{M}$. Therefore, Algorithm 1 is correct.

With respect to the time complexity, all the operations of both loops take $O(1)$ time, thus the total time complexity of Algorithm 1 is $O(n)$.

To test if \mathcal{M} is equivalent to an interval model when \mathcal{M} is not NHCA, we compute the intersection graph G of \mathcal{M} and we invoke the $O(n + m)$ time certified algorithm in [28]. Unfortunately, we were unable to find an $O(n)$ time algorithm to obtain an interval model from \mathcal{M} .

Algorithm 1 Authentication of an NHCA model.**Input:** A circular-arc model \mathcal{M} .**Output:** If \mathcal{M} is not NHCA, then two or three arcs that cover \mathcal{M} . Otherwise, there is no output.

1. Let A_1 be any arc of \mathcal{M} and set $N := A_1$.
2. Traverse $C(\mathcal{M})$ twice from $s(A_1)$ and apply the following operation when an extreme e of an arc A is reached.
3. If $e = s(A)$ and $t(A)$ reaches farther than $t(N)$, then set $N := A$.
4. If $e = t(A)$, then set $NEXT(A) := N$.
5. Traverse $C(\mathcal{M})$ once again from $s(A_1)$ and apply the following evaluation when $s(A)$ is reached.
6. If $NEXT(A)$ crosses $s(A)$, then output $A, NEXT(A)$.
7. If $NEXT^2(A)$ crosses $s(A)$, then output $A, NEXT(A)$ and $NEXT^2(A)$.

We now discuss the certification and the authentication procedures. When \mathcal{M} is an NHCA model, the positive certificate is just \mathcal{M} . If \mathcal{M} is not an NHCA model, but it is equivalent to an interval model, the certificate is provided by the certified interval graph recognition algorithm in $O(n + m)$ time [28]. If \mathcal{M} is neither NHCA nor equivalent to an interval model, then the negative certificate is obtained by combining the certificate of the interval graph recognition algorithm with the two or three arcs that cover the circle. This certificate is enough by Corollary 3. The negative certificate can be authenticated in $O(n)$ time as in [28]. To authenticate the positive certificates it is enough to test that the output model \mathcal{M}' is NHCA and equivalent to \mathcal{M} . For the NHCA authentication use Algorithm 1, and for the isomorphism authentication use the $O(n)$ time algorithm of [6]. The complete certified procedure is summarized in Algorithm 2.

Algorithm 2 Recognition of NHCA graphs.**Input:** A circular-arc model \mathcal{M} .**Output:** Either an NHCA model equivalent to \mathcal{M} or a subset of arcs whose intersection graph is not NHCA.

1. Execute Algorithm 1 to authenticate if \mathcal{M} is an NHCA model. If so, output \mathcal{M} .
2. Otherwise, execute the algorithm in [28] to the intersection graph G of \mathcal{M} . If G is an interval graph, then output the interval model obtained by the algorithm. Otherwise, output the two or three arcs covering $C(\mathcal{M})$ together with the negative certificate obtained by the interval graph recognition algorithm.

5.2. Recognition of PHCA graphs

In this section we show two algorithms that can be used to recognize PHCA graphs. The first one transforms an NHCA model into a PHCA model in $O(n)$ time, if possible. The second one transforms a PCA model into a PHCA model in $O(n)$ time, if possible. The algorithm to transform a PCA model into a PHCA model that we present is based on the work by the authors [32], while the algorithm for transforming an NHCA model into a PHCA model uses some interesting arc operations. We observe that these algorithms have been superseded by the algorithm of Nussbaum [43] that transforms a CA model into an equivalent PCA model. Such an algorithm outputs a PIG model whenever possible. Thus, by Theorem 19, it is enough to check that the output PCA model is NHCA with Algorithm 2. Nevertheless, given the restricted nature of their input, the algorithms described in this section use simpler techniques for some of the tasks in [43] which are worth describing.

To transform an NHCA model into a PHCA model we need only to sort the extreme sequences as in Theorem 9. If the model so obtained is not PHCA, then we can search for an induced $K_{1,3}$. We begin by describing how to sort all the extreme sequences in $O(n)$ time.

Let \mathcal{M} be an NHCA model. For arcs $A_i, A_j \in \mathcal{A}(\mathcal{M})$ with nonempty intersection, say that $s(A_i)$ appears before $s(A_j)$ if $s(A_i)$ appears before $s(A_j)$ in a traversal of $C(\mathcal{M})$ from some point $p \in C(\mathcal{M}) \setminus (A_i \cup A_j)$. Observe that the point p must always exist because A_i and A_j do not cover $C(\mathcal{M})$. Similarly, say that $t(A_i)$ appears before $t(A_j)$ if $t(A_i)$ appears before $t(A_j)$ in a traversal of $C(\mathcal{M})$ from the same point p . The sorting of the extremes sequences in Theorem 9 can be rephrased as follows. First, sort each t -sequence T so that, for $t(A_i), t(A_j) \in T$, if $s(A_i)$ appears before $s(A_j)$, then $t(A_i)$ appears before $t(A_j)$. Next, sort each s -sequence S so that, for $s(A_i), s(A_j) \in S$, if $t(A_i)$ appears before $t(A_j)$, then $s(A_i)$ appears before $s(A_j)$. The algorithms to sort the t -sequences and s -sequences are symmetric, so we only describe how to sort the t -sequences.

Let T_1, \dots, T_k be the t -sequences of \mathcal{M} and T be the set of all the ending points corresponding to arcs that cross some fixed beginning point s . Consider the t -sequence T'_i that results from sorting the t -sequence T_i , for some $1 \leq i \leq k$. In T'_i , all the ending points of $T_i \cap T$ appear before all the ending points of $T_i \setminus T$. Thus, we can sort all the t -sequences with four traversals of $C(\mathcal{M})$ as in Algorithm 3. In the first traversal of $C(\mathcal{M})$, Algorithm 3 marks all those arcs that cross the fixed beginning point $s(A_1)$. Thus, the ending points of T are precisely those ending points corresponding to the marked arcs. The second traversal is used to find all the t -sequences T_1, \dots, T_k of \mathcal{M} . The third traversal computes and sorts the sequences $T_{i,1} = T_i \cap T$ and $T_{i,2} = T_i \setminus T$, for every $1 \leq i \leq k$. Note that, in Step 9, $t(A)$ is stored at the end of either $T_{i,1}$ or $T_{i,2}$, and

all the ending points corresponding to arcs whose beginning point appears before $s(A)$ were already stored. Finally, the last traversal of $C(\mathcal{M})$ replaces each t -sequence with the sorted t -sequence. Thus, the algorithm is correct.

Algorithm 3 Sorting of the t -sequences of an NHCA model.

Input: An NHCA model \mathcal{M} .

Output: An NHCA model \mathcal{M}' equivalent to \mathcal{M} in which every t -sequence is sorted. That is, if $t(A_i)$, $t(A_j)$ are extremes of the t -sequence T of \mathcal{M}' and $s(A_i)$ appears before $s(A_j)$ then $t(A_i)$ appears before $t(A_j)$.

