Weighted inequalities and a.e. convergence for Poisson integrals in light-cones *

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Abstract

We show that the Poisson maximal operator for the tube over the light-cone, P^*, is bounded in the weighted space $L^p(w)$ if and only if the weight $w(x)$ belongs to the Muckenhoupt class A_p. We also characterize with a geometric condition related to the intrinsic geometry of the cone the weights $v(x)$ for which P^* is bounded from $L^p(v)$ into $L^p(u)$, for some other weight $u(x) > 0$. Some applications to *a.e.* restricted convergence of Poisson integrals are given.

1 Introduction

Let $\Omega = \{y \in \mathbb{R}^n \ : \ y_1 > \sqrt{y_2^2 + \ldots + y_n^2}\}$ be the *forward light-cone* in $\mathbb{R}^n$, $n \geq 3$. We also denote the corresponding *Lorentz form* by

$$\Delta(z) := z_1^2 - (z_2^2 + \ldots + z_n^2), \quad z \in \mathbb{C}^n.$$

The *Poisson kernel* associated with the tube domain $T_\Omega := \mathbb{R}^n + i\Omega$ is defined by

$$P_y(x) = \frac{\Delta(y)^{n/2}}{|\Delta(x+iy)|^n}, \quad x + iy \in T_\Omega. \tag{1.1}$$

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This kernel arises in the study of Hardy spaces of holomorphic functions in tube domains [16, Ch.3], and also in the theory of harmonic functions in the symmetric space T_Ω [7, 9, 8]. A major question in this field concerns the validity of Fatou-type theorems, that is, the search of suitable conditions on a function f in $\mathbb{R}^n$ so that

$$\lim_{y\to 0} P_y * f(x) = f(x), \quad a.e.\, x \in \mathbb{R}^n. \tag{1.2}$$

A classical result of E. Stein and N. Weiss [18, 17] establishes such convergence for all $f \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, provided $y \to 0$ *restricted* to a proper subcone of Ω (i.e. $|y'| < \delta y_1$ for fixed $\delta < 1$). In fact, restricted convergence is essentially equivalent to the L^p-boundedness of the *(vertical) Poisson maximal operator*:

$$P^* f(x) = \sup_{t>0} \big| P_{t\mathbf{e}} * f(x) \big| = \sup_{t>0} \Big| \int_{\mathbb{R}^n} P_{t\mathbf{e}}(x-u) f(u)\, du \Big|, \tag{1.3}$$

where $\mathbf{e}$ denotes the fixed point $(1, 0, \dots, 0) \in \Omega$ (see [18, Lemma 4.3]).

It should be noted that the stronger notion of *unrestricted convergence* (i.e. when $\Omega \ni y \to 0$) fails to satisfy (1.2) even for bounded functions (see [15, p. 459]). There are also some intermediate notions such as the *admissible semirestricted convergence* in the sense of Korányi [10], which are related to the "strong" maximal function

$$P^{**} f(x) = \sup_{(y_1, y_2, 0) \in \Omega} \big| P_y * f(x) \big|.$$

In this last case we refer to [2, 13, 3] for positive results in L^p $(p > 1)$, and negative results in L^1 in the different contexts of T_Ω, symmetric domains and general homogeneous Siegel domains.

In this paper we shall be interested in restricted convergence and the boundedness properties of the operator P^*. In L^p spaces these have been studied by different methods (see the above mentioned papers [18, 17], or a general procedure for symmetric domains in [14]). A particularly simple approach in the case of T_Ω is based on *vector-valued Calderón-Zygmund theory*, or what is the same on suitable decay and smoothness estimates on the kernel $P(x) = P_{\mathbf{e}}(x)$. With this machinery it is not difficult to establish the boundedness of P^* in $L^p(\mathbb{R}^n)$ for $1 < p \le \infty$, and the weak boundedness in $L^1(\mathbb{R}^n)$ (see e.g. [15, p. 82]).

Our goal in this paper is to pursue this approach, by considering finer estimates in the kernels which lead to new boundedness properties of P^* in weighted L^p spaces.

In particular, this will produce large classes of functions which satisfy the restricted pointwise limit in (1.2). In fact, we shall actually *characterize* the $L^p(v)$ spaces which admit restricted pointwise convergence as in (1.2). Our theorems in this direction can be stated as follows. Below, A_p denotes the usual Muckenhoupt class (as defined e.g. in [15, Ch.5]).

THEOREM 1.4 *Let $w(x) > 0$ and $1 < p < \infty$. Then P^* is bounded in $L^p(w)$ if and only if $w \in A_p$. Similarly, P^* is bounded from $L^1(w)$ into $L^{1,\infty}(w)$ if and only if $w \in A_1$.*

A surprising consequence of this theorem is that P^* is bounded in the same $L^p(w)$ spaces as the classical Hardy-Littlewood maximal operator, even though $P^*f(x)$ is typically much larger than $Mf(x)$. We do not know whether this result remains true in higher rank cones. We also remark a main difference with the standard approach to weighted inequalities since P^* is not a "regular" Calderón-Zygmund operator. We shall handle this difficulty with a fine computation of the smoothness of $P(x)$, and an application of the refined vector-valued Calderón-Zygmund theory developed in [12].

Our second result is very much related to the intrinsic geometry of the cone. To state it we introduce the following subsets of $\mathbb{R}^n$:

$$E_k = \{x \in \mathbb{R}^n \ : \ \text{dist}\,(x, \pm\partial\Omega) \leq 2^k\}, \quad k \geq 1. \tag{1.5}$$

THEOREM 1.6 *Let $v(x) > 0$ and $1 < p < \infty$. Then, the following are equivalent:*

(*a*) *There exists $u(x) > 0$ such that $P^* : L^p(v) \to L^p(u)$ is bounded;*

(*b*) *For all $f \in L^p(v)$ it holds that $P^*f(x) < \infty$, a.e. $x \in \mathbb{R}^n$;*

(*c*) *For all $f \in L^p(v)$ it holds that*

$$\lim_{t\to 0} P_{t\mathbf{e}} * f(x) = f(x) \quad \textit{and} \quad \lim_{t\to\infty} P_{t\mathbf{e}} * f(x) = 0, \quad \textit{a.e.}\, x \in \mathbb{R}^n; \tag{1.7}$$

(*d*) *The weight $v(x)$ satisfies the property*

$$\|v\|_*^{p'} := \sup_{k\geq 1} \int_{E_k} \frac{v^{-\frac{1}{p-1}}(y)}{(2^k + |y|)^{np'}}\, dy < \infty. \tag{1.8}$$

The equivalence between $(a), (b)$ and (c) is a standard consequence of Nikishin-Stein type theorems. They are in fact also equivalent to the weak L^p boundedness of P^*, as we shall see in §4.5. The main contribution of the previous theorem is the characterization in terms of the geometric condition (d). This new condition lies in between the known conditions for the Hardy-Littlewood maximal operator and the Riesz transforms, which respectively take the form

$$\sup_{k\geq 1} \frac{1}{2^{knp'}} \int_{|y|\leq 2^k} v^{-\frac{1}{p-1}}(y)\,dy < \infty \quad \text{and} \quad \int_{\mathbb{R}^n} \frac{v^{-\frac{1}{p-1}}(y)}{(1+|y|)^{np'}}\,dy < \infty \tag{1.9}$$

(see e.g. [5, Ch.6]). Observe that (1.8) contains specific information about cones, via size conditions for $v(x)$ in the sets E_k. This can be used, for instance, to construct examples of functions f so that $Mf < \infty$ and $P^*f \equiv \infty$ (see §4.1 below). In contrast with Theorem 1.4, this result illustrates a quantitatively different behavior between Poisson integrals and classical approximations of the identity regarding the convergence problem in (1.2).

