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WEAK TYPE INEQUALITY FOR A FAMILY OF SINGULAR INTEGRAL OPERATORS RELATED WITH THE GAUSSIAN MEASURE

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Abstract In this paper we study a family of singular integral operators that generalizes the higher order Gaussian Riesz Transforms and find the right weight w to make them continuous from $L^1(wd\gamma)$ into $L^{1,\infty}(d\gamma)$, being $d\gamma(x) = e^{-|x|^2}dx$. Some boundedness properties of these operators had already been derived by W. Urbina in [10] and S. Pérez in [8].

Key words Ornstein-Uhlenbeck operator functional calculus Gaussian measure

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1 Introduction

We start by introducing a family of operators that we are going to deal with.
Let $d\gamma(x) = e^{-|x|^2}dx$ be the Gaussian measure and $F \in C^1(\mathbf{R}^n)$ such that

i) $\int_{\mathbf{R}^n} F(x)\, d\gamma(x) = 0$,
ii) $\forall \epsilon : 0 < \epsilon < 1$ there exists $C_\epsilon > 0$ such that $|F(x)| \leq C_\epsilon e^{\epsilon|x|^2}$ and $|\nabla F(x)| \leq C_\epsilon e^{\epsilon|x|^2}$.

Let us remark that property ii) provides a function ψ satisfying the property

iii) $|F(x)| \leq \psi(|x|)$ for some continuous function $\psi : [0, +\infty) \to [0, +\infty)$ for which there exists a $\delta > 0$ with $1-2/n < \delta < 1$, such that $\psi(t)e^{-(1-\delta)t^2}$ is a non-increasing function for all $t \geq 0$.

Indeed, for $0 < \epsilon < 2/n$ we set $\psi(t) = C_\epsilon e^{\epsilon t^2}$ and $\delta = 1 - \epsilon$. In what follows we denote by ψ any function satisfying property iii). As we shall see the smaller the function ψ is taken the better is the result obtained in Theorem 2. For instance, for F equals to any Hermite polynomial of degree k we might take $\psi(t) \simeq 1 + t^k$.
Remark: The hypothesis on the monotonicity of $\psi(t)e^{-(1-\delta)t^2}$ can be relaxed by assuming such a monotonicity just for $t \geq N$ for some positive constant N. Indeed, if such a function ψ exists, then by defining

$$\phi(t) = \begin{cases} \max_{0 \leq s \leq N} \psi(s) & \text{if } \ 0 \leq t \leq N \\ \\ \frac{\max_{0 \leq s \leq N} \psi(s)}{\psi(N)} \psi(t) & \text{if } \qquad t \geq N \end{cases}$$

this turns out to have all the properties described in (iii).

Given a real number $m > 0$ and F as above, we define

$$T_{F,m} f(x) = p.v. \int_{\mathbf{R}^n} K_{F,m}(x,y) f(y)\, dy$$

with

$$\begin{aligned} K_{F,m}(x,y) &= \int_0^1 r^{m-1} \left(\frac{-\log r}{1-r^2} \right)^{(m-2)/2} F\left(\frac{y - rx}{\sqrt{1-r^2}} \right) \frac{e^{-|y-rx|^2/(1-r^2)}}{(1-r^2)^{n/2+1}}\, dr \\ &= \frac{1}{2} \int_0^1 (1-t)^{(m-2)/2} \left(\frac{-\log \sqrt{1-t}}{t} \right)^{(m-2)/2} F\left(\frac{y - \sqrt{1-t}x}{\sqrt{t}} \right) \\ &\qquad \frac{e^{-u(t)}}{t^{n/2+1}}\, dt, \end{aligned} \tag{1}$$

where this last equation was obtained by the change of variables $t = 1 - r^2$, and

$$u(t) = \frac{|y - \sqrt{1-t}x|^2}{t}.$$

This operator was firstly introduced by W. Urbina in [10] and later taken up by S. Pérez in [8] where she proved under the hypotheses (i) and (ii) on F that it is strong type (p,p), $1 < p < \infty$ and its "local part" is weak type (1,1). It is a generalization of the Gaussian Riesz transforms. In fact, when $F(x) = H_\alpha(x)$ the n-dimensional Hermite polynomial of degree $|\alpha|$ and $m = |\alpha|$, the operator $T_{F,m}$ is the Gaussian Riesz transform of order m. It is known that the weak type $(1,1)$ for these operators holds true if and only if $m \leq 2$ (see [6], [1], [2], [4], and [9]).
A rather intricate proof of the weak type $(1,1)$ for the Gaussian Riesz Transforms of order $m = 2$ and $n > 1$ is given in [9]. There is another proof of this result in [4] but it contains a mistake. An interesting consequence of the

results in this paper is a different and simpler proof of the above mentioned weak type inequality.
Our goal in this paper is to answer the question: what are the precise conditions needed on F and on m to guarantee the weak type $(1, 1)$ of the associated singular integral operator $T_{F,m}$?
We present here results concerning with this question in the two sections below which will be called *the negative result* and *the positive result*.
On section 2, Theorem 1 roughly says that if the function $\psi(t)$ controlling F increases at infinity more than t^2, then the operator $T_{F,m}$ fails to be weak type $(1, 1)$.
On section 3, in order to get sufficient conditions on F for the weak type $(1, 1)$ of $T_{F,m}$ we solve a more general problem. In fact, under the hypotheses (i) – (iii) on F we find precise weights w, depending on the function ψ, in order to ensure that $T_{F,m}$ is bounded from $L^1(wd\gamma)$ into $L^{1,\infty}(d\gamma)$.

