

Generalized Bessel potentials on Lipschitz type spaces

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We introduce generalizations of Bessel potentials by considering operators of the form $\varphi[(I - \Delta)^{-1/2}]$ where the functions φ extend the classical power case. The kernel of such an operator is subordinate to a growth function η . We explore conditions on η in such a way that these operators become isomorphisms between generalized Lipschitz spaces.

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1 Introduction

One of the main achievements of modern theories of function spaces is the unified approach of all classical spaces through the harmonic, caloric or more general extensions to the upper half space. This point of view leads to the notions of Triebel-Lizorkin and Besov spaces. The corner stones of such a development is the work of O. Besov [2], M. Taibleson contained in [9], [10], [11] and H. Triebel [12], [13], [14] where the Lipschitz spaces of any power order, Bessel potentials and operator theory have been deeply studied.

In 1965 S. Spanne, [6] studying generalized BMO spaces introduced the classes Lip_ϕ . Spaces of integrable functions with non-power control were introduced even earlier giving rise to the theory of Orlicz spaces.

Generalized Lipschitz spaces with moduli of continuity different from powers were considered essentially by Janson [4] and also by Ansorena-Blasco [1]. As it was pointed out by our referee, there are also relevant results in these directions proved by Kaljabin (see for example [5]).

One of the central tools in the modern theory of Sobolev, the Bessel potential of fractional order, defines a natural isomorphism between Lipschitz spaces. In fact, being isomorphisms between Lipschitz spaces $\Lambda(\alpha; p, q)$, allows to solve particular problems by reducing them to the case $0 < \alpha \leq 1$, or, in other words, they provide another characterization of these classes. A deep result of Stein and Zygmund shows that, in some sense, every translation invariant operator improving Lipschitz spaces improves also Lebesgue spaces.

In the way of extending this result from generalized Lipschitz to generalized Lebesgue (Orlicz) spaces, it becomes a central point dealing with some kind of appropriate Bessel potentials as isomorphisms between generalized Lipschitz spaces.

In this paper, we introduce generalizations of Bessel potentials by considering operators which are formally of type $\varphi[(I - \Delta)^{-1/2}]$, where φ includes the usual power case. This is achieved by introducing the right Fourier multiplier (see Section 4). While in the classical case, Bessel potentials define a semigroup in the sense that $J_\alpha \circ J_\beta = J_{\alpha+\beta}$, the composition of two of our potentials $J_{\eta_1} \circ J_{\eta_2}$, leads to the generalized potential corresponding to a special product $\eta_1 \otimes \eta_2$ (see Section 5). This operation provides a semigroup structure for the class of admissible functions η , as it is shown in Section 3. The main purpose of this paper is the search of conditions as general as possible on the functions η in order to have isomorphisms J_η between generalized Lipschitz spaces (see Sections 6 and 7). A point that deserves special care and actually the one imposing restrictions on η is the proof that our operators are onto. The precise class \mathcal{M} of functions η for which J_η is an isomorphism is fully described in Section 3. Let us point out here that \mathcal{M} contains some typical continuity moduli as $t^\alpha(1 + \log^+ \frac{1}{t})$.

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Even though the basic theory of generalized Lipschitz spaces can be found in the above mentioned literature [4] and [1], we start our paper with an introduction to these spaces relying on the historical harmonic approach through the Poisson kernel, which will be more appropriate to our purposes.

2 Generalized Lipschitz spaces

For related results and definitions see [4] and [1]. The basic difference in our approach is that a characterization of Lipschitz spaces is given in terms of the harmonic extensions of functions.

Since we are interested in function spaces defined in terms of *continuity moduli* η more general than powers, we shall start by introducing the basic properties of such functions η . We shall say that a non-negative function η defined on the positive real numbers is a *modulus of continuity* if η is non-decreasing and $\eta(0^+) = 0$ in the sense that $\lim_{t \rightarrow 0^+} \eta(t) = 0$. We say that two continuity moduli η_1 and η_2 are *equivalent* if there exist two constants c_1 and c_2 such that the inequalities $c_1 \leq \frac{\eta_1(t)}{\eta_2(t)} \leq c_2$ hold for every positive t . Given a non-negative function η defined on the set \mathbb{R}^+ of non-negative real numbers and a real number α , we shall say that η has *lower type* α or that η is of lower type α or even that α is a lower type for η if there exists a constant C such that the inequality $\eta(st) \leq Cs^\alpha \eta(t)$ holds for every $s \leq 1$ and every $t > 0$. If such an inequality holds for $s \geq 1$ we shall say that η has *upper type* α . Let us first notice that if η has lower type α and $\beta < \alpha$ then η has lower type β . So that given a function η with finite lower type, there is a left half line associated to it corresponding to all the lower types for η . Also, given a function η with finite upper type, there is a right half line associated to η corresponding to all the upper types for η . Since every lower type is less than or equal to every upper type, given such a function η we have a partition of the real numbers of the form $\mathbb{R} = L \cup I \cup U$, where L and U are the half lines of the lower and upper types and I is an interval which could be empty, closed, open or neither one of these, if I is not empty this partition is disjoint. Let us observe that since η is non-decreasing, then η has lower type 0. The supremum of L is called the Orlicz-Maligranda lower index of η . The expression η is of lower type greater than γ makes sense by saying that γ belongs to $\overset{\circ}{L}$ the interior of L . Similarly, η is of upper type less than γ if $\gamma \in \overset{\circ}{U}$. Let us notice that for a given η , having finite upper type is equivalent to the Orlicz Δ_2 condition $\eta(2t) \leq A\eta(t)$. Continuity moduli satisfying Δ_2 are usually called *growth functions*. Up to equivalence smoothness of growth functions can be assumed. Indeed, in case η has positive lower type we can take $\tilde{\eta}(t) = \int_0^t \frac{\eta(s)}{s} ds$ as a regularization of η . In the general case, pick a real number ϵ less than a lower type of η , then

$$\eta_\epsilon(t) = t^\epsilon \int_0^t \frac{\eta(s)}{s^{1+\epsilon}} ds,$$

is a differentiable function which is equivalent to η .

Since, we are interested in the description of local continuity properties of functions in terms of η and, on the other hand, functions like $\eta_1(t) = \min(t^{1/2}, t^{1/3})$ and $\eta_2(t) = \max(t^{1/2}, t^{1/3})$ have the same types even when they behave in quite different ways near zero, we are lead to the idea of *local types* by restricting t to the range $0 \leq t \leq 1$ in the definition of lower and upper types given before.

Actually, local types can be extended to global types by changing the continuity modulus only for large values of t , in fact if η is a growth function with local lower type α and local upper type β , then $\tilde{\eta}(t) = \eta(t)$ for $t < 1$ and $\tilde{\eta}(t) = \frac{1}{\eta(\frac{1}{t})}$ for $t \geq 1$ has the desired properties.

Given a modulus of continuity η we shall denote by \mathcal{L}_η the vector space of those measurable, real valued, essentially bounded functions defined on the euclidean space \mathbb{R}^n such that $\omega_\infty(t) = \text{ess sup}_{x \in \mathbb{R}^n} |f(x+t) - f(x)| = \|f(\cdot+t) - f(\cdot)\|_\infty$ is bounded by $A\eta(|t|)$ for some constant A and every $t \in \mathbb{R}^n$. The space \mathcal{L}_η becomes a normed space with

$$\|f\|_{\mathcal{L}_\eta} = \|f\|_\infty + \sup_{t \neq 0} \frac{\omega_\infty(t)}{\eta(t)}.$$

If η and $\tilde{\eta}$ are equivalent continuity moduli, then $\mathcal{L}_\eta = \mathcal{L}_{\tilde{\eta}}$ with equivalent norms. An extension of the argument given by E. Stein in [7] allows us to show that \mathcal{L}_η is a space of continuous functions since $\eta(0^+) = 0$. It is easy to show that if η has upper type less than one, and $\tilde{\eta}$ is a growth function such that $L_\eta = L_{\tilde{\eta}}$, then $\eta(t)$ and $\tilde{\eta}(t)$ are equivalent for $t \in [0, 1]$.

The basic characterizations of \mathcal{L}_η in terms of Besov type spaces is contained in the next theorem, which is an extension of Theorem 7 in [7].

Theorem 2.1 *Let η be a growth function of positive lower type α and upper type β less than one. Then a bounded function f belongs to $\mathcal{L}_\eta(\mathbb{R}^n)$ if and only if its Poisson integral $u(x, y) = (P_y * f)(x)$ satisfies an inequality of the type*

$$\left\| \frac{\partial u(x, y)}{\partial y} \right\|_\infty \leq A \frac{\eta(y)}{y}, \tag{2.1}$$

for some constant A and every $y > 0$. Moreover if A_1 is the smallest constant A for which (2.1) holds, then

$$\|f\|_{\eta; \infty} = \|f\|_\infty + A_1 \quad \text{and} \quad \|f\|_{\mathcal{L}_\eta}$$

give equivalent norms.

