J Geom Anal (2009) 19: 1–18
DOI 10.1007/s12220-008-9048-5

Discrete Approximation of Spaces of Homogeneous Type

Hugo Aimar · Marilina Carena · Bibiana Iaffei

Received: 12 October 2007 / Published online: 2 October 2008

Abstract In this note we combine the dyadic families introduced by M. Christ in (Colloq. Math. 60/61(2):601–628, 1990) and the discrete partitions introduced by J.M. Wu in (Proc. Am. Math. Soc. 126(5):1453–1459, 1998) to get approximation of a compact space of homogeneous type by a uniform sequence of finite spaces of homogeneous type. The convergence holds in the sense of a metric built on the Hausdorff distance between compact sets and on the Kantorovich-Rubinshtein metric between measures.

Keywords Quasi-metric space · Doubling measure · Space of homogeneous type

Mathematics Subject Classification (2000) 28A33 · 60B10

1 Introduction

In this note we aim to build uniform discrete approximations of compact spaces of homogeneous type.

Communicated by Steven Krantz.

The authors were supported by CONICET, CAI+D (UNL) and ANPCyT.

H. Aimar · M. Carena (✉)
Departamento de Matemática (FIQ-UNL), Instituto de Matemática Aplicada del Litoral (CONICET-UNL), Santa Fe, Argentina
e-mail: mcarena@santafe-conicet.gov.ar

H. Aimar
e-mail: haimar@santafe-conicet.gov.ar

B. Iaffei
Departamento de Matemática (FHUC-UNL), Instituto de Matemática Aplicada del Litoral (CONICET-UNL), Santa Fe, Argentina
e-mail: biaffei@santafe-conicet.gov.ar

The problem leading us to this construction has its starting point in a basic technique for the study of weak type $(1, 1)$ of maximal operators (see [9] and [6]). In [5] one of the authors gives an extension of that technique to some spaces of homogeneous type. Moreover, the weak type $(1, 1)$ of such a maximal operator follows from the uniform weak type $(1, 1)$ of its restriction to discrete approximations of the space.

The main result of this paper is contained in Theorem 4.1. We shall introduce the problem considered here with two classical and elementary examples.

If for each positive integer n we define on the Borel sets of the real numbers the normalized counting measure supported on $S_n = \{i/n : i = 0, 1, 2, \dots, n\}$, given by $\mu_n(A) = \frac{1}{n+1}\,\mathrm{card}(\{i : 0 \le i \le n \text{ and } i/n \in A\})$, we have that

$$S_n \xrightarrow{d_H} [0, 1] \quad \text{and} \quad \mu_n \xrightarrow{w*} m,$$

where the d_H-convergence is the Hausdorff convergence of compact sets, the $w*$-convergence is the weak star convergence of measures, and m is the Lebesgue measure on the closed interval $[0, 1]$. In other words, perhaps the most elementary probability space of homogeneous type $([0, 1], |\cdot|, m)$, where $|\cdot|$ denotes the usual distance on $\mathbb{R}$, can be approximated in the Hausdorff-Kantorovich sense by a sequence of finite spaces. Moreover, the spaces $(S_n, |\cdot|, \mu_n)$ are themselves spaces of homogeneous type with doubling constants bounded above by a fixed number independent of n. In fact given n, $x \in S_n$ and $0 < r \le 1$, choosing an integer j such that $j/n < r \le (j + 1)/n$ we have

$$\begin{aligned}(n+1)\mu_n(B(x, 2r)) &\le 2(2j + 1) + 1\\ &< 4(j + 1)\\ &\le 4(n + 1)\mu_n(B(x, r)),\end{aligned}$$

where $B(x, r) = \{y : |x - y| < r\}$.

More interesting is the case of the classical Cantor set C. Let F be the Cantor function extended to $\mathbb{R}$ as a continuous function by defining $F(x) = 1$ for $x \ge 1$ and $F(x) = 0$ for $x \le 0$. Let μ the unique probability Borel measure on $\mathbb{R}$ such that $\mu_F((a, b]) = F(b) - F(a)$ for every $a < b$. It is well known (see [12], [16] and [19]), realizing the Cantor set as the attractor of an iterated function system, that $(C, |\cdot|, \mu)$ is a space of homogeneous type and even normal in the sense of [14]. To get a discrete approximation of this space, let us write

$$C = \bigcap_{n=1}^{\infty} C_n, \quad C_n = \bigcup_{j=1}^{2^n} I_n^j,\ I_n^j = [a_n^j, b_n^j],$$

where C_n is the union of disjoint intervals in the n-th step in the construction of the Cantor set. For each positive integer n, set $L_n = \{a_n^j : j = 1, 2, \dots, 2^n\}$, in other words, L_n is the collection of all the left points of each interval in C_n. Notice that for each n we have that $d_H(L_n, C) \le 2/3^n$. Then $L_n \xrightarrow{d_H} C$ when $n \to \infty$. Let μ_n be the discrete measure defined on L_n by $\mu_n(\{x\}) = 2^{-n}$ for each $x \in L_n$. Then $\mu_n \xrightarrow{w*} \mu$.

In fact, for $\varphi \in \mathcal{C}([0,1])$ we have

$$\int_{[0,1]} \varphi(x)\, d\mu_n(x) = \frac{1}{2^n} \sum_{j=1}^{2^n} \varphi(a_n^j).$$

On the other hand, for fixed n, the partition of $[0,1]$ given by

$$P_n = \{x_\ell = \ell/3^n : \ell = 0, 1, 2, \ldots, 3^n\}$$

contains L_n. From the construction of F as a limit of the continuous and piecewise linear functions F_k, one easily see that $F_k(x_\ell) = F_n(x_\ell)$ for every $\ell = 0, 1, \ldots, 3^n$ and every $k \geq n$. Then $F(x_\ell) = F_n(x_\ell)$ for every $\ell = 0, 1, \ldots, 3^n$, so that

$$\begin{aligned}\sum_{\ell=0}^{3^n-1} \varphi(x_\ell)[F(x_{\ell+1}) - F(x_\ell)] &= \sum_{\ell=0}^{3^n-1} \varphi(x_\ell)[F_n(x_{\ell+1}) - F_n(x_\ell)] \\ &= \frac{1}{2^n} \sum_{j=1}^{2^n} \varphi(a_n^j).\end{aligned}$$

The last expression follows from the fact that $F_n(x_{\ell+1}) - F_n(x_\ell) = 2^{-n}$ if $x_\ell \in L_n$ and it vanishes if $x_\ell \notin L_n$. Hence

$$\int_{[0,1]} \varphi(x)\, d\mu_n(x) = \sum_{\ell=0}^{3^n-1} \varphi(x_\ell)[F(x_{\ell+1}) - F(x_\ell)] \underset{n\to\infty}{\longrightarrow} \int_{[0,1]} \varphi(x)\, dF(x),$$

so that $\mu_n \stackrel{w*}{\longrightarrow} \mu$.