2 The negative result

The following theorem is a generalization of what is already known about the behaviour on $L^1(d\gamma)$ of the higher order Gaussian Riesz transforms.

Theorem 1 *Let* $\Omega_t = \{z \in \mathbf{R}^n : \min_{1\le i\le n} |z_i| \ge t\}$ *and* $\beta(t) = \dfrac{\inf_{\Omega_t} F(z)}{t^2}$, *if* $\limsup_{t\to\infty} \beta(t) = +\infty$, *then the operator* $T_{F,m}$ *is not of weak type (1,1) with respect to the Gaussian measure.*

Proof To see that $T_{F,m}f$ need not satisfy the weak type $(1, 1)$ inequality, we refer to [4] where it is shown that the higher order Riesz transforms need not be weak type $(1, 1)$ with respect to γ if their order is greater than 2. There they take $y \in \mathbf{R}^n$ such that $|y|$ is large and $y_i \ge c|y|, \quad i = 1, \dots, n$, and define $J = \left\{\xi \frac{y}{|y|} + v : \ \frac{1}{2}|y| < \xi < \frac{3}{4}|y|, v \perp y, \ |v| < 1\right\}$. It follows that for $x \in J$ there is a $c > 0$ so that $\frac{y_i - rx_i}{\sqrt{1-r^2}} \ge \frac{c|y|}{\sqrt{1-r^2}} \ge c|y|, \quad i = 1, \dots, n$, and therefore

$$F\left(\frac{y - rx}{\sqrt{1 - r^2}}\right) \ge c|y|^2\beta(c|y|).$$

Thus, for $x \in J$

$$\begin{aligned}
K_{F,m}(x, y) &\ge c\ |y|^2\beta(c|y|) \int_{1/4}^{3/4} \frac{e^{-|rx-y|^2/(1-r^2)}}{(1 - r^2)^{n/2+1}}\ dr \\
&\ge c\ |y|^2\beta(c|y|) \int_{1/4}^{3/4} e^{\xi^2-|y|^2} e^{-((\xi-r|y|)^2+r^2|v|^2)/(1-r^2)}\ dr \\
&\ge c\ |y|^2\beta(c|y|)e^{\xi^2-|y|^2} \int_{1/4}^{3/4} e^{-c(\xi-r|y|)^2}\ dr \\
&\ge c\ |y|\beta(c|y|)e^{\xi^2-|y|^2}.
\end{aligned}$$

Now, if we take $f = \delta_y e^{|y|^2}$ (by δ_y we mean the delta measure at the point y) we get for $x \in J$

$$\begin{aligned} T_{F,m}(\delta_y e^{|y|^2})(x) &\geq |y|\beta(c|y|)e^{\xi^2} \\ &\geq c\ |y|\beta(c|y|)e^{(|y|/2)^2}. \end{aligned}$$

Let us assume that $T_{F,m}$ is weak type $(1,1)$ with respect to γ. Then

$$\begin{aligned} \gamma(J) &\leq \gamma\{x \in \mathbf{R}^n : T_{F,m}f(x) > c\ |y|\beta(c|y|)e^{(|y|/2)^2}\} \\ &\leq C\frac{e^{-(|y|/2)^2}}{|y|\beta(c|y|)}, \end{aligned}$$

but $\gamma(J) \sim e^{-(|y|/2)^2}|y|^{-1}$; therefore β is bounded for $|y|$ large, which is a contradiction with the assumption on β.

3 The positive result

Since the weak type (1,1) need not be true, the natural question that arises is what weights should we put in the domain in order to get a weak type inequality with respect to the Gaussian measure? For the case of the Gaussian Riesz transforms this can be deduced from the proof of a theorem in Pérez 's paper [7], that is, for $|\alpha| > 2$

$$\mathcal{R}_\alpha : L^1((1 + |y|^{|\alpha|-2})d\gamma) \to L^{1,\infty}(d\gamma).$$

Moreover, it can be proved that for every $0 < \epsilon < |\alpha| - 2$, there exists a function $f \in L^1((1 + |y|^\epsilon)d\gamma)$ such that $\mathcal{R}_\alpha f \notin L^{1,\infty}(d\gamma)$, see [2].
It was the above result for the higher order Gaussian Riesz transforms that led us to study this type of weighted inequalities for this family of singular integral operators. The method of proof we use is based upon a refinement of several inequalities used by S. Pérez in [7] and the application of a technique developed by García-Cuerva et al in [3] which allows us to get rid of the classical technique called "forbidden regions technique".