Finally, we would like to point out that we restrict our results to light-cones because of the explicit estimates of the kernels and the simple statements of the characterization theorems. The situation in general symmetric spaces is necessarily more subtle, as it happens already in the unweighted case [14, 13]. A more detailed investigation of such situations will be presented elsewhere.

Throughout the paper, the notation $A \lesssim B$ means that $A \leq cB$, for a non relevant constant $c > 0$. Likewise, $A \sim B$ means that $c_1 A \leq B \leq c_2 A$ for two such constants c_1 and c_2.

2 Estimates for Poisson kernels

Throughout this section we denote

$$P(x) = P_{\mathbf{e}}(x) = |\Delta(x + i\mathbf{e})|^{-n} \quad \text{and} \quad P_t(x) = t^{-n}P(x/t) \quad \text{for } t > 0.$$

In the next proposition we list some elementary and well-known properties of $P(x)$.

PROPOSITION 2.1 *The following properties hold:*

(i) $P_t(x) = \left(\frac{\Delta(x)^2}{t^2} + 2|x|^2 + t^2\right)^{-n/2}, \quad t > 0.$

(ii) $c_1\,(1+|x|)^{-2n} \;\le\; P(x) \;\le\; c_2\,(1+|x|)^{-n}.$

(iii) *For each* $x \in \mathbb{R}^n, \quad r \in (0,\infty) \mapsto P(rx)$ *is decreasing.*

(iv) $P \in L^\alpha(\mathbb{R}^n)$ *if and only if* $\alpha > \frac{n-1}{n}$.

(v) $|\nabla P(x)| \le c\,P(x), \quad x \in \mathbb{R}^n.$

PROOF: The proofs of (i)-(iii) are completely elementary, following from the definition of $\Delta(x+i\mathbf{e})$. Property (iv) can be found, e.g. in [1, Lemma 3.4]. Finally, (v) can be checked by direct computation of the gradient.

A slightly more general proof of (v) which is also valid for higher rank cones is as follows. Using the notation in [4, Ch.3], we can diagonalize $x = \lambda_1\mathbf{c}_1 + \lambda_2\mathbf{c}_2$ for a system of idempotents $\{\mathbf{c}_1, \mathbf{c}_2\}$ in $\mathbb{R}^n$ and a pair of eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$. Then $\Delta(x+i\mathbf{e}) = \det(x+i\mathbf{e}) = \Pi_{j=1}^2(\lambda_j + i)$, and therefore

$$\nabla\big(|\det(x+i\mathbf{e})|^{-n}\big) \;=\; -\tfrac{n}{2}\left(|\det(x+i\mathbf{e})|^2\right)^{-\frac{n}{2}-1}\nabla\left[\,\Pi_{j=1}^2(\lambda_j^2+1)\,\right].$$

Since $\lambda_j = (x|\mathbf{c}_j)$, it follows that $\nabla\lambda_j = \mathbf{c}_j$, which leads to the expression

$$\begin{aligned}\nabla P(x) \;&=\; -\tfrac{n}{2}\,P(x)\,|\det(x+i\mathbf{e})|^{-2}\left[\,\Sigma_{j=1}^2 \tfrac{2\lambda_j}{\lambda_j^2+1}\,\Pi_{j=1}^2(\lambda_j^2+1)\mathbf{c}_j\,\right]\\ &=\; -\tfrac{n}{2}\,P(x)\,\left[\,\Sigma_{j=1}^2 \tfrac{2\lambda_j}{\lambda_j^2+1}\,\mathbf{c}_j\,\right].\end{aligned}$$

Taking modulus of this last quantity one easily sees that $|\nabla P(x)| \le nP(x)$.

□

REMARK 2.2 To illustrate the anisotropy of the kernel we remark that (ii) gives the best possible radial estimates for $P(x)$. In fact, the best decay is attained at points in the axis of the cone $x = x_1\mathbf{e}$, while the worse decay corresponds to the boundary $\pm\partial\Omega = \{\Delta(x) = 0\}$.

REMARK 2.3 Observe also from (v) that the decay of $P(x)$ is too poor to give a *regular* Calderón-Zygmund kernel (in the sense e.g. of [5, p.204]). In fact, it is possible to show that one actually has

$$\sup_{|h|<\frac{|x|}{2}} \frac{|P(x+h) - P(x)|}{|h|^\varepsilon/|x|^{n+\varepsilon}} = \infty, \quad \forall\,\varepsilon > 0.$$

In our next proposition we shall give a key decay estimate for the vector-valued kernel $\mathcal{P}(x) = \{P_{2^m}(x)\}_{m\in\mathbb{Z}}$. We shall use the notation $S_k(|h|)$ for the spherical shell $2^k|h| \le |x| \le 2^{k+1}|h|$.

Proposition 2.4 *Let $1 \le s < \infty$. Then there exist constants $C, \gamma > 0$ so that*

$$\Big[\int_{S_k(|h|)} \sup_{m\in\mathbb{Z}} |P_{2^m}(x+h) - P_{2^m}(x)|^s\, dx\Big]^{\frac{1}{s}} \le C\, 2^{-\gamma k}\, \big|S_k(|h|)\big|^{-\frac{1}{s'}}, \tag{2.5}$$

for all for all $k \ge 1$ and all $h \ne 0$

Remark 2.6 When $s = 1$, (2.5) gives the classical Hörmander condition for $\mathcal{P} = \{P_{2^m}\}_{m\in\mathbb{Z}}$, namely

$$\int_{|x|\ge 2|h|} \big|\{P_{2^m}(x+h) - P_{2^m}(x)\}_m\big|_{\ell^\infty}\, dx \le C, \quad \forall\, h \ne 0. \tag{2.7}$$