1. Let A_1 be any arc of \mathcal{M} .
//Find the arcs that cross $s(A_1)$.
 2. Traverse \mathcal{M} from $s(A_1)$ and apply the following operation when an extreme e of an arc A is reached:
 3. If $e = s(A)$, then mark A .
 4. If $e = t(A)$, then clear the mark of A .
//Sort the extremes
 5. Traverse \mathcal{M} to compute the family T_1, \dots, T_k of t -sequences of \mathcal{M} .
 6. For each $i := 1, \dots, k$, define $T_{i,1}$ and $T_{i,2}$ as empty sequences.
 7. Traverse \mathcal{M} from $s(A_1)$ and apply the following operations each time a beginning point $s(A)$ is reached:
 8. Find the t -sequence T_i that contains $t(A)$.
 9. If A is marked, insert $t(A)$ at the end of $T_{i,1}$; otherwise, insert $t(A)$ at the end of $T_{i,2}$.
 10. Replace T_i by $T_{i,1}, T_{i,2}$ in \mathcal{M} for every $1 \leq i \leq k$.
 11. Output \mathcal{M} .
-

With respect to the time complexity, all the operations of both loops take $O(1)$ time, while the computation of the t -sequences can be easily done in $O(n)$ time. Therefore, the total time complexity of the sorting algorithm is $O(n)$.

After sorting the extremes, we must check whether its output model \mathcal{M}' is PHCA or not. Algorithm 3 does not modify the elements that compose each t -sequence, thus \mathcal{M}' is NHCA. So, it is enough to check whether \mathcal{M}' is PCA. That is, we ought to check if the beginning points of the arcs appear in the same order as the ending points. If affirmative, then \mathcal{M}' is a PHCA model equivalent to the input model \mathcal{M} . Otherwise, there are two arcs A_i and A_j such that $s(A_i)$, $s(A_j)$, $t(A_j)$ and $t(A_i)$ appear in this order in a traversal $C(\mathcal{M}')$. Let L be the arc whose ending point appears immediately after $s(A_i)$ and R be the arc whose beginning point appears immediately before $t(A_i)$. Arcs A_i , A_j , L , and R are taken as the negative certificate since, as in the proof of Theorem 9, they induce a circular-arc model of $K_{1,3}$. As for the authentication, the negative certificate can be tested to be an induced submodel of $K_{1,3}$ in \mathcal{M} in $O(1)$ time, if it is implemented as a set of four pointers. To authenticate the positive certificate we need to verify that the output model \mathcal{M}' is normal, proper, Helly and equivalent to \mathcal{M} . The NHCA properties can be checked with Algorithm 1, while the test of whether \mathcal{M}' is PCA or not is done as in the PHCA recognition algorithm (Steps 3–5 of Algorithm 4). Finally, if \mathcal{M}' is PCA, then the equivalence of \mathcal{M} and \mathcal{M}' can be tested by running the PCA isomorphism algorithm of [33]. Algorithm 4 summarizes the complete procedure.

Algorithm 4 Recognition of PHCA graphs from NHCA models.

Input: An NHCA model \mathcal{M} .

Output: Either a PHCA model equivalent to \mathcal{M} or an induced submodel of $K_{1,3}$.

1. Apply Algorithm 3 to sort the t -sequences, and the symmetric algorithm to sort the s -sequences.
 2. Let $A_1 := (s_1, t_1)$ be any arc of \mathcal{M} .
 3. For $i := 1, \dots, n$ do:
 4. Find the arc $A_{i+1} = (s_{i+1}, t_{i+1})$ whose beginning point is the first after s_i .
 5. If $A_i \supset A_{i+1}$, then output A_i, A_{i+1} , the arc whose ending point appears first from s_i , and the arc whose beginning point appears first from t_i in a counterclockwise traversal of $C(\mathcal{M})$.
 6. Output \mathcal{M} .
-

We now proceed to describe the algorithm that transforms a PCA model \mathcal{M} into a PHCA model \mathcal{M}' , when possible. The algorithm is a direct consequence of Theorem 16. That is, it verifies if either $U_1(\mathcal{M})$ is an HCA model or $U_0(\mathcal{M})$ is an interval model. If affirmative, then \mathcal{M} is equivalent to a PHCA model, and one such PHCA model can be obtained as in Theorem 16. Otherwise, the algorithm finds an induced submodel of W_4 or S_3 .

The PHCA recognition algorithm is obtained by gluing together several parts of the algorithms developed so far. The first step is to compute $U_1(\mathcal{M})$ using Lemma 2 as in [26]. The second step is to verify whether $U_1(\mathcal{M})$ is HCA. For this, it is enough

to invoke Algorithm 1 so as to verify if $U_1(\mathcal{M})$ is NHCA, because $U_1(\mathcal{M})$ is NCA by Lemma 1. However, we can simplify the computation of *NEXT* so that it takes only one traversal of $C(\mathcal{M})$. Let $t(A_1), \dots, t(A_k)$ be a t -sequence of \mathcal{M} . Since \mathcal{M} is PCA, then $NEXT(A_i)$ is the arc of \mathcal{M} whose beginning point is closest to $t(A_i)$ in the counterclockwise direction. Hence $NEXT(A_i) = NEXT(A_1)$, for every $1 \leq i \leq k$. Therefore, with only one traversal we can find $NEXT(A)$ for every $A \in \mathcal{A}(\mathcal{M})$. The last step is to test whether $U_0(\mathcal{M})$ is a PIG model or not, whenever $U_1(\mathcal{M})$ has a universal arc A . Instead of doing this, we can check that no beginning point appears before an ending point inside A , as it is done in Theorem 16. All these steps can be implemented so as to run in $O(n)$ time with techniques similar to those discussed so far.

The algorithm described above can be modified so as to produce certificates in $O(n)$ time. When $U_1(\mathcal{M})$ is a PHCA model, we can obtain a PHCA model equivalent to \mathcal{M} by duplicating the universal so as to insert the arcs that were possibly removed by Lemma 2. To obtain the positive certificate when $U_0(\mathcal{M})$ is a PIG model, we refer to the proof of Theorem 16, in particular, the implication (iii) \implies (i). In this situation, $U_1(\mathcal{M})$ contains a universal arc A . To obtain the required model, we include in \mathcal{M}_0 the arc \bar{A} , and then we can include the possible remaining universal arcs as before. The authentication of these certificates can be done in $O(n)$ time, as discussed for Algorithm 4.

