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How to find the sunrise and sunset times from a Sun clock and calendar

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Abstract
The purpose of this article is to describe the construction of a Sun clock and calendar (SCandC) that will allow an observer to not only see the time but also the symmetry properties of the Sun–Earth relative movement. A set of circles drawn on the SCandC will allow the observer to see their associated dates as well as to perform a visual interpolation between any pair of consecutive circles to estimate an intermediate date. By introducing the sunrise and sunset horizon lines in the SCandC the observer will be able to find the sunrise and sunset times during most of the year. The observer will also be able to appreciate the difference in time duration between the spring–summer and autumn–winter periods, as a consequence of Kepler's second law, as well as to observe that there is a small difference between the circle radii of equidistant dates from the solstices in the spring–summer versus autumn–winter periods as a consequence also of Kepler's second law. The equations derived in the present article will be tested against data from a particular lighthouse.

Introduction

The purpose of this article is to describe the construction of a Sun clock and calendar (SCandC) that will allow an observer to not only see the time but also the symmetry properties of the Sun–Earth relative movement. A set of circles drawn on the SCandC will allow the observer to see their associated dates as well as to perform a visual interpolation between any pair of consecutive circles to estimate an intermediate date. By introducing sunrise and sunset horizon lines in the SCandC the observer will be able to find the sunrise and sunset times during most of the year. The observer will also be able to see the difference in time duration between the spring–summer and autumn–winter periods, as a consequence of Kepler's second law.

In the section 'Evaluation of the Sun rays' incidence (declination) angle on the equatorial plane' we determine an analytical equation to evaluate the Sun rays' declination angle. In the section 'How long is a geometrical day during

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the year and at different latitudes?' we derive an equation to determine (i) how long a geometric day is, that is, the time that would elapse between the centre of the Sun's disc rising from and setting into the horizon if Sun ray diffraction due to the atmosphere was absent, and (ii) to determine how long a geometric day is during the year and at different latitudes. In the section 'Kepler's second law' we discuss the consequences of Kepler's second law on the design of our SCandC. In the section 'Adjusting a particular date during the year on the Sun clock and calendar' we explain how to adjust a particular date during the year on the SCandC. In the section 'Evaluating the geometric day duration for a particular date' equations are derived to evaluate the geometric day duration (GDD) of a particular date. The equations derived in this section are successfully tested with data from a particular lighthouse. Finally in the section 'Conclusions' our conclusions are given.

Evaluation of the Sun rays' incidence (declination) angle on the equatorial plane

Let λ be the Sun rays' incidence (declination) angle on the equatorial plane, see figure 1(*a*). Taking into account that the eccentricity of the Earth orbit in its movement around the Sun is rather small (less than 3%), for simplicity reasons, in this section we will consider it as circular. Eccentricity considerations will be introduced in the section 'Kepler's second law'. Let θ be the angle that determines the Earth's position (see figure 1(*b*)) in that circular orbit with respect to some reference initial time $\theta_0 = 0$. Without loss of generality let us fix that initial time when the value of λ reaches one of its extreme values (solstices). That is, when the Sun rays are perpendicular to the Tropics of either Cancer or Capricorn, then $\lambda = \lambda_0$. After some time the Earth will have rotated an angle θ with respect to the initial position at θ_0.

Let i, j and k be orthogonal versors oriented along a system of orthogonal coordinates (x, y, z) respectively with its origin at the Earth's centre, so that i, j belong to the equatorial plane whereas k is perpendicular to that plane. Let us define two vectors a and b of unitary modulus with components

$$a = \cos\lambda_0 i + \sin\lambda_0 k \qquad b = j \tag{1}$$

that will help us to determine $\lambda = \lambda(\theta)$, that is, how λ depends on θ. After some time, the Earth will have rotated an angle θ ($0 \leqslant \theta \leqslant 2\pi$) in its trajectory around the Sun. A linear combination c of vectors a and b will determine the plane where the Sun rays are impinging perpendicularly on the Earth's surface:

$$c = \cos\theta a + \sin\theta b \tag{2}$$

or

$$c = \cos\lambda_0 \cos\theta i + \sin\theta j + \sin\lambda_0 \cos\theta k. \tag{3}$$

Therefore, according to figure 2,

$$e = \cos\lambda_0 \cos\theta i + \sin\theta j \qquad f = \sin\lambda_0 \cos\theta k. \tag{4}$$

We can conclude that

$$\tan\lambda = \frac{\sin\lambda_0 \cos\theta}{\sqrt{\sin^2\theta + \cos^2\theta \cos^2\lambda_0}}. \tag{5}$$

Equation (5) allows the determination of how the incidence (declination) angle λ depends on time (represented by the angle θ). Table 1 shows some particular values of λ.