Proof. Let us first assume that $f \in \mathcal{L}_\eta$. Since the integral on \mathbb{R}^n of $\frac{\partial P_y}{\partial y}$ vanishes, we have that

$$\left\| \frac{\partial u}{\partial y} \right\|_\infty \leq \|f\|_{\mathcal{L}_\eta} \int_{\mathbb{R}^n} \left| \frac{\partial P_y(t)}{\partial y} \right| \eta(|t|) dt.$$

Using polar coordinates for the integral on the right, splitting the radial integral at $r = y$ and finally applying the lower type property of η in the bounded region and the upper type property of η in the unbounded region, we get

$$\begin{aligned} \left\| \frac{\partial u}{\partial y} \right\|_\infty &\leq \|f\|_{\mathcal{L}_\eta} \left[C' |S^{n-1}| y^{-n-1} \int_0^y r^n \frac{\eta(r)}{r} dr + C' |S^{n-1}| \int_y^\infty r^{-n-1} r^{n-1} \eta(r) dr \right] \\ &= \|f\|_{\mathcal{L}_\eta} C' |S^{n-1}| \left[y^{-n-1} \int_0^1 y^n s^n \frac{\eta(ys)}{s} ds + \int_1^\infty \frac{1}{ys} \frac{\eta(ys)}{s} ds \right] \\ &\leq \|f\|_{\mathcal{L}_\eta} C'' |S^{n-1}| \left[y^{-1} \int_0^1 s^{n+\alpha-1} \eta(y) ds + y^{-1} \int_1^\infty \eta(y) s^{\beta-2} ds \right] \\ &\leq C \frac{\eta(y)}{y}. \end{aligned}$$

In order to show that (2.1) implies that $f \in \mathcal{L}_\eta(\mathbb{R}^n)$, let us first state and prove the next lemma.

Lemma 2.2 *Let η be a growth function of upper type β less than one and let f be an essentially bounded measurable function defined on \mathbb{R}^n . Then, (2.1) is equivalent to the existence of a positive constant A' such that the inequalities*

$$\left\| \frac{\partial u}{\partial x_j} \right\|_\infty \leq A' \frac{\eta(y)}{y}, \tag{2.2}$$

hold for every $y > 0$ and every $j = 1, \dots, n$. The smallest constants A and A' for which (2.1) and (2.2) hold are comparable.

Proof. The proof follows the basic lines of that of Lemma 4 in [7]. Let us first show that (2.1) implies (2.2). Using the semigroup property of Poisson integrals and the estimates for the $L^1(\mathbb{R}^n)$ norm of $\frac{\partial P_j}{\partial x_j}$ we obtain $\left\| \frac{\partial^2 u}{\partial y \partial x_j} \right\|_\infty \leq A_1 \frac{\eta(y)}{y^2}$. On the other hand, since $\frac{\partial u(x, y)}{\partial x_j} = - \int_y^\infty \frac{\partial^2}{\partial y' \partial x_j} u(x, y') dy'$, from the upper type property of η we get

$$\begin{aligned} \left\| \frac{\partial u(x, y)}{\partial x_j} \right\|_\infty &\leq A_1 \int_y^\infty \frac{\eta(y')}{y'^2} dy' \\ &= A_1 \frac{1}{y} \int_1^\infty \frac{\eta(sy)}{s^2} ds \leq CA_1 \frac{1}{y} \eta(y) \int_1^\infty s^{\beta-2} ds \leq A_2 \frac{\eta(y)}{y}. \end{aligned}$$

The converse is proved by using that u solves the Laplace equation. □

Let us now finish the proof of Theorem 2.1. Since we are now assuming that $\|\frac{\partial u}{\partial y}\|_\infty \leq A \frac{\eta(y)}{y}$, Lemma 2.2 implies that $\|\frac{\partial u}{\partial x_j}\|_\infty \leq A' \frac{\eta(y)}{y}$. For x and t in \mathbb{R}^n , taking $y = |t|$ we have that $f(x+t) - f(x)$ can be written as $\{u(x+t, y) - u(x, y)\} + \{f(x+t) - u(x+t, y)\} - \{f(x) - u(x, y)\}$. Notice now that the absolute value of the first term in the last expression is bounded by $|t| \sum_{j=1}^n \|\frac{\partial u(x, |t|)}{\partial x_j}\|_\infty \leq nA'\eta(|t|)$. On the other hand the second and the third terms are bounded by $\int_0^{|t|} \|\frac{\partial u}{\partial y}\|_\infty dy \leq A \int_0^{|t|} \frac{\eta(y)}{y} dy$. The conclusion follows from the fact that η is of positive lower type. \square

Let us observe that in the proof of Theorem 2.1 only that (2.1) implies (2.2) was actually used.

Let us point out that if we only have a growth function η with no further conditions on its lower or upper types, we still have the inequality

$$y \left\| \frac{\partial u}{\partial y} \right\|_\infty \leq C \left(\int_0^y \frac{\eta(r)}{r} dr + y \int_y^\infty \frac{\eta(r)}{r^2} dr \right),$$

where the right-hand side is generally not bounded by a constant times $\eta(y)$.

The result of Theorem 2.1 suggests a way to define Besov type spaces for any growth function η . Let $\beta < \infty$ be an upper type for η and take k the smallest integer greater than β , we define $\Lambda(\eta; \infty)$ as the vector space of all essentially bounded functions f whose Poisson integral satisfies the inequality

$$\left\| \frac{\partial^k u(x, y)}{\partial y^k} \right\|_\infty \leq A \frac{\eta(y)}{y^k}, \tag{2.3}$$

for some constant A and every $y > 0$. The space $\Lambda(\eta; \infty)$ becomes normed by $\|f\|_\infty + A_k$, where A_k is the infimum of those constants A for which (2.3) holds. Let us point out that from the rate of decreasing at infinity of the Poisson kernel and its derivatives, inequality (2.3) is true for $y > 1$, for every $f \in L^\infty(\mathbb{R}^n)$. So that (2.3) is actually a local condition in the sense that the only values of η that matter are those $\eta(t)$ with $0 < t \leq 1$. It is not difficult to show that if β is an upper type for η , then the usual Lipschitz space Λ_β (see [7]) is continuously embedded into $\Lambda(\eta; \infty)$.

The precise choice of k as the smallest integer greater than β is irrelevant as the following lemma shows.

Lemma 2.3 *Let η be a growth function of upper type β and let f be an essentially bounded measurable function defined on \mathbb{R}^n . Let m and l be two integers, both greater than β . Then the two conditions*

$$\left\| \frac{\partial^m u(x, y)}{\partial y^m} \right\|_\infty \leq A_m \frac{\eta(y)}{y^m} \quad \text{and} \quad \left\| \frac{\partial^l u(x, y)}{\partial y^l} \right\|_\infty \leq A_l \frac{\eta(y)}{y^l}$$

are equivalent. Moreover, the smallest A_m and A_l holding in the above inequalities are comparable. In particular, we obtain that

$$\|f\|_\infty + \sup_{y>0} \left(\frac{y^m}{\eta(y)} \left\| \frac{\partial^m u(x, y)}{\partial y^m} \right\|_\infty \right)$$

defines in $\Lambda(\eta; \infty)$ a norm equivalent to $\|f\|_{\eta; \infty}$.

In order to prove this lemma, we just mention that it is possible to extend the arguments of Stein applying the techniques used to prove Theorem 2.1 and Lemma 2.2. In the same spirit we only state the main results for these spaces: Zygmund type characterization in terms of second differences and a Sobolev type characterization in terms of derivatives of f . The full details can be found in [3].

Theorem 2.4 *Let η be a growth function of positive lower type α and upper type β less than 2. A function f belongs to $\Lambda(\eta; \infty)$ if and only if f is a measurable essentially bounded function satisfying*

$$\|f(x+t) + f(x-t) - 2f(x)\|_\infty \leq A\eta(|t|).$$

Moreover the number

$$\|f\|_\infty + \sum_{j=1}^n \left\| \frac{\partial f}{\partial x_j} \right\|_{\tilde{\eta}; \infty}$$

defines a norm on $\Lambda(\eta; \infty)$ which is equivalent to $\|\cdot\|_{\eta; \infty}$.