The algorithm fails to transform the input PCA model into a PHCA model when $U_1(\mathcal{M})$ is not HCA and $U_0(\mathcal{M})$ is not PIG. According to Theorem 16, an induced submodel of \mathcal{M} whose intersection graph is either isomorphic to W_4 or to S_3 can be obtained as follows. Let A_1, A_2 , and A_3 be the three arcs that together cover the circle of $U_1(\mathcal{M})$. If none of these arcs is universal, then, as in Theorem 16, we know that there are three arcs B_1, B_2, B_3 , such that B_i intersects A_j and not A_i , for all $1 \leq i, j \leq 3, i \neq j$. In this case, the arcs A_1, A_2, A_3, B_1, B_2 , and B_3 either induce a model of S_3 or contain a model of W_4 . On the other hand, if one among A_1, A_2, A_3 , say A_1 , is a universal arc then there are arcs L, R , such that $s(R)$ precedes $t(L)$ in A_1 . In this situation, a negative certificate can be obtained as above by replacing A_1 with either R or L when $R = A_2$ or $L = A_3$. Finally, when $R \neq A_2$ and $L \neq A_3$, the arcs A_1, A_2, A_3, L and R induce the model of a forbidden W_4 . The authentication takes $O(1)$ time if the forbidden submodel is stored as a set of five or six pointers to the corresponding arcs of the model. We summarize the above discussion in Algorithm 5.

Algorithm 5 Recognition of PHCA graphs from PCA models.

Input: A PCA model \mathcal{M} .

Output: Either a PHCA model equivalent to \mathcal{M} or an induced submodel of W_4 or S_3 .

1. Compute $U_1(\mathcal{M})$ as in Lemma 2.
 2. Execute Algorithm 1 (simplified for PCA graphs), to verify whether $U_1(\mathcal{M})$ contains three arcs that cover the circle. If not, output the model obtained by duplicating the universal arc of $U_1(\mathcal{M})$ so as to include the arcs removed in Step 1.
 3. Let A_1, A_2 and A_3 be the three arcs of $U_1(\mathcal{M})$ that were obtained in the previous step.
 4. If A_1, A_2 and A_3 are not universal, then:
 5. Let B_i be the arc whose beginning point is the first from $t(A_i)$, for $i \in \{1, 2, 3\}$.
 6. If B_1, B_2 and B_3 are pairwise disjoint, then output $\{A_i, B_i\}_{1 \leq i \leq 3}$; otherwise, output A_1, A_2, A_3 and two intersecting arcs of B_1, B_2 and B_3 .
 7. Sort A_1, A_2 and A_3 so that A_1 is the universal arc and $s(A_2) \in A_1$.
 8. Traverse \mathcal{M} from $s(A_1)$ to $t(A_1)$ to find if there are two arcs L and R such that $s(R)$ appears before $t(L)$. If L and R are found then:
 9. If $R = A_2$, then set $A_1 := R$ and goto Step 5. If $L = A_3$, then set $A_1 := L$ and goto Step 5. Output A_1, A_2, A_3, L and R .
 10. Let $\mathcal{M}' := (\mathcal{M} \setminus \{A_1\}) \cup \{\bar{A}_1\}$.
 11. Duplicate \bar{A}_1 in \mathcal{M}' as many times as arcs were removed in Step 1, and return the model so obtained.
-

5.3. Recognition of UHCA graphs

In this section we briefly discuss how to transform either a UCA or a PHCA model \mathcal{M} into a UHCA model, when possible. We begin with the case in which \mathcal{M} is PHCA. In this case, apply the algorithm in [36] to transform \mathcal{M} into a UCA model \mathcal{M}' , if possible. Since this algorithm preserves the order of the extremes of \mathcal{M} then \mathcal{M}' is both UCA and PHCA, i.e., \mathcal{M}' is UHCA. This algorithm takes $O(n)$ time and the model \mathcal{M}' so generated can be authenticated to be UCA in $O(n)$ time. If \mathcal{M} is not equivalent to a UHCA model, then apply the algorithm in [26] to generate a negative certificate in $O(n)$ time. This negative certificate can also be authenticated in $O(n)$ time.

Finally, consider the case in which \mathcal{M} is a UCA model. If \mathcal{M} has two arcs that cover the circle, then the intersection graph of \mathcal{M} is a complete graph and an equivalent UIG model is easily obtained in $O(n)$ time. Suppose, then, that \mathcal{M} is an NCA model, and consider the model $U_1(\mathcal{M})$. If $U_1(\mathcal{M})$ is HCA then it is also UHCA and a UHCA model equivalent to \mathcal{M} can be easily obtained in $O(n)$ time by duplicating the universal arc of $U_1(\mathcal{M})$, if it exists. If $U_1(\mathcal{M})$ is not HCA but $U_0(\mathcal{M})$ is an interval model then a PIG model equivalent to \mathcal{M} can be obtained by applying Algorithm 5. A UIG model equivalent to \mathcal{M} can be constructed in $O(n)$ time as in [4,34,41]. In the last case, if $U_1(\mathcal{M})$ is not an HCA model and $U_0(\mathcal{M})$ is not an interval

Table 3

Time complexities of the transformation algorithms. All algorithms proposed in this section are certified.

From	To	Time complexity	References
CA	NCA	open	
co-bipartite CA	NCA	$O(n^5 m^6 \log m)$	[42,18]
CA	HCA	$O(n)$	[24]
CA	PCA	$O(n)$	[43]
PCA	UCA	$O(n)$	[36]
CA (HCA, NCA)	NHCA	$O(n + m)$	[28] and Section 5.1
NHCA	PHCA	$O(n)$	[43] and Section 5.2
PCA	PHCA	$O(n)$	[32,43] and Section 5.2
UCA	UHCA	$O(n)$	Sections 5.2 and 5.3
PHCA	UHCA	$O(n)$	[36] and Section 5.2
NHCA	IG	$O(n)^a$	Theorem 7
PHCA	PIG	$O(n)^a$	Theorem 7
IG	PIG	$O(n)$	[38] (also Algorithm 3)
PIG	UIG	$O(n)$	[4,34,41]

^a The transformation phase is trivial because any NHCA model of an interval graph is an interval model by Theorem 7. However, it takes $O(n)$ time to authenticate that the input model has some point not contained in any arc.

model then, by Theorem 16, \mathcal{M} is not equivalent to a UHCA model and a negative certificate is obtained as in Algorithm 5. All the certificate authentications take $O(n)$ time as before.

5.4. Summary of the transformation algorithms

In Table 3 we summarize the complexities of the transformation algorithms between models for circular-arc subclasses. The main open problems are how to transform a circular-arc model into an NCA model in polynomial time, and how to transform a circular-arc model into an NHCA model in $O(n)$ time. We recall that this last problem can be reduced to transforming an HCA model into an interval model in $O(n)$ time.

6. Some additional properties of the NHCA subclasses

In this section we prove some properties about the NHCA subclasses that might be useful from an algorithmic point of view. Some of these were presented in Section 3 as natural generalizations of properties about interval graphs.

6.1. Counting NPHCA models of PHCA graphs

The first problem is to count how many NPHCA models a PHCA graph admits. This question of how many representations of a graph there are has been solved for both PIG graphs and co-connected PCA graphs; Roberts [44] proved that the PIG model of a connected PIG graph is unique up to full reversal, while Huang [22] proved that the PCA model of a connected and co-connected PCA graph is unique up to full reversal.

