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ON THE MAXIMAL FUNCTION FOR THE GENERALIZED ORNSTEIN-UHLENBECK SEMIGROUP.

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Abstract. In this note we consider the maximal function for the generalized Ornstein-Uhlenbeck semigroup in $\mathbb{R}$ associated with the generalized Hermite polynomials $\{H_n^\mu\}$ and prove that it is weak type (1,1) with respect to $d\lambda_\mu(x) = |x|^{2\mu} e^{-|x|^2} dx$, for $\mu > -1/2$ as well as bounded on $L^p(d\lambda_\mu)$ for $p > 1$.

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1. Introduction and preliminaries.

The generalized Hermite polynomials were defined by G. Szëgo in [16] (see problem 25, page 380) as being orthogonal

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polynomials with respect to the measure $d\lambda(x) = d\lambda_\mu(x) = |x|^{2\mu}e^{-|x|^2}dx$, with $\mu > -1/2$. In his doctoral thesis T. S. Chihara [2] (see also [3]) studied them in detail. In this paper we consider the definition of the generalized Hermite polynonials given by M. Rosenblum in [11].
Let us denote by H_n^μ this generalized Hermite polynomial of degree n. Then for n even

$$H_{2m}^\mu(x) = (-1)^m(2m)!\frac{\Gamma(\mu+\frac{1}{2})}{\Gamma(m+\mu+\frac{1}{2})}L_m^{\mu-\frac{1}{2}}(x^2) \tag{1.1}$$

and for n odd

$$H_{2m+1}^\mu(x) = (-1)^m(2m+1)!\frac{\Gamma(\mu+\frac{3}{2})}{\Gamma(m+\mu+\frac{3}{2})}xL_m^{\mu+\frac{1}{2}}(x^2), \tag{1.2}$$

L_m^γ being the γ-Laguerre polynomial of degree m.
Thus, for every $n \in \mathbb{N}$,

$$\|H_n^\mu\|_{L^2(d\lambda)} = \left(\frac{2^n(n!)^2\Gamma(\mu+1/2)}{\gamma_\mu(n)}\right)^{1/2},$$

where $\gamma_\mu(n)$ is a generalized factorial defined for n even or odd by,

$$\gamma_\mu(2m) = \frac{2^{2m}m!\Gamma(m+\mu+\frac{1}{2})}{\Gamma(\mu+\frac{1}{2})} = (2m)!\frac{\Gamma(m+\mu+\frac{1}{2})}{\Gamma(\mu+\frac{1}{2})}\frac{\Gamma(\frac{1}{2})}{\Gamma(m+\frac{1}{2})},$$

$$\gamma_\mu(2m+1) = \frac{2^{2m+1}m!\Gamma(m+\mu+\frac{3}{2})}{\Gamma(\mu+\frac{1}{2})} = (2m)!\frac{\Gamma(m+\mu+\frac{3}{2})}{\Gamma(\mu+\frac{1}{2})}\frac{\Gamma(\frac{1}{2})}{\Gamma(m+\frac{3}{2})}.$$

The generalized Hermite polynomials $\{H_n^\mu\}$ have a generating function (2.5.8) of [11]) which involves the generalized exponential function e_μ defined by

$$e_\mu(z) = \sum_{m=0}^\infty \frac{z^m}{\gamma_\mu(m)}, \quad z \in \mathbb{R}. \tag{1.3}$$

On the other hand each generalized Hermite polynomial H_n^μ satisfies the following differential equation, see [3],

$$(H_n^\mu)''(x) + 2(\frac{\mu}{x} - x)(H_n^\mu)'(x) + 2(n - \mu\frac{\theta_n}{x^2})H_n^\mu(x) = 0, \tag{1.4}$$

with

$$\theta_n = \begin{cases} 1 & \text{if} \quad n \text{ is odd}, \\ 0 & \text{if} \quad n \text{ is even}. \end{cases}$$