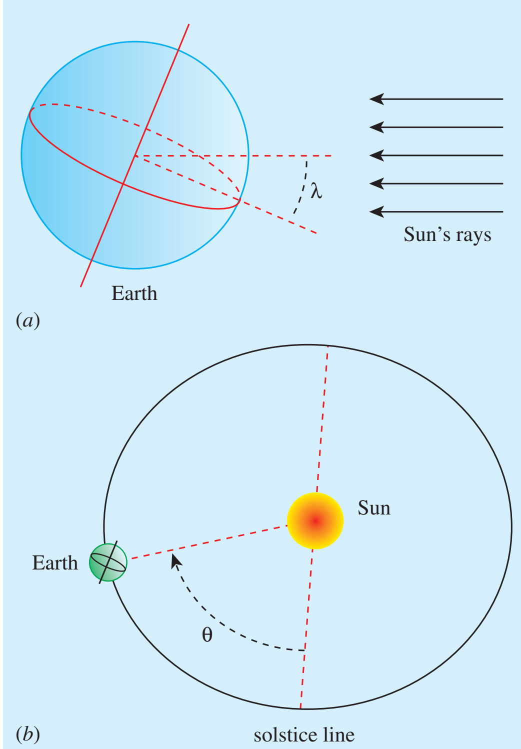

Figure 1. (*a*) Declination angle λ is the Sun's rays incidence angle on the equatorial plane. (*b*) Earth position angle θ with respect to the solstice line ($\theta_0 = 0$).

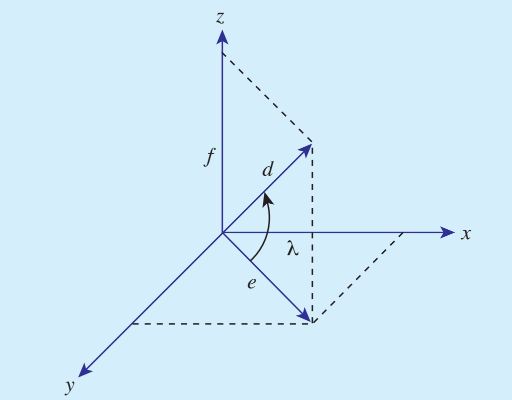

Figure 2. Derivation of the declination angle λ.

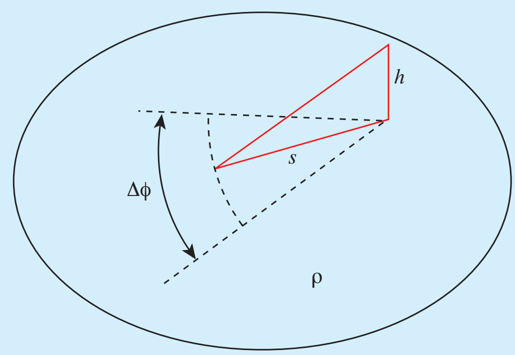

Figure 3. SCandC: plane ρ, needle *h* and needle's shadow angle swept along Δϕ.

Table 1. Declination angle values λ for a particular set of angles (dates) θ.

θ	λ
0	λ_0
$\pi/6$	$\arctan \frac{\sqrt{3}\sin\lambda_0}{\sqrt{1+3\cos^2\lambda_0}}$
$\pi/3$	$\arctan \frac{\sin\lambda_0}{\sqrt{3+\cos^2\lambda_0}}$
$\pi/2$	0

How long is a geometrical day during the year and at different latitudes?