Theorem 18 ([44]). *Every connected PIG graph admits at most two PIG models, one the reverse of the other.*

Theorem 19 ([22]). *Every connected PCA graph whose complement is either connected or non-bipartite admits at most two PCA models, one the reverse of the other.*

Besides the theoretical interest behind these questions, it turns out that the uniqueness of PIG models is strongly used to solve the recognition problem. In fact, PIG graphs can be recognized in $O(1)$ time per edge insertion due to this property (see [7,20]). Because of the strong relationship between PIG and PHCA graphs, it should come at no surprise that there is a unique NPHCA model of every PHCA graph, up to full reversal. Even more, this property can be exploited so as to generalize the PIG recognition algorithms to the PHCA case, with not as much effort as for PCA graphs (see [47]).

Theorem 20. *Every connected PHCA graph admits at most two NPHCA models, one the reverse of the other.*

Proof. If \bar{G} is connected or non-bipartite then the result follows from Theorem 19. Then, it is enough to deal with the case in which \bar{G} has $k > 1$ components H_1, \dots, H_k , all of which are bipartite. We denote by H_i the subgraph of G induced by the vertices of \bar{H}_i , for $1 \leq i \leq k$, and we assume $|H_i| \geq |H_j|$ for every $1 \leq i < j \leq k$. Analyze the following two cases:

Case 1: $|H_2| > 1$. In this case, H has a hole formed by two non-adjacent vertices of H_1 and two non-adjacent vertices of H_2 . Hence, since G is W_4 -free by Corollary 5, it follows that $k = 2$. Similarly, if H_1 contains an induced path of four vertices v_1, v_2, v_3, v_4 then v_1, v_2, v_3 together with a pair of non-adjacent vertices of H_2 induce a W_4 in G , which also

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

Fig. 8. From left to right, the clique matrices of $K_{1,3}$, W_4 , W_5 , and S_3 are shown.

contradicts Corollary 5. Then, since $\overline{H_1}$ is bipartite and H_1 contains no paths of four vertices, it follows that H_1 is the co-bipartite graph formed by two disjoint cliques. A similar analysis holds for H_2 . Consequently, G is isomorphic to a graph that can be obtained from C_4 by inserting twin vertices in an iterative manner. It is not hard to see that G admits a unique PCA model that is obtained from the unique PCA model of C_4 by duplicating arcs.

Case 2: $|H_2| = 1$. In this case G contains no hole, or otherwise the hole and the vertex of H_2 would induce a wheel. Let \mathcal{M} be an NPHCA model of G . Since G contains no hole, it follows that some point of $C(\mathcal{M})$ is not covered by the arcs of \mathcal{M} . In other words, every NPHCA model of G is a PIG model. Hence, by Theorem 18, G admits two NPHCA models, one the reverse of the other. \square

6.2. Circular-ones properties of the clique matrix

For the second part of this section, we study the relationship between PHCA graphs and the circular-ones properties of the clique matrix. A $(0, 1)$ -matrix M has the consecutive-ones property for rows if its columns can be ordered so that, in every row, the ones are consecutive. Matrix M has the circular-ones property for rows if the columns can be ordered so that, in every row, either the zeros or the ones are consecutive. The consecutive and circular-ones properties for columns are defined analogously. That is, M has the consecutive-ones property for columns if M^T has the consecutive-ones property for rows, while M has the circular-ones property for columns if M^T has the circular-ones property for rows. Here M^T is the transpose matrix of M .

Let C_1, \dots, C_k be the cliques of a graph G and v_1, \dots, v_n be its vertices. The clique-vertex incidence matrix of G , or simply the clique matrix of G , is the $(0, 1)$ -matrix $Q(G)$ of order $k \times n$ where $Q(G)_{i,j} = 1$ if and only if vertex v_j belongs to clique C_i , for every $1 \leq i \leq k, 1 \leq j \leq n$.

We mentioned in Section 3 that $Q(G)$ has the consecutive-ones (resp. circular-ones) property for columns if and only if G is an interval graph (resp. HCA graph). The stronger condition of $Q(G)$ having also the consecutive-ones property for rows is equivalent to the condition of G being a proper interval graph. We prove the analogous result for the circular-ones property, i.e., $Q(G)$ has the circular-ones property for both rows and columns if and only if G is a PHCA graph.

Theorem 21. A graph G is a PHCA graph if and only if $Q(G)$ has the circular-ones property for both rows and columns.

Proof. Let \mathcal{M} be a PHCA model of G and let A_1, \dots, A_n be the arcs of \mathcal{M} in order of appearance of its beginning points. Since \mathcal{M} is HCA, it follows that each clique is represented by some clique point. Let p_1, \dots, p_k be the clique points in circular order. Define Q as the $(0, 1)$ -matrix of order $k \times n$ where $Q_{i,j} = 1$ if A_j crosses p_i , and 0 otherwise. By definition, Q is a clique matrix of G . Since we used the same construction as in [13], it follows that Q has the circular-ones property for columns. We now show that Q has also the circular-ones property for rows. Let r be some row of Q and represent by r_i the i -th column of r . Suppose that the ones in r are not all consecutive. Then, there exist a, b, c such that $r_a = r_c = 1, r_b = 0$ and $1 \leq a < b < c \leq n$. In other words, the clique point p_i is crossed by A_a and A_c , but not by A_b . Since $s(A_a), s(A_b), s(A_c)$ appear in this order in \mathcal{M} and \mathcal{M} is proper, then $s(A_a), p_i, t(A_c), s(A_b), t(A_b)$ appear in this order in \mathcal{M} . Even more, since \mathcal{M} is proper, it follows that A_d crosses p_i for every d such that $1 \leq d \leq a$ or $c \leq d \leq n$. Thus, every zero in r is consecutive, and so Q has the circular-ones property for rows.

For the converse, we show that if G is not a PHCA graph, then $Q(G)$ does not have the circular-ones property for either the rows or the columns. If G is not an HCA graph, then $Q(G)$ does not have the circular-ones property for columns [13]. Suppose then that G is HCA and it is not PHCA. Then, by Corollary 2 and Theorem 9, G contains a $K_{1,3}$, a W_4 , a W_5 , or an S_3 as an induced subgraph. Clique matrices for these graphs are depicted in Fig. 8. By inspection, none of these matrices has the circular-ones property for rows. Thus, as $Q(G)$ contains at least one (permutation) of these matrices as a submatrix, it follows that $Q(G)$ does not have the circular-ones property for rows. \square

6.3. The local interval property for graphs

Recall that, by the local interval property, \mathcal{M} is an NHCA model if and only if $N[A]$ is an interval model, for every arc $A \in \mathcal{A}(\mathcal{M})$. As a consequence, we obtain that if G is an NHCA graph, then $N[v]$ is an interval graph for every $v \in V(G)$. The converse does not hold; the umbrella is not an NHCA graph, but the closed neighborhood of each of its vertices is an interval graph. Nevertheless, it is possible to give a necessary and sufficient condition that reflects the fact that NHCA graphs are local interval, independent of how their models look like. This local interval property about graphs is based on straight and round oriented graphs. Unlike the previous properties described in this paper, these characterizations can be generalized to

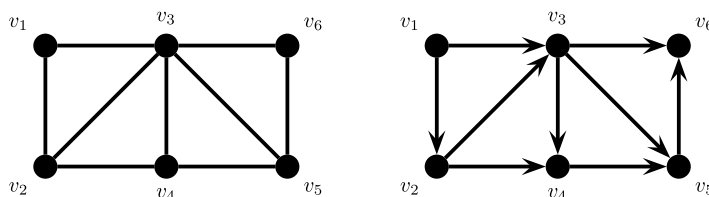


Fig. 9. An interval graph and one of its corresponding out-straight orientations. The ordering v_1, \dots, v_6 is an out-straight enumeration of the orientation.