Theorem 2.5 Let η be a growth function of lower type α greater than one and upper type β with $\beta - \alpha < 1$. A function f belongs to $\Lambda(\eta; \infty)$ if and only if $f \in L^\infty(\mathbb{R}^n)$ and $\frac{\partial f}{\partial x_j} \in \Lambda(\tilde{\eta}; \infty)$ for every $j = 1, \dots, n$, with $\tilde{\eta}(t) = \frac{\eta(t)}{t}$. The norms $\|f\|_{\eta; \infty}$ and $\|f\|_\infty + \sum_{j=1}^n \|\frac{\partial f}{\partial x_j}\|_{\tilde{\eta}; \infty}$ are equivalent.

In the study of operators acting on the $\Lambda(\eta; \infty)$ arise kernels that belong to more general Lipschitz classes, defined in terms of L^p modulus of continuity for $p \geq 1$. In analogy with our definition of $\Lambda(\eta; \infty)$ we define the spaces $\Lambda(\eta; p)$, $p \geq 1$; substituting the infinity norm by p norm. The basic properties of the spaces, $\Lambda(\eta; \infty)$, given in Theorem 2.1, (2.4) and (2.5) hold with the obvious modifications for the space $\Lambda(\eta; p)$.

3 Some special classes of growth functions

We introduce in this section some definitions of certain classes of growth functions. To these classes will belong the functions associated to generalized Bessel potentials introduced in the next section. More precisely for functions belonging to one of these classes we will show that their potentials are well defined and they work as isomorphisms between generalized Lipschitz spaces.

Definition 3.1 A non negative function η is said to belong to the \mathcal{J} class if it is of positive lower type and finite upper type. Let us observe that a function $\eta \in \mathcal{J}$ is necessarily quasi-increasing.

In this class we can define a special product $\eta_1 \otimes \eta_2$ in the following way

$$\eta_1 \otimes \eta_2(v) = \int_0^1 \frac{\eta_1(vt)}{t} \frac{\eta_2(v\sqrt{1-t^2})}{1-t^2} dt. \tag{3.1}$$

The reason to introduce this special product will be clear in Section 5 where the composition of generalized Bessel potentials is defined. In what follows we give a basic property which implies that this product es closed in \mathcal{J} .

Proposition 3.2 For η_1 and η_2 in \mathcal{J} , the function $\eta = \eta_1 \otimes \eta_2$ defined by (3.1) belongs to \mathcal{J} and is equivalent to $\eta_1 \eta_2$, the usual pointwise product function of η_1 and η_2 .

Proof. Let $\alpha_1 > 0$ and $\alpha_2 > 0$ be lower types of η_1 and η_2 respectively. Then

$$\eta(v) \leq C \eta_1(v) \eta_2(v) \int_0^1 t^{\alpha_1-1} (\sqrt{1-t^2})^{\alpha_2-2} dt \leq C \eta_1(v) \eta_2(v). \tag{3.2}$$

Similarly, if β_1 and β_2 are the upper types of η_1 and η_2 respectively, we have

$$\eta(v) \geq c \eta_1(v) \eta_2(v) \int_0^1 t^{\beta_1-1} (\sqrt{1-t^2})^{\beta_2-2} dt, \tag{3.3}$$

and the equivalence $\eta_1 \otimes \eta_2 \sim \eta_1 \eta_2$ is proved. This in turn, implies the first part of the statement. □

Definition 3.3 A function η of the \mathcal{J} class belongs to \mathcal{J}_o if it has an upper type less than one.

Definition 3.4 We denote by \mathcal{M}_o the class of functions η that can be written as

$$\eta(s) = \eta(1) e^{\int_1^s \frac{\phi(\tau)}{\tau} d\tau} \tag{3.4}$$

with ϕ satisfying the following conditions

$$0 < c_1 \leq \phi \leq c_2 < 1, \tag{3.5}$$

$$\phi \in \mathcal{C}^1((0, \infty)) \text{ and} \tag{3.6}$$

$$\phi' \text{ is absolutely integrable in a neighborhood of the origin.} \tag{3.7}$$

Next we explore how \mathcal{M}_o is related to \mathcal{J}_o .

Lemma 3.5 *Let η be given by (3.4) above with ϕ satisfying only condition (3.5). Then η belongs to \mathcal{J}_o with upper and lower type constants one. Conversely, given $\eta \in \mathcal{J}_o$ with upper and lower type constants one, there exists an equivalent growth function given by (3.4) with a function ϕ satisfying (3.5).*

Proof. Let η to be as in (3.4) above with ϕ satisfying only condition (3.5). Let us show that η is of positive lower type and of upper type less than one. In fact, for $s < 1$ we have

$$\begin{aligned}\eta(st) &= \eta(1)e^{\int_1^t \frac{\phi(\tau)}{\tau} d\tau + \int_t^{st} \frac{\phi(\tau)}{\tau} d\tau} = \eta(t)e^{\int_1^s \frac{\phi(ut)}{u} du} \\ &= \eta(t)e^{-\int_s^1 \frac{\phi(ut)}{u} du} \leq \eta(t)e^{-c_1(\log 1 - \log s)} = \eta(t)s^{c_1}.\end{aligned}$$

This means that η is of lower type $c_1 > 0$ with constant 1; and, in a similar way it can be proved that η is of upper type $c_2 < 1$ with constant 1. Now, we show that for every $\eta \in \mathcal{J}_o$ with upper and lower type α and β , both with constants one, there exists an equivalent growth function as (3.4) above with ϕ satisfying condition (3.5). For such an η it is easy to check that $\frac{\eta(s)}{s^\beta}$ is quasi-decreasing, but perhaps not a decreasing function. However we can consider another function $\tilde{\eta}$ equivalent to η , with $\frac{\tilde{\eta}(s)}{s^\beta}$ decreasing, by defining

$$\frac{\tilde{\eta}(s)}{s^\beta} = \sup_{t \geq s} \frac{\eta(t)}{t^\beta}.$$

Let us verify that $\tilde{\eta}$ and η are equivalent

$$\tilde{\eta}(s) = s^\beta \sup_{t \geq s} \frac{\eta(t)}{t^\beta} \geq s^\beta \frac{\eta(s)}{s^\beta} = \eta(s).$$

Regarding the opposite inequality,

$$\tilde{\eta}(s) = s^\beta \sup_{t \geq s} \frac{\eta(t)}{t^\beta} \leq s^\beta \sup_{t \geq s} \frac{C\eta(s)}{s^\beta} = C\eta(s),$$

where we have used that $\frac{\eta(s)}{s^\beta}$ is quasi-decreasing. Therefore, without loss of generality we may assume that η is of positive lower type α and of upper type $\beta < 1$ with $\frac{\eta(s)}{s^\beta}$ decreasing. Now the function

$$\eta_a(s) = s^a \int_0^s \frac{\eta(t)}{t^{1+a}} dt,$$

$0 \leq a < \alpha \leq \beta$, is equivalent to η and moreover it satisfies (3.4) and (3.5) with

$$\frac{\phi(\tau)}{\tau} = \frac{(\eta_a)'(\tau)}{\eta_a(\tau)}.$$

In fact,

$$\eta_a(s) \geq s^a \frac{\eta(s)}{s^\beta} \int_0^s \frac{dt}{t^{1-\beta+a}} = \frac{\eta(s)}{\beta - a}.$$

So $\eta(s) \leq \eta_a(s)(\beta - a)$. We get the opposite inequality of the equivalence using the positive lower type of η

$$\eta_a(s) = \int_0^1 \frac{\eta(us) du}{u^{1+a}} \leq \frac{C}{\alpha - a} \eta(s).$$

On the other hand, let us observe that

$$(\eta_a)'(s) = a \frac{\eta_a(s)}{s} + \frac{\eta(s)}{s}$$

thus using the above estimates it follows easily that

$$a \frac{\eta_a(s)}{s} \leq (\eta_a)'(s) \leq \beta \frac{\eta_a(s)}{s}$$

giving (3.5) with $c_1 = a$ and $c_2 = \beta$. □

Next we denote by \mathcal{M} the class of those functions in \mathcal{J} that can be obtained as a \otimes -product of a function in \mathcal{M}_o with a non negative power function $t^\nu, \nu \geq 0$. So

$$\mathcal{M} = \mathcal{P} \otimes \mathcal{M}_o, \tag{3.8}$$

where \mathcal{P} is the set of power functions with non negative exponent. Similarly we denote by \mathcal{N} the functions that can be obtained as the pointwise product of functions in \mathcal{P} by \mathcal{M}_o . So

$$\mathcal{N} = \mathcal{P}\mathcal{M}_o. \tag{3.9}$$

Notice that \mathcal{N} can also be described in exponential terms as in (3.4) with ϕ satisfying (3.6), (3.7) and changing (3.5) by

$$0 < c_1 \leq \phi \leq c_2 < \infty, \quad \text{with} \quad 0 \leq c_2 - c_1 < 1. \tag{3.10}$$

It is easy to see that a result similar to Lemma 3.5 holds for the class \mathcal{N} and those growth functions η of lower type α and upper type β such that $0 < \alpha \leq \beta < \infty, \beta - \alpha < 1$ and type constants one.