Let us next prove that there exists $A \geq 1$ such that $(L_n, |\cdot|, \mu_n)$ is a space of homogeneous type with doubling constant bounded by A, for every n. Notice that L_n can be obtained by dividing by 3^n all the non-negative integers whose expansion in basis 3 do not contain the digit 1 and having at most n digits. So that each point $x \in L_n$ can be identified with an n-tuple $(x_1, x_2, \ldots, x_n)$ where each x_i is zero or two. With this notation, following [2], define $d_n : L_n \times L_n \to \mathbb{R}^+ \cup \{0\}$ by

$$d_n(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 3^{-j}, & \text{if } x_i = y_i \text{ for every } i < j \text{ and } x_j \neq y_j. \end{cases}$$

It is easy to see that d_n is a distance on L_n. Let us first show that (L_n, d_n, μ_n) is a uniform family of spaces of homogeneous type, in the sense that there exists a constant A such that the inequalities

$$0 < \mu_n(B_{d_n}(x, 2r)) \leq A\mu_n(B_{d_n}(x, r)) < \infty \tag{1}$$

hold for each $x \in L_n$, $r > 0$ and every positive integer n, where $B_{d_n}(x, r) = \{y \in L_n : d_n(x, y) < r\}$. Notice that for $x \in L_n$ and j a positive integer, we have

$$B_{d_n}(x, 3^{-j}) = \{y \in L_n : y_i = x_i,\ i = 1, 2, \ldots, j\},$$

hence

$$\operatorname{card}(B_{d_n}(x, 3^{-j})) = \begin{cases} 2^{n-j}, & j \le n, \\ 1, & j \ge n. \end{cases}$$

So that

$$\mu_n(B_{d_n}(x, 3^{-j})) = \begin{cases} 2^{-j}, & j \le n, \\ 2^{-n}, & j \ge n. \end{cases}$$

From this estimate, for a given $0 < r < 1$, choosing j such that $3^{-j} < r \le 3^{1-j}$ we have (1) with $A = 4$. Observe that given a positive integer n and $x, y \in L_n$, $x \neq y$, with $d_n(x, y) = 3^{-j}$, we necessarily have that

$$x - y = \sum_{i=j}^{n} 3^{-i}(x_i - y_i),$$

from which we obtain the inequalities

$$d_n(x, y) \le |x - y| \le 3 d_n(x, y),$$

for every n and every $x, y \in L_n$. Hence also $(L_n, |\cdot|, \mu_n)$ is a uniform sequence of spaces of homogeneous type. In fact, for a given n, $x \in L_n$ and $r > 0$ we have

$$\begin{aligned} \mu_n(B(x, 2r)) &\le \mu_n(B_{d_n}(x, 2r)) \\ &\le 4^3 \mu_n(B_{d_n}(x, r/3)) \\ &\le 4^3 \mu_n(B(x, r)). \end{aligned}$$

Hence the Cantor set $(C, |\cdot|, \mu)$ can be approximated in the Hausdorff-Kantorovich distance by the uniform sequence of discrete spaces $(L_n, |\cdot|, \mu_n)$.

The aim of this paper is to show that the situation of the above examples is typical. More precisely, we shall prove that each probability compact space of homogeneous type can be approximated in the Hausdorff-Kantorovich sense by a sequence of finite spaces of homogeneous type with a uniform bound for the doubling constant.

To prove our result we use the techniques introduced by J.M. Wu in [18] to produce partitions on the discrete approximation, and those introduced by M. Christ in [7] to build dyadic type families on spaces of homogeneous type.

In Sect. 2 we introduce the Hausdorff-Kantorovich distance and in Sect. 3 we prove a completeness type property for the families of spaces of homogeneous type with bounded doubling constant. The main result, providing the discrete approximation of a given space of homogeneous type, is contained in Sect. 4.

2 The Hausdorff-Kantorovich Quasi-Metric

Let X be a given set. A function $\rho : X \times X \to \mathbb{R}^+ \cup \{0\}$ is called a *quasi-distance* if ρ is symmetric, ρ vanishes on the diagonal of $X \times X$, $\rho(x, y) = 0$ implies $x = y$, and there exists a constant $\Lambda \ge 1$ such that the inequality $\rho(x, y) \le \Lambda(\rho(x, z) +$

$\rho(z, y))$ holds for every $x, y, z \in X$. The family $\mathcal{N}_x$ of subsets E of X for which, for some $r > 0$, $B_\rho(x, r) := \{y \in X : \rho(x, y) < r\} \subseteq E$ is a neighborhood system for a topology τ on X. The sets $B(x, r)$ are called the ρ-balls or simply the balls in X. The basic result concerning quasi-metric spaces is a theorem due to Macías and Segovia [15] which states that for each quasi-distance ρ on X there exist a distance d on X and a number $\xi \geq 1$ depending only on Λ such that $\rho \simeq d^\xi$. In other words, there exist constants c_1 and c_2 which depend only on Λ such that the inequalities

$$c_1 \rho(x, y) \leq d^\xi(x, y) \leq c_2 \rho(x, y), \tag{2}$$

hold for every $x, y \in X$. In particular the topology τ introduced through the neighborhood system $\mathcal{N}_x$ given by the ρ-balls, is the metric topology induced on X by d. Hence each topological concept introduced further can be regarded as a metric one.

Throughout this paper (X, ρ) shall be a compact quasi-metric space. With d we shall always denote a distance for which there exist ξ, c_1 and c_2 constants such that (2) holds. For any closed subset Y of X, the quasi-metric space (Y, ρ) is a compact subspace of (X, ρ).

To accomplish our aims we start by introducing a quasi-metric structure on the closed probability subspaces (Y, ρ, μ) of (X, ρ). This topology involves the Hausdorff convergence of compact sets and the Kantorovich weak star convergence of probabilities.

Let $\mathcal{K} = \{K \subseteq X : K \neq \emptyset,\ K \text{ compact}\}$. With $[A]_\epsilon$ we shall denote the ϵ-enlargement of the set $A \subset X$; i.e. $[A]_\epsilon = \bigcup_{x \in A} B_\rho(x, \epsilon) = \{y \in X : \rho(y, A) < \epsilon\}$. Here $\rho(x, A) = \inf\{\rho(x, y) : y \in A\}$. Given A and B two sets in $\mathcal{K}$ the Hausdorff quasi-distance from A to B is given by

$$\delta_H(A, B) = \inf\{\epsilon > 0 : A \subseteq [B]_\epsilon \text{ and } B \subseteq [A]_\epsilon\}.$$

Of course δ_H is the usual Hausdorff distance when ρ itself is a metric. The next result is a corollary of the completeness of the Hausdorff distance (see [11]) and of the above mentioned theorem of Macías and Segovia.