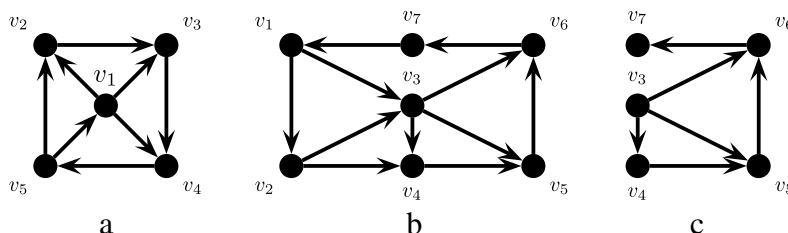


Fig. 10. Examples of round oriented graphs. An out-round orientation of the 4-wheel graph is depicted in (a). Fig. (b) shows an orientation of a non-interval graph together with a locally out-straight enumeration $\phi = v_1, \dots, v_7$. The scope of v_6 in ϕ appears at (c).

NCA and PCA graphs and, for this reason, they fill a gap by showing exactly what is lost in the jump from interval graph to NCA graphs and from PIG graphs to PCA graphs (Bang-Jensen and Gutin [1] present many classes of graphs defined as those graphs that can be oriented with some particularity). We begin by introducing some useful notation and terminology.

For an ordering $X = x_1, \dots, x_n$, we say that x_i is to the left of x_j and x_j is to the right of x_i , for every $1 \leq i < j \leq n$. The elements x_1 and x_n are respectively called the leftmost and rightmost of X . For $x_i, x_j \in X$, the circular range $\text{CIR}[x_i, x_j]$ is defined as the ordering x_i, x_{i+1}, \dots, x_j where, as usual, all the operations are taken modulo n . Similarly, the range $\text{CIR}(x_i, x_j)$ is obtained by removing the rightmost element from $\text{CIR}[x_i, x_j]$, the range $\text{CIR}(x_i, x_j)$ is obtained by removing the leftmost element from $\text{CIR}[x_i, x_j]$, and $\text{CIR}(x_i, x_j)$ is obtained by removing both the leftmost and rightmost elements from $\text{CIR}[x_i, x_j]$. For $1 \leq i \leq j \leq n$, define the linear range $\text{LIN}[x_i, x_j]$ as $[x_i, x_j]$, while for $i > j$ define $\text{LIN}[x_i, x_j]$ as the empty ordering. The linear ranges $\text{LIN}[x_i, x_j]$, $\text{LIN}(x_i, x_j)$ and $\text{LIN}(x_i, x_j)$ are defined analogously.

An oriented graph D is out-straight if there is an ordering v_1, \dots, v_n of $V(D)$ such that, for every vertex v_i , $N^+[v_i] = \text{LIN}[v_i, v_{i+r}]$, where $r = d^+(v_i)$. The ordering v_1, \dots, v_n is referred to as an out-straight enumeration of D . It is not hard to see that a graph G is an interval graph if and only if it admits an out-straight orientation D . Indeed, $\{(s_i, t_i) \mid v_i \in V(G)\}$ is an interval model of a graph G such that $s_1 < \dots < s_n$ if and only if v_1, \dots, v_n is an out-straight enumeration of the digraph with vertex set $\{v_1, \dots, v_n\}$ and edge set $\{v_i \rightarrow v_j \mid i < j \text{ and } t_i > s_j\}$. In Fig. 9, an interval graph, together with one of its out-straight orientations, is depicted.

The most common way to generalize the definition of out-straight enumerations is to exchange the linear range $\text{LIN}[\bullet, \bullet]$ with a circular range $\text{CIR}[\bullet, \bullet]$. That is, an oriented graph D is out-round if there is an ordering v_1, \dots, v_n of $V(D)$ such that, for every vertex v_i , $N^+[v_i] = \text{CIR}[v_i, v_{i+r}]$, where $r = d^+(v_i)$. The ordering v_1, \dots, v_n is referred to as an out-round enumeration of D . Fig. 10 shows three examples of out-round oriented graphs.

Although out-round oriented graphs are a natural generalization of out-straight oriented graphs, there is a key property about the scopes (cf. below) of the vertices that is completely lost. Let $\phi = v_1, \dots, v_n$ be an ordering of $V(D)$ for an oriented graph D . For $v_i \in V(D)$, define the leftmost neighbor of v_i in ϕ as the vertex $v_l \in N^-[v]$ that appears last when traversing ϕ from v_i in reverse order. Similarly, define the rightmost neighbor of v_i in ϕ as the vertex $v_r \in N^+[v]$ that appears last when traversing ϕ from v_i in forward order. The scope of v_i in ϕ is the range $\text{CIR}[v_l, v_r]$ where v_l and v_r are the leftmost and rightmost neighbors of v_i in ϕ (see Fig. 10(c)). The scope of v_i can be thought as the unique range $S = \text{CIR}[v_l, v_r]$ such that $N[v_i] \subseteq S$, $v_l \in N^-[v_i]$ and $v_r \in N^+[v_i]$ (the inconvenience is that such a range S is not well defined when $v_i \notin \text{CIR}[v_l, v_r]$).

It is easy to see that if $\phi = v_1, \dots, v_n$ is an out-straight enumeration of an oriented graph D , then the scope S of the vertex v_i is an out-straight enumeration of $D[S]$, for every $v_i \in V(D)$. This property does not hold for out-round oriented graphs. The oriented 4-wheel graph D in Fig. 10(a) is out-round, but the scope S of its universal vertex contains an induced hole and, therefore, $D[S]$ is not out-straight. We define the locally out-straight oriented graphs specifically to restore this property back. That is, an oriented graph D is locally out-straight if there is an out-round enumeration $\phi = v_1, \dots, v_n$ of D such that S is an out-straight enumeration of $D[S]$, for every $v_i \in V(D)$ with scope S in ϕ . As before, the enumeration ϕ is referred to as a locally out-straight enumeration of D . The graph depicted in Fig. 10(b) is locally out-straight and it is not out-straight.

The next two theorems relate the out-round and the locally out-straight oriented graphs with the NCA and NHCA graphs, respectively.

Theorem 22. A graph is an NCA graph if and only if it admits an out-round orientation.