Therefore, by considering the (differential-diference) operator

$$L_\mu = \frac{1}{2}\frac{d^2}{dx^2} + (\frac{\mu}{x} - x)\frac{d}{dx} - \mu\frac{I - \tilde{I}}{2x^2}, \tag{1.5}$$

where $If(x) = f(x)$ and $\tilde{I}f(x) = f(-x)$, for every $n \in \mathbb{N}$, H_n^μ turns out to be an eigenfunction of L_μ with eigenvalue $-n$. This operator is one example of a Dunkl operator in one dimension. The theory of Dunkl operators originated in [4] and it is nowdays a very robust theory, see for instance M. Rösler [12].

Now we let us define the associated Markov semigroup, see D. Bakry [1], as

$$P_t(x,dy) = \sum_{n=0}^{\infty} \frac{\gamma_\mu(n)}{2^n(n!)^2} H_n^\mu(x) H_n^\mu(y) e^{-nt} \lambda(dy). \tag{1.6}$$

This semigroup is entirely characterized by the action on positive or bounded measurable functions by

$$T_\mu^t f(x) = \int_{-\infty}^{\infty} f(y) P_t(x,dy).$$

Thus the family of operators $\{T_\mu^t\}_{t\geq 0}$ is then a conservative semigroup of operators with generator L_μ, that we will call the generalized Ornstein-Uhlenbeck semigroup which also is called the associated heat-diffusion semigroup, since

$$\frac{\partial T_\mu^t f(x)}{\partial t} = L_\mu T_\mu^t f(x).$$

For $\mu = 0$, $\{T_\mu^t\}_{t\geq 0}$ reduces to the Ornstein-Uhlenbeck semigroup whose behavior on L^p was studied by B. Muckenhoupt in [8] for the one-dimensional case. By using the generalized Mehler's formula (2.6.8) of [11]: for x, $y \in \mathbb{R}$ and $|z| < 1$,

$$\sum_{n=0}^{\infty} \frac{\gamma_\mu(n)}{2^n(n!)^2} H_n^\mu(x) H_n^\mu(y) z^n = \frac{1}{(1-z^2)^{\mu+1/2}} e^{-\frac{z^2(x^2+y^2)}{1-z^2}} e_\mu\left(\frac{2xyz}{1-z^2}\right), \tag{1.7}$$

we can obtain the following integral expression of this generalized Ornstein-Uhlenbeck semigroup T_μ^t, $t > 0$,

$$T_\mu^t f(x) = \frac{1}{(1-e^{-2t})^{\mu+1/2}} \int_{-\infty}^{\infty} e^{-\frac{e^{-2t}(x^2+y^2)}{1-e^{-2t}}} e_\mu\left(\frac{2xye^{-t}}{1-e^{-2t}}\right) f(y)|y|^{2\mu} e^{-|y|^2} dy. \tag{1.8}$$

In the following section we will consider the maximal operator associated with $\{T_\mu^t\}_{t\geq 0}$, and prove this maximal operator is weak type $(1,1)$ with respect to the measure λ, bounded in L^∞ and therefore bounded in L^p with respect to λ, for $1 < p < \infty$. It is important to observe that since $\{T_\mu^t\}_{t\geq 0}$ is not a convolution semigroup, its associated maximal operator is not bounded by the Hardy-Littlewood maximal operator. Therefore in order to prove the weak $(1,1)$ inequality with respect to λ it is needed to develop new techniques. The case $\mu = 0$, that as we already said corresponds to the maximal operator of the Ornstein-Uhlenbeck semigroup, was proved by B. Muckenhoupt [8] in one dimension and by P. Sjögren in [13] in any dimension.
We will use repeatedly that

$$|x|^k e^{-x^2} \leq C e^{-x^2/2} \quad x \in \mathbb{R}. \tag{1.9}$$

The constant C which will appear throughout this paper may be different on each occurrence.