Theorem 2 *The operator $T_{F,m}$ maps continuously $L^1(wd\gamma)$ into $L^{1,\infty}(d\gamma)$ with $w(y) = 1 \vee \max_{1 \leq t \leq |y|} \eta(t)$ and*

$$\eta(s) = \begin{cases} \dfrac{\psi(s)}{s} & \text{if } 1 \leq m < 2 \\ \\ \dfrac{\psi(s)}{s^2} & \text{if } \quad m \geq 2 \end{cases}$$

where $a \vee b = \max\{a, b\}$.

As an immediate consequence we get the following

Corollary 1 *if for s large either $\psi(s) \leq Cs$ when $1 \leq m < 2$ or $\psi(s) \leq Cs^2$ when $m \geq 2$, then the operator $T_{F,m}$ is of weak type $(1,1)$ with respect to the Gaussian measure.*

Proof of Theorem 2
In order to prove this theorem for each $x \in \mathbf{R}^n$ let us split up $\mathbf{R}^n$ into five regions
$R_1 = B(x, n(1 \wedge 1/|x|))$, usually called the "local region",

$R_2 = \{y \notin R_1$ such that $x \cdot y \leq 0\}$,

$R_3 = \{y \notin R_1$ such that $x \cdot y > 0$ and $|y| \leq |x|\}$,

$R_4 = \{y \notin R_1$ such that $x \cdot y > 0$ and $|x| < |y| < 2|x|\}$,

$R_5 = \{y \notin R_1$ such that $x \cdot y > 0$ and $|y| \geq 2|x|\}$,

where $a \wedge b = \min\{a, b\}$.
Now for each $i = 1, \ldots, 5$ we define $T^i_{F,m}$ as the integral operator with kernel $\chi_{R_i} K_{F,m}$ where χ_{R_i} is the indicator function of the set R_i. Therefore in order to get the result it will be enough to prove that each $T^i_{F,m}$ maps continuously $L^1(w d\gamma)$ into $L^{1,\infty}(d\gamma)$.
Observe that for the operator $T^1_{F,m}$, ususally called the local part, the result follows from [8] where S. Pérez proved that $T^1_{F,m}$ is of weak type $(1,1)$ with respect to the Gaussian measure.

Boundedness of the operator $T^2_{F,m}$.

We will prove that for $y \in R_2$ we get

$$|K_{F,m}(x,y)| \leq Ce^{-|y|^2} w(y), \tag{2}$$

whence it will follow that $T^2_{F,m}$ maps continuously $L^1(w\, d\gamma)$ into $L^1(d\gamma)$.

Let us call $a = |x|^2 + |y|^2$, and $b = 2x \cdot y$, then $a \geq n/2$ and

$$u(t) = \frac{a}{t} - |x|^2 - \frac{\sqrt{1-t}}{t} b.$$

Since $b \leq 0$,

$$\frac{a}{t} - |x|^2 \leq u(t). \tag{3}$$

By applying property iii), after the change of variables $s = \frac{a}{t} - a$, taking into account that $\frac{t}{-\log \sqrt{1-t}} \leq C$, we obtain that for $1 \leq m < 2$

$$\begin{aligned} |K_{F,m}(x,y)| &\leq C \int_0^1 (1-t)^{(m-1)/2} \psi\left(\sqrt{\frac{a}{t} - |x|^2}\right) \frac{e^{-(a/t - |x|^2)}}{t^{n/2+1}\sqrt{1-t}} dt \\ &\leq \frac{C}{a^{n/2}} \int_0^\infty (s+a)^{n/2-1} \left(\frac{s}{s+a}\right)^{(m-2)/2} e^{-(s+|y|^2)} \end{aligned}$$

$$
\begin{aligned}
&\psi\left(\sqrt{s+|y|^2}\right)\, ds\\
&\leq Ce^{-(1-\delta)|y|^2}\ \frac{\psi(|y|)}{a}\int_0^\infty \left(\frac{s+a}{a}\right)^{n/2-1}\left(\frac{s}{s+a}\right)^{(m-2)/2}\\
&\quad e^{-\delta(s+|y|^2)}ds\\
&\leq Ce^{-|y|^2}\ \frac{w(|y|)}{a^{1/2}}\int_0^\infty \left(\frac{s+a}{a}\right)^{n/2-1}\left(\frac{s}{s+a}\right)^{(m-2)/2} e^{-\delta s}ds,
\end{aligned}
$$

and for $m \geq 2$, since $(1-t)\dfrac{-\log\sqrt{1-t}}{t} \leq C$,

$$
\begin{aligned}
|K_{F,m}(x,y)| &\leq C\int_0^1 \psi\left(\sqrt{\frac{a}{t}-|x|^2}\right)\frac{e^{-(a/t-|x|^2)}}{t^{n/2+1}}dt\\
&\leq \frac{C}{a^{n/2}}\int_0^\infty (s+a)^{n/2-1}e^{-(s+|y|^2))}\psi\left(\sqrt{s+|y|^2}\right)\, ds\\
&\leq Ce^{-(1-\delta)|y|^2}\ \frac{\psi(|y|)}{a}\int_0^\infty \left(\frac{s+a}{a}\right)^{n/2-1} e^{-\delta(s+|y|^2)}ds\\
&\leq Ce^{-|y|^2}w(|y|)\int_0^\infty \left(\frac{s+a}{a}\right)^{n/2-1} e^{-\delta s}ds.
\end{aligned}
$$