Proposition 2.1 *$(\mathcal{K}, \delta_H)$ is a complete quasi-metric space.*

Proof Set d_H to denote the usual Hausdorff distance on $\mathcal{K}$ associated to d. In other words if $[A]_{\epsilon,d} = \{x \in X : d(x, A) < \epsilon\}$ denotes the ϵ-neighborhood of A with respect to d, then $d_H(A, B) = \inf\{\epsilon > 0 : A \subseteq [B]_{\epsilon,d} \text{ and } B \subseteq [A]_{\epsilon,d}\}$, for $A, B \in \mathcal{K}$. Since for every $\epsilon > 0$ we have that

$$[A]_{(c_1\epsilon)^{1/\xi},d} \subseteq [A]_\epsilon \subseteq [A]_{(c_2\epsilon)^{1/\xi},d}$$

then

$$\begin{aligned} \delta_H(A, B) &\geq \inf\{\epsilon > 0 : A \subseteq [B]_{(c_2\epsilon)^{1/\xi},d} \text{ and } B \subseteq [A]_{(c_2\epsilon)^{1/\xi},d}\} \\ &= \frac{1}{c_2} d_H^\xi(A, B), \end{aligned}$$

for every A and B in $\mathcal{K}$. With a similar argument we can show that $\delta_H(A,B) \le d_H^\xi(A,B)/c_1$. Hence $\delta_H \simeq d_H^\xi$, with the same constants c_1 and c_2 in (2). Since $(\mathcal{K}, d_H)$ is a complete metric space, we have that δ_H is a quasi distance on $\mathcal{K}$ and $(\mathcal{K}, \delta_H)$ is a complete quasi-metric space. □

Let us now introduce the Kantorovich-Rubinshtein distance (known also as the Hutchinson distance, see [1, 13]) on the set of all Borel regular probability measures on the quasi-metric space (X, ρ). Let

$$\mathcal{P}(X) = \{\mu : \mu \text{ is a positive Borel measure on } X \text{ and } \mu(X) = 1\},$$

and let $\mathcal{C}(X)$ be the space of continuous real valued functions on X. Since the Borel σ-algebra induced by the quasi-distance ρ is the same as the one induced by d, we have that every measure μ in $\mathcal{P}(X)$ is regular (see [4]). For $c > 0$, let us denote by Lip_c the space of all d-Lipschitz continuous functions defined on X with Lipschitz constant at most c, i.e. $f \in \mathrm{Lip}_c$ if and only if $|f(x) - f(y)| \le c d(x,y)$ for every x and $y \in X$.

Since (X, ρ) is compact, $\delta_K(\mu, \nu) = \sup\{|\int f\,d\mu - \int f\,d\nu| : f \in \mathrm{Lip}_1\}$ gives a distance on $\mathcal{P}(X)$ such that the δ_K-convergence of a sequence is equivalent to its weak star convergence to the same limit. Hence, in our situation, the metric space $(\mathcal{P}(X), \delta_K)$ becomes complete.

Even when the results stated in the above paragraph are well known, specially for subsets of the Euclidean space, for the sake of completeness, we shall briefly sketch their proofs.

Let us remind that $\mu_n \xrightarrow{w*} \mu$ if and only if $\int \varphi\,d\mu_n \to \int \varphi\,d\mu$ for every $\varphi \in \mathcal{C}(X)$. Notice that weak star convergence depends only on the topology of X, not on the specific metric or quasi-metric that generates it. Since X is compact, $\mathcal{P}(X)$ is sequentially compact by Prohorov's Theorem (see for example [4]), that is, for every sequence $\{\mu_n\}$ in $\mathcal{P}(X)$ there exist a subsequence $\{\mu_{n_i}\}$ and a measure $\mu \in \mathcal{P}(X)$ such that $\mu_{n_i} \xrightarrow{w*} \mu$. This fact implies that $\mathcal{P}(X)$ is complete with the weak star topology.

Lemma 2.2 *Let $\mu_1, \mu_2 \ldots$ and μ be measures in $\mathcal{P}(X)$. Then $\mu_n \xrightarrow{w*} \mu$ if and only if $\delta_K(\mu_n, \mu) \to 0$ when $n \to \infty$.*

The proof follows the lines of Lemma 1.10 in [10], actually the fact that the weak star convergence implies that $\delta_K(\mu_n, \mu) \to 0$ is valid with no changes.

For the converse suppose that $\delta_K(\mu_n, \mu) \to 0$. Notice that since X is compact, then the class $\mathcal{A} = \bigcup_{c>0} \mathrm{Lip}_c$ is a subalgebra of $\mathcal{C}(X)$. Also $\mathcal{A}$ separate points; that is, given two distinct points x and y in X, we can find an f in $\mathcal{A}$ such that $f(x) \ne f(y)$. In fact, given $x, y \in X$ with $x \ne y$, it is enough to take $f(z) = d(x,z)$, which belongs to Lip_1. Since $\mathcal{A}$ contains the constant functions, then from the Stone-Weierstrass theorem for compact metric spaces (see [17]) we have that $\mathcal{A}$ is dense in $\mathcal{C}(X)$. Then given $\varphi \in \mathcal{C}(X)$ and $\epsilon > 0$, there exists $f \in \mathrm{Lip}_c$ for some $c > 0$ such that $|\varphi(x) - f(x)| < \epsilon/3$ for all $x \in X$. Let $n_0 = n_0(\epsilon)$ be such that if $n \ge n_0$ then $\delta_K(\mu_n, \mu) <$

$\epsilon/(3c)$. Then if $n \geq n_0$ we have

$$\left|\int \varphi\, d\mu_n - \int \varphi\, d\mu\right| \leq \int |\varphi - f|\, d\mu_n + \left|\int f\, d\mu_n - \int f\, d\mu\right| + \int |\varphi - f|\, d\mu$$
$$< 2\epsilon/3 + c\delta_K(\mu_n, \mu) < \epsilon.$$

Hence $\mu_n \xrightarrow{w*} \mu$.

We are now in position to describe the basic quasi-metric space whose structure and convergence properties are of our interest. Let $\mathcal{X}$ be the set of all couples (Y, μ) such that Y is a closed, and hence compact, subset of X, and μ is a regular Borel probability measure on X. In other words, $\mathcal{X} = \mathcal{K} \times \mathcal{P}$. Given two elements (Y_i, μ_i) of $\mathcal{X}$, $i = 1, 2$, define

$$\delta((Y_1, \mu_1), (Y_2, \mu_2)) = \delta_H(Y_1, Y_2) + \delta_K(\mu_1, \mu_2),$$

so that $(\mathcal{X}, \delta)$ becomes a complete quasi-metric space. Let $\mathcal{E}$ be the set of all $(Y, \mu) \in \mathcal{X}$ such that the support of μ is contained in Y, in other words

$$\mathcal{E} = \{(Y, \mu) \in \mathcal{X} : \operatorname{supp} \mu \subseteq Y\}.$$

Here $\operatorname{supp} \mu$ is the complementary of the largest open set G in X for which $\int \varphi\, d\mu = 0$ for every $\varphi \in \mathcal{C}(X)$ with $\operatorname{supp} \varphi \subseteq G$, and $\operatorname{supp} \varphi$ is the closure of the set $\{\varphi \neq 0\}$.