The next lemma shows that locally the behavior of a function η belonging to \mathcal{N} is determined by the number $\phi(0) = \lim_{t \rightarrow 0^+} \phi(t)$ which always exists.

Lemma 3.6 *Let ϕ be a function satisfying (3.7) and define η by (3.4). Then $\lim_{t \rightarrow 0^+} \phi(t) = \phi(0)$ exists and for every positive ϵ , there is a positive number $\delta(\epsilon)$ such that $\eta(t)s^{\phi(0)+\epsilon} \leq \eta(st) \leq \eta(t)s^{\phi(0)-\epsilon}$ for every $0 < s < 1$ and $0 < t < \delta(\epsilon)$.*

Proof. For $s < t$ we certainly have that

$$|\phi(t) - \phi(s)| \leq \int_s^t |\phi'(x)| dx, \tag{3.11}$$

then $\lim_{t \rightarrow 0^+} \phi(t)$ exists, call this number $\phi(0)$. Let us take $0 < s < 1$ and $t < \delta(\epsilon)$, where δ is chosen in such a way that, $|\phi(\tau) - \phi(0)| < \epsilon$ for every $0 < \tau < \delta(\epsilon)$. Then

$$\begin{aligned} \eta(st) &= \eta(1)e^{\int_1^{st} \frac{\phi(\tau)}{\tau} d\tau} = \eta(1)e^{[\int_1^t \frac{\phi(\tau)}{\tau} d\tau + \int_t^{st} \frac{\phi(\tau)}{\tau} d\tau]} \\ &\leq \eta(t)e^{-\int_{st}^t \frac{\phi(0)-\epsilon}{\tau} d\tau} = \eta(t)e^{-(\phi(0)-\epsilon)(\log t - \log(st))} = \eta(t)s^{\phi(0)-\epsilon}. \end{aligned}$$

Similarly

$$\eta(st) \geq \eta(t)e^{-(\phi(0)+\epsilon)(\log t - \log(st))} = \eta(t)e^{\phi(0)+\epsilon \log s} = \eta(t)s^{\phi(0)+\epsilon}. \quad \square$$

We would like to point out that condition (3.7) in Lemma 3.6 is essential. In fact it is possible to exhibit an example of a function η given by (3.4) with ϕ satisfying (3.5) and having different upper and lower local indexes.

It is clear that even when conditions (3.4), (3.6), and (3.7) are preserved by pointwise multiplication, this is not the case for (3.5) neither for (3.10). Nevertheless given η_1 and η_2 in \mathcal{N} , changing the functions $\phi_i, i = 1, 2$, outside a small neighborhood of the origin, we obtain $\tilde{\eta}_i, i = 1, 2$, such that the pointwise product $\tilde{\eta}_1 \tilde{\eta}_2$ belongs to \mathcal{N} and it coincides with $\eta_1 \eta_2$ in a neighborhood of the origin.

Let us finish this section with some examples about these classes of functions. Of course power functions $\eta(t) = t^\alpha$ correspond to the constant functions $\phi(\tau) = \alpha$. For $\eta \in \mathcal{N}$ and β positive, $t^\beta \eta(t)$ also belongs to \mathcal{N} . Also $t^{-\alpha} \eta(t)$ belongs to \mathcal{N} provided that α is less than a lower type for η . Typical functions of the Orlicz setting are also examples here

$$\eta_1(t) = t^\alpha \log \frac{1}{t}, \quad \eta_2(t) = t^\alpha \frac{1}{\log \frac{1}{t}}, \quad \eta_3(t) = t^\alpha \log(\log \frac{1}{t}),$$

and, in each case, the corresponding function ϕ_i has the general form

$$\phi_i(t) = \alpha + g_i(t),$$

for g_i a monotone smooth function with $g_i(0^+) = 0$.

4 Generalized Bessel potentials

The Bessel potential J_α of order $\alpha > 0$ is the convolution operator with kernel given by

$$G_\alpha(x) = C \int_0^\infty e^{-\pi \frac{|x|^2}{\delta}} e^{-\frac{\delta}{4\pi}} \delta^{-\frac{n+\alpha}{2}} \frac{d\delta}{\delta}.$$

This operator solves the formal expression $(I - \Delta)^{-\alpha/2}$. It seems natural to introduce a generalization dealing with the formal operators of the type $\varphi[(I - \Delta)^{-1/2}]$ where φ belongs to some class of positive functions containing the power functions. Actually given a function $\eta \in \mathcal{J}$ if we take the kernel

$$G_\eta(x) = C \int_0^\infty e^{-\pi \frac{|x|^2}{\delta}} e^{-\frac{\delta}{4\pi}} \delta^{-\frac{n}{2}} \eta(\sqrt{\delta}) \frac{d\delta}{\delta}, \tag{4.1}$$

the integral defining $G_\eta(x)$ converges and satisfies the following basic properties which are themselves natural generalizations of the classical ones (see [9]).

$$G_\eta \text{ is a non negative radial and integrable function on } \mathbb{R}^n; \tag{4.2}$$

$$(G_\eta)^\wedge(x) = \varphi[(1 + 4\pi^2 |x|^2)^{-1/2}] \quad \text{where} \quad \varphi(u) = C(\eta) \mathcal{L} \left[\frac{\eta(\sqrt{\delta})}{\delta} \right] \left(\frac{1}{4\pi u^2} \right), \tag{4.3}$$

\mathcal{L} denotes the Laplace transform and $C(\eta)$ is a constant depending only on η ;

$$\widehat{G_\eta} \text{ and its multiplicative inverse } (\widehat{G_\eta})^{-1}, \text{ both belong to the space } \mathcal{O} \text{ of all } C^\infty \text{ functions with slowly increasing derivatives.} \tag{4.4}$$

In other words we have that the convolution operator J_η with kernel G_η , that is, $J_\eta f(x) = (G_\eta * f)(x)$, gives the desired generalized Bessel potential induced by η , $J_\eta(f) = \varphi[(I - \Delta)^{-1/2}](f)$ in the sense that as distributions, for any test function ψ we have

$$(J_\eta(f))^\wedge(\psi) = \varphi((1 + 4\pi^2 |x|^2)^{-1/2}) \hat{f}(\psi).$$

Let us now sketch the proof of (4.2), (4.3) and (4.4). Notice first that assuming (4.2) and (4.3), it becomes clear that both $\widehat{G_\eta}$ and $(\widehat{G_\eta})^{-1}$ are C^∞ functions on \mathbb{R}^n . In order to estimate the growth of their derivatives let us introduce the function

$$\Phi(\xi) = \int_0^\infty e^{-\delta \frac{1+4\pi^2|\xi|^2}{4\pi}} \frac{\eta(\sqrt{\delta})}{\delta} d\delta = C(\eta) \varphi[(1 + 4\pi^2 |\xi|^2)^{-1/2}].$$

Since the derivatives of Φ are linear combinations of powers of ξ multiplied by functions of the same kind that Φ , where aside from $\frac{\eta(\sqrt{\delta})}{\delta}$ there appear other powers of δ , it is easy to see that the derivatives of $\Phi(\xi)$ never grow faster than polynomials of ξ . In order to show that the same is true for $(\widehat{G_\eta})^{-1}$ we only need to prove that $\Phi(x)$ is bounded below by a negative power of a polynomial. But this is true since changing variables in the definition of $\Phi(\xi)$, the finiteness of the upper type of η gives

$$\Phi(\xi) \geq \frac{C}{(1 + 4\pi^2 |\xi|)^{-m}}$$

for some constant C and any integer m greater than a half of an upper type for η .

Properties (4.2) and (4.3) will immediately follow from the next lemma.