Proof: Let I denote the s-th power of the left-hand side of (2.5), and write $I \le I_1 + I_2$ where

$$\begin{aligned} I_1 &= \int_{S_k(|h|)} \sum_{2^m>|h|} |P_{2^m}(x+h) - P_{2^m}(x)|^s\, dx \\ \text{and} \qquad I_2 &= \int_{S_k(|h|)} \sum_{2^m\le|h|} |P_{2^m}(x+h) - P_{2^m}(x)|^s\, dx. \end{aligned}$$

To estimate the first term we shall use the inequality

$$|P(\tfrac{x+h}{2^m}) - P(\tfrac{x}{2^m})| \le |\tfrac{h}{2^m}| \int_0^1 |\nabla P(\tfrac{x+\theta h}{2^m})|\, d\theta.$$

Then, controlling the gradient with (v) above, using Hölder's inequality in $d\theta$ and changing variables we obtain

$$\begin{aligned} I_1 &\lesssim \sum_{2^m>|h|} |\tfrac{h}{2^m}|^s\, 2^{-nms} \int_{S_k(|h|)} \int_0^1 P(\tfrac{x+\theta h}{2^m})^s\, d\theta\, dx \\ &\le |h|^s \sum_{2^m>|h|} 2^{-(n+1)ms} 2^{nm} \int_{|u|\sim 2^{k-m}|h|} P(u)^s\, du. \end{aligned} \tag{2.8}$$

For the computations that follow the next elementary lemma will be useful.

Lemma 2.9 *For any real numbers $\alpha \in (\frac{n-1}{n}, 1)$ and $\beta \in (0,1)$ we have*

$$\int_{|u|\sim 2^\ell A} P(u)^s\, du \,\leq\, C\, \frac{(2^\ell A)^{n(1-\beta)}}{(1+2^\ell A)^{n(s-\alpha\beta)}}\ , \tag{2.10}$$

where the constant C does not depend on $A>0$ or $\ell \in \mathbb{Z}$.

Proof: By Hölder's inequality with $1/\beta$ we see that

$$\int_{|u|\sim 2^\ell A} P(u)^s\, du \lesssim \Big(\int_{|u|\sim 2^\ell A} P(u)^{s/\beta}\, du\Big)^\beta \,(2^\ell A)^{n(1-\beta)}.$$

Next, we use (ii) to the estimate the integrand by $P(u)^{\frac{s}{\beta}} \lesssim P(u)^\alpha\,(1+|u|)^{-n(\frac{s}{\beta}-\alpha)}$, and therefore obtain

$$\int_{|u|\sim 2^\ell A} P(u)^s\, du \lesssim \frac{(2^\ell A)^{n(1-\beta)}}{(1+2^\ell A)^{n(s-\alpha\beta)}}\Big(\int_{\mathbb{R}^n} P(u)^\alpha\, du\Big)^\beta.$$

Finally, (iv) implies that the last integral is finite, hence establishing (2.10).

□

Continuing with the estimation of I_1 in (2.8), we write

$$I_1 = \sum_{2^m>|h|} \ldots \leq \sum_{2^m>2^k|h|} \ldots + \sum_{|h|<2^m\leq 2^k|h|} \ldots = S_1 + S_2.$$

For the first term, using the lemma,

$$\begin{aligned}
S_1 &\lesssim |h|^s \sum_{2^m>2^k|h|} 2^{-(n+1)ms} 2^{nm}\,(2^{k-m}|h|)^{n(1-\beta)}\\
&= |h|^s\,(2^k|h|)^{n(1-\beta)} \sum_{2^m>2^k|h|} 2^{-(n+1)sm} 2^{n\beta m}\\
&\lesssim 2^{-ks}\,(2^k|h|)^{-ns/s'},
\end{aligned}$$

where in the last inequality we used $(n+1)s > n\beta$ (since $s \geq 1 > \beta$). On the other hand, the second term is controlled by

$$\begin{aligned}
S_2 &\lesssim |h|^s \sum_{2^m>|h|} 2^{-(n+1)ms} 2^{nm}\, \frac{(2^{k-m}|h|)^{n(1-\beta)}}{(2^{k-m}|h|)^{n(s-\alpha\beta)}}\\
&= |h|^s\,(2^k|h|)^{n(1-s-\beta(1-\alpha))} \sum_{2^m>|h|} 2^{-sm}\, 2^{n\beta(1-\alpha)m}\\
&\lesssim 2^{-kn\beta(1-\alpha)}\,(2^k|h|)^{-ns/s'},
\end{aligned}$$

provided β is small enough so that $n\beta(1-\alpha) < s$. Thus, choosing $\gamma = n\beta(1-\alpha)/s$ we have $I_1 \lesssim 2^{-k\gamma s}\,|S_k(|h|)|^{-s/s'}$.

We now turn to the summand I_2, for which we use the crude estimate

$$\int_{S_k(|h|)} |P_{2^m}(x+h) - P_{2^m}(x)|^s\,dx \lesssim \int_{|u|\sim 2^k|h|} P_{2^m}(u)^s\,du.$$

Inserting this into I_2 and using the lemma we obtain a series similar to S_2 above:

$$\begin{aligned} I_2 &\lesssim \sum_{2^m\le|h|} 2^{-nms}2^{nm}\,\frac{(2^{k-m}|h|)^{n(1-\beta)}}{(2^{k-m}|h|)^{n(s-\alpha\beta)}} \\ &= (2^k|h|)^{n(1-s-\beta(1-\alpha))} \sum_{2^m\le|h|} 2^{n\beta(1-\alpha)m} \\ &\lesssim 2^{-kn\beta(1-\alpha)}\,(2^k|h|)^{-ns/s'} = c\,2^{-k\gamma s}\,|S_k(|h|)|^{-s/s'}, \end{aligned}$$

where to sum the series we used $\beta(1-\alpha) > 0$. This concludes the proof of the proposition.

$\square$

To conclude this section, observe from property (iii) above that

$$P^*f(x) \sim \sup_{m\in\mathbb{Z}} P_{2^m} * f(x), \quad \forall\, f \ge 0.$$

Thus, the L^p-boundedness of P^* is controlled by the vector-valued linear operator $\mathbf{P} : L^\infty \to L^\infty_{\ell^\infty}$ with convolution kernel $\mathcal{P}(x) = \{P_{2^m}(x)\}_{m\in\mathbb{Z}}$. As a corollary of the estimate (2.7) for $\mathcal{P}(x)$ (i.e., (2.5) with $s=1$) we obtain an improved version of the classical result of Stein and Weiss.