2. The maximal function of the generalized Ornstein Uhlenbeck semi-group. Let us define the generalized Ornstein-Uhlenbeck maximal function as

$$T_\mu^* f(x) \quad = \quad \sup_{t>0} |T_\mu^t f(x)|, \tag{2.1}$$

for each $x \in \mathbb{R}$. Taking $r = e^{-t}$, we can write

$$T_\mu^* f(x) = \sup_{0<r<1} \left| \int_{-\infty}^{\infty} K_r(x,y) f(y)\, d\lambda(y) \right|,$$

with

$$K_r(x,y) = \frac{1}{(1-r^2)^{\mu+\frac{1}{2}}} e^{-(x^2+y^2)\frac{r^2}{1-r^2}} e_\mu\left(\frac{2xyr}{1-r^2}\right), \;\; 0<r<1, \;\; x,y \in \mathbb{R}.$$

The main result of this paper is summarized in

THEOREM 2.1. *For $\mu > -1/2$,*

i) *T_μ^* is weak type $(1,1)$ with respect to λ, i.e. there exists a real constant $C > 0$ such that for every $\eta > 0$*

$$\lambda\{x \in \mathbb{R} : T_\mu^* f(x) > \eta\} \leq \frac{C}{\eta} \|f\|_{1,\lambda}, \tag{2.2}$$

where $\|f\|_{1,\lambda} = \int_{\mathbb{R}} |f(y)| d\lambda(y)$.

ii) *T_μ^* is bounded in L^∞, i. e. there exists a real constant $C > 0$ such that*

$$||T_\mu^* f||_\infty \leq C ||f||_\infty \tag{2.3}$$

where $\|f\|_\infty$ represents the L^∞ norm.

COROLLARY 2.2. *For $\mu > -1/2$ and $p > 1$,*

$$\|T_\mu^* f\|_{p,\lambda} \leq C\, \|f\|_{p,\lambda}, \tag{2.4}$$

where $\|f\|_{p,\lambda}^p = \int_{\mathbb{R}} |f(y)|^p d\lambda(y)$.

This corollary follows from Marcinkiewicz interpolation theorem between the weak type $(1,1)$ and the boundedness in L^∞ which will be proved in Theorem 2.1. In order to prove Theorem 2.1 we will introduce well known bounds for the functions e_μ and prove two propositions. The first one, due to I.P. Natanson and B. Muckenhoupt ([9] and [8]), can be seen a generalized Young's inequality for Borel measures. We will write it only for the particular case of the measure λ. The other one has to do with the biggest function whose density distribution as a function of η with respect to the measure λ is bounded by C/η.

2.1. Properties of e_μ. It can be proved, see (2.2.3) of [11], that the generalized exponential function e_μ can be written as,

$$e_\mu(x) = \Gamma(\mu + 1/2)(2/x)^{\mu-1/2}(I_{\mu-1/2}(x) + I_{\mu+1/2}(x)),$$

where I_ν denotes the ν-th modified Bessel function. Then, according to [17, (2), p. 77, and (2), p. 203], we have the following estimates that will be useful in the sequel

$$|e_\mu(x)| \leq e_\mu(|x|) \leq C(1 + |x|)^{-\mu} e^{|x|}, \;\; x \in \mathbb{R}. \tag{2.5}$$

Also, e_μ admits the following integral representations depending on the values of μ [11],

1. if $\mu > 0$ then
$$e_\mu(x) = \frac{1}{B(\frac{1}{2}, \mu)} \int_{-1}^{1} e^{xt}(1-t)^{\mu-1}(1+t)^\mu \, dt, \tag{2.6}$$
2. if $\mu = 0$ then
$$e_0(x) = e^x, \tag{2.7}$$
3. if $-\frac{1}{2} < \mu < 0$ then
$$e_\mu(x) = e^x + \frac{\mu}{\mu + 1/2} \frac{1}{B(1/2, \mu+1)} \int_{-1}^{1} (e^{xt} - e^x)(1-t)^{\mu-1}(1+t)^\mu dt. \tag{2.8}$$