We shall call the time that would elapse between the centre of the Sun's disc rising from and setting into the horizon if Sun ray diffraction due to the atmosphere was absent the 'geometrical day'. Let us now consider an SCandC made up of a plane ρ parallel to the equatorial plane plus a needle that crosses that plane (emerging a distance h from both sides) and perpendicular to it. The intersection of the plane ρ with the horizontal plane (i) must determine a straight line in the east–west direction, (ii) must determine an angle $\frac{\pi}{2} - \alpha$ and (iii) must point the needle towards the sky pole of the hemisphere where the SCandC is placed.

The Sun's rays will project the needle's shadow s onto ρ whose length will depend on the Earth orbit position θ, and the latitude α as well as on the day time ϕ, that is therefore $s = s(\theta, \alpha, \phi)$, see figure 3. The needle's shadow will describe (for practical purposes) a circle of radius s:

$$s = h \cot \lambda. \qquad (6)$$

Actually the shadow describes an expanding and contracting spiral with as many arms as there are days in a year. The apparent movement of the Sun around the Earth will produce the rotation of the shadow s around the intersection point of h with ρ. The angle $\Delta\phi_{max}$ swept by the shadow during a whole day is the geometric day duration we are looking for and it will be enclosed between ϕ_{min} and ϕ_{max} values, corresponding to the sunrise and sunset times. The $\Delta\phi_{max}$ value depends on the latitude α as well as the time θ elapsed since the initial time θ_0. A student placed on the equator ($\alpha = 0$) will see that $\Delta\phi_{max} = \pi$ (or 12 h), whereas another one placed on one of the poles ($\alpha = \pi$), will see that $\Delta\phi_{max} = 2\pi$ or 0, depending whether the Sun's rays are impinging perpendicularly to the hemisphere to which that pole belongs, or to the other one respectively.

Let μ be a plane parallel to the horizontal plane where our SCandC is placed and such that the extreme of the needle belongs to it. There will be two such planes for $\alpha \neq 0$ and $\frac{\pi}{2}$, for both northern and southern latitudes. We will call these μ_{ss} and μ_{aw} depending on whether the face illuminated by the Sun is the spring–summer or autumn–winter face, respectively. Those planes μ_{ii} will intersect the plane ρ determining a straight line in the east–west direction at a distance ω,

$$\omega = h \tan \alpha, \qquad (7)$$

from the east–west straight line passing through the intersection of the needle with the plane ρ. The intersection straight line will be (with respect to the intersection of the needle with the plane ρ) oriented to the north on the plane μ_{ss} and to the south on μ_{aw} in the Southern Hemisphere, and in the opposite direction in the Northern one.

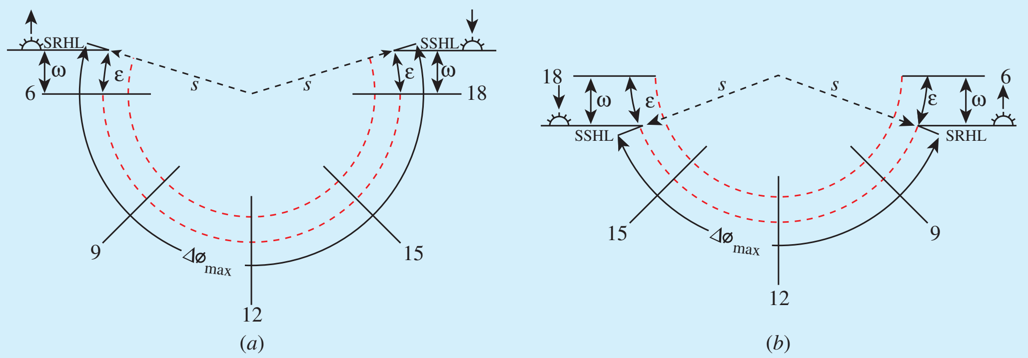

Figure 4. An SCandC if $\alpha < \pi/2 - \lambda_0$. The angle $\Delta\phi_{max}$ for a particular date (second circular sector). The inner circle represents the shadow trajectory during the corresponding solstice. The sunrise and sunset horizon lines (SRHL and SSHL, respectively) are also indicated for a Southern Hemisphere SCandC. (*a*) The spring–summer face of the SCandC is shown, (*b*) the autumn–winter one. Shadow length *s* (equation (6)), angle ε (equation (8)) and ω (equation (7)) are also indicated.