Thus (2) will follow once we prove that for $1 \leq m < 2$

$$
\int_0^\infty \left(\frac{s+a}{a}\right)^{n/2-1}\left(\frac{s}{s+a}\right)^{(m-2)/2} e^{-\delta s}ds \leq Ca^{1/2}, \tag{4}
$$

while for $m \geq 2$

$$
\int_0^\infty \left(\frac{s+a}{a}\right)^{n/2-1} e^{-\delta s}ds \leq C, \tag{5}
$$

since $a \geq \dfrac{n}{2}$. In order to prove inequality (5), we divide the integral into two terms, $\int_0^1 + \int_1^\infty$. For the first one we use that $\frac{s+a}{a} \simeq C$, and for the second one we use that $\frac{s+a}{a} \leq 3s$ and we are through with (5). For (4) we again split the integral into two terms as before and call them I and II. Then

$$
I \simeq Ca^{(2-m)/2}\int_0^1 s^{(m-2)/2}e^{-\delta s}ds \leq Ca^{1/2}
$$

since $1 \leq m < 2$ and

$$
\begin{aligned}
II &\leq C\int_1^\infty s^{n/2-1}(1+\frac{a}{s})^{(2-m)/2}e^{-\delta s}ds\\
&\leq Ca^{(2-m)/2}\int_1^\infty s^{n/2-1}e^{-\delta s}ds\\
&\leq Ca^{1/2}.
\end{aligned}
$$

With this last estimate we are done with the study of $T^2_{F,m}$.

For the remaining regions we need to prove that if $x \cdot y > 0$ the following estimate for the kernel

$$|K_{F,m}(x,y)| \le Ce^{-u_0}\eta(\sqrt{u_0})\left(\frac{|x+y|}{|x-y|}\right)^{n/2}\left(\frac{2|x||y|}{|x|^2+|y|^2}u_0^{1/2}+1\right) \qquad (6)$$

holds, where

$$u_0 = \frac{|y|^2-|x|^2}{2} + \frac{|x-y||x+y|}{2} = \min_{0<t<1} u(t).$$

Also as it was shown in [5], $u_0 = u(t_0)$ with

$$t_0 = 2\frac{\sqrt{a^2-b^2}}{a+\sqrt{a^2-b^2}} \sim \frac{|x-y|}{|x+y|}. \qquad (7)$$

Let us prove (6). By using (1) and property iii), we get

$$|K_{F,m}(x,y)| \le C\begin{cases} \int_0^1 \psi(\sqrt{u(t)})\frac{e^{-u(t)}}{t^{n/2+1}}\frac{dt}{\sqrt{1-t}} \text{ if } 1\le m<2 \\ \\ \int_0^1 \psi(\sqrt{u(t)})\frac{e^{-u(t)}}{t^{n/2+1}}dt \quad \text{if} \quad m\ge 2\end{cases}. \qquad (8)$$

Since $x \cdot y > 0$ we have the following estimates due to S. Pérez

$$\int_0^1 \frac{u^{1/2}(t)}{\sqrt{1-t}}e^{-\nu u(t)}\frac{dt}{t^2} + \int_0^1 u(t)e^{-\nu u(t)}\frac{dt}{t^2} \le Ce^{-\nu u_0}\frac{|x+y|}{|x-y|} \left(\frac{2|x||y|}{|x|^2+|y|^2}u_0^{1/2}+1\right) \qquad (9)$$

and

$$\int_0^1 u^{1/2}(t)e^{-\nu u(t)}\frac{dt}{t^{3/2}\sqrt{1-t}} \le C\frac{e^{-\nu u_0}}{t_0^{1/2}} \qquad (10)$$

hold for $\nu > 0$ and $n \ge 1$, see [7, Lemma 2 on page 41, inequality (1) on page 53], and [9, Lemma 2.3];
and for $n > 1$

$$e^{-((n-2)/n)u(t)}t^{-(n-2)/2} \le Ce^{-((n-2)/n)u_0}\left(\frac{|x+y|}{|x-y|}\right)^{(n-2)/2} \qquad (11)$$

also holds, see [5].
Now the proof of (6) will be split into two cases:
Case $n = 1$.
For $1 \le m < 2$, using (iii) and (10) with $\nu = \delta > 0$, we obtain

$$\begin{aligned}|K_{F,m}(x,y)| &\le C\frac{\psi(\sqrt{u_0})}{\sqrt{u_0}}e^{-(1-\delta)u_0}\int_0^1 u^{1/2}(t)e^{-\delta u(t)}\frac{dt}{t^{3/2}\sqrt{1-t}} \\ &\le C\eta(\sqrt{u_0})e^{-(1-\delta)u_0}\frac{e^{-\delta u_0}}{t_0^{1/2}} \\ &\le Ce^{-u_0}\eta(\sqrt{u_0})\left(\frac{|x+y|}{|x-y|}\right)^{1/2}.\end{aligned}$$