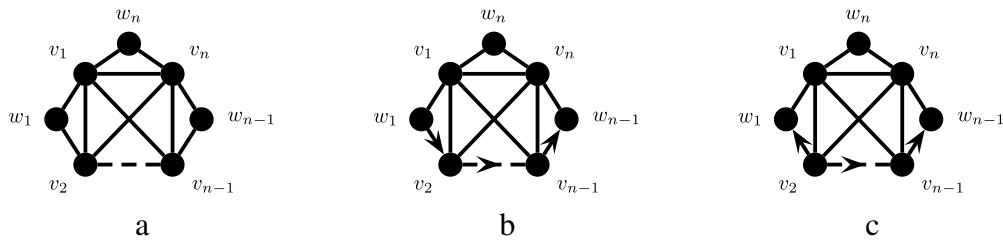


Fig. 11. The n -rising sun graph and the orientations corresponding to ρ and γ .

Proof. Let \mathcal{M} be an NCA model of a graph G with arcs A_1, \dots, A_n , where $s(A_1), \dots, s(A_n)$ appear in this order in a traversal of $C(\mathcal{M})$. For $1 \leq i \leq n$, call v_i to the vertex of G that corresponds to A_i . Define D as the digraph with vertex set $V(G)$, where $v_i \rightarrow v_j$ if and only if A_i crosses $s(A_j)$, for every $1 \leq i, j \leq n$. We claim that D is an out-round orientation of G with $\phi = v_1, \dots, v_n$ as an out-round enumeration. Fix i and j such that $1 \leq i, j \leq n$. First, notice that $s(A_i) \in A_j$ only if $s(A_j) \notin A_i$ because \mathcal{M} is NCA. Thus, $v_i \rightarrow v_j$ if and only if v_i is adjacent to v_j and $v_j \not\rightarrow v_i$, i.e., D is an orientation of G . Second, if $v_i \rightarrow v_j$ then, since $s(A_j) \in A_i$, it follows that $v_i \rightarrow v_k$ for every $v_k \in \text{CIR}(v_i, v_j)$. Hence $N^+[v_i] = \text{CIR}[v_i, v_{i+r}]$, where $r = d^+(v_i)$, implying that ϕ is an out-round enumeration of D .

For the converse, let $\phi = v_1, \dots, v_n$ be an out-round enumeration of some orientation D of G . Pick n points $s(1), \dots, s(n)$ of a circle C in such a way that $s(1), \dots, s(n)$ appear in this order in a traversal of C . For each vertex $v_i \in V(D)$, define A_i as the arc of C whose beginning point is $s(i)$ and whose ending point lies in $(s(i + r_i), s(i + r_i + 1))$, where $r_i = d^+(v_i)$. We claim that $\mathcal{M} = (C, \{A_i\}_{1 \leq i \leq n})$ is an NCA model of G . Fix i and j such that $1 \leq i, j \leq n$. By definition, $v_i \rightarrow v_j$ if and only if $i < j \leq i + r_i$, thus A_i crosses $s(j) = s(A_j)$ if and only if $v_i \rightarrow v_j$, i.e., \mathcal{M} is a circular-arc model of G . On the other hand, if $v_i \rightarrow v_j$, then $v_j \not\rightarrow v_i$, hence if A_i crosses $s(A_j)$, then A_j does not cross $s(A_i)$. That is, \mathcal{M} is an NCA model of G . \square

Theorem 23. A graph is an NHCA graph if and only if it admits a locally out-straight orientation.

Proof. Let \mathcal{M} be an NHCA model of a graph G with arcs A_1, \dots, A_n , where $s(A_1), \dots, s(A_n)$ appear in this order in a traversal of $C(\mathcal{M})$. Define the set $\{v_i\}_{1 \leq i \leq n}$, the orientation D of G , and the out-round enumeration ϕ of D as in the proof of Theorem 22. We claim that D is actually a locally out-straight orientation of G with ϕ as a locally out-straight enumeration. Fix i such that $1 \leq i \leq n$, and take a small enough ϵ . Let A_l be the arc crossing $s_i + \epsilon$ whose beginning point is farthest from s_i in a counterclockwise traversal of $C(\mathcal{M})$. Similarly, let A_r be the arc crossing $t_i - \epsilon$ whose beginning point is nearest to t_i in a counterclockwise traversal of $C(\mathcal{M})$. By the way ϕ is defined, it follows that v_l and v_r are the leftmost and rightmost neighbors of v_i in ϕ . Hence, $S = \text{CIR}[v_l, v_r]$ is the scope of vertex v_i in ϕ .

Since $s(A_1), \dots, s(A_n)$ appear in this order in \mathcal{M} , it follows that $v_j \in S$ if and only if $s(A_j) \in (s_l - \epsilon, t_i + \epsilon)$. Thus, A_j cannot cross s_l , or otherwise A_j together with A_l and A_i would cover the circle. Therefore, the submodel \mathcal{M}' of \mathcal{M} induced by the arcs A_l, \dots, A_r is an interval model of $G[S]$. This implies that S is a locally out-straight enumeration of $D[S]$. Consequently, ϕ is a locally out-straight enumeration of D as claimed.

For the converse, let G be a graph that admits a locally out-straight orientation. By Theorem 22, G must be an NCA graph. So, it is enough to prove that G contains no wheels, 3-suns, umbrellas, rising suns, or tents as induced subgraphs, by Corollary 4. It is not hard to see that the class of graphs that admit a locally out-straight orientation is hereditary, hence we need only prove that wheels, 3-suns, umbrellas, rising suns, and tents admit no locally out-straight orientations. We shall do this for the proof.

For the first case, let D be an out-round orientation of a wheel and take ϕ as an out-round enumeration of D . Clearly, if v is the universal vertex of the wheel and S is its scope, then $N[v] = V(D) \subseteq S$. Consequently, $D[S]$ is not out-straight because it contains a hole, implying that ϕ is not a locally out-straight enumeration. Therefore, D is not locally out-straight.