According to (2.6) it is clear that $e_\mu(x) \geq 0$, for $\mu \geq 0$, $x \in \mathbb{R}$. However, this one is not the case when $-1/2 < \mu < 0$. Indeed, assume that $-1/2 < \mu < 0$. Since $e^u - 1 \geq u$, $u > 0$, we can write

$$\begin{aligned} e^{-x} e_\mu(x) &= 1 + \frac{\mu}{\mu + 1/2} \frac{1}{B(1/2, \mu+1)} \int_{-1}^{1} (e^{x(t-1)} - 1)(1-t)^{\mu-1}(1+t)^\mu dt \\ &\leq 1 - \frac{x\mu}{\mu + 1/2} \frac{1}{B(1/2, \mu+1)} \int_{-1}^{1} (1-t)^\mu (1+t)^\mu dt, \;\; x < 0. \end{aligned}$$

Hence, there exists $x_0 > 0$ such that $e_\mu(x) < 0$ for every $x < -x_0$.

From the above we infer that the generalized Ornstein-Uhlenbeck semigroup $\{T_\mu^t\}_{t>0}$ is positive when $\mu \geq 0$ but it is not when $-1/2 < \mu < 0$.

PROPOSITION 2.3. (Natanson) *Let f and g be two $L^1(d\lambda)$ functions. Let us assume that g(y) is nonnegative and there is an $x \in \mathbb{R}$ such that $g(y)$ is monotonically increasing for $y \leq x$ and monotonically decreasing for $x \leq y$, then*

$$\left| \int g(y) f(y) \, d\lambda(y) \right| \leq \|g\|_{1,\lambda} \mathcal{M}_\lambda f(x), \tag{2.9}$$

where

$$\mathcal{M}_\lambda f(x) = \sup_{I \supset \{x\}} \frac{1}{\lambda(I)} \int_I |f(y)| \, d\lambda(y)$$

is the Hardy-Littlewood maximal fuction of f with respect to λ. Moreover the Hardy-Littlewood maximal fuction $\mathcal{M}_\lambda f$ is weak type (1,1) and strong type (p,p) for $p > 1$ with respect to the measure λ.

A proof of this proposition can be found in [8].

PROPOSITION 2.4. *For $\mu > -1/2$, there is a real constant $C > 0$ such that the distribution function with respect to λ of the function*

$$h(x) = \max\left(\frac{1}{|x|}, |x|\right)\frac{e^{x^2}}{|x|^{2\mu}}$$

satisfies the inequality

$$\lambda\{x \in \mathbb{R} : h(x) > \eta\} \leq \frac{C}{\eta},$$

for any $\eta > 0$.

Proof. Since λ is a finite measure, it is enough to prove this result for $\eta \geq e$. Besides, due to the fact that h is even and λ is symmetric, then $\lambda\{x \in \mathbb{R} : h(x) > \eta\} = 2\lambda\{x > 0 : h(x) > \eta\}$. Now

$$\begin{aligned}
\lambda\{x > 0 : h(x) > \eta\} &\leq \lambda\left\{0 < x < 1 : \frac{1}{x^{2\mu+1}} > \eta/e\right\} \\
&\quad + \lambda\left\{x > 1 : \frac{e^{x^2}}{x^{2\mu-1}} > \eta\right\} \\
&= \int_0^{(e/\eta)^{\frac{1}{2\mu+1}}} x^{2\mu} e^{-x^2}\, dx \\
&\quad + \int_{x_0}^{\infty} x^{2\mu} e^{-x^2}\, dx \\
&= I + II,
\end{aligned}$$

with $x_0 > 1$ and $\frac{e^{x_0^2}}{x_0^{2\mu-1}} = \eta$. Let us observe that

$$I \leq \int_0^{(e/\eta)^{1/(2\mu+1)}} x^{2\mu} dx = \frac{e}{(1+2\mu)\eta},$$

and

$$II \leq C x_0^{2\mu-1} e^{-x_0^2} = \frac{C}{\eta}.$$

For last inequality see [6]. From these two bounds the conclusion of this proposition follows. □