Theorem 2.3 *The set $\mathcal{E}$ is closed in $(\mathcal{X}, \delta)$. Hence $(\mathcal{E}, \delta)$ is a complete quasi-metric subspace of $(\mathcal{X}, \delta)$.*

Proof Let $\{(Y_n, \mu_n) : n \in \mathbb{N}\}$ be a sequence in $\mathcal{E}$ with $(Y_n, \mu_n) \xrightarrow{\delta} (Y, \mu)$. We only have to prove that $\operatorname{supp} \mu \subseteq Y$. Let us show that $\int \varphi\, d\mu = 0$ for every $\varphi \in \mathcal{C}(X)$ with $\operatorname{supp} \varphi \cap Y = \emptyset$. Take $\epsilon = \rho(\operatorname{supp} \varphi, Y) > 0$ and notice that $\operatorname{supp} \varphi \cap [Y]_\epsilon = \emptyset$. Since, on the other hand, $Y_n \xrightarrow{\delta_H} Y$, for the same value of ϵ, there must exist $N = N(\epsilon)$ such that $Y_n \subseteq [Y]_\epsilon$ whenever $n \geq N$. Hence $\operatorname{supp} \varphi \cap Y_n = \emptyset$ for every $n \geq N$, so that

$$\int \varphi\, d\mu = \lim_{n\to\infty} \int \varphi\, d\mu_n = 0. \qquad \square$$

3 Subspaces of $\mathcal{E}$: the Doubling Property

Let (X, ρ) and d be as in Section 1. It is not difficult to construct a translation invariant quasi-distance $\rho(x, y)$ on $\mathbb{R}$ generating the usual topology and equivalent to $|x - y|$ for which the ρ-balls are not even Lebesgue measurable sets. Hence for a Borel measure μ on (X, ρ) it could happen that the expression $\mu(B_\rho(x, r))$ is not well defined. To avoid this difficulty we shall keep assuming that every ρ-ball is a Borel set.

Let A be a given real number with $A \geq 1$. Let $\mathcal{D}(A)$ be the set of all couples (Y, μ) in $\mathcal{E}$ such that the inequalities

$$0 < \mu(B_\rho(y, 2r)) \leq A\mu(B_\rho(y, r)) \tag{3}$$

hold for every $y \in Y$ and every $r > 0$. Such a couple (Y, μ) is usually called a space of homogeneous type if we understand that the quasi-metric is the one inherited from the environment X.

Theorem 3.1 *Let $\{(Y_n, \mu_n)\}$ be a sequence in $\mathcal{E}$ such that $(Y_n, \mu_n) \xrightarrow{\delta} (Y, \mu)$. If there exists $A \geq 1$ such that $(Y_n, \mu_n) \in \mathcal{D}(A)$ for every n, then there exists A' depending only on A and Λ such that $(Y, \mu) \in \mathcal{D}(A')$.*

Proof Let φ be the continuous function defined on the non-negative real numbers as $\varphi \equiv 1$ on $[0, 1]$, $\varphi \equiv 0$ on $[2, \infty)$ which is linear in the interval $[1, 2]$. Take $y \in Y$ and $r > 0$. Since $Y_n \xrightarrow{\delta_H} Y$, let us take $y_n \in Y_n$ such that $d(y_n, y) \to 0$ as $n \to \infty$. Then, since $\mathcal{X}_{[0,1]} \leq \varphi$, the following inequality follows easily

$$\mu(B_d(y, 2r)) \leq \int \varphi\left(\frac{d(x, y)}{2r}\right) d\mu(x).$$

Also, for y and r fixed, $\varphi(\frac{d(x,y)}{2r})$ is a continuous function of $x \in X$, and since $\mu_n \xrightarrow{w*} \mu$ we have

$$\mu(B_d(y, 2r)) \leq \lim_{n\to\infty} \int \varphi\left(\frac{d(x, y)}{2r}\right) d\mu_n(x).$$

Now, since $y_n \to y$ and $\varphi \leq \mathcal{X}_{[0,2]}$ we have the inequalities

$$\begin{aligned}
\mu(B_d(y, 2r)) &\leq \liminf_{n\to\infty} \mu_n(B_d(y, 4r)) \\
&\leq \liminf_{n\to\infty} \mu_n(B_d(y_n, 5r)) \\
&\leq \liminf_{n\to\infty} A^4 \mu_n\left(B_d\left(y_n, \frac{5r}{16}\right)\right) \\
&\leq A^4 \liminf_{n\to\infty} \mu_n\left(B_d\left(y, \frac{r}{2}\right)\right) \\
&\leq A^4 \lim_{n\to\infty} \int \varphi\left(\frac{2d(x, y)}{r}\right) d\mu_n(x) \\
&= A^4 \int \varphi\left(\frac{2d(x, y)}{r}\right) d\mu(x) \\
&\leq A^4 \mu(B_d(y, r)).
\end{aligned}$$

Since for every $s > 0$ we have $B_\rho(y, 2s) \subseteq B_d(y, (2c_2)^{1/\xi} s^{1/\xi})$ and $B_\rho(y, s) \supseteq B_d(y, c_1^{1/\xi} s^{1/\xi})$, applying k times the above inequality, with $2^k \geq (\frac{2c_2}{c_1})^{1/\xi}$, we obtain

$$\begin{aligned}\mu(B_\rho(y,2s)) &\le \mu(B_d(y,(2c_2)^{1/\xi}s^{1/\xi}))\\ &\le A^{4k}\mu(B_d(y,c_1^{1/\xi}s^{1/\xi}))\\ &\le A^{4k}\mu(B_\rho(y,s)),\end{aligned}$$

which is the desired inequality since k depends only on Λ. □

Notice that, since $(\mathcal{E},\delta)$ is complete, given a Cauchy sequence $\{(Y_n,\mu_n)\}$ in $\mathcal{D}(A)$, we have a limit couple $(Y,\mu)\in\mathcal{E}$ for that sequence. The above theorem shows that $(Y,\mu)\in\mathcal{D}(A')$, which is a kind of completeness of the doubling classes. Let us also remark that the class $\bigcup_{A\ge 1}\mathcal{D}(A)\subseteq\mathcal{E}$ is not complete. In fact, consider $X=[0,1]$ with ρ the usual distance. Take $Y_n=[0,1]$ for each n and μ_n the measure with density $f_n(t)=n-1+1/n$ on $[0,1/n]$ and $f_n(t)=1/n$ on $(1/n,1]$. It is easy to see that $\mu_n\overset{\delta_K}{\to}\delta_0$, and that each $(Y_n,\mu_n)\in\mathcal{D}(A_n)$, with $A_n=2n(n-1+1/n)$ as a possible doubling constant. Since in any space of homogeneous type atoms are isolated (see [15]), the limit space $([0,1],|\cdot|,\delta_0)$ cannot be a space of homogeneous type. From the above theorem we have in particular that $\sup_{x,r}\frac{\mu_n(x,2r)}{\mu_n(x,r)}$ is unbounded.

Finally, observe that the well known doubling property for the Hausdorff measure of order $\log 2/\log 3$ on the Cantor set is a consequence of the uniform estimates obtained in the introduction and of Theorem 3.1.