For the second case, let D be an out-round orientation of the n -rising sun graph, for $n \geq 4$. Recall that the n -rising sun is the graph obtained by inserting two universal vertices v_1 and v_n into a path $P = v_2, \dots, v_{n-1}$, and then inserting three vertices w_1, w_{n-1}, w_n such that w_i is adjacent only to v_i and v_{i+1} , for $i \in \{1, n - 1, n\}$ (see Fig. 11(a)). Suppose, to obtain a contradiction, that D admits a locally out-straight enumeration ϕ . It is not hard to see that the induced path w_1, P, w_{n-1} admits only the following out-round enumerations, all of which are out-straight:

$$\begin{aligned} \rho &= w_1, v_2, \dots, v_{n-1}, w_{n-1}, \\ \gamma &= v_2, w_1, v_3, \dots, v_{n-1}, w_{n-1}, \\ \rho' &= w_{n-1}, v_{n-1}, \dots, v_2, w_1, \quad \text{and} \\ \gamma' &= v_{n-1}, w_{n-1}, v_{n-2}, \dots, v_2, w_1. \end{aligned}$$

Each of these enumerations corresponds to one of the four possible out-straight orientations of a path (see Fig. 11 (b) and (c)). So, one of these enumerations must appear in this order inside ϕ ; call it ϕ' . Furthermore, a vertex z_1 in ϕ' has a directed edge to another vertex z_2 in ϕ' only if z_1 is to the left of z_2 in ϕ' . Enumerations ρ and ρ' are symmetric, in the sense that the former is obtained from the latter by exchanging the labels of the vertices. This is also true for enumerations

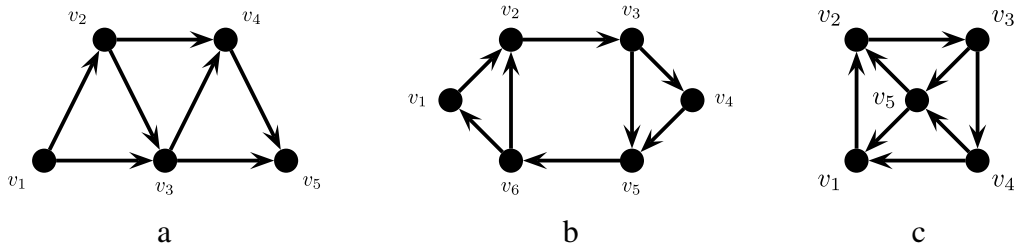


Fig. 12. Examples of straight (a), locally straight (b), and round (c) oriented graphs.

γ and γ' , so we need to consider only two cases, either $\phi' = \rho$ or $\phi' = \gamma$. For the sake of simplicity, define $u_1 = w_1$ and $u_2 = v_2$ in ρ , and $u_1 = v_2$ and $u_2 = w_1$ in γ . Thus, $\phi' = u_1, u_2, v_3, \dots, v_{n-1}, w_{n-1}$. Denote by $l(v)$ and $r(v)$ the leftmost and rightmost neighbors of v in ϕ for every $v \in V(D)$. The following claims analyze the positions of w_n and v_n in ϕ .

Claim 1: $w_n \notin \text{Cir}[u_1, w_{n-1}]$ in ϕ . Otherwise, there would be two adjacent vertices z_1 and z_2 such that z_1, z_2 appear in this order in ϕ' and z_1, w_n, z_2 appear in this order in ϕ . But this is impossible, because $z_1 \rightarrow z_2$ and $z_1 \not\rightarrow w_n$.

Claim 2: $v_n \in \text{Cir}[w_1, w_{n-1}]$ in ϕ . Again, suppose otherwise, and observe that $v_n \notin \text{Cir}[w_1, v_3]$ in ϕ . Vertices v_3 and v_n are adjacent, so either $v_n \rightarrow v_3$ or $v_3 \rightarrow v_n$. The former is impossible because it implies that $w_1 \in [v_n, r(v_n)]$, contradicting the fact that v_n is not adjacent to w_1 . In the latter case, $v_2 \not\rightarrow v_n$ because $w_{n-1} \notin [v_2, r(v_2)]$. But this is also impossible since $v_3 \in [\ell(v_n), v_n]$, $v_2 \in [v_n, r(v_n)]$, and $v_2 \rightarrow v_3$, contradicting the fact that $[\ell(v_n), r(v_n)]$ is out-straight.

By Claims 1 and 2, w_1, v_n, w_{n-1}, w_n must appear in this order in a traversal of ϕ . So, $v_n \rightarrow w_n$ in D because $w_1 \notin [w_n, r(w_n)]$ or, otherwise, ϕ would not be out-round. Hence, v_1, w_{n-1}, w_n cannot appear in this order in the range $[\ell(v_n), r(v_n)]$, because v_1 is adjacent to w_n and not to w_{n-1} . Consequently, v_n, w_{n-1}, v_1 appear in this order in $[\ell(v_n), r(v_n)]$. Analogously, v_2, w_{n-1}, v_1 cannot appear in this order in $[\ell(v_n), r(v_n)]$, so v_n, w_{n-1}, v_2 must appear in this order in $[\ell(v_n), r(v_n)]$. Recall that $v_2 \rightarrow v_3$ in D , thus v_n, w_{n-1}, v_2, v_3 must appear in this order in $[v_n, r(v_n)]$ and, therefore, $v_n \rightarrow v_3$. But since ϕ' is either ρ or γ , it follows that $v_n, w_{n-1}, u_1, u_2, v_3$ appear in this order in ϕ . This is a contradiction to the fact that ϕ is out-round, because $v_n \rightarrow v_3$ but $v_n \not\rightarrow w_1 \in \{u_1, u_2\}$.

Finally, by using backtracking arguments, it can be proved that there are no locally out-straight orientations of the 3-sun, the tent, and the umbrella. \square

PIG, PCA, and PHCA graphs can also be described in terms of orientations and enumerations. The analog of an out-straight oriented graph, for PIG graphs, is called a straight oriented graph. An oriented graph D is *straight* if there is an ordering v_1, \dots, v_n of $V(D)$ such that, for every vertex v_i , $N^-[v_i] = \text{LIN}[v_{i-\ell}, v_i]$ and $N^+[v_i] = \text{LIN}[v_i, v_{i+r}]$, where $\ell = d^-(v_i)$ and $r = d^+(v_i)$ (see Fig. 12(a)). As before, the ordering v_1, \dots, v_n is called a *straight enumeration* of D . Deng et al. [7] (see also [23]) proved that G is a PIG graph if and only if it admits a straight orientation.

As out-straight oriented graphs, straight oriented graphs can be generalized either as locally straight oriented graphs or as round oriented graphs. An oriented graph D is *round* if there is an ordering v_1, \dots, v_n of $V(D)$ such that $N^-[v_i] = \text{Cir}[v_{i-\ell}, v_i]$ and $N^+[v_i] = \text{Cir}[v_i, v_{i+r}]$, for every vertex v_i with $\ell = d^-(v_i)$ and $r = d^+(v_i)$. When the scope S of each vertex is a straight enumeration of $D[S]$, then D is also a *locally straight oriented graph*. As usual, the circular orderings corresponding to the round and locally straight oriented graphs are called *round enumerations* and *locally straight enumerations*, respectively. Examples of straight, locally straight and round oriented graphs are depicted in Fig. 12. Hell and Huang [19] (see also [46]) proved that the class of PCA graphs is exactly the class of graphs that admit a round orientation. As with NCA graphs, we can restore the Helly condition of PIG graphs by restricting the attention to locally straight graphs.

Theorem 24. *A graph is a PHCA graph if and only if it admits a locally straight orientation.*

Proof. Let \mathcal{M} be an NPHCA model of a graph G and define D and $\phi = v_1, \dots, v_n$ as in the proof of Theorem 23. By Theorem 23, the scope S of each vertex v_i in ϕ is an out-straight enumeration of $D[S]$, for every $1 \leq i \leq n$. Also, as in the proof of Theorem 23, the submodel \mathcal{M}' of \mathcal{M} induced by the vertices of S is an interval model of $G[S]$. Since \mathcal{M}' is also PCA then \mathcal{M}' is a PIG model and so S is in fact a straight enumeration of $D[S]$. Therefore, D is locally straight.