Within this geometry the shadow projected by the needle will describe every day on the plane ρ (without much error) a circular sector of radius s given by equation (6). That circular sector begins at a point belonging to the straight line determined by the intersection of μ_{ii} with ρ. The shadow of the needle's point starts and finishes its travel on ρ at a point on that straight line. The hemistraight line on the left (right) will be called the sunrise horizon of the SCandC, whereas the hemistraight line on the right (left) will be called the sunset horizon of the SCandC for the spring–summer (autumn–winter) faces of ρ in the Southern Hemisphere, whereas at the Northern Hemispere the relations will be the opposite. A student, therefore, will be able to 'read', with an error of a few minutes, the sunrise (sunset) time during the year.

The length of a day during the year can be divided into two parts depending on the latitude α:

(a) $0 \leqslant \alpha < \frac{\pi}{2} - \lambda_0$;
(b) $\frac{\pi}{2} - \lambda_0 \leqslant \alpha \leqslant \frac{\pi}{2}$.

We consider α as a positive defined quantity that can be either southern or northern latitude. In case (a) the day's duration will be determined by the angle $\Delta\phi_{max}$ swept by the needle's shadow on that particular day, see figure 4. From that figure we conclude the following.

If $\alpha < \frac{\pi}{2} - \lambda_0$

$$\sin\varepsilon = \frac{\omega}{s} = \frac{h\tan\alpha}{h\cot\lambda} = \tan\alpha\tan\lambda, \quad \text{or } \varepsilon = \arcsin\tan\alpha\tan\lambda \tag{8}$$

where $\lambda = f(\lambda_0, \theta)$, see equation (5). Therefore the geometric day duration (GDD) in hours will be

$$\text{GDD (h)} = 12 + \tfrac{2}{15}\arcsin\tan\alpha\tan\lambda. \tag{9}$$

Let us consider that $\alpha > 0$ either at the Southern or Northern Hemisphere and that λ can be either positive or negative depending on the date during the year represented by the value of θ.

If $\frac{\pi}{2} - \lambda_0 \leqslant \alpha \leqslant \frac{\pi}{2}$, sunrise time (SRT) and sunset time (SST) will be

$$\text{SRT (h)} = \frac{\xi - \xi_0}{15} + 12 - \frac{1}{15}\arcsin\tan\alpha\tan\lambda,$$
$$\text{SST (h)} = \frac{\xi - \xi_0}{15} + 12 + \frac{1}{15}\arcsin\tan\alpha\tan\lambda. \tag{10}$$

Equation (10) is the answer to the article's title, where ξ and ξ_0 are the longitudes of the place where the SCandC is located and the one used to define the civil time, respectively.

From figure 5 we conclude that the day's duration will be 24 h if

$$s \leqslant h\tan\alpha \qquad \text{or} \qquad \cot\lambda \leqslant \tan\alpha. \tag{11}$$

When the previous inequality is not fulfilled, that is, for those λ values that do not satisfy