Lemma 4.1 *The operator which transforms the measurable function h of the positive real variable δ , into the function*

$$\tilde{h}(x) = \int_0^\infty e^{-\pi \frac{|x|^2}{\delta}} e^{-\frac{\delta}{4\pi}} \delta^{-n/2} \frac{h(\sqrt{\delta})}{\delta} d\delta$$

is bounded from $L^1(\mathbb{R}^+, e^{-u^2/4\pi} du/u)$ to $L^1(\mathbb{R}^n)$. In fact

$$\int_{\mathbb{R}^n} |\tilde{h}(x)| dx \leq 2 \int_0^\infty |h(u)| e^{-\frac{u^2}{4\pi}} \frac{du}{u}.$$

Moreover, the Fourier transform of $\tilde{h}(x)$ is given by the Laplace transform of $\frac{h(\sqrt{\delta})}{\delta}$ evaluated at $\frac{1+4\pi|\xi|^2}{4\pi}$, in other words

$$\widehat{\tilde{h}}(\xi) = \mathcal{L} \left[\frac{h(\sqrt{\delta})}{\delta} \right] \left(\frac{1+4\pi^2|\xi|^2}{4\pi} \right).$$

Proof. Since $\int_{\mathbb{R}^n} e^{-\pi\frac{|x|^2}{\delta}} dx = \delta^{n/2}$ for $\delta > 0$, we have from Tonelli's theorem that

$$\begin{aligned} \int_{\mathbb{R}^n} \left\{ \int_0^\infty e^{-\pi\frac{|x|^2}{\delta}} e^{-\frac{\delta}{4\pi}} \delta^{-n/2} \frac{|h(\sqrt{\delta})|}{\delta} d\delta \right\} dx &= \int_0^\infty \delta^{n/2} e^{-\frac{\delta}{4\pi}} \delta^{-n/2} \frac{|h(\sqrt{\delta})|}{\delta} d\delta \\ &= 2 \int_0^\infty e^{-\frac{u^2}{4\pi}} |h(u)| \frac{du}{u}, \end{aligned}$$

from which the first part of the lemma follows readily. In particular $\tilde{h}(x)$ is finite almost everywhere. Notice that since $h \in L^1(\mathbb{R}^+, e^{-\frac{u^2}{4\pi}} \frac{du}{u})$, we have that $\mathcal{L} \left[\frac{|h(\sqrt{\delta})|}{\delta} \right] \left(\frac{1}{4\pi} \right)$ is finite, so that also $\mathcal{L} \left[\frac{|h(\sqrt{\delta})|}{\delta} \right] \left(\frac{1+4\pi^2|\xi|^2}{4\pi} \right)$ is a well defined for every $\xi \in \mathbb{R}^n$. Hence $\int_0^\infty e^{-\delta\frac{1+4\pi^2|\xi|^2}{4\pi}} \frac{h(\sqrt{\delta})}{\delta} d\delta$ is absolutely convergent. Let us compute the Fourier transform of \tilde{h} , using the multiplication Plancherel formula: given a function f in $L^1(\mathbb{R}^n)$ its Fourier transform is the only $L^\infty(\mathbb{R}^n)$ function \hat{f} for which

$$\int_{\mathbb{R}^n} f(x)\hat{\phi}(x) dx = \int_{\mathbb{R}^n} \hat{f}(\xi)\phi(\xi) d\xi,$$

holds for every ϕ in the Schwartz class \mathcal{S} . Since the Fourier transform of $e^{-\pi\delta|x|^2}$ is given by $\delta^{-\frac{n}{2}} e^{-\pi\frac{|\xi|^2}{\delta}}$ we have that

$$\int_{\mathbb{R}^n} e^{-\frac{\delta}{4\pi}} e^{-\pi\delta|x|^2} \hat{\phi}(x) dx = \int_{\mathbb{R}^n} e^{-\frac{\delta}{4\pi}} e^{-\pi\frac{|x|^2}{\delta}} \delta^{-\frac{n}{2}} \phi(x) dx,$$

holds for every positive δ . Let us now multiply by $h(\sqrt{\delta})$ both sides of this equality and integrate from 0 to ∞ with respect to the measure $\frac{d\delta}{\delta}$. After changing the order of integration we get

$$\begin{aligned} \int_{\mathbb{R}^n} \hat{\phi}(x) \left\{ \int_0^\infty e^{-\frac{\delta}{4\pi}(1+4\pi^2|x|^2)} \frac{h(\sqrt{\delta})}{\delta} d\delta \right\} dx \\ = \int_{\mathbb{R}^n} \phi(x) \left\{ \int_0^\infty e^{-\frac{\delta}{4\pi}} e^{-\pi\frac{|x|^2}{\delta}} \delta^{-\frac{n}{2}} \frac{h(\sqrt{\delta})}{\delta} d\delta \right\} dx = \int_{\mathbb{R}^n} \phi(x)\tilde{h}(x) dx, \end{aligned} \tag{4.5}$$

from which, since we are dealing with real functions, we get

$$\widehat{\tilde{h}}(\xi) = \mathcal{L} \left(\frac{h(\sqrt{\delta})}{\delta} \right) \left(\frac{1+4\pi^2|\xi|^2}{4\pi} \right). \quad \square$$

In order to prove (4.2) and (4.3) it only remains to notice that $\eta \in \mathcal{J}$ implies that $\eta \in L^1(\mathbb{R}^+, e^{-\frac{u^2}{4\pi}} \frac{du}{u})$. In fact, the positive lower type of η guarantees the integrability of $\frac{\eta(u)}{u}$ in every neighborhood of the origin, and the finiteness of the upper type of η leads the integrability of $\frac{\eta(u)e^{-\frac{u^2}{4\pi}}}{u}$ for $u \geq 1$. With these properties of G_η we are in position to give the precise definition of the potential.

Definition 4.2 Let η be a function in the class \mathcal{J} . We define on each $L^p(\mathbb{R}^n)$ with $1 \leq p \leq \infty$ the *generalized Bessel potential* as the convolution

$$J_\eta(f) = G_\eta * f \quad (4.6)$$

with $f \in L^p(\mathbb{R}^n)$.

Of course, since $G_\eta \in L^1(\mathbb{R}^n)$ and $\|G_\eta\|_1 = 1$, we have $\|J_\eta(f)\|_p \leq \|f\|_p$, $1 \leq p \leq \infty$.

A basic fact about the properties of G_η that can be described in terms of the generalized Lipschitz classes introduced in Section 2 is contained in the next result.

Theorem 4.3 Let $\eta \in \mathcal{J}$ with upper type $b < 1$. Then $G_\eta \in \Lambda(\eta; 1)$.

Proof. Let us start by observing that under the current hypothesis in η , we have that

$$|G_\eta(x)| \leq C \frac{\eta(|x|)}{|x|^n}, \quad (4.7)$$

$$\left| \frac{\partial}{\partial x_j} G_\eta(x) \right| \leq C \frac{\eta(|x|)}{|x|^{n+1}}, \quad (4.8)$$

for some constant C . Since we already know that $G_\eta \in L^1(\mathbb{R}^n)$, to prove the theorem we only have to show that

$$\int_{\mathbb{R}^n} |G_\eta(x+t) - G_\eta(x)| dx \leq A\eta(|t|),$$

for some constant A . Since

$$\int_{\mathbb{R}^n} |G_\eta(x+t) - G_\eta(x)| dx = \int_{|x| \leq 2|t|} |\cdot| dx + \int_{|x| > 2|t|} |\cdot| dx,$$

we can use standard arguments to estimate each term on the right-hand side. For the first we use (4.7) and the lower type of η . For the second we can apply (4.8) and the upper type condition in η . \square

5 Composition of generalized Bessel potentials

As we pointed out in Section 1, while in the classical case the Bessel potentials define a semigroup in the sense that $\mathcal{J}_\alpha \circ \mathcal{J}_\beta = \mathcal{J}_{\alpha+\beta}$ or $J_{t^\alpha} \circ J_{t^\beta} = J_{t^{\alpha+\beta}}$ the composition of these new operators \mathcal{J}_{η_1} and \mathcal{J}_{η_2} is the potential $\mathcal{J}_{\eta_1 \otimes \eta_2}$ for the special product $\eta_1 \otimes \eta_2$, defined by (3.1) which in fact is equivalent to the usual product $\eta_1 \eta_2$. It suffices to prove that given η_1 and η_2 in \mathcal{J} , $G_{\eta_1} * G_{\eta_2}$ is $G_{\eta_1 \otimes \eta_2}$, with $\widehat{G_{\eta_i}}(x) = \varphi_i[(1 + 4\pi^2|x|^2)^{-1/2}]$, where the relationship between η_i and φ_i is $\mathcal{L}\left[\frac{\eta_i(\sqrt{\delta})}{\delta}\right](s) = \varphi_i\left(\frac{1}{\sqrt{4\pi s}}\right)$, for $i = 1, 2$.