For the converse, let G be a locally straight orientable graph. Since locally straight oriented graphs are locally out-straight, then G is an NHCA graph by Theorem 23. Thus, it is enough to see that G contains no induced $K_{1,3}$ by Theorem 9. But, since the class of locally straight orientable graphs is hereditary, it suffices to prove that no orientation of $K_{1,3}$ is locally straight. This is clearly true because the $K_{1,3}$ graph admits no round orientations at all. \square

In Fig. 13, we depict the class hierarchy of NCA graphs. This hierarchy uses the same notation as the hierarchy depicted in Fig. 7. This time, however, the label beside the edge between \mathcal{C}_1 and \mathcal{C}_2 indicates the additional condition that a round enumeration of an orientation of a graph $G \in \mathcal{C}_1$ has to satisfy in order to also prove that $G \in \mathcal{C}_2$.

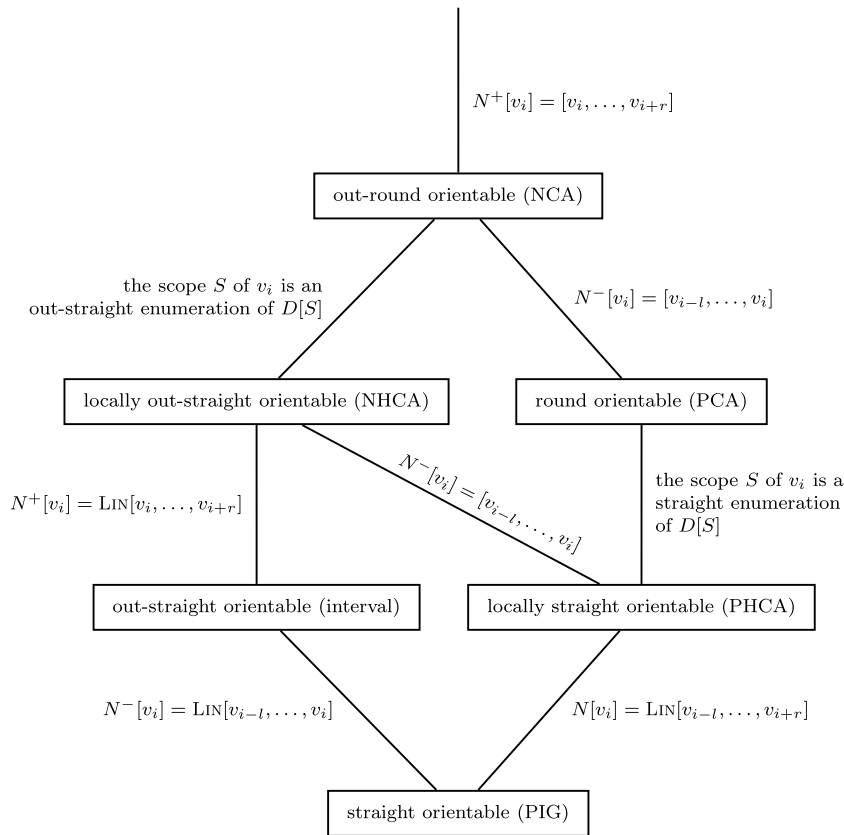


Fig. 13. The NCA class hierarchy, depicted in terms of round enumerations.

7. Conclusions

In this paper we studied the NHCA, PHCA, and UHCA classes of graphs. In Section 3 we showed that these classes arise naturally when some properties are generalized from a family of intervals on the real line to a family of arcs of a circle. NHCA, PHCA, and UHCA also appear when studying some problems on circular-arc graphs that are not so well solved for the general case but easy for interval graphs. Given the close relationship between interval and NHCA graphs, we could expect to find good results for such problems, restricted to NHCA or PHCA graphs. The coloring problem on circular-arc graphs is perhaps the most well known example in which NHCA and PHCA play a major role. Let χ and ω be the chromatic number and clique number of a circular-arc graph G , and r be the maximum number of arcs that share a common point of a circular-arc model \mathcal{M} of G . Tucker [49] proved that $\chi \leq 3/2\omega$ when G is NHCA, while $\chi \leq 3/2r \leq 3/2\omega$ when \mathcal{M} is a PHCA model. Given the close relationship between interval and NHCA graphs, we expect that many properties and algorithms on (proper) interval graphs can be easily generalized for (PHCA) NHCA graphs.

In Section 4 we described characterizations for the classes of NHCA, PHCA, and UHCA graphs. These characterizations imply a complete family of forbidden induced subgraphs for the PHCA and UHCA classes. That is, adding W_4 and the 3-sun to Tucker’s forbidden subgraphs for PCA graphs, we obtain the complete list of forbidden subgraphs for PHCA graphs, while adding W_4 to Tucker’s forbidden family for UCA graphs, we obtain all the subgraphs forbidden for UHCA graphs. On the other hand, the characterization for NHCA graphs is partial; we know which circular-arc graphs are not NHCA, but the complete family of forbidden subgraphs for NHCA is still unknown.

On the algorithmic side, Nussbaum [43] showed how to transform a circular-arc model into an equivalent PCA model in $O(n)$ time. Combining this result with those in Section 5, we obtain $O(n)$ time algorithms for transforming a circular-arc model into an equivalent PHCA or UHCA model. For NHCA graphs we did not find such an algorithm. Furthermore, the $O(n + m)$ time algorithm for the recognition of NHCA graphs is not based in transformations on the input circular-arc model. The problem is that we did not find how to transform an HCA model into an interval model in $O(n)$ time.

The recognition problem for NHCA, PHCA, and UHCA graphs is well solved when the input is given by the adjacency lists of the graph. However, the algorithm for the recognition of NHCA graphs given in Section 5 is hard, because it requires the construction of a circular-arc model of the input graph. Therefore, it would be nice to find a direct algorithm for the recognition of NHCA graphs. The local interval property could play an important part in such an algorithm. In fact, many ideas that were successful for other recognition algorithms can perhaps be used for the recognition of NHCA graphs. For

instance, we could test whether G is an interval graph. If so, output the interval model of G . Otherwise, try to split G into several interval graphs, resembling the process done by Deng et al. [7]. Then, recognize each part as an interval graph and “rebuild” the model. Another possibility is to find the cliques of each part so as to build a PC -tree later (see [21]), as Booth and Lueker [3] do. The major drawback of this approach is that finding the cliques of an NHCA graph directly seems difficult. The third possibility is to extend the incremental data structure used by Korte and Möhring [27] in their interval graph recognition algorithm. Since the insertion of a vertex depends only on its neighborhood, then it is perhaps possible to exploit the local interval property. Also, it could be interesting to generalize the incremental algorithm by Crespelle [5], so as to insert each vertex in $O(n)$ time.

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