4 Density of Finite Spaces in $\mathcal{D}(A)$

Let us denote by $\mathcal{F}$ the family of all couples (Y,μ) in $\mathcal{E}$ for which Y is finite. In other words

$$\mathcal{F}=\{(Y,\mu)\in\mathcal{E}:\operatorname{card}(Y)<\infty\}.$$

Observe that each $(Y,\mu)\in\mathcal{F}$ for which $\mu(\{y\})>0$ for every $y\in Y$, belongs to $\mathcal{D}(A)$ for some A. The main result of this paper, which is contained in the next statement, shows that every space $(Y,\mu)\in\mathcal{D}(A)$ can be approximated in the metric δ by a sequence in $\mathcal{F}$ with uniform doubling constant.

Theorem 4.1 *Let (X,ρ) be a compact quasi-metric space with $\Lambda\ge 1$ such that $\rho(x,z)\le\Lambda(\rho(x,y)+\rho(y,z))$ for every x, y and z in X. Assume that each ρ-ball in X is a Borel set. Given $A\ge 1$ set $\mathcal{D}(A)$ to denote the family of all couples (Y,μ), where Y is a compact subset of X and μ is a Borel probability measure supported on Y, satisfying*

$$0<\mu(B_\rho(y,2r))\le A\mu(B_\rho(y,r))$$

for every $y\in Y$ and every $r>0$. Then there exists $A'\ge 1$ depending only on A and on Λ, such that for every $(Y,\mu)\in\mathcal{D}(A)$ there exists a sequence $\{(Y_n,\mu_n)\}\subseteq\mathcal{D}(A')$ with $\operatorname{card}(Y_n)<\infty$ for every n, satisfying

$$(Y_n,\mu_n)\overset{\delta}{\longrightarrow}(Y,\mu)$$

as $n\to\infty$.

Before starting with the proof of the theorem we shall state some properties of spaces of homogeneous type. The basic property that we shall need is actually contained in the first systematic treatment of spaces of homogeneous type due to R. Coiffman and G. Weiss [8], and reflects the fact that spaces of homogeneous type have finite uniform metric (or Assouad [3]) dimension. In what follows a set E is said to be r-disperse or r-separated provided that the distance between two different points of E is larger than or equal to r.

Lemma 4.2 *For each space of homogeneous type there exists a geometric constant N such that every r-disperse subset E has at most N^m points in each ball of radius $2^m r$, with m a positive integer.*

Notice that from (2) we only have to prove the theorem for the metric space (X, d).

Since (X, d) is compact, we can normalize the distance d in such a way that the d-diameter of X is less that one.

Let $A \geq 1$ be a given constant and take $(Y, \mu) \in \mathcal{E}$ such that

$$0 < \mu(B_d(y, 2r)) \leq A\mu(B_d(y, r))$$

holds for every $y \in Y$ and every $r > 0$.

We shall combine the idea used by J.M. Wu in [18] and the construction of dyadic sets given by M. Christ in [7] in order to define the approximating sets and the approximating measures.

For each non-negative integer n, let $S_n = \{x_{n,k} : 1 \leq k \leq K_n\}$ be a 10^{-n}-net in Y (this means that S_n is a maximal 10^{-n}-disperse subset of Y) with

$$S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n \subseteq S_{n+1} \subseteq \cdots$$

Notice that since $\operatorname{diam}(Y) < 1$, the net S_0 contains only one point $x_{0,1}$, and that $K_n = \operatorname{card}(S_n) < \infty$ for every n.

Let us construct a finite partition $\{T_{n,k} : 1 \leq k \leq K_n\}$ of S_{n+1} given by

$$T_{n,1} = S_{n+1} \cap B_d(x_{n,1}, 10^{-n}) - \bigcup_{k=2}^{K_n} B_d(x_{n,k}, 10^{-n}/2),$$

$$T_{n,\ell} = S_{n+1} \cap B_d(x_{n,\ell}, 10^{-n}) - \bigcup_{h=1}^{\ell-1} T_{n,h} - \bigcup_{k=\ell+1}^{K_n} B_d(x_{n,k}, 10^{-n}/2),$$

for $2 \leq \ell < K_n$, and

$$T_{n,K_n} = S_{n+1} \cap B_d(x_{n,\ell}, 10^{-n}) - \bigcup_{h=1}^{\ell-1} T_{n,h}.$$

The following inclusions are easy to check and shall be used in the sequel

$$S_{n+1} \cap B_d(x_{n,k}, 10^{-n}/2) \subseteq T_{n,k} \subseteq S_{n+1} \cap B_d(x_{n,k}, 10^{-n}), \tag{4}$$

for each non-negative integer n and $1 \le k \le K_n$.

The partitions introduced above give rise to a partial order $\preceq$ on the set

$$\mathcal{A} = \{(n,k) : n \in \mathbb{N} \cup \{0\},\ 1 \le k \le K_n\}.$$

The elements of $\mathcal{A}$ are related according to the following rules

(o.1) $(n,k) \preceq (n,k)$ for every $(n,k) \in \mathcal{A}$, and (n,k) is not related to any other (n,i), $i = 1, 2, \ldots, K_n$, $i \ne k$;
(o.2) $(n+1,q) \preceq (n,k)$ if and only if $x_{n+1,q} \in T_{n,k}$;
(o.3) extension by transitivity of (o.2): $(\ell,i) \preceq (n,k)$ if and only if $\ell \ge n+1$ and there exist $i_1, i_2, \ldots, i_{\ell-n-1}$ such that $(\ell,i) \preceq (\ell-1,i_1) \preceq (\ell-2,i_2) \preceq \cdots \preceq (n+1,i_{\ell-n-1}) \preceq (n,k)$.

In the next lemma for a given non-negative integer n and $\ell \ge n+1$, following [18, page 1455] we shall write $T^{\ell}_{n,k} = \{x_{\ell,i} : (\ell,i) \preceq (n,k)\}$. With this notation we readily see that

$$T^{\ell}_{n,k} \subseteq S_\ell \cap B_d(x_{n,k}, 10^{-n+1}/9), \tag{5}$$

and that the family $\{T^{\ell}_{n,k} : k = 1, \ldots, K_n\}$ is a disjoint partition of S_ℓ.

Lemma 4.3 *Let (X,d), (Y,μ) and $\{S_n : n = 0,1,2,\ldots\}$ as before. Assume that for each n a probability measure μ_n on the Borel subsets of Y, with support S_n is given. If the sequence $\{(S_n,\mu_n) : n = 0,1,2,\ldots\}$ satisfies*

(1) *$\mu_\ell(T^{\ell}_{n,k}) = \mu_n(\{x_{n,k}\})$, for every non-negative integer n, $1 \le k \le K_n$ and $\ell \ge n+1$;*
(2) *there exists a constant C_1 such that for $x_{n,k} \in S_n$ and $x_{n+1,\ell}$ with $(n+1,\ell) \preceq (n,k)$ we have $\mu_n(\{x_{n,k}\}) \le C_1 \mu_{n+1}(\{x_{n+1,\ell}\})$;*
(3) *there exists a constant C_2 such that if $x_{n,k}$ and $x_{n,i}$ are points in S_n satisfying $d(x_{n,k},x_{n,i}) < 10^{-n+3}$, then $\mu_n(\{x_{n,k}\} \le C_2 \mu_n(\{x_{n,i}\})$,*

then there exists a constant $\widetilde{A}$ which depends only on C_1, C_2 and A, such that $(S_n,\mu_n) \in \mathcal{D}(\widetilde{A})$ for each non-negative integer n.