Proof of Theorem 2.1. In order to prove this theorem it suffices to show that there exists $C > 0$ such that

$$\lambda\{x \in (0,\infty) : T^*_{\mu,+}f(x) > \eta\} \leq \frac{C}{\eta}\|f\|_{1,\lambda}, \ \ \eta > 0, \tag{2.10}$$

and

$$\|T^*_{\mu,+}f\|_\infty \leq C\|f\|_\infty, \tag{2.11}$$

for every $f \geq 0$. Here

$$T^*_{\mu,+}f(x) = \sup_{t>0} |T^\mu_{t,+}f(x)|,$$

and

$$T^t_{\mu,+}f(x) = \frac{1}{(1-e^{-2t})^{\mu+1/2}} \int_0^\infty e^{-\frac{e^{-2t}(x^2+y^2)}{1-e^{-2t}}} e_\mu\left(\frac{2xye^{-t}}{1-e^{-2t}}\right) f(y)|y|^{2\mu}e^{-|y|^2}dy.$$

Indeed, let us write $r = e^{-t}$, with $t > 0$. By (2.5), we have that

$$K_r(x,y) \leq K_r(|x|,|y|), \ \ x,y \in \mathbb{R}.$$

Then

$$|T^t_\mu f(x)| \leq T^t_{\mu,+}|f|(|x|) + T^t_{\mu,+}|\tilde{f}|(|x|), \ \ x \in \mathbb{R},$$

being $\tilde{f}(x) = f(-x)$, $x \in \mathbb{R}$. Hence,

$$T^*_\mu f(x) \leq T^*_{\mu,+}|f|(|x|) + T^*_{\mu,+}|\tilde{f}|(|x|), \ \ x \in \mathbb{R},$$

and we can write, for every $\eta > 0$,

$$\begin{aligned}\lambda\{x \in \mathbb{R} : T^*_\mu f(x) > \eta\} &\leq \lambda\{x \in \mathbb{R} : T^*_{\mu,+}|f|(|x|) > \eta/2\} \\ &\quad + \lambda\{x \in \mathbb{R} : T^*_{\mu,+}|\tilde{f}|(|x|) > \eta/2\} \\ &\leq 2(\lambda\{x \in (0,\infty) : T^*_{\mu,+}|f|(x) > \eta/2\} \\ &\quad + \lambda\{x \in (0,\infty) : T^*_{\mu,+}|\tilde{f}|(x) > \eta/2\}).\end{aligned}$$

Thus (2.2) follows from (2.10) and the fact that $\|f\|_{1,\lambda} = \|\tilde{f}\|_{1,\lambda}$. Moreover (2.3) is deduced from (2.11) because $\|f\|_\infty = \|\tilde{f}\|_\infty$.

From now on let us assume $f \geq 0$ and $x > 0$. First let us prove the weak type $(1,1)$ inequality.

(1) Case $\mu = 0$. This case corresponds to the Ornstein-Uhlenbeck maximal operator which was proved to be weak type $(1,1)$ by B. Muckenhoupt in [8].