Corollary 2.11 *The maximal Poisson operator P^* is bounded $L^p(\mathbb{R}^n)$, for all $1 < p \le \infty$, and is weakly bounded in $L^1(\mathbb{R}^n)$. Moreover, for all $1 < q < \infty$ the following vector-valued inequalities hold*

$$\Big\| \big(\sum_{j=1}^\infty |P^*f_j|^q\big)^{\frac1q} \Big\|_p \le C \Big\| \big(\sum_{j=1}^\infty |f_j|^q\big)^{\frac1q} \Big\|_p;$$

$$\Big| \big\{ x\in\mathbb{R}^n \;:\; (\Sigma_{j=1}^\infty |P^*f_j(x)|^q)^{\frac1q} > \lambda \big\} \Big| \le \frac{C}{\lambda} \Big\| \big(\sum_{j=1}^\infty |f_j|^q\big)^{\frac1q} \Big\|_1, \quad \forall\,\lambda > 0.$$

Proof: First apply to the operator $\mathbf{P}$ the vector-valued Calderón-Zygmund theory (see e.g. Theorems V.3.4 and V.3.9 in [5]), and then use the pointwise estimate $P^*f(x) \le c\,|\mathbf{P}(|f|)(x)|_{\ell^\infty}$.

$\square$

3 The proof of Theorem 1.4

In this section we turn to the weighted inequalities in Theorem 1.4. These will be a direct consequence of the estimate of the kernel $P(x)$ in Proposition 2.4 and the results in the paper [12]. In fact, one can prove a stronger statement than Theorem 1.4 above.

Theorem 3.1 *For all $1 < p < \infty$ and all $w \in A_p$ the maximal Poisson operator P^* is bounded in $L^p(w)$, and moreover it holds the vector valued inequality:*

$$\Big\| \big(\sum_{j=1}^{\infty} |P^* f_j|^q\big)^{\frac{1}{q}} \Big\|_{L^p(w)} \leq C \Big\| \big(\sum_{j=1}^{\infty} |f_j|^q\big)^{\frac{1}{q}} \Big\|_{L^p(w)}, \quad \forall\, q \in (1, \infty).$$

Likewise, if $w \in A_1$ then P^* is weakly bounded in $L^1(\mathbb{R}^n)$, and for all $1 < q < \infty$

$$w\big\{x \in \mathbb{R}^n \ : \ (\Sigma_{j=1}^{\infty} |P^* f_j(x)|^q)^{\frac{1}{q}} > \lambda\big\} \leq \frac{C}{\lambda} \Big\| \big(\sum_{j=1}^{\infty} |f_j|^q\big)^{\frac{1}{q}} \Big\|_{L^1(w)}, \quad \forall\, \lambda > 0.$$

Conversely, if P^* is bounded from $L^p(\mu)$ into $L^{p,\infty}(\mu)$ for some $1 \leq p < \infty$ and some positive Borel measure μ, then necessarily $\mu = w(x)\,dx$ and $w \in A_p$.

Proof: By classical results on Muckenhoupt classes, if $1 < p < \infty$ and $w \in A_p$ then there exists a real number $\sigma \in (1, p)$ (sufficiently close to 1) so that $w \in A_{p/\sigma}$. Similarly, if $w \in A_1$ there is some $\sigma > 1$ so that $w^\sigma \in A_1$ [15, Ch. 5]. In either case, the kernel $\mathcal{P}(x) = \{P_{2^m}(x)\}_{m\in\mathbb{Z}}$ of the vector-valued operator $\mathbf{P}$ satisfies the regularity condition (2.5) with $s = \sigma'$ (the conjugate index of σ). Thus, we are under the conditions of Theorem II.1.6 in the paper [12], which implies the boundedness of $\mathbf{P}$ from $L^p(w)$ into $L^p_{\ell^\infty}(w)$, as well as the corresponding weak boundedness for $p = 1$ and their ℓ^q-valued counterparts. This and the trivial estimate $P^*f(x) \lesssim |\mathbf{P}(|f|)(x)|_{\ell^\infty}$ are enough to establish the first part of the theorem.

The converse follows easily from the inequality

$$Mf(x) \leq c\, P^*f(x), \quad \forall\, f \geq 0. \tag{3.2}$$

Indeed assuming (3.2), if P^* is weakly bounded in $L^p(\mu)$, then so is the Hardy-Littlewood maximal operator M. But then the characterization theorem of A_p weights gives $\mu = w(x)\,dx$ and $w \in A_p$ (see [15, p.198]).

Finally, to see (3.2) one uses the left-hand estimate of $P(x)$ in (ii) of the previous section, so that

$$P^* f(x) \gtrsim \sup_{t>0} t^{-n} \int_{B_t(x)} \frac{f(u)}{(1+|x-u|/t)^{2n}}\, du \sim Mf(x).$$

$\square$

4 The two weight problem

In this section we give a complete proof of Theorem 1.6. We begin with an example which illustrates the different behaviors at infinity of the maximal functions Mf and P^*f.

4.1 A first example

Let $f(y) = \frac{y_1^2}{\log y_1}\chi_E(y)$, where the set E is given by

$$E = \{y \in \Omega \ : \ 1 \leq \Delta(y) \leq 2, \ \ |y| \geq 2\}.$$

This function has a critical growth along the singular directions of $P(x)$. More precisely, we have the following lemma.

Lemma 4.1 *The function $f(y) = \frac{y_1^2}{\log y_1}\chi_E(y)$ satisfies $P^*f \equiv \infty$ and $Mf < \infty$.*

Proof: For the first assertion we will actually show that $P * f \equiv \infty$ (which by monotonicity even implies $P_t * f \equiv \infty$ for any $t > 0$). Given $x \in \mathbb{R}^n$ recall that

$$P * f(x) = \int_E \frac{f(y)}{(|\Delta(x-y)|^2 + 2|x-y|^2 + 1)^{\frac{n}{2}}}\, dy.$$

Observe that $\Delta(x-y) = \Delta(x) + \Delta(y) + 2(x_1 y_1 - x' \cdot y')$. So, if we restrict the region of integration to $E \cap \{|y| \geq 2|x|\}$, we will have

$$|\Delta(x-y)| + |x-y| + 1 \lesssim |x|^2 + |x||y| + |y| + 1 \leq c_{|x|}(|y|+1).$$

Also, since in the cone $y_1 \sim |y|$ we have

$$P * f(x) \gtrsim c_{|x|} \int_{E\cap\{|y|\geq 2|x|\}} \frac{f(y)}{(y_1+1)^n}\, dy_1.$$

Now a simple computation shows that, for each $y_1 \gg 1$, the $(n-1)$-dimensional Lebesgue measure of the set $\{y' \,:\, 1 \le y_1^2 - |y'|^2 \le 2\}$ is comparable to y_1^{n-3}. Thus

$$P * f(x) \gtrsim c_{|x|} \int_{2|x|+2}^{\infty} \frac{y_1^2}{\log y_1} \frac{y_1^{n-3}}{(y_1+1)^n}\, dy = \infty.$$