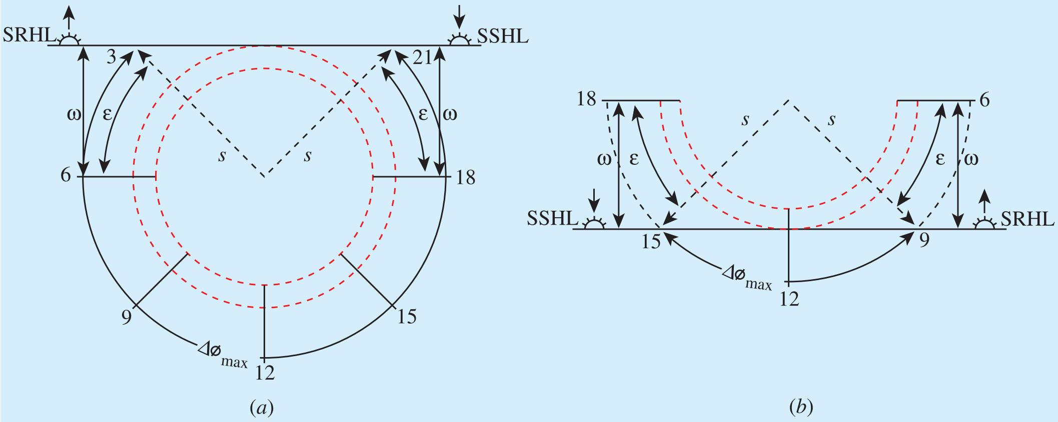

Figure 5. An SCandC if $\pi/2 - \lambda_0 \leqslant \alpha \leqslant \pi/2$. (*a*) Spring–summer face: the angle $\Delta\phi_{max}$ will be 2π (or GDD = 24 hr) for an interval of days centred on the summer solstice date and ending on the dates indicated by the second circle. For the rest of the year $\Delta\phi_{max}$ is shown for a particular date (third circular sector) and can be calculated with equation (9). The inner circle represents the needle shadow trajectory during the summer solstice. (*b*) Autumn–winter face: on this face there are no full circular trajectories. $\Delta\phi_{max}$ is calculated with equation (9). The sunrise and sunset horizon lines, SRHL and SSHL, are also indicated for a Southern Hemisphere SCandC. Shadow length *s* (equation (6)), angle ε (equation (8)) and ω (equation (7)) are also indicated.

equation (11), the day's duration will be less than 24 h and the value of ε will again be determined by equation (8). It is interesting to note that within this interval of latitudes the two horizon lines converge into a unique continuous straight line.

Kepler's second law

Although the Earth's orbital eccentricity is small ($\approx$3%), it is enough to introduce variations in the durations of seasons. The Earth reaches the aphelion (its largest distance from the Sun) a few days after the winter solstice (SH) and the perihelion a few days after the summer solstice (SH). This fact will introduce a change in the Earth's 'apparent circular angular velocity' around the Sun compared to the $\xi = 2\pi/\text{year}$ value for an actual circular trajectory because of Kepler's second law. Therefore the apparent angular velocity will be lower than ξ from the (SH) autumnal equinox to the (SH) vernal equinox and greater than ξ from the (SH) vernal equinox to the (SH) autumn equinox. Consequently the (SH) autumn–winter period will be longer than the (SH) spring–summer period. From the equinox and solstice dates provided by the US Naval Observatory for the 2000–10 period we estimate that the (SH) autumn–winter period lasts on average 186.3934 ($\pm$0.0500) days whereas the (SH) spring–summer period lasts on average only 178.8591 ($\pm$0.0549) days.

Therefore any observer of this SCandC will see the effect of Kepler's second law when noticing that the upper face, (SH) spring–summer, of the plane ρ remains illuminated on average 7.5343 days less than the lower face, (SH) autumn–winter.

Adjusting a particular date during the year on the Sun clock and calendar

Our SCandC provides calendar information during most of the year, except during the periods around the equinoxes when the shadow length tends to infinity. The calendar information is provided by a set of circles adjusted to a set of dates that can be chosen according to a previously fixed criterion. The first step in evaluation of the radius of the circle associated to a particular date is to evaluate the angle θ (to use in equations (1)–(5)) for that particular date. Let us assume that our chosen date is $\pm n$ days with respect to a particular solstice, then

$$\theta = \pm n \xi_{ii}, \qquad (12)$$

where ξ_{ii} is either the Earth's angular velocity during the (SH) autumn–winter period