(2) Case $\mu > -1/2$. By using (2.5) we can write

$$\begin{aligned}
&T^t_{\mu,+}f(x)\\
&\le \frac{C}{(1-r^2)^{\mu+1/2}} \int_0^\infty e^{-\frac{(x^2+y^2)r^2}{1-r^2}+\frac{2xyr}{1-r^2}} \left(1+\frac{2xyr}{1-r^2}\right)^{-\mu} f(y)\,d\lambda(y)\\
&= \frac{Ce^{x^2}}{(1-r^2)^{\mu+1/2}} \int_0^\infty e^{-\frac{|x-ry|^2}{1-r^2}} \left(1+\frac{2xyr}{1-r^2}\right)^{-\mu} f(y)\,d\lambda(y)\\
&= \frac{Ce^{x^2}}{(1-r^2)^{\mu+1/2}} \left(\int_0^{x/2r}+\int_{x/2r}^{4x/r}+\int_{4x/r}^\infty\right) e^{-\frac{|x-ry|^2}{1-r^2}} \left(1+\frac{2xyr}{1-r^2}\right)^{-\mu} f(y)\,d\lambda(y)\\
&= C(K_{1,r}f(x)+K_{2,r}f(x)+K_{3,r}f(x)).
\end{aligned}$$

Observe that if $0 < y < x/2r$, then $x - ry > x/2$ and

$$\frac{1}{(1-r^2)^{\mu+1/2}}\left(1+\frac{2rxy}{1-r^2}\right)^{-\mu} \le C\Big(\frac{1}{(1-r^2)^{\mu+1/2}}+\frac{x^{-2\mu}}{(1-r^2)^{1/2}}\Big).$$

Then

$$K_{1,r}f(x) \le Ce^{x^2}\left(\frac{1}{(1-r^2)^{\mu+1/2}}+\frac{x^{-2\mu}}{(1-r^2)^{1/2}}\right) e^{-\frac{x^2}{4(1-r^2)}}\|f\|_{1,\lambda} \le C\frac{e^{x^2}}{x^{2\mu+1}}\|f\|_{1,\lambda},$$

where last inequality is obtained as an application of (1.9).

On the other hand, if $y > \frac{4x}{r}$, then $ry - x > x$, and again by applying (1.9) repeatedly in the sequel below

$$\begin{aligned}
\frac{e^{-\frac{|x-ry|^2}{1-r^2}}}{(1-r^2)^{\mu+1/2}}\left(1+\frac{2rxy}{1-r^2}\right)^{-\mu} &= \frac{e^{-\frac{|x-ry|^2}{1-r^2}}}{(1-r^2)^{\mu+1/2}}\left(1+\frac{2x(ry-x)+2x^2}{1-r^2}\right)^{-\mu}\\
&\le C\,e^{-\frac{x^2}{2(1-r^2)}}\left(\frac{1}{(1-r^2)^{\mu+1/2}}+\frac{x^{-\mu}}{(1-r^2)^{\frac{\mu+1}{2}}}+\right.\\
&\qquad\left.\frac{x^{-2\mu}}{(1-r^2)^{1/2}}\right)\\
&\le \frac{C}{x^{2\mu+1}},
\end{aligned}$$

and we obtain

$$K_{3,r}f(x) \le C\frac{e^{x^2}}{x^{2\mu+1}}\|f\|_{1,\lambda}.$$

Finally for $\frac{x}{2r} \le y \le \frac{4x}{r}$ we have the following estimate

$$\frac{1}{(1-r^2)^{\mu+1/2}}\left(1+\frac{2rxy}{1-r^2}\right)^{-\mu} \le C\Big(\frac{1}{x^{2\mu+1}}+\frac{x^{-2\mu}}{(1-r^2)^{1/2}}\Big), \tag{2.12}$$

which is immediate for $\mu \ge 0$ and for $\mu < 0$ one can argue consideing $\frac{2rxy}{1-r^2} \le 1$ and $\frac{2rxy}{1-r^2} \ge 1$, separately. Now by taking into account inequality (2.12) we are ready to estimate $K_{2,r}f(x)$. We analyze two diferent cases. If $0 < r \leq 1/2$ we have

$$K_{2,r}f(x) \leq C\left(\frac{1}{x}+1\right)\frac{e^{x^2}}{x^{2\mu}}\|f\|_{1,\lambda},$$