To establish the second assertion we will construct a weight $v(y) > 0$ so that $f \in L^2(v)$ and the Hardy-Littlewood maximal operator is bounded from $L^2(v)$ into $L^2(u)$ (for some other weight $u > 0$). Recall that for such boundedness to hold it is necessary and sufficient that $v(y)$ satisfies the left-hand condition in (1.9) for $p = 2$ (see e.g. [5, Th. VI.6.10]). In this setting we only need to choose

$$v(y) = \frac{1}{y_1^{n+2}} \chi_E(y) \,+\, \chi_{E^c}(y).$$

Then, a similar reasoning as above gives

$$\int_E |f(y)|^2 v(y)\, dy \,\sim\, \int_2^{\infty} \frac{y_1^4}{(\log y_1)^2} \frac{y_1^{n-3}}{y_1^{n+2}}\, dy < \infty.$$

On the other hand, the left-hand condition in (1.9) holds trivially when we integrate along E^c, where $v \equiv 1$. For the other part observe that

$$\frac{1}{R^{2n}} \int_{B_R(0)\cap E} v^{-1}(y)\, dy \lesssim \frac{1}{R^{2n}} \int_2^R y_1^{n+2}\, y_1^{n-3}\, dy \,\sim\, 1, \quad \forall\, R \ge 1.$$

This completes the proof of the lemma. □

Remark 4.2 The previous example also illustrates a feature of the theory of weights. This theory shows that the Hardy-Littlewood maximal function behaves well in a "large" space such as $L^2(v)$, even though this space contains functions like f which do not belong to $L^p(\mathbb{R}^n)$ for any $1 \le p \le \infty$.

Remark 4.3 The function $f(y)$ defined above is also "critical" for P^* in the sense that a slightly slower growth such as

$$f_\varepsilon(y) = \frac{y_1^2}{(\log y_1)^{1+\varepsilon}} \chi_E(y), \quad \varepsilon > 0$$

implies that $P^* f_\varepsilon(x) < \infty$ *a.e.* $x \in \mathbb{R}^n$. To see this one can proceed as in the last part of the lemma, showing that $f_\varepsilon \in L^2(v_\varepsilon)$ for the slightly better weight $v_\varepsilon(y) = (\log y_1)^{1+\varepsilon} y_1^{-(n+2)} \chi_E \,+\, \chi_{E^c}$. Now this weight satisfies the right-hand condition in (1.9) for $p = 2$. Thus the vector-valued Calderón-Zygmund operator $\mathbf{P}$ is bounded from $L^2(v_\varepsilon)$ into $L^2_{\ell^\infty}(u)$ for some weight u (see [5, Th. VI.6.4 and remark in p. 563]). Hence so is P^* from $L^2(v_\varepsilon)$ into $L^2(u)$, which implies $P^* f_\varepsilon < \infty$.

4.2 The sufficient condition

In this subsection we show the key implication "$(d) \Rightarrow (a)$" in Theorem 1.6. Our strategy follows the ideas developed by J.L. Rubio de Francia in [11], based on the equivalence between vector-valued and weighted inequalities. We shall use the following factorization result, valid for a general Banach space $\mathbb{B}$:

Theorem 4.4 : see [5, Th. VI.4.2]. *Let $0 < r < p < \infty$ and T be a sublinear operator defined on $\mathbb{B}$ such that, for some constant $C > 0$*

$$\Big\| \big(\sum_{j=1}^{\infty} |Tf_j|^p\big)^{\frac{1}{p}} \Big\|_{L^r(\mathbb{R}^n)} \leq C \big(\sum_{j=1}^{\infty} \|f_j\|_{\mathbb{B}}^p\big)^{\frac{1}{p}}, \quad \forall \{f_j\} \subset \mathbb{B}. \tag{4.5}$$

Then, there exists $u > 0$ satisfying $\int u(x)^{-\frac{r}{p-r}}\, dx \leq 1$ and $\|Tf\|_{L^p(u)} \leq C\|f\|_{\mathbb{B}}$.

In our application we shall take $\mathbb{B} = L^p(v)$, so that the right-hand side of (4.5) actually equals $\|\big(\sum_{j=1}^{\infty} |f_j|^p\big)^{\frac{1}{p}}\|_{L^p(v)}$. We shall also decompose $\mathbb{R}^n$ into $S_0 = B_1(0)$ and the spherical shells $S_k = \{2^{k-1} \leq |y| < 2^k\}$, $k \geq 1$, and consider each of the corresponding operators

$$T^{(k)}f := (P^*f)\chi_{S_k}, \quad k \geq 0.$$

Our goal now is to show that: *for any $r < 1 < p$ and $k \geq 0$ there exists a constant $c_{p,r}$ so that*

$$\Big\| \big(\sum_{j=1}^{\infty} |P^*f_j|^p\big)^{\frac{1}{p}} \Big\|_{L^r(S_k)} \leq c_{p,r}\, |S_k|^{\frac{1}{r}} \Big\| \big(\sum_{j=1}^{\infty} |f_j|^p\big)^{\frac{1}{p}} \Big\|_{L^p(v)}. \tag{4.6}$$

Assuming this, we can apply Theorem 4.4 to find corresponding weights u_k so that

$$\|T^{(k)}f\|_{L^p(u_k)}^p = \int_{S_k} |P^*f(y)|^p u_k(y)\, dy \leq c_{p,r}^p\, |S_k|^{\frac{p}{r}}\, \|f\|_{L^p(v)}^p.$$

Thus, for any $\gamma > 0$ we can define $u(y) = \sum_{k=0}^{\infty} 2^{-\gamma k} 2^{-nkp/r} u_k(y)\chi_{S_k}(y)$, so that

$$\int_{\mathbb{R}^n} |P^*f(y)|^p\, u(y)\, dy \leq c'_{p,r,\gamma}\, \|f\|_{L^p(v)}^p,$$

which is the desired part (a) of Theorem 1.6.

Remark 4.7 Observe that Theorem 4.4 gives as well some size information on the weights $u_k(x)$. In fact, it follows easily from the previous argument that for any $\varepsilon > 0$, one can a find a weight u so that $\int u(y)^{-\frac{1}{p-1}+\varepsilon}(1+|y|)^{-np'}dy < \infty$.

Let us now turn to the proof of the vector-valued inequalities (4.6). Given a fixed $k \geq 0$ and function $f \in L^p(v)$, we shall split it in three summands $f = f' + f'' + f'''$ as follows

$$f' = f\,\chi_{B_{2^{k+1}}(0)}, \quad f'' = f\,\chi_{E_{k+1}\setminus B_{2^{k+1}}(0)} \quad \text{and} \quad f''' = f\,\chi_{E^c_{k+1}},$$

where the sets E_k were defined in (1.5). We proceed now differently in each of these cases.