For $m \geq 2$ we apply property iii) to inequality (8) and get the following estimate

$$|K_{F,m}(x,y)| \leq C\frac{\psi(\sqrt{u_0})}{u_0^{1/2}}e^{-(1-\delta)u_0}\int_0^1 u^{1/2}(t)e^{-\delta u(t)}\frac{dt}{t^{3/2}}. \qquad (12)$$

Now let us proceed to bound $\displaystyle\int_0^1 u^{1/2}(t)e^{-\delta u(t)}\frac{dt}{t^{3/2}}$. First we split $\int_0^1 = \int_0^{t_0} + \int_{t_0}^1 = I + II$

$$\begin{aligned} I &\leq \frac{\sqrt{t_0}}{u_0^{1/2}}\int_0^1 u(t)e^{-\delta u(t)}\frac{dt}{t^2} \\ &\leq \left(\frac{|x-y|}{|x+y|}\right)^{1/2}\frac{e^{-\delta u(t)}}{u_0^{1/2}}\frac{|x+y|}{|x-y|}\left(\frac{2|x||y|}{|x|^2+|y|^2}u_0^{1/2}+1\right) \end{aligned}$$

where in order to get these bounds we applied inequality (9) with $\nu = \delta > 0$. As for the estimate of II by using (10) with $\nu = \delta > 0$ we get

$$II \leq \sqrt{1-t_0}\int_{t_0}^1 u^{1/2}(t)e^{-\delta u(t)}\frac{dt}{t^{3/2}\sqrt{1-t}} \leq C\frac{|x||y|}{|x|^2+|y|^2}\frac{e^{-\delta u_0}}{t_0^{1/2}}.$$

Now combining these two estimates with (12), inequality (6) follows in this case too.
Therefore the proof of (6) is complete for the case $n = 1$.
Case $n > 1$.
We use (8), (11), property iii) and (9) with $\nu = 2/n - (1-\delta) > 0$, and we get for $1 \leq m < 2$

$$\begin{aligned} |K_{F,m}(x,y)| &\leq Ce^{-((n-2)/n)u_0}\left(\frac{|x+y|}{|x-y|}\right)^{(n-2)/2}\int_0^1 \psi(\sqrt{u(t)}) \quad e^{-(2/n)u(t)} \\ &\qquad \frac{1}{t^2\sqrt{1-t}}\ dt \\ &\leq Ce^{-((n-2)/n)u_0-(1-\delta)u_0}\frac{\psi(\sqrt{u_0})}{\sqrt{u_0}}\left(\frac{|x+y|}{|x-y|}\right)^{(n-2)/2} \\ &\qquad \int_0^1 \frac{u^{1/2}(t)}{\sqrt{1-t}}e^{-(2/n-(1-\delta))u(t)}\frac{dt}{t^2} \\ &\leq Ce^{-u_0}\eta(\sqrt{u_0})\left(\frac{|x+y|}{|x-y|}\right)^{n/2}\left(\frac{2|x||y|}{|x|^2+|y|^2}u_0^{1/2}+1\right) \end{aligned}$$

and for $m \geq 2$

$$|K_{F,m}(x,y)| \leq Ce^{-((n-2)/n)u_0}\left(\frac{|x+y|}{|x-y|}\right)^{(n-2)/2}\int_0^1 \psi(\sqrt{u(t)})e^{-(2/n)u(t)}\frac{dt}{t^2}$$

$$\leq Ce^{-((n-2)/n)u_0-(1-\delta)u_0}\frac{\psi(\sqrt{u_0})}{u_0}\left(\frac{|x+y|}{|x-y|}\right)^{(n-2)/2}$$
$$\int_0^1 u(t)e^{-(2/n-(1-\delta))u(t)}\frac{dt}{t^2}$$
$$\leq Ce^{-u_0}\eta(\sqrt{u_0})\left(\frac{|x+y|}{|x-y|}\right)^{n/2}\left(\frac{2|x||y|}{|x|^2+|y|^2}u_0^{1/2}+1\right).$$

This finishes the proof of (6). Before getting into the study of the remaining operators let us observe that

$$u_0 \leq |y|^2. \tag{13}$$

Indeed, after setting $g = x - y$ and $h = x + y$, we deduce immediately this inequality from $2|g||h| \leq |g|^2 + |h|^2$.

Boundedness of the operator $T^3_{F,m}$.