and, if $1/2 < r < 1$ then

$$K_{2,r}f(x) \leq C\left(\frac{e^{x^2}}{x^{2\mu+1}}\|f\|_{1,\lambda} + \frac{e^{x^2}}{(1-r^2)^{1/2}x^{2\mu}}\int_0^\infty N(r,x,y)f(y)d\lambda(x)\right),$$

with

$$N(r,x,y) = \begin{cases} 1 & \text{if} \quad y \in \left[x, \frac{x}{r}\right] \\ e^{-\frac{|x-ry|^2}{1-r^2}} & \text{if} \quad y \in \left[\frac{x}{2r}, \frac{4x}{r}\right] \setminus \left[x, \frac{x}{r}\right] \\ 0 & \quad \text{otherwise.} \end{cases} \tag{2.13}$$

Since $N(r,x,.)$ is a Natanson kernel (see (2.9)), we obtain

$$K_{2,r}f(x) \leq C\left(\frac{e^{x^2}}{x^{2\mu+1}}\|f\|_{1,\lambda} + \frac{e^{x^2}}{x^{2\mu}(1-r^2)^{1/2}}\|N(r,x,.)\|_{1,\lambda}\,\mathcal{M}_\lambda f(x)\right).$$

Let us prove that

$$\|N(r,x,.)\|_{1,\lambda} \leq Cx^{2\mu}(1-r^2)^{1/2}e^{-x^2}. \tag{2.14}$$

Indeed,

$$\begin{aligned}
\int_{\mathbb{R}} N(r,x,y)\,d\lambda(y) &= \int_x^{x/r} e^{-y^2}y^{2\mu}\,dy + \int_{x/2r}^{x} e^{-\frac{|x-ry|^2}{1-r^2}}e^{-y^2}y^{2\mu}\,dy \\
&\quad + \int_{x/r}^{4x/r} e^{-\frac{|x-ry|^2}{1-r^2}}e^{-y^2}y^{2\mu}\,dy \\
&\leq Cx^{2\mu}\left(\int_x^{x/r} e^{-y^2}\,dy + e^{-x^2}\int_{x/2r}^{x} e^{-\frac{|rx-y|^2}{1-r^2}}\,dy\right. \\
&\quad \left. + e^{-x^2}\int_{x/r}^{4x/r} e^{-\frac{|rx-y|^2}{1-r^2}}\,dy\right) \\
&\leq C\,x^{2\mu}e^{-x^2}\left(\min\left(\frac{1}{x},(1-r)x\right)\right. \\
&\quad \left. + \int_{\mathbb{R}} e^{-\frac{|rx-y|^2}{1-r^2}}\,dy\right) \\
&\leq C\,x^{2\mu}(1-r^2)^{1/2}e^{-x^2}.
\end{aligned}$$

Now gathering together all the bounds obtained above, we get

$$T^t_{\mu,+}f(x) \leq C(h(x)\|f\|_{1,\lambda} + \mathcal{M}_\lambda f(x)),$$

for all x, $t > 0$, where h is the function defined in Proposition 2.4. Thus the weak type $(1,1)$ of $T^*_{\mu,+}$ follows from Propositions 2.3 and 2.4.

Now let us take care of the boundedness of $T^*_{\mu,+}$ in L^∞. For the case $\mu \geq 0$ this boundedness is immediate since its kernel is non-negative and its integral equals 1. Therefore let us study just the case $-1/2 < \mu < 0$. By using (2.5) and proceeding like in case 2 of the proof of the weak type $(1,1)$ inequality it follows