Since the kernels G_η are radial functions, the convolution $G_{\eta_1} * G_{\eta_2}$ is also radial and so is its Fourier's transform; this observation allows us think the $(G_{\eta_1} * G_{\eta_2})^\wedge$ as a function of the variable $|x|$. Applying the convolution theorem for Laplace's transform we get that for $s = \frac{1+4\pi^2|x|^2}{4\pi}$

$$\begin{aligned} (G_{\eta_1} * G_{\eta_2})^\wedge(|x|) &= \varphi_1\left(\frac{1}{\sqrt{4\pi s}}\right) \varphi_2\left(\frac{1}{\sqrt{4\pi s}}\right) \\ &= C(\eta_1) \mathcal{L}\left[\frac{\eta_1(\sqrt{\delta})}{\delta}\right](s) C(\eta_2) \mathcal{L}\left[\frac{\eta_2(\sqrt{\delta})}{\delta}\right](s) \\ &= C(\eta_1, \eta_2) \mathcal{L}\left[\int_0^\delta \frac{\eta_1(\sqrt{\xi})}{\xi} \frac{\eta_2(\sqrt{\delta-\xi})}{\delta-\xi} d\xi\right](s) \\ &= C(\eta_1, \eta_2) \mathcal{L}\left[\frac{1}{\delta} \int_0^\delta \frac{\eta_1(\sqrt{\xi})}{\xi} \frac{\eta_2(\sqrt{\delta-\xi})}{\delta-\xi} \frac{\delta}{2} d\xi\right](s) = \end{aligned} \quad (5.1)$$

$$= C(\eta_1, \eta_2) \mathcal{L} \left[\frac{1}{\delta} \int_0^1 \frac{\eta_1(\sqrt{\delta}t)}{t} \frac{\eta_2(\sqrt{\delta}\sqrt{1-t^2})}{1-t^2} dt \right] (s) = C(\eta_1, \eta_2) \mathcal{L} \left[\frac{(\eta_1 \otimes \eta_2)(\sqrt{\delta})}{\delta} \right] (s),$$

where for the last equality we use an appropriate change of variables in the definition of $\eta_1 \otimes \eta_2$ given in (3.1). So that $G_{\eta_1} * G_{\eta_2} = G_{\eta_1 \otimes \eta_2}$. This identity is the reason for introducing the special \otimes -product given in Section 3.

Proposition 5.1 *Given a function η_1 of positive lower type and upper type less than one, and a positive number β , the generalized Bessel kernel G_η associated to $\eta = t^\beta \otimes \eta_1$ belongs to the class $\Lambda(\eta; 1)$.*

Proof. Since $\eta = t^\beta \otimes \eta_1$, then $\mathcal{J}_\eta = \mathcal{J}_\beta \circ \mathcal{J}_{\eta_1}$ where \mathcal{J}_β is the classical Bessel potential of order β . In other words $G_\eta = G_\beta * G_{\eta_1}$. So that its Poisson extension can be written as $G_\eta(\cdot, y) = G_\beta(\cdot, y_1) * G_{\eta_1}(\cdot, y_2)$ where $y_1 + y_2 = y$. Let us denote by b the finite upper type of η and observe that $b > \beta$. Let l be an integer greater than $b + 1$, since $l - 1 > \beta$, taking $l - 1$ derivatives of $G_\eta(\cdot, y_1 + y_2)$ with respect to y_1 and one derivative with respect to y_2 , by doing $y_1 = y_2 = \frac{y}{2}$ we get that

$$\frac{\partial^l}{\partial y^l} G_\eta(x, y) = \frac{\partial^{l-1}}{\partial y_1^{l-1}} G_\beta(x, y/2) * \frac{\partial}{\partial y_2} G_{\eta_1}(x, y/2).$$

Now applying Young’s inequality and the estimate about the Poisson integral of the kernel G_β and (4.8) we have

$$\left\| \frac{\partial^l}{\partial y^l} G_\eta(x, y) \right\|_1 \leq A \frac{\eta(y)}{y^l}. \tag{5.2}$$

□

6 The action of J_η on $\Lambda(\Psi, p)$

Since, from (4.2), the kernel G_η belongs to $L^1(\mathbb{R}^n)$, we certainly have that J_η preserves generalized Lipschitz spaces, but it is important to observe that the smoothness of G_η allows us to expect that J_η is regularity improving. In fact we have the following result

Theorem 6.1 *Let $\Psi \in \mathcal{J}$ and $\eta \in \mathcal{J}_o$ be two given growth functions. Then J_η is a one to one continuous operator from $\Lambda(\Psi, p)$ into $\Lambda(\eta\Psi, p)$ for every $1 \leq p \leq \infty$.*

Proof. That the mapping J_η is one to one can be proved exactly in the same way than for the classical Bessel potentials (see [7]). To prove that the image of $\Lambda(\Psi, p)$ under J_η lies in $\Lambda(\eta\Psi, p)$ and that the mapping is continuous, let us take $f \in \Lambda(\Psi, p)$ and argue as follows. Let u denote the Poisson integral of f , and U be the Poisson integral of $J_\eta(f) = G_\eta * f$. Then $u = P_y * f$ and $U = P_y * G_\eta * f$. Thus $U(x, y) = (G_\eta(\cdot, y) * f)(x)$, where $G_\eta(x, y)$ is the Poisson integral of $G_\eta(x)$. Now, from Theorem 4.3 and Lemma 2.3 we see that for every $m \geq 1$

$$\left\| \frac{\partial^m G_\eta(x, y)}{\partial y^m} \right\|_1 \leq A \frac{\eta(y)}{y^m}. \tag{6.1}$$

For $y = y_1 + y_2$ we see that

$$U(x, y) = U(x, y_1 + y_2) = (P_{y_1} * G_\eta) * (P_{y_2} * f)(x) = (G_\eta(\cdot, y_1) * u(\cdot, y_2))(x).$$

Taking l derivatives with respect to y_1 and k derivatives with respect to y_2 , where $l \geq 1$ and k is the smallest integer greater than the upper type of Ψ , we get

$$\frac{\partial^{k+l} U(x, y)}{\partial y^{k+l}} = \frac{\partial^l}{\partial y_1^l} G_\eta(x, y_1) * \frac{\partial^k}{\partial y_2^k} u(x, y_2), \quad y = y_1 + y_2.$$

Now, taking $y_1 = y_2 = y/2$ since $f \in \Lambda(\Psi, p)$, it follows from (6.1) that

$$\begin{aligned} \left\| \frac{\partial^{k+l} U(x, y)}{\partial y^{k+l}} \right\|_p &\leq \left\| \frac{\partial^l}{\partial y_1^l} G_\eta(x, y_1) \right\|_1 \left\| \frac{\partial^k}{\partial y_2^k} u(x, y_2) \right\|_p \\ &\leq A_1 \frac{\eta(y/2)}{y^l} \|f\|_{\Psi; p} \frac{\Psi(y/2)}{y^k} \\ &\leq C \|f\|_{\Psi; p} \frac{(\eta\Psi)(y)}{y^{l+k}}. \end{aligned} \tag{6.2}$$

Observe that $k + l$ is an integer greater than the upper type of Ψ_η . To establish the continuity it only remains to show that $J_\eta f$ belongs to $L^p(\mathbb{R}^n)$. This fact is an immediate consequence of Young's inequality for $J_\eta f = G_\eta * f$, since G_η belongs to $L^1(\mathbb{R}^n)$ and $f \in L^p(\mathbb{R}^n)$. \square

Corollary 6.2 For $\beta > 0$ and $\Psi \in \mathcal{J}$, the Bessel Potentials J_β are one to one continuous operators from $\Lambda(\Psi; p)$ into $\Lambda(\Psi t^\beta; p)$ for every $1 \leq p \leq \infty$.

Proof. If $0 < \beta < 1$ the proof is direct because it is a particular case of η in the theorem above. For any β we know that the kernels G_β of Bessel Potentials satisfies one inequality like (6.1) (see [7]), with $\eta(y) = y^\beta$. In other words, for m any integer greater than β , we have

$$\left\| \frac{\partial^m G_\eta}{\partial y^m}(x, y) \right\|_1 \leq A \frac{y^\beta}{y^m}. \tag{6.3}$$

Then the proof follows with the same argument used in the proof of the above theorem. \square

Corollary 6.3 If $\eta = \eta_1 \otimes t^\beta$ where $\eta_1 \in \mathcal{J}_o$, then J_η is a one to one continuous operator from $\Lambda(\Psi; p)$ into $\Lambda(\eta\Psi; p)$.

Proof. Since $J_\eta = J_{\eta_1} \circ J_{t^\beta}$ and both of them are one to one continuous operators, so is J_η . \square

7 J_η as an isomorphism between generalized Lipschitz spaces

This section contains the main result of this paper.