We may assume that $|x| \geq 1$ since otherwise $R_3 = \emptyset$.
We claim that on R_3

$$|K_{F,m}(x,y)| \leq C|x|^n e^{|x|^2}e^{-(|x||x-y|)/2}w(y)e^{-|y|^2}. \tag{14}$$

In order to prove this inequality let us consider two cases:
If $u_0 \geq 1$, then from inequality (6)

$$\begin{aligned}|K_{F,m}(x,y)| &\leq Ce^{-u_0}\max_{1\leq t\leq |y|}\eta(t)\left(\frac{|x+y|}{|x-y|}\right)^{n/2}u_0^{1/2}\\ &\leq Ce^{-u_0}\left(\frac{|x+y|}{|x-y|}\right)^{n/2}(|x-y||x+y|)^{1/2}w(y)\\ &\leq Ce^{-u_0}|x|^n w(y),\end{aligned}$$

since $u_0 \leq \frac{|x-y||x+y|}{2}$ and $|x-y||x+y| \geq n$.
If $u_0 \leq 1$, let us prove that $|K_{F,m}(x,y)| \leq Ce^{-u_0}|x|^n$.
From (8)

$$|K_{F,m}(x,y)| \leq C\int_0^1 \psi(\sqrt{u(t)})\frac{e^{-u(t)}}{t^{n/2+1}}\frac{dt}{\sqrt{1-t}}$$

and using property iii) we have

$$|K_{F,m}(x,y)| \leq C\psi(\sqrt{u_0})e^{-(1-\delta)u_0}\int_0^1 \frac{e^{-\delta u(t)}}{t^{n/2+1}}\frac{dt}{\sqrt{1-t}}.$$

To end the proof of this inequality we shall see that

$$\int_0^1 \frac{e^{-\delta u(t)}}{t^{n/2+1}}\frac{dt}{\sqrt{1-t}} \leq C\ |x|^n.$$

Let us split $\int_0^1 = \int_0^{1/(2|x|^2)} + \int_{1/(2|x|^2)}^{1/2} + \int_{1/2}^1 = I + II + III$. Observe that $III \le C$, and

$$II \le C \int_{1/(2|x|^2)}^{1/2} \frac{dt}{t^{n/2+1}} \le C|x|^n.$$

As for I, taking into account that $|\sqrt{1-t}x - y| \ge (1 - \frac{1}{2n})|x-y|$ if $0 \le t \le \frac{1}{2|x|^2}$, we have after an appropriate change of variables that

$$I \le \frac{C}{|x-y|^n} \le C|x|^n.$$

Now inequality (14) follows from the two cases above and taking into account that since $|x+y| \ge |x|$ and $|x| \ge |y|$, $e^{-u_0} \le e^{|x|^2} e^{-(|x||x-y|)/2} e^{-|y|^2}$.
Let us check that $T^3_{F,m}$ in this case maps $L^1(wd\gamma)$ continuously into $L^1(d\gamma)$.

$$\begin{aligned}\int_{\mathbf{R}^n} |T^3_{F,m} f(x)| d\gamma(x) &\le C \int_{\mathbf{R}^n} \int_{|x|>|y|} e^{-(|x||y-x|)/2} \ |x|^n dx \ |f(y)| \ w(y) \ d\gamma(y) \\ &\le C \int_{\mathbf{R}^n} |f(y)| \ w(y) \ d\gamma(y).\end{aligned}$$

Boundedness of the operators $T^4_{F,m}$ and $T^5_{F,m}$

Before taking care of the remaining operators we prove the following remark which will be used to estimate their kernels.
Remark: If $y \notin B(x, n(1 \wedge \frac{1}{|x|}))$, $|y| > |x|$, and $x \cdot y \ge 0$, then $u_0 \ge 1$.
In fact, under these hypotheses on x and y, $|x-y||x+y| \ge |x+y| n(1 \wedge 1/|x|) \ge (1 \vee |x|) n(1 \wedge 1/|x|) \ge n$. Thus

$$u_0 = \frac{|y|^2 - |x|^2}{2} + \frac{|x-y||x+y|}{2} \ge n/2$$

and so the result follows when $n \ge 2$. As for $n = 1$, $u_0 = y^2 - x^2 \ge 1$.

Let $\alpha(x,y)$ denote the sine of the angle between x and y. We start by showing the following estimates for the kernel $K_{F,m}(x,y)$.
On R_4

$$|K_{F,m}(x,y)| \le C|x|^n e^{|x|^2} \left(1 \wedge \frac{e^{-c|x|^4 \alpha^2(x,y)/(|y|^2 - |x|^2 + \alpha(x,y)|x|^2)}}{(|y|^2 - |x|^2 + \alpha(x,y)|x|^2)^{(n-1)/2}} \right) w(y) e^{-|y|^2}. \quad (15)$$

And on R_5

$$|K_{F,m}(x,y)| \le C(1+|x|) e^{|x|^2} e^{-c\alpha^2(x,y)|x|^2} w(y) e^{-|y|^2}. \quad (16)$$