$$
\begin{aligned}
&T^*_{\mu,+}f(x)\\
&\leq \frac{C}{(1-r^2)^{\mu+1/2}}\int_0^\infty e^{-\frac{(x^2+y^2)r^2}{1-r^2}+\frac{2xyr}{1-r^2}}\left(1+\frac{2xyr}{1-r^2}\right)^{-\mu} f(y)\,d\lambda(y)\\
&\leq \frac{C}{(1-r^2)^{\mu+1/2}}\int_0^\infty e^{-\frac{|rx-y|^2}{1-r^2}}\left(1+\frac{2xyr}{1-r^2}\right)^{-\mu} y^{2\mu}\,dy\ \|f\|_\infty\\
&= \frac{C}{(1-r^2)^{\mu+1/2}}\int_0^\infty e^{-\frac{|rx-y|^2}{1-r^2}}\left(1+\frac{2(rx-y)y}{1-r^2}+\frac{2y^2}{1-r^2}\right)^{-\mu} y^{2\mu}\,dy\ \|f\|_\infty\\
&\leq C\Big(\int_0^\infty \frac{e^{-\frac{|rx-y|^2}{1-r^2}}}{(1-r^2)^{\mu+1/2}}\left(1+\frac{2|rx-y|y}{1-r^2}\right)^{-\mu} y^{2\mu}\,dy\\
&\qquad+\int_0^\infty \frac{e^{-\frac{|rx-y|^2}{1-r^2}}}{(1-r^2)^{1/2}}dy\Big)\|f\|_\infty.
\end{aligned}
$$

In order to prove that the first integral of last inequality is bounded by a constant independent of r, y, and x firstly we use (1.9) to obtain the inequality

$$
\left(\frac{2|rx-y|y}{1-r^2}\right)^{-\mu} e^{-\frac{|rx-y|^2}{1-r^2}} \leq C\left(\frac{y}{(1-r^2)^{1/2}}\right)^{-\mu} e^{-\frac{|rx-y|^2}{2(1-r^2)}}.
$$

Then we split the integral in two subintervals one from 0 to $\sqrt{1-r^2}$ and the other from $\sqrt{1-r^2}$ to ∞ and we call them I and II. Now we proceed to bound each part in the following way

$$
\begin{aligned}
I &= \int_0^{\sqrt{1-r^2}} \frac{e^{-\frac{|rx-y|^2}{2(1-r^2)}}}{(1-r^2)^{\mu+1/2}}\left(1+\left(\frac{y}{(1-r^2)^{1/2}}\right)^{-\mu}\right) y^{2\mu}\,dy\\
&\leq \int_0^{\sqrt{1-r^2}} \frac{y^{2\mu}}{(1-r^2)^{\mu+1/2}}\,dy + \int_0^{\sqrt{1-r^2}} \frac{y^{\mu}}{(1-r^2)^{(\mu+1)/2}}\,dy \leq C,
\end{aligned}
$$

and

$$
\begin{aligned}
II &= \int_{\sqrt{1-r^2}}^\infty \frac{e^{-\frac{|rx-y|^2}{2(1-r^2)}}}{(1-r^2)^{\mu+1/2}}\left(1+\left(\frac{y}{(1-r^2)^{1/2}}\right)^{-\mu}\right) y^{2\mu}\,dy\\
&\leq \int_{\sqrt{1-r^2}}^\infty \frac{e^{-\frac{|rx-y|^2}{2(1-r^2)}}}{(1-r^2)^{\mu+1/2}}(\sqrt{1-r^2})^{2\mu}\,dy + \int_{\sqrt{1-r^2}}^\infty \frac{e^{-\frac{|rx-y|^2}{2(1-r^2)}}}{(1-r^2)^{\mu+1/2}}\frac{y^{\mu}}{(1-r^2)^{-\mu/2}}\,dy\\
&\leq 2\int_0^\infty \frac{e^{-\frac{|rx-y|^2}{2(1-r^2)}}}{(1-r^2)^{1/2}}dy \leq C.
\end{aligned}
$$

This ends the proof of the boundedness of $T^*_{\mu,+}$ in L^∞ and at the same time the proof of Theorem 2.1. □

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