Step 1.- *Proof of (4.6) for $\{f'_j\}$.*

This is the "local part", which follows essentially from the vector-valued inequality in Corollary 2.11. Indeed, since $r < 1$, by Kolmogorov's inequality [5, p. 485],

$$\begin{aligned} \Big\| \big(\sum_j |P^* f'_j|^p\big)^{\frac{1}{p}} \Big\|_{L^r(S_k)} &\lesssim |S_k|^{\frac{1}{r}-1} \Big\| \big(\sum_j |P^* f'_j|^p\big)^{\frac{1}{p}} \Big\|_{L^{1,\infty}(\mathbb{R}^n)} \\ \text{(Corollary 2.11)} \quad &\lesssim |S_k|^{\frac{1}{r}-1} \Big\| \big(\sum_j |f'_j|^p\big)^{\frac{1}{p}} \Big\|_{L^1(\mathbb{R}^n)} \\ (\text{Supp}\, f'_j \subset B_{2^{k+1}}) \quad &\lesssim |S_k|^{\frac{1}{r}-1} \Big\| \big(\sum_j |f_j|^p\big)^{\frac{1}{p}} \Big\|_{L^p(v)} \Big(\int_{B_{2^{k+1}}} v^{-\frac{p'}{p}}(y)\,dy\Big)^{\frac{1}{p'}}. \end{aligned}$$

The claim now follows from the condition on v in (d) since $\frac{p'}{p} = \frac{1}{p-1}$ and

$$|S_k|^{-1}\Big(\int_{B_{2^{k+1}}} v^{-\frac{1}{p-1}}(y)\,dy\Big)^{\frac{1}{p'}} \lesssim \Big(\int_{B_{2^{k+1}}} \frac{v^{-\frac{1}{p-1}}(y)}{(2^k+|y|)^{np'}}\,dy\Big)^{\frac{1}{p'}} \leq \|v\|_*. \tag{4.8}$$

Step 2.- *Proof of (4.6) for $\{f''_j\}$.*

For the "global part" it suffices to prove a pointwise estimate of the form

$$\sup_{x\in S_k} P^* f''(x) \lesssim \|f\|_{L^p(v)}, \quad \forall\, f \in L^p(v). \tag{4.9}$$

Indeed, assuming (4.9) for the collection $\{f_j\}$ and summing in j we will also have

$$\sup_{x\in S_k}\big(\sum_j |P^* f''_j(x)|^p\big)^{\frac{1}{p}} \lesssim \big(\sum_j \|f_j\|^p_{L^p(v)}\big)^{\frac{1}{p}} = \big\|\big(\sum_j |f_j|^p\big)^{\frac{1}{p}}\big\|_{L^p(v)},$$

and therefore $\big\| \, (\sum_j |P^* f_j''|^p)^{\frac{1}{p}} \, \big\|_{L^r(S_k)} \lesssim |S_k|^{\frac{1}{r}} \, \big\| (\sum_j |f_j|^p)^{\frac{1}{p}} \, \big\|_{L^p(v)}$ which is (4.6).

To show (4.9) we use a crude estimate in the kernel

$$P_t(x-y) \lesssim \frac{1}{|x-y|^n} \sim \frac{1}{(1+|y|)^n}, \quad \text{when} \quad x \in S_k, \; y \in B^c_{2^{k+1}}.$$

Thus, when $x \in S_k$

$$P^* f''(x) \lesssim \int \frac{|f''(y)|}{(1+|y|)^n} \, dy \, \leq \, \|f\|_{L^p(v)} \Big(\int_{E_{k+1} \setminus B_{2^{k+1}}} \frac{v^{-\frac{p'}{p}}(y)}{(1+|y|)^{np'}} \, dy \Big)^{\frac{1}{p'}}, \qquad (4.10)$$

and this last integral is clearly majorized by a constant multiple of $\|v\|_*$.

Step 3.- *Proof of (4.6) for* $\{f_j'''\}$.

For this part we shall also prove a pointwise estimate as in (4.9) (with f'' replaced by f'''), although this time we shall need a finer analysis on the kernel. We shall use the following elementary lemma.

Lemma 4.11 *For all* $y \in \mathbb{R}^n$, $\sqrt{2}\,|y| \, dist\,(y, \pm\partial\Omega) \, \leq \, |\Delta(y)| \, \leq 2\,|y| \, dist\,(y, \pm\partial\Omega)$.

Proof: Reducing to the case $y = (|y_1|, |y'|, 0)$ (by rotation invariance), it is easy to see that $\operatorname{dist}(y, \pm\partial\Omega) = \big||y_1| - |y'|\big|/\sqrt{2}$. Thus, the result follows by writing $\Delta(y) = (|y_1| + |y'|)(|y_1| - |y'|)$.

□

For the estimation of $P^* f'''(x)$ we define the dyadic hyperboloidal shells

$$D_\ell = \{y \in \mathbb{R}^n \;:\; 2^{\ell-1} < \operatorname{dist}\,(y, \pm\partial\Omega) \leq 2^\ell\}, \quad \ell \in \mathbb{Z}.$$

Recall that f''' is supported in $E^c_{k+1} = \cup_{\ell \geq 2} D_{k+\ell}$. Using the lemma, for any $x \in S_k$ and $y \in D_{k+\ell}$ with $\ell \geq 2$ we have

$$|\Delta(x-y)| \, \geq \, |\Delta(y)| - |\Delta(x)| - 2|x||y| \, \geq \, c\, 2^{k+\ell}|y|.$$

Thus, if $x \in S_k$

$$\begin{aligned} P^* f'''(x) \;&\leq\; \sup_{0<t<2^{k+1}} P_t|f'''|(x) + \sup_{t \geq 2^{k+1}} P_t|f'''|(x) \\ &\lesssim\; \sup_{0<t<2^{k+1}} \sum_{\ell=2}^{\infty} \int_{D_{k+\ell}} \frac{|f(y)|}{(2^{k+\ell}|y|/t)^n} \, dy + \sup_{m \geq k+1} P_{2^m}|f|(x). \qquad (4.12) \end{aligned}$$

The first term in (4.12) can be estimated by Hölder's inequality

$$\sum_{\ell=2}^{\infty} 2^{-\ell n} \int_{D_{k+\ell}} \frac{|f(y)|}{|y|^n}\, dy \;\leq\; \sum_{\ell=2}^{\infty} 2^{-\ell n}\, \|f\|_{L^p(v)} \Big(\int_{D_{k+\ell}} \frac{v^{-\frac{p'}{p}}(y)}{|y|^n}\, dy\Big)^{\frac{1}{p'}}. \tag{4.13}$$

Observe that in $D_{k+\ell}$ we have $|y| \sim (2^{k+\ell}+|y|)$, so the integral inside the parenthesis is actually bounded by $\|v\|_*$. To deal with the second term in (4.12) we fix $m \geq k+1$ and split the integral defining $P_{2^m}|f|(x)$ in three parts, majorizing the kernel accordingly:

$$P_{2^m}|f|(x) \;\lesssim\; \int_{B_{2^m}} \frac{|f(y)|}{2^{mn}}\, dy \;+\; \int_{E_{m+1}\setminus B_{2^m}} \frac{|f(y)|}{|y|^n}\, dy \;+\; \sum_{\ell=2}^{\infty} \int_{D_{m+\ell}} \frac{|f(y)|}{(2^\ell |y|)^n}\, dy.$$

Now, each of these three terms has been handled respectively in (4.8), (4.10) and (4.13) with m replaced by k. Thus, the same arguments with Hölder's inequality lead to the uniform bound $\|v\|_*\|f\|_{L^p(v)}$. This establishes step 3, completing the proof of "$(d) \Rightarrow (a)$" in Theorem 1.6.