Theorem 7.1 Given two functions Φ and Ψ in the class \mathcal{J} such that Ψ/Φ is locally equivalent to a function η belonging to \mathcal{M} , then J_η is a Banach space isomorphism between $\Lambda(\Phi; p)$ and $\Lambda(\Psi; p)$. In particular for any $\eta \in \mathcal{M}$, J_η is an isomorphism from $\Lambda(\Phi; p)$ onto $\Lambda(\eta\Phi; p)$.

For the classical Lipschitz spaces and Bessel potentials we have that the image of $\Lambda(\alpha; p)$ under J_β is all of $\Lambda(\alpha + \beta; p)$ (see [7], Chap. V). This result has a somehow trivial extension to our more general setting: $J_\beta : \Lambda(\Psi; p) \rightarrow \Lambda(t^\beta\Psi; p)$ onto. So a special case of Theorem 7.1 can be proved, following the classical approach of [7].

Lemma 7.2 Let $\beta > 0$, $1 \leq p \leq \infty$ and $\Psi \in \mathcal{J}_o$. Then J_β maps $\Lambda(\Psi; p)$ isomorphically onto $\Lambda(\Psi t^\beta; p)$.

Proof. As in the usual case it would be enough to prove that J_2 defines an operator from $\Lambda(\Psi; p)$ onto $\Lambda(t^2\Psi; p)$. In fact we only have to observe that for a given $f \in \Lambda(t^2\Psi; p)$ we have that $f \in \Lambda(\Psi; p)$ and $\Delta f \in \Lambda(\Psi; p)$; therefore arguing as in [7] p. 150, we get the result. \square

The proof of Theorem 7.1 amounts to improve the arguments in [7] in order to get that an appropriate perturbation J_ν of J_2 applies $\Lambda(\Psi; p)$ onto $\Lambda(t^2\Psi; p)$ for $\Psi \in \mathcal{J}$. Precisely we shall prove the following

Proposition 7.3 Let $H(t)$ be a function in the class $\mathcal{C}^1[(0, \infty)]$ between two positive constants such that $H'(t)$ is absolutely integrable in a neighborhood of the origin, then $\nu(t) = H(t)t^2 \in \mathcal{J}$ and J_ν is an isomorphism of $\Lambda(\Psi; p)$ onto $\Lambda(\Psi t^2; p)$; moreover

$$J_\nu = J_2 \circ T,$$

where T is a convolution operator giving an isometric mapping in $\Lambda(\Psi; p)$, with multiplier

$$m(x) = K + C\widehat{h}(x),$$

where K and C are constants and \tilde{h} is the integrable function from Lemma 4.1 associated to the function $h(t) = tH'(t)$.

First we gather two lemmas needed in the proof of Proposition 7.3.

Lemma 7.4 a) A function $K \in L^1(\mathbb{R}^n)$ defines a convolution continuous operator on $\Lambda(\Psi; p)$, namely

$$\|K * f\|_{\Psi; p} \leq \|K\|_1 \|f\|_{\Psi; p}.$$

b) If also \widehat{K} is C^∞ we have

$$(K * f)^\wedge = \widehat{K} \cdot \widehat{f}$$

in the distributional sense.

Proof. The proof of (a) follows the same lines of that of Theorem 6.1. The proof of part (b) is based on the fact that since $K * f$ belongs to $L^p(\mathbb{R}^n)$, the involved equalities can be justified through Fubini's theorem. \square

We shall need the following trivial modification of a Wiener's result (see [7], p. 134).

Wiener's Lemma Let ϕ_1 be an integrable function and λ a number with $\lambda \neq 0$ such that $\widehat{\phi_1} + \lambda$ is nowhere zero, then there exists $\phi_2 \in L^1(\mathbb{R}^n)$ so that $(\widehat{\phi_1} + \lambda)^{-1} = \widehat{\phi_2}(x) + 1/\lambda$.

Proof of Proposition 7.3. Let us notice first that since Laplace transform applies equivalent positive functions into equivalent positive function we see at once that

$$\widehat{G_\nu} = m \widehat{G_2},$$

where m is a C^∞ function bounded above and below. Let us now get an explicit formula for $m(x)$,

$$\begin{aligned} m(x) &= \left(\frac{\widehat{G_\nu}}{\widehat{G_2}} \right)(x) = C \frac{\mathcal{L}[H(\sqrt{\delta})] \left(\frac{1+4\pi^2|x|^2}{4\pi} \right)}{(1+4\pi^2|x|^2)^{-1}} \\ &= C(1+4\pi^2|x|^2) \int_0^\infty e^{-\left(\frac{1+4\pi^2|x|^2}{4\pi}\right)\delta} H(\sqrt{\delta}) d\delta \\ &= -C \int_0^\infty \left[-\frac{(1+4\pi^2|x|^2)}{4\pi} e^{-\left(\frac{1+4\pi^2|x|^2}{4\pi}\right)\delta} \right] H(\sqrt{\delta}) d\delta \\ &= -C \int_0^\infty \frac{d}{d\delta} \left[e^{-\left(\frac{1+4\pi^2|x|^2}{4\pi}\right)\delta} \right] H(\sqrt{\delta}) d\delta. \end{aligned}$$

The existence of $\lim H(t)$ for $t \rightarrow 0^+$ is a consequence of the integrability of H' in a neighborhood of the origin. Integrating by parts and setting $h(t) = tH'(t)$ we obtain

$$m(x) = C \left[\kappa + \int_0^\infty e^{-\left(\frac{1+4\pi^2|x|^2}{4\pi}\right)\delta} h(\sqrt{\delta}) \frac{d\delta}{\delta} \right] = C \left[\kappa + \mathcal{L} \left(\frac{h(\sqrt{\delta})}{\delta} \right) \left(\frac{1+4\pi^2|\xi|^2}{4\pi} \right) \right],$$

where $\kappa \neq 0$ is the limit of H in the origin. Since $h \in L^1(\mathbb{R}^+, e^{-\frac{u^2}{4\pi}} \frac{du}{u})$, by Lemma 4.1 we can write

$$m(x) = C \left[\kappa + \widehat{h}(x) \right]$$

with $\tilde{h} \in L^1(\mathbb{R}^n)$. Also $\widehat{\tilde{h}} \in C^\infty$ and therefore, by Lemma 7.4 b), the following equality in the sense of \mathcal{S}' is valid for all $f \in \Lambda(\Psi, p)$,

$$m\widehat{f} = c_1\widehat{f} + c_2\widehat{\tilde{h}}\widehat{f} = [(c_1\delta + c_2\tilde{h}) * f]^\wedge.$$

It follows that $J_\nu = J_2 \circ T$ with

$$Tf = (c_1\delta + c_2\tilde{h}) * f.$$

From Lemma 7.4 a), we can assert that T is a continuous operator in $\Lambda(\Psi; p)$. Since m does not vanish in a neighborhood of the origin, we see that T is one to one. Moreover, by Wiener's Lemma we get

$$\frac{1}{m} = c + \widehat{K}$$

with $K \in L^1$. Therefore we get that T^{-1} , the inverse operator of T , has the same structure of T itself: T^{-1} is the convolution with $c\delta + K$ and K belongs to $L^1(\mathbb{R}^n)$. This fact proves that T^{-1} is continuous on $\Lambda(\Psi; p)$. Then we have shown that T is an isomorphism in $\Lambda(\Psi; p)$ and thus J_η is an isomorphism between $\Lambda(\Psi; p)$ and $\Lambda(t^2\Psi; p)$ by invoking Lemma 7.2 for $\beta = 2$. \square

Proposition 7.5 *Let Ψ be a function in the class \mathcal{J} and let η be a function in the class \mathcal{M}_o , then the operator $J_{\frac{t}{\eta} \otimes t \otimes \eta}$ is an isomorphism between $\Lambda(\Psi; p)$ and $\Lambda(t^2\Psi; p)$.*

Proof. Let us first rewrite the function $\frac{t}{\eta} \otimes t \otimes \eta$,

$$\begin{aligned} \left(\frac{t}{\eta} \otimes t \otimes \eta\right)(t) &= 2t^2 \int_0^t \frac{\left(\frac{t}{\eta} \otimes t\right)(u)}{u} \frac{\eta(\sqrt{t^2 - u^2})}{t^2 - u^2} du \\ &= 4t^2 \int_0^1 v\eta(t\sqrt{1 - v^2}) \int_0^1 \frac{\sqrt{1 - u^2}}{\eta(tv\sqrt{1 - u^2})} \frac{du}{1 - u^2} \frac{dv}{1 - v^2} \\ &= 4t^2 \int_0^1 \int_0^1 \frac{\eta(tv)}{\eta(t\sqrt{1 - v^2}\sqrt{1 - u^2})} \frac{du}{\sqrt{1 - u^2}} \frac{dv}{\sqrt{1 - v^2}}. \end{aligned}$$