Proof Notice that for $n = 0$, S_0 reduces to the only point $x_{0,1}$. Hence, since μ_0 is supported on S_0, we necessarily have $\mu_0(B_d(x_{0,1},2r)) = \mu_0(B_d(x_{0,1},r)) = \mu_0(\{x_{0,1}\}) = 1$ for every $r > 0$. Then μ_0 is trivially doubling with any $\widetilde{A} \ge 1$. For $n \ge 1$, any $x \in S_n$, $x = x_{n,k}$ for some $k = 1,2,\ldots,K_n$, and $r > 0$ we have to estimate $\mu_n(B_d(x,2r))$ in terms of $\mu_n(B_d(x,r))$. Observe that since diam $X < 1$ we only have to consider the case $0 < r < 1$. We shall divide our analysis in three steps, according to the relation between r and n:

i. $0 < r \le 10^{-n}/2$;
ii. $10^{-n}/6 < r/3 \le 10^{-n+1}$;
iii. $r/3 > 10^{-n+1}$.

Case i: $0 < r \le 10^{-n}/2$. Since S_n is 10^{-n}-disperse, we have $B_d(x,2r) \cap S_n = B_d(x,r) \cap S_n = \{x\}$, and any $\widetilde{A} \ge 1$ works.

Case ii: $10^{-n}/6 < r/3 \leq 10^{-n+1}$. Let $\mathcal{Q}$ be the set defined as

$$\mathcal{Q} = \{q : x_{n-1,q} \in B_d(x, 22r)\}.$$

Notice that if $N = N(A)$ is the constant provided by Lemma 4.2, we have

$$\begin{aligned} \operatorname{card}(\mathcal{Q}) &= \operatorname{card}(S_{n-1} \cap B_d(x, 22r)) \\ &\leq \operatorname{card}(S_{n-1} \cap B_d(x, 2^7 10^{-n+1})) \\ &\leq N^7. \end{aligned} \tag{6}$$

Next we prove that

$$S_n \cap B_d(x, 2r) \subseteq \bigcup_{q \in \mathcal{Q}} T_{n-1,q}. \tag{7}$$

Take $x_{n,i} \in B_d(x, 2r)$, and let q be the unique index in $\{1, 2, \ldots, K_n\}$ such that $x_{n,i} \in T_{n-1,q}$. Let us prove that $q \in \mathcal{Q}$. In fact, from (4)

$$\begin{aligned} d(x_{n-1,q}, x) &\leq d(x_{n-1,q}, x_{n,i}) + d(x_{n,i}, x) \\ &< 10^{-n+1} + 2r \\ &< 20r + 2r \\ &= 22r, \end{aligned}$$

which proves (7). Let p be such that $x \in T_{n-1,p}$. If $q \in \mathcal{Q}$, then

$$\begin{aligned} d(x_{n-1,q}, x_{n-1,p}) &\leq d(x_{n-1,q}, x) + d(x, x_{n-1,p}) \\ &< 22r + 10^{-n+1} \\ &< 66\, 10^{-n+1} + 10^{-n+1} \\ &< 10^{-n+3}, \end{aligned}$$

and we can apply (3) to get

$$\mu_{n-1}(\{x_{n-1,q}\}) \leq C_2 \mu_{n-1}(\{x_{n-1,p}\}), \quad \text{for every } q \in \mathcal{Q}.$$

Hence using (7), (1), the inequality above, (6) and (2) in that order, we have the desired inequality since

$$\begin{aligned} \mu_n(B_d(x, 2r)) &\leq \sum_{q \in \mathcal{Q}} \mu_n(T_{n-1,q}) \\ &= \sum_{q \in \mathcal{Q}} \mu_{n-1}(\{x_{n-1,q}\}) \\ &\leq \sum_{q \in \mathcal{Q}} C_2 \mu_{n-1}(\{x_{n-1,p}\}) \end{aligned}$$

$$\begin{aligned}&\le N^7 C_2 \mu_{n-1}(\{x_{n-1,p}\})\\&\le N^7 C_2 C_1 \mu_n(\{x\})\\&\le N^7 C_2 C_1 \mu_n(B_d(x,r)).\end{aligned}$$

Case iii: $r/3 > 10^{-n+1}$. Since $r < 1$, this case is only possible if $n \ge 2$. Let $0 < \ell \le n-1$ such that $10^{-\ell} < r/3 \le 10^{-\ell+1}$, and define the set $\mathcal{J} = \{j : x_{\ell,j} \in S_\ell \cap B_d(x,3r)\}$. Then

$$S_n \cap B_d(x,2r) \subseteq \bigcup_{j\in\mathcal{J}} T^n_{\ell,j}.$$

In fact, take $x_{n,i} \in B_d(x,2r)$ and $x_{\ell,j}$ such that $x_{n,i} \in T^n_{\ell,j}$. Then, from (5)

$$\begin{aligned}d(x_{\ell,j},x) &\le d(x_{\ell,j},x_{n,i}) + d(x_{n,i},x)\\&< \frac{10^{-\ell+1}}{9} + 2r\\&< 3r,\end{aligned}$$

and thus $j \in \mathcal{J}$. Hence, from (a) we have

$$\mu_n(B_d(x,2r)) \le \sum_{j\in\mathcal{J}} \mu_n(T^n_{\ell,j}) = \sum_{j\in\mathcal{J}} \mu_\ell(\{x_{\ell,j}\}) = \mu_\ell(B_d(x,3r)). \tag{8}$$

On the other hand

$$\bigcup_{x_{\ell,i}\in B_d(x,r/2)} T^n_{\ell,i} \subseteq S_n \cap B_d(x,r).$$

In fact, if $x_{\ell,i} \in B_d(x,r/2)$ and $x_{n,p} \in T^n_{\ell,i}$, then applying (5) again

$$\begin{aligned}d(x,x_{n,p}) &\le d(x,x_{\ell,i}) + d(x_{\ell,i},x_{n,p})\\&< \frac{r}{2} + \frac{10^{-\ell+1}}{9}\\&< r.\end{aligned}$$

From the above inclusion we obtain

$$\begin{aligned}\mu_n(B_d(x,r)) &\ge \sum_{x_{\ell,i}\in B_d(x,r/2)} \mu_n(T^n_{\ell,i})\\&= \sum_{x_{\ell,i}\in B_d(x,r/2)} \mu_\ell(\{x_{\ell,i}\})\\&= \mu_\ell(B_d(x,r/2)).\end{aligned} \tag{9}$$