4.3 The necessary condition

In this subsection we show the implication "$(a) \Rightarrow (d)$" in Theorem 1.6. The key estimate is contained in the following lemma.

Lemma 4.14 *Let $\gamma > 0$ be fixed. Then, there exists $c = c(\gamma) > 0$ so that for all for all $k \geq 1$ and $f \geq 0$ supported in E_k we have*

$$P^* f(x) \;\geq\; c \int_{E_k} \frac{f(y)}{(2^k + |y|)^n}\, dy, \quad \forall\, x \in B_\gamma(0).$$

Proof: If $x \in B_\gamma(0)$ and $y \in E_k$, then by Lemma 4.11

$$|\Delta(x-y)| \leq |\Delta(x)| + |\Delta(y)| + 2|x||y| \lesssim 1 + 2^k|y| + |y|.$$

Thus, using (i) in §2 we can majorize the denominator of $P_{2^k}(x-y)$ by

$$\big(2^{-k}|\Delta(x-y)| + |x-y| + 2^k\big)^n \;\lesssim\; (|y| + 2^k)^n.$$

Therefore,

$$P^* f(x) \;\geq\; P_{2^k} f(x) \;\gtrsim\; \int_{E_k} \frac{f(y)}{(2^k + |y|)^n}\, dy, \quad \forall\, x \in B_\gamma(0).$$

□

From the previous result, and using a more or less standard method, we can prove a stronger version of "$(a) \Rightarrow (d)$".

Proposition 4.15 *Let $1 < p < \infty$ and $v(x) > 0$. Suppose that P^* is bounded from $L^p(v)$ into $L^{p,\infty}(\mu)$ for some positive Borel measure μ. Then*

$$\|v\|_*^{p'} := \sup_{k\geq 1} \int_{E_k} \frac{v^{-\frac{1}{p-1}}(y)}{(2^k+|y|)^{np'}}\, dy < \infty.$$

Proof: We first choose a constant $\gamma > 0$ so that $\mu(B_\gamma(0)) > 0$. Let $k \geq 1$ be fixed and consider an arbitrary function $f \geq 0$ supported in E_k. Then, by the previous lemma

$$\mu\big\{x \in \mathbb{R}^n \ : \ P^*f(x) > c \int_{E_k} \tfrac{f(y)}{(2^k+|y|)^n}\, dy\big\} \ \geq \ \mu\big(B_\gamma(0)\big).$$

By the assumed weak boundedness of P^* we can also estimate from above the left-hand side of the previous inequality. This leads to

$$c\,\mu\big(B_\gamma(0)\big)^{\frac{1}{p}} \int_{E_k} \tfrac{f(y)}{(2^k+|y|)^n}\, dy \ \leq \ C \left(\int |f(y)|^p v(y)\, dy\right)^{\frac{1}{p}}.$$

Observe that $\mu(B_\gamma(0))$ can be absorbed in the previous inequality as a positive constant. Now, taking $f = gv^{-1/p}$, for arbitrary $g \geq 0$, supported in E_k and with $\int |g|^p = 1$ we can write the previous as

$$\int_{E_k} g(y)\, \frac{v(y)^{-\frac{1}{p}}}{(2^k+|y|)^n}\, dy \ \leq \ C' \left(\int |g(y)|^p\, dy\right)^{\frac{1}{p}} \ = \ C'.$$

By duality this implies that $v(y)^{-1/p}(2^k+|y|)^{-n} \in L^{p'}(E_k)$ with norm bounded a the constant C' (which is independent of k). Thus, $\|v\|_* \leq C'$ and we have proved the proposition.

□

4.4 Nikishin type theorems

The following implications in Theorem 1.6 are easy to verify: "$(a) \Rightarrow (c) \Rightarrow (b)$". Indeed, the first one is an application of Banach's principle (see e.g. [6, p.11]), since (1.7) holds for functions $f \in C_c^\infty(\mathbb{R}^n)$. The second implication is trivial using the monotonicity of the kernel. The main point is therefore to show that "$(b) \Rightarrow (a)$". We will prove an apparently weaker result.

Proposition 4.16 *Let $1 < p < \infty$ and $v(x) > 0$. Suppose that*

$$P^* f(x) < \infty, \quad a.e.\, x \in \mathbb{R}^n, \quad \forall\, f \in L^p(v). \tag{4.17}$$

Then, there exists $u(x) > 0$ so that P^* is bounded from $L^p(v)$ into $L^{p,\infty}(u)$.

Assuming that (b) holds and assuming the previous proposition we are exactly in the hypothesis of Proposition 4.15 of the previous subsection. Thus $\|v\|_* < \infty$ and henceforth, by the implication shown in §4.2, P^* must be bounded from $L^p(v)$ into $L^p(u)$ for some u. This will establish the step "$(b) \Rightarrow (a)$" and complete the proof of Theorem 1.6.

Proof of Proposition 4.16:

By mononicity, we only need to prove the result for $\tilde{P}^* f = \sup_{m\in\mathbb{Z}} |P_{2^m} * f|$. By Nikishin's theorem (see e.g. [5, Corol. VI.2.7]) it suffices to show that the sublinear operator $\tilde{P}^*$ is continuous in measure from $L^p(v)$ into $L^0(dx)$ (the space of all Lebesgue measurable functions which are finite *a.e.*). We observe that Nikishin's theorem can be applied in the whole range $1 < p < \infty$ since $\tilde{P}^*$ is positive (i.e. $|\tilde{P}^* f| \leq \tilde{P}^*|f|$). Banach's continuity principle (see [5, Prop. VI.1.4]) reduces matters to show that each of the convolution operators P_{2^m} is continuous in measure from $L^p(v)$ into $L^0(dx)$. That is, for each fixed $m \in \mathbb{Z}$ we have to show that, given a sequence $f_k \to 0$ in $L^p(v)$, for every $R \geq 1$ and $\lambda > 0$ we have

$$\lim_{k\to\infty} \Big| \Big\{ x \in B_R(0) \ : \ |P_{2^m} * f_k(x)| > \lambda \Big\} \Big| \ = \ 0. \tag{4.18}$$

There is no loss of generality if we assume $f_k \geq 0$. Moreover, since $P_{2^m} * f = \sup_{\ell\geq 1} P_{2^m} * (f\chi_{B_{2^\ell}})$ for $f \geq 0$, another application of Banach's continuity principle lets us assume in addition that the f_k's are supported in a fixed ball $B_{2^\ell}(0)$.