To prove inequality (15) we start from (6) and observe that $1+\frac{2|x||y|}{|x|^2+|y|^2}u_0^{1/2} \le Cu_0^{1/2}$ and $\eta(\sqrt{u_0}) \le \max_{1\le t\le |y|}\eta(t) \le w(y)$ since $1 \le \sqrt{u_0} \le |y|$ according to (13) and the above remark. In this way we get the bound

$$Ce^{-u_0}w(y)\frac{|x+y|^n}{(|x-y||x+y|)^{n/2}}u_0^{1/2}. \tag{17}$$

Now, taking into account that

$$(|x-y||x+y|)^2 = (|y|^2-|x|^2)^2 + 4|x|^2|y|^2\alpha^2(x,y), \tag{18}$$

$|x| \sim |y|$, and $|x-y||x+y| \sim |y|^2-|x|^2+|\alpha(x,y)||x|^2$, we get

$$\begin{aligned} u_0 &= |y|^2-|x|^2 + \frac{|x-y||x+y|-(|y|^2-|x|^2)}{2} \\ &= |y|^2-|x|^2 + \frac{2|x|^2|y|^2\alpha^2(x,y)}{|y|^2-|x|^2+|x-y||x+y|} \\ &\ge |y|^2-|x|^2 + \frac{c|x|^4\alpha^2(x,y)}{|y|^2-|x|^2+\alpha(x,y)|x|^2} \end{aligned} \tag{19}$$

and also $u_0 \le |x-y||x+y|$. These remarks applied to (17) give (15).
On R_5 first we observe that (13) implies that $u_0^{1/2} \le |y|$ which implies that $1+\frac{|x||y|}{|x|^2+|y|^2}u_0^{1/2} \le C(1+|x|)$. Also $\frac{|x+y|}{|x-y|} \le C$ and from the expression of u_0 given in (19) together with (18)

$$u_0 \ge |y|^2-|x|^2 + \frac{c|x|^2|y|^2\alpha^2(x,y)}{|y|^2-|x|^2+|x||y||\alpha(x,y)|},$$

and by taking into account that in this region $|x| \le \frac{1}{2}|y|$, u_0 is bounded below by $|y|^2-|x|^2+c|x|^2\alpha^2(x,y)$. Therefore with these estimates on R_5 we get from (6)

$$|K_{F,m}(x,y)| \le C(1+|x|)e^{|x|^2}e^{-c\alpha^2(x,y)|x|^2}w(y)e^{-|y|^2}.$$

From (15) and (16) it will be enough to prove that the operators

$$S_0f(x) = |x|^n e^{|x|^2}\int_{R_4}\left(1 \wedge \frac{e^{-c|x|^4\alpha^2(x,y)/(|y|^2-|x|^2+\alpha(x,y)|x|^2)}}{(|y|^2-|x|^2+\alpha(x,y)|x|^2)^{(n-1)/2}}\right)f(y)w(y)d\gamma(y)$$

and

$$S_1f(x) = (1+|x|)e^{|x|^2}\int_{R_5} e^{-c\alpha^2(x,y)|x|^2}f(y)w(y)d\gamma(y)$$

map $L^1(wd\gamma)$ into $L^{1,\infty}(d\gamma)$. Let us point out here that the idea of this proof was taken from [3, Lemma 4.3].
Without loss of generality we may assume that $f \ge 0$. For $\lambda > 0$ let E_i be the level set $\{x \in \mathbf{R}^n : \ S_if(x) > \lambda\}$, for $i = 0, 1$. We shall prove

that $\gamma(E_i) \leq \frac{C}{\lambda}||f||_{1,wd\gamma}$. Let r_0 and r_1 be the unique positive roots of the equations

$$r_0^n e^{r_0^2}||f||_{1,wd\gamma} = \lambda \quad \text{and} \quad (1+r_1)e^{r_1^2}||f||_{1,wd\gamma} = \lambda.$$

Therefore $E_i \cap \{x \in \mathbf{R}^n : |x| < r_i\} = \emptyset$. On the other hand, since we are working on a space of finite measure, it is enough to take $\lambda > K||f||_{1,wd\gamma}$ and by choosing K large enough we may assume that both r_0 and r_1 are larger than one. Hence $\gamma\{x \in \mathbf{R}^n : |x| > 2r_i\} \leq Cr_i^{n-2}e^{-4r_i^2} \leq \frac{C}{\lambda}||f||_{1,wd\gamma}$. Thus we only need to estimate $\gamma\{x \in E_i : r_i \leq |x| \leq 2r_i\}$.
Let $E_i' = \{x' \in S^{n-1} : \exists \rho \in [r_i, 2r_i] \text{ with } \rho x' \in E_i\}$ and for $x' \in E_i'$ let $\rho_i(x')$ be the smallest such ρ. Then $S_i f(\rho_i(x')x') = \lambda$ by the continuity of $S_i f(x)$. This implies that for $i = 0$

$$Ce^{\rho_0(x')^2} r_0^n \int_{|y|\geq r_0} \left(1 \wedge \frac{e^{-cr_0^4\alpha^2(x',y)/(|y|^2-r_0^2+\alpha(x',y)r_0^2)}}{(|y|^2 - r_0^2 + \alpha(x',y)r_0^2)^{(n-1)/2}}\right) f(y)w(y)d\gamma(y) \geq \lambda, \tag{20}$$

and for $i = 1$ since $r_1 > 1$

$$Ce^{\rho_1(x')^2} r_1 \int_{|y|\geq r_1} e^{-c\alpha^2(x',y)r_1^2} f(y)w(y)d\gamma(y) \geq \lambda. \tag{21}$$