Now, changing the variables (u, v) to (φ, θ) with $u = \sin \varphi$ and $v = \sin \theta$, we have

$$\left(\frac{t}{\eta} \otimes t \otimes \eta\right)(t) = 4t^2 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\eta(t \sin \theta)}{\eta(t \cos \theta \cos \varphi)} d\varphi d\theta.$$

Then we can write

$$\left(\frac{t}{\eta} \otimes t \otimes \eta\right)(t) = 4t^2 H(t),$$

where $H(t)$ is defined by

$$H(t) = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\eta(t \sin \theta)}{\eta(t \cos \theta \cos \varphi)} d\varphi d\theta.$$

Then we check that this function H satisfies the conditions of Proposition 7.3. In order to show that H belongs to $C^1(0, \infty)$ it would be enough to prove that the derivative with respect to t of $\frac{\eta(t \sin \theta)}{\eta(t \cos \theta \cos \varphi)}$ is absolutely integrable in φ and θ . This derivative is given by

$$\frac{\sin \theta \eta'(t \sin \theta) \eta(t \cos \theta \cos \varphi) - \cos \theta \cos \varphi \eta(t \sin \theta) \eta'(t \cos \theta \cos \varphi)}{\eta^2(t \cos \theta \cos \varphi)}.$$

We can see that each term above is bounded by an integrable function, we shall only work the first one, because the second is similar. The first term is given by

$$\frac{\sin \theta \eta'(t \sin \theta)}{\eta(t \cos \theta \cos \varphi)}.$$

Since η is in the class \mathcal{M}_o , η' is a function of lower type $\alpha - 1$, if α is a lower type for η . As a consequence the above expression is bounded by

$$\frac{C \sin^\alpha \theta \eta'(t)}{\cos^\beta \theta \cos^\beta \varphi \eta(t)},$$

where we also used that η is a function of upper type β . It is easy to see that the integral with respect to θ of $\sin^\alpha \theta \cos^{-\beta} \theta$ is less than or equal to

$$2^{\beta/2} \int_0^{\frac{\pi}{4}} \sin^\alpha \theta \, d\theta + \int_0^{\frac{\sqrt{2}}{2}} u^{-\beta} \, du,$$

which is finite because $\alpha > 0$ and $\beta < 1$. Then

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\sin^\alpha \theta}{\cos^\beta \theta \cos^\beta \varphi} \, d\varphi \, d\theta \leq C \left\{ \frac{2}{\sqrt{3}} \int_0^{1/2} \frac{du}{u^\beta} + 2^\beta \int_{1/2}^1 \frac{du}{\sqrt{1-u^2}} \right\}.$$

Now we check that H' is absolutely integrable in a neighborhood of the origin. For the sake of simplicity we substitute in the expression of H' , $\frac{t\eta'(t)}{\eta(t)}$ by $\phi(t)$ of (3.4), then we get

$$\int_0^\epsilon |H'(t)| \, dt = \int_0^\epsilon \left| \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\eta(t \sin \theta)}{\eta(t \cos \varphi \cos \theta)} \frac{\phi(t \sin \theta) - \phi(t \cos \varphi \cos \theta)}{t} \, d\theta \, d\varphi \right| \, dt.$$

Since η is positive we can apply Tonelli's theorem to obtain that the above expression is less than or equal to

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^\epsilon \frac{\eta(t \sin \theta)}{\eta(t \cos \varphi \cos \theta)} \left| \frac{\phi(t \sin \theta) - \phi(t \cos \varphi \cos \theta)}{t} \right| \, dt \, d\theta \, d\varphi.$$

Notice now that for every (θ, φ) in the integration domain and every fixed $t \in (0, \epsilon)$, we have $\frac{\eta(t \sin \theta)}{\eta(t \cos \varphi \cos \theta)} \leq C \left(1 + \left(\frac{\sin \theta}{\cos \varphi \cos \theta} \right)^\beta \right)$, since for $\sin \theta \leq \cos \varphi \cos \theta$ is $\eta(t \sin \theta) \leq \eta(t \cos \varphi \cos \theta)$ and for $\sin \theta > \cos \varphi \cos \theta$, from the upper type property of η , we have $\frac{\eta(t \sin \theta)}{\eta(t \cos \varphi \cos \theta)} \leq C \left(\frac{\sin \theta}{\cos \varphi \cos \theta} \right)^\beta$. So that, interchanging orders of integration

$$\begin{aligned} & \int_0^\epsilon |H'(t)| \, dt \\ & \leq C \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^\epsilon \left[1 + \left(\frac{\sin \theta}{\cos \varphi \cos \theta} \right)^\beta \right] \left| \frac{1}{t} \int_{t \cos \varphi \cos \theta}^{t \sin \theta} \phi'(s) \, ds \right| \, dt \, d\theta \, d\varphi \\ & = C \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \left[1 + \left(\frac{\sin \theta}{\cos \varphi \cos \theta} \right)^\beta \right] \left(\int_0^\epsilon \left| \frac{1}{t} \int_{t \cos \varphi \cos \theta}^{t \sin \theta} \phi'(s) \, ds \right| \, dt \right) \, d\theta \, d\varphi \\ & \leq C \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \left[1 + \left(\frac{\sin \theta}{\cos \varphi \cos \theta} \right)^\beta \right] \left(\int_0^{(\sin \theta \wedge \cos \varphi \cos \theta) \epsilon} |\phi'(s)| \int_{(\sin \theta \vee \cos \varphi \cos \theta)^{-1} s}^{(\sin \theta \wedge \cos \varphi \cos \theta)^{-1} s} \frac{dt}{t} \, ds \right. \\ & \quad \left. + \int_{(\sin \theta \wedge \cos \varphi \cos \theta) \epsilon}^{(\sin \theta \vee \cos \varphi \cos \theta) \epsilon} |\phi'(s)| \int_{(\sin \theta \vee \cos \varphi \cos \theta)^{-1} s}^\epsilon \frac{dt}{t} \, ds \right) \, d\theta \, d\varphi \\ & \leq C \left(\int_0^\epsilon |\phi'(s)| \, ds \right) \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \left[1 + \left(\frac{\sin \theta}{\cos \varphi \cos \theta} \right)^\beta \right] \log \frac{\cos \theta \cos \varphi \vee \sin \theta}{\cos \theta \cos \varphi \wedge \sin \theta} \, d\theta \, d\varphi, \end{aligned}$$

which, from (3.7) and the fact that $\beta < 1$, finishes the proof of the proposition. □

Proof of Theorem 7.1. Since $\eta \in \mathcal{M}$ can be written as $\eta_1 \otimes t^\beta$, with $\eta_1 \in \mathcal{M}_o$, then J_η is the composition $J_{\eta_1} \circ J_\beta$. By Theorem 7.2 we know that J_β is a isomorphism between $\Lambda(\Phi; p)$ and $\Lambda(t^\beta \Phi; p)$. If we assume that J_{η_1} is an isomorphism between $\Lambda(t^\beta \Phi; p)$ and $\Lambda(t^\beta \Phi \eta_1; p)$ then we are done, because being the function $\Phi t^\beta \eta_1$ locally equivalent to the function Ψ we have that the spaces $\Lambda(t^\beta \Phi \eta_1; p)$ and $\Lambda(\Psi; p)$ coincide. Now we must show the validity the assumption above. In fact, invoking Theorem 6.1 we know that J_{η_1} is one-one and applies continuously $\Lambda(t^\beta \Phi; p)$ into $\Lambda(t^\beta \Phi \eta_1; p)$. Let us finally show that J_{η_1} is onto. Let us consider the function $\eta_2 = \frac{t}{\eta_1} \otimes t \otimes \eta_1$. Since $\eta_1 \in \mathcal{M}_o$, from Proposition 7.5 we see that J_{η_2} is an isomorphism between $\Lambda(t^\beta \Phi; p)$ and $\Lambda(t^{\beta+2} \Phi; p)$. On the other hand writting $J_{\eta_2} = J_{\frac{t}{\eta_1} \otimes t} \circ J_{\eta_1}$ with $J_{\frac{t}{\eta_1} \otimes t}$ one to one, we see that J_{η_1} must be onto as an operator from $\Lambda(t^\beta \Phi; p)$ to $\Lambda(t^\beta \Phi \eta_1; p)$. Finally the fact that J_{η_1} is an isomorphism follows from the open mapping theorem. □

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