Then, from (8) and (9) the desired result will with $\widetilde{A} = C_1C_2N^8$ if we prove

$$\mu_\ell(B_d(x,3r)) \le C_1C_2N^8 \mu_\ell(B_d(x,r/2)). \tag{10}$$

Notice that the current situation is similar to the one worked out in Case ii considered before, but here the center x of the ball does not necessarily belong to the net S_ℓ. $\square$

Proof of (10) If we define

$$\mathcal{Q} = \{q : x_{\ell,q} \in B_d(x, 7r)\},$$
$$\mathcal{J} = \{j : x_{\ell-1,j} \in B_d(x, 7r)\},$$

we have

$$\begin{aligned}\operatorname{card}(\mathcal{Q}) &\leq \operatorname{card}(S_\ell \cap B_d(x, 21.10^{-\ell+1}))\\ &\leq \operatorname{card}(S_\ell \cap B_d(x, 2^8 10^{-\ell}))\\ &\leq N^8.\end{aligned}$$

Notice that

$$S_\ell \cap B_d(x, 3r) \subseteq \bigcup_{j\in\mathcal{J}} T_{\ell-1,j}.$$

In order to prove the above inclusion, take $x_{\ell,i} \in B_d(x, 3r)$ and $x_{\ell-1,j}$ such that $x_{\ell,i} \in T_{\ell-1,j}$. We have to check that $j \in \mathcal{J}$, which is equivalent to show that $x_{\ell-1,j} \in B_d(x, 7r)$. In fact

$$\begin{aligned}d(x_{\ell-1,j}, x) &\leq d(x_{\ell-1,j}, x_{\ell,i}) + d(x_{\ell,i}, x)\\ &< 10^{-\ell+1} + 3r\\ &< 7r.\end{aligned}$$

On the other hand, since $x = x_{n,k} \in S_n$, there exists only one $p \in \{1, \ldots, K_\ell\}$ such that $x \in T^n_{\ell,p}$. In order to use (c) to compare the μ_ℓ measure of the singletons $\{x_{\ell,p}\}$ and $\{x_{\ell,q}\}$ for every $q \in \mathcal{Q}$, we have to check that $d(x_{\ell,p}, x_{\ell,q}) < 10^{-\ell+3}$. In fact, for $q \in \mathcal{Q}$ we have

$$\begin{aligned}d(x_{\ell,p}, x_{\ell,q}) &\leq d(x_{\ell,p}, x) + d(x, x_{\ell,q})\\ &< \frac{10^{-\ell+1}}{9} + 7r\\ &\leq \frac{10^{-\ell+1}}{9} + 21.10^{-\ell+1}\\ &< 10^{-\ell+3}.\end{aligned}$$

So that applying (3) we obtain

$$\mu_\ell(\{x_{\ell,q}\}) \leq C_2 \mu_\ell(\{x_{\ell,p}\}), \quad \text{for every } q \in \mathcal{Q}.$$

From the considerations above, (a) and (b), we have

$$
\begin{aligned}
\mu_\ell(B_d(x,3r)) &\le \sum_{j\in\mathcal{J}} \mu_\ell(T_{\ell-1,j}) \\
&= \sum_{j\in\mathcal{J}} \mu_{\ell-1}(\{x_{\ell-1,j}\}) \\
&\le C_1 \sum_{q\in\mathcal{Q}} \mu_\ell(\{x_{\ell,q}\}) \\
&\le C_1C_2 \sum_{q\in\mathcal{Q}} \mu_\ell(\{x_{\ell,p}\}) \\
&= C_1C_2\,\mathrm{card}(\mathcal{Q})\mu_\ell(\{x_{\ell,p}\}) \\
&\le C_1C_2N^8\mu_\ell(\{x_{\ell,p}\}).
\end{aligned}
$$

Finally since

$$
d(x_{\ell,p},x) < \frac{10^{-\ell+1}}{9} < \frac{10}{27}r < \frac{r}{2}
$$

we have that $\mu_\ell(\{x_{\ell,p}\}) \le \mu_\ell(B_d(x,r/2))$, which finishes the proof of (10). □

Notice now that the partial order $\preceq$ defined on $\mathcal{A}$ satisfies the following tree properties:

(t.1) $(n_1,k_1) \preceq (n_2,k_2)$ implies $n_2 \le n_1$;
(t.2) for every $(n_1,k_1) \in \mathcal{A}$ and every $n_2 \le n_1$, there exists a unique $1 \le k_2 \le K_{n_2}$ such that $(n_1,k_1) \preceq (n_2,k_2)$;
(t.3) if $(n,k) \preceq (n-1,i)$, then $d(x_{n,k},x_{n-1,i}) < 10^{-n+1}$;
(t.4) if $d(x_{n,k},x_{n-1,i}) < \frac{10^{-n+1}}{2}$, then $(n,k) \preceq (n-1,i)$.

Following M. Christ (see [7]), the sets

$$
Q_k^n = \bigcup_{(\ell,i)\preceq(n,k)} B_d(x_{\ell,i},10^{-\ell-1}),
$$

share with the classical dyadic cubes in $\mathbb{R}^n$ at least the properties contained in the next lemma.

Lemma 4.4 (Christ [7], Theorem 11, p. 607)

(d.1) *Q_k^n is an open set for every $(n,k) \in \mathcal{A}$;*
(d.2) *$B_d(x_{n,k},10^{-n-1}) \subseteq Q_k^n$ for every $(n,k) \in \mathcal{A}$;*
(d.3) *$Q_k^n \subseteq B_d(x_{n,k},10^{-n+1}/9)$ for every $(n,k) \in \mathcal{A}$;*
(d.4) *for each non-negative integer n, $Q_k^n \cap Q_i^n \ne \emptyset$ implies $k=i$;*
(d.5) *for every $(n,k) \in \mathcal{A}$ and every $\ell < n$ there exists a unique $1 \le i \le K_n$ such that $Q_k^n \subseteq Q_i^\ell$;*

(d.6) *if* $n \geq \ell$, *for every* $1 \leq k \leq K_n$, $1 \leq i \leq K_\ell$ *we have that* $Q_k^n \subseteq Q_i^\ell$ *or* $Q_k^n \cap Q_i^\ell = \emptyset$;
(d.7) $\mu(Y \setminus \bigcup_{1 \leq k \leq K_n} Q_k^n) = 0$, *for every non-negative integer* n;
(d.8) $\mu(Q_k^n) = \sum_{i:(\ell,i) \preceq (n,k)} \mu(Q_i^\ell)$, *for every* n, $\ell \geq n+1$ *and* $1 \leq k \leq K_n$.

From now on Q_k^n will denote the Christ's sets defined by the order $\preceq$. After an adequate choice of μ_n supported on $Y_n = S_n$, Theorem 4.1 shall be a consequence of Lemma 4.3.