We shall use the following lemma.

Lemma 4.19 *In the conditions of Proposition 4.16, if the weight v satisfies (4.17) then $v^{-\frac{1}{p-1}} \in L^1_{\ell oc}(\mathbb{R}^n)$. In particular, $L^p(v) \hookrightarrow L^1_{\ell oc}(\mathbb{R}^n)$ with continuous inclusion.*

Proof: This follows essentially from the inequality $M|f| \lesssim P^*|f|$ in (3.2), which together with (4.17) implies that all functions f in $L^p(v)$ must be locally integrable. Now, for each compact set $K \subset \mathbb{R}^n$ and each $m \geq 1$, we define the functionals in $(L^p(v))^*$

$$f \in L^p(v) \ \longmapsto \ T_m(f) := \int_{K\cap\{v>1/m\}} f(x)\, dx.$$

For every $f \in L^p(v)$ we have the local uniform bound $\sup_{m\geq 1} |T_m(f)| \leq \int_K |f| < \infty$, which by the Banach-Steinhaus principle gives $\sup_{m\geq 1} \|T_m\|_{(L^p(v))^*} < \infty$. Thus, by monotone convergence

$$\int_K v(x)^{-\frac{1}{p-1}}\, dx \;=\; \lim_{m\to\infty} \int_{K\cap\{v>1/m\}} v(x)^{-p'+1}\, dx \;=\; \lim_{m\to\infty} \|T_m\|^{p'}_{(L^p(v))^*} < \infty.$$

□

We now turn to the proof of (4.18). By Chebichev's inequality

$$\Big|\big\{x \in B_R(0) \;:\; P_{2^m} * (f_k\chi_{B_{2\ell}})(x) > \lambda\big\}\Big| \;\leq\; \frac{1}{\lambda}\int_{B_R} P_{2^m} * (f_k\chi_{B_{2\ell}})(x)\, dx$$

$$= \frac{1}{\lambda}\int_{\mathbb{R}^n} f_k(y)\chi_{B_{2\ell}}(y)\, P_{2^m} * (\chi_{B_R})(y)\, dy.$$

The second factor in the last integral is majorized by the constant $\int P_{2^m} = \int P$. Thus using Hölder's inequality and the previous lemma we obtain

$$LHS \;\leq\; c\,\lambda^{-1}\, \|f_k\|_{L^p(v)} \Big(\int_{B_{2\ell}} v(y)^{-\frac{p'}{p}}\, dy\Big)^{\frac{1}{p'}} \;\longrightarrow 0, \quad \text{as } k\to\infty.$$

This establishes Proposition 4.16 and completes the proof of Theorem 1.6.

□

4.5 Some further remarks

1.- The previous proof actually shows that $(a)-(d)$ are also equivalent to the boundedness of

$$P^*\colon L^p(v) \;\longrightarrow\; L^{p,\infty}(\mu) \quad \text{for some positive Borel measure } \mu. \qquad (a')$$

Indeed, clearly $(a) \Rightarrow (a')$, while in Proposition 4.15 we have shown that $(a') \Rightarrow (d)$.

2.- We can also replace the "vertical" pointwise convergence in (1.7) by *restricted pointwise convergence*. That is, for all $f \in L^p(v)$ and for all $\delta \in [0,1)$ it holds that

$$\lim_{\substack{y\to 0\\ |y'|<\delta y_1}} P_y * f(x) = f(x) \quad \text{and} \quad \lim_{\substack{y\to\infty\\ |y'|<\delta y_1}} P_y * f(x) = 0, \quad a.e.\, x \in \mathbb{R}^n. \qquad (c')$$

One implication is trivial: $(c') \Rightarrow (c)$. Conversely, by Banach's principle (c') will hold if we can show the boundedness from $L^p(v)$ into $L^p(u)$ of the corresponding maximal function

$$P^*_{(\delta)} f(x) = \sup_{\substack{y\in\Omega\\ |y'|<\delta y_1}} P_y * f(x).$$

Now, it is not difficult to see that $P_y(x) \le c_\delta P_{y_1}(x)$ (see [18, Lemma 4.3]). Hence if (c) holds, P^* is bounded from $L^p(v)$ into $L^p(u)$ for some u, and by the previous estimate so is the maximal operator $P^*_{(\delta)}$ for each $\delta \in [0,1)$.

3.- For $p = 1$ there is also a natural version of Theorem 1.6.

Proposition 4.20 *For a positive weight $v(x)$ the following are equivalent:*

(a'') *There exists $u(x) > 0$ such that $P^* : L^1(v) \to L^{1,\infty}(u)$ is bounded;*

(d'') $\sup_{y\in\mathbb{R}^n} v^{-1}(y)(1+|y|)^{-n} < \infty$.

This result is easier to establish and essentially contained in [5]. The reason is that for $p = 1$ condition (d'') is necessary and sufficient for both the Hardy-Littlewood maximal operator and the Riesz transforms [5, p.565]. In fact, the necessity in the proposition follows from $Mf \lesssim P^*f$, and the sufficiency by writing P^* as a vector-valued Calderón-Zygmund operator and using Remark VI.6.12.(b) in [5]. The reader can also give a direct proof of the sufficient condition by using Nikishin's theorem and reasoning as in (4.18).

4.- Finally, we mention that the dual problem of Theorem 1.6 is simpler and also contained in [5]. Namely, one has the following.

Proposition 4.21 *Given a positive weight $u(x)$ the following are equivalent:*

(a^*) *There exists $v(x) > 0$ such that $P^* : L^p(v) \to L^p(u)$ is bounded;*

(d^*) $\int_{\mathbb{R}^n} u(y)(1+|y|)^{-np} < \infty$.

Indeed, this result follows again from the fact that condition (d^*) is necessary and sufficient for both the Hardy-Littlewood maximal operator and the Riesz transforms. We refer to Theorems VI.6.4 and VI.6.10 in [5]. Also in this case, for each fixed $\varepsilon > 0$ one can find a weight $v(x)$ with the size condition $\int v(y)^{1-\varepsilon}(1+|y|)^{-np}dy < \infty$.

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