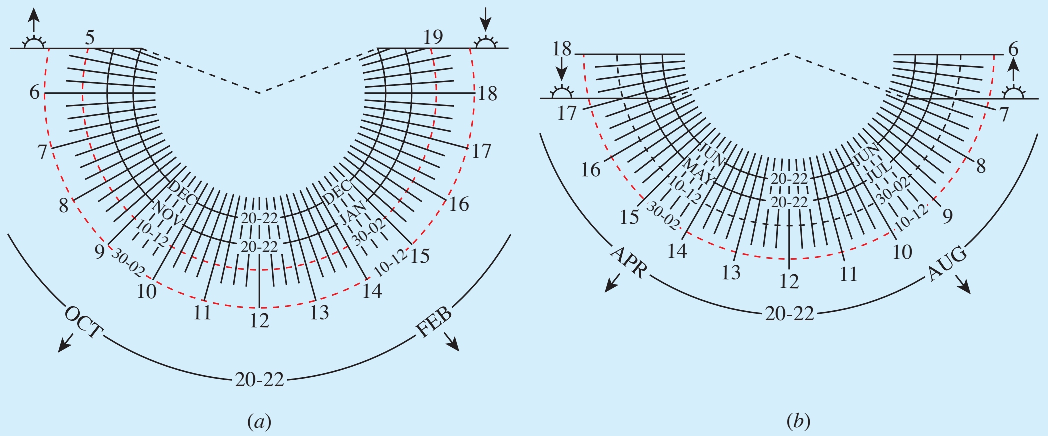

Figure 6. The SCandC adjusted at the latitude of the Stella Maris Chapel (α = 38.985°) at the small town of Monte Hermoso, Buenos Aires Province, Argentina. (*a*) Spring–summer face of the SCandC and (*b*) the autumn–winter one.

$\xi_{aw} = 0.9657°/\text{day}$ $(=\pi/186.3934)$ or the Earth's angular velocity during the (SH) spring–summer period $\xi_{ss} = 1.0064°/\text{day}(=\pi/178.8591)$. Once the value of θ for a particular date is determined, then the declination angle λ is determined from equation (5). The radius of the circle associated with that particular date is evaluated from equation (6).

Evaluating the geometric day duration for a particular date

Once the declination angle λ is evaluated for a particular date, see the previous section, either the sunrise and sunset times or the GDD can be calculated from equation (9), if the latitude α where the SCandC is placed is known.

As an example let us evaluate the GDD for a couple of days ($n = 45$) placed after the 2009 (SH) summer solstice, 4 February 2010 and after the (SH) winter solstice, 5 August 2010 (see [1]). Let us suppose that our SCandC is placed at a southern latitude $\alpha = 38.990°$. Recala Light House is at this latitude and at a western longitude of 61.260°. The Argentinean Naval Hydrographic Service, ANHS, provides (see [2]) the sunrise and sunset times for every Argentinean lighthouse. For those two days we deduce from equation (5) that $\lambda(4\text{ Feb}) = 16.2499°$ and $\lambda(5\text{ Aug}) = 16.7812°$. From these data and equation (9) we derive that GDD(4 Feb) = 13 h 49 min and GDD(5 Aug) = 10 h 7 min. These values should be compared with those provided by the ANHS which are 13 h 58 min and 10 h 15 min, respectively. The differences of 9 and 8 min, respectively, observed should be attributed to the refraction of the Sun's rays by the atmosphere during sunrise and sunset that enlarge both steps by about 4 min.

Conclusions

We chose that particular lighthouse to compare our equations because an SCandC like the one described in this article (see figure 6) will be placed at the Stella Maris Chapel ($\alpha = 38.985$ and western longitude 61.290°) which is in a nearby small town called Monte Hermoso, on the southern coast of Buenos Aires province, Argentina, see figure 6. On that part of the Atlantic coast the Sun rises from and sets into the sea during most of the year. As there are no physical obstacles to the Sun's rays, the SCandC will provide full information. Figure 7 shows the Stella Maris Chapel with the SCandC adjusted to the civil time.

The SCandC described in this article will allow an observer (i) to learn about the symmetry properties of the relative movements of the Sun–Earth system, and (ii) to find the time with an error of less than ±2 min during the whole day. (iii) A set of circles drawn on the SCandC will allow the observer to see their associated dates as well as to perform a visual interpolation between any pair of consecutive circles to estimate an intermediate date, (iv) to observe that the (NH) spring–summer period is on average seven and a half days longer than the (SH) spring–summer

Figure 7. The Stella Maris Chapel with the SCandC adjusted to the civil time.

period and (v) to observe that there is a small difference between the circle radii of equidistant dates from the solstices in the spring–summer versus autumn–winter periods as a consequence of Kepler's second law. (vi) The introduction of the sunrise and sunset horizons on the SCandC will allow the observer to find the sunrise and sunset times as well as the GDD during most of the year and (vii) to appreciate the few minutes (≈4 min) by which a day is lengthened because of Sun ray diffraction by the atmosphere.

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