Proof of Theorem 4.1 Let us define the measure μ_n on Y supported on S_n by

$$\mu_n(\{x_{n,k}\}) = \mu(Q_k^n),$$

for every non-negative integer n and $1 \leq k \leq K_n$. Notice that from (d.7) we have $\mu_n(S_n) = 1$ for every n. To check that $\mu_n \xrightarrow{w*} \mu$, take a continuous function φ on Y, and let $\epsilon > 0$ be given. Since Y is compact, φ es uniformly continuous, hence there exists $\eta > 0$ such that $|\varphi(x) - \varphi(y)| < \epsilon$, for every $x, y \in Y$ such that $d(x, y) < \eta$. Observe that

$$\int_Y \varphi \, d\mu_n = \sum_{k=1}^{K_n} \varphi(x_{n,k})\mu(Q_k^n),$$

and on the other hand

$$\int_Y \varphi \, d\mu = \sum_{k=1}^{K_n} \int_{Q_k^n} \varphi \, d\mu.$$

Thus, from (d.3),

$$\left| \int_Y \varphi \, d\mu_n - \int_Y \varphi \, d\mu \right| \leq \sum_{k=1}^{K_n} \int_{Q_k^n} |\varphi(x_{n,k}) - \varphi(x)| \, d\mu(x) < \epsilon,$$

choosing n large enough to get $10^{-n+1}/9 < \eta$.

Since S_n is 10^{-n}-disperse and $S_n \subseteq Y$, then $\delta_H(S_n, Y) \leq 10^{-n}$, so that $S_n \xrightarrow{\delta_H} Y$. Thus $(S_n, \mu_n) \xrightarrow{\delta} (Y, \mu)$.

It only remains to prove that (S_n, d, μ_n) is a uniform family of spaces of homogeneous type. We only need to check that μ_n satisfies the conditions of Lemma 4.3. In fact, from (d.8) we have

$$\mu_\ell(T_{n,k}^\ell) = \sum_{i:x_{\ell,i} \in T_{n,k}^\ell} \mu_\ell(\{x_{\ell,i}\}) = \sum_{i:(\ell,i) \preceq (n,k)} \mu(Q_i^\ell) = \mu_n(\{x_{n,k}\}).$$

for every n, $1 \leq k \leq K_n$ and $\ell \geq n+1$, and then (1) holds.

In order to check (2), notice that for $x_{n,k} \in S_n$ and $x_{n+1,\ell}$ with $(n+1, \ell) \preceq (n, k)$ we have $d(x_{n,k}, x_{n+1,\ell}) < 10^{-n}$. Then (d.3) implies $Q_k^n \subseteq B_d(x_{n+1,\ell}, 10^{-n+1}/4)$. So that

$$\begin{aligned}\mu_n(\{x_{n,k}\}) &= \mu(Q_k^n)\\ &\le \mu(B_d(x_{n+1,\ell}, 10^{-n+1}/4))\\ &\le A^8\mu(B_d(x_{n+1,\ell}, 10^{-n-2}))\\ &\le A^8\mu(Q_\ell^{n+1})\\ &= A^8\mu_{n+1}(\{x_{n+1,\ell}\}),\end{aligned}$$

and (2) holds with $C_1 = A^8$.

Finally, if $x_{n,k}$ and $x_{n,i}$ are points in S_n such that $d(x_{n,k}, x_{n,i}) < 10^{-n+3}$, then $Q_k^n \subseteq B_d(x_{n,i}, 101.10^{-n+1})$, and so

$$\begin{aligned}\mu_n(\{x_{n,k}\}) &\le \mu(B_d(x_{n,i}, 101.10^{-n+1}))\\ &\le A^{14}\mu(B_d(x_{n,i}, 10^{-n-1}))\\ &\le A^{14}\mu(Q_i^n)\\ &= A^{14}\mu_n(\{x_{n,i}\}),\end{aligned}$$

and we get (3) taking $C_2 = A^{14}$. □

References

1. Åkerlund-Biström, C.: A generalization of the Hutchinson distance and applications. Random Comput. Dynam. **5**(2-3), 159–176 (1997)
2. Ambrosio, L., Miranda, M. Jr., Pallara, D.: Special functions of bounded variation in doubling metric measure spaces. In: Calculus of Variations: Topics from the Mathematical Heritage of E. De Giorgi. Quad. Mat., vol. 14, pp. 1–45. Seconda Univ. Napoli, Caserta (2004)
3. Assouad, P.: Étude d'une dimension métrique liée à la possibilité de plongements dans $\mathbf{R}^n$. C. R. Acad. Sci. Paris Sér. A-B **288**(15), A731–A734 (1979)
4. Billingsley, P.: Convergence of Probability Measures. Wiley Series in Probability and Statistics: Probability and Statistics, Wiley, New York (1999)
5. Carena, M.: Aproximación y convergencia de espacios de tipo homogéneo. Problemas analíticos y geométricos. Tesis Doctoral, Universidad Nacional del Litoral (2008)
6. Carrillo, M.T., de Guzmán, M.: Maximal convolution operators and approximations. In: Functional Analysis, Holomorphy and Approximation Theory, Rio de Janeiro, 1980. North-Holland Math. Stud., vol. 71, pp. 117–129. North-Holland, Amsterdam (1982)
7. Christ, M.: A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral. Colloq. Math. **60/61**(2), 601–628 (1990)
8. Coifman, R.R., Weiss, G.: Analyse harmonique non-commutative sur certains espaces homogènes. In: Étude de Certaines Intégrales Singulières. Lecture Notes in Mathematics, vol. 142, Springer, Berlin (1971)
9. de Guzmán, M.: Real Variable Methods in Fourier Analysis. North-Holland Mathematics Studies, vol. 46, North-Holland, Amsterdam (1981)
10. Falconer, K.: Techniques in Fractal Geometry. Wiley, Chichester (1997)
11. Federer, H.: Geometric Measure Theory. Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer, New York (1969).
12. Hutchinson, J.E.: Fractals and self-similarity. Indiana Univ. Math. J. **30**(5), 713–747 (1981)
13. Kravchenko, A.S.: Completeness of the space of separable measures in the Kantorovich-Rubinshteĭn metric. Sibirsk. Mat. Zh. **47**(1), 85–96 (2006)

14. Macías, R.A., Segovia, C.: A decomposition into atoms of distributions on spaces of homogeneous type. Adv. Math. **33**(3), 271–309 (1979)
15. Macías, R.A., Segovia, C.: Lipschitz functions on spaces of homogeneous type. Adv. Math. **33**(3), 257–270 (1979)
16. Mosco, U.: Variational fractals. Ann. Sc. Norm. Supper. Pisa Cl. Sci. (4) **25**(3–4), 683–712 (1998), 1997. Dedicated to Ennio De Giorgi
17. Royden, H.L.: Real Analysis. Macmillan, New York (1963)
18. Wu, J.-M.: Hausdorff dimension and doubling measures on metric spaces. Proc. Am. Math. Soc. **126**(5), 1453–1459 (1998)
19. Yung, P.-L.: Doubling properties of self-similar measures. Indiana Univ. Math. J. **56**(2), 965–990 (2007)