Clearly

$$\begin{aligned}\gamma\{x \in E_i : r_i \leq |x| \leq 2r_i\} &\leq \int_{E_i'} d\sigma(x') \int_{\rho_i(x')}^{2r_i} e^{-\rho^2}\rho^{n-1}d\rho \\ &\leq C \int_{E_i'} e^{-\rho_i^2(x')} r_i^{n-2} d\sigma(x').\end{aligned}$$

Combining this estimate with (20), we get

$$\gamma\{x \in E_0 : r_0 \leq |x| \leq 2r_0\} \leq \frac{C}{\lambda} \int_{E_0'} r_0^{2n-2} d\sigma(x')(I_0 + II_0); \tag{22}$$

with

$$I_0 = \int_{\{|y|\geq r_0,\ \alpha(x',y)r_0^2 \geq c\}} \frac{e^{-cr_0^4\alpha^2(x',y)/(|y|^2-r_0^2+\alpha(x',y)r_0^2)}}{(|y|^2 - r_0^2 + \alpha(x',y)r_0^2)^{(n-1)/2}} f(y)w(y)d\gamma(y)$$

and

$$II_0 = \int_{\{\alpha(x',y)r_0^2 \leq c\}} f(y)w(y)d\gamma(y);$$

and for $i = 1$

$$\gamma\{x \in E_1 :\ r_1 \leq |x| \leq 2r_1\} \leq \frac{C}{\lambda} \int_{E_1'} r_1^{n-1} d\sigma(x')(I_1 + II_1) \tag{23}$$

with

$$I_1 = \int_{\{|y|\geq r_1,\ \alpha(x',y)r_1 \geq c\}} e^{-c\alpha^2(x',y)r_1^2} f(y)w(y)d\gamma(y)$$

and

$$II_1 = \int_{\{\alpha(x',y)r_1 \leq c\}} f(y)w(y)d\gamma(y).$$

It is immediate to verify that

$$r_0^{2n-2} \int_{\{\alpha(x',y)r_0^2 \leq c\}} d\sigma(x') \leq C \text{ and } r_1^{n-1} \int_{\{\alpha(x',y)r_1 \leq c\}} d\sigma(x') \leq C, \quad (24)$$

which give, after changing the order of integration in (22) and (23), the desired estimates for the terms involving II_0 and II_1 respectively.
Now let us prove that for $|y| > r_0$

$$r_0^{2n-2} \int_{\{\alpha(x',y)r_0^2 \geq c\}} \frac{e^{-cr_0^4\alpha^2(x',y)/(|y|^2-r_0^2+\alpha(x',y)r_0^2)}}{(|y|^2 - r_0^2 + \alpha(x',y)r_0^2)^{(n-1)/2}} d\sigma(x') \leq C \quad (25)$$

and for $|y| > r_1$

$$r_1^{n-1} \int_{\{\alpha(x',y)r_1 \geq c\}} e^{-c\alpha^2(x',y)r_1^2} d\sigma(x') \leq C. \quad (26)$$

Remark: We observe that for $n = 1$ the sets $\{\alpha(x',y)r_0^2 \geq c\}$ and $\{\alpha(x',y)r_1 \geq c\}$ are empty.
For any fixed $y \in \mathbf{R}^n$ with $n > 1$, we choose coordinates on S^{n-1} in such a way that the north pole is on the direction of y. Then the left hand side of (25) can be written as

$$r_0^{2n-2} \int_{\{\sin\theta r_0^2 \geq c\}} e^{-cr_0^4 \sin^2\theta/(|y|^2 - r_0^2 + \sin\theta r_0^2)} \frac{\sin^{n-2}\theta}{(|y|^2 - r_0^2 + \sin\theta\ r_0^2)^{(n-1)/2}} d\theta.$$

The boundedness of this integral when restricted to the angles θ such that $\sin\theta \geq 1/2$ follows easily by using that $|t|^{n-1}e^{-ct^2} \leq C$. For the remaining integral we introduce the factor $\cos\theta$ in the integral, make the change of variables $\alpha = \sin\theta$, and get

$$r_0^{2n-2} \int_{\{\alpha r_0^2 \geq c\}} e^{-cr_0^4\alpha^2/(|y|^2-r_0^2+\alpha r_0^2)} \frac{\alpha^{n-3}}{(|y|^2 - r_0^2 + \alpha r_0^2)^{(n-3)/2}} \frac{\alpha d\alpha}{|y|^2 - r_0^2 + \alpha r_0^2}. \quad (27)$$

Performing the change of variables

$$u = \frac{r_0^4\alpha^2}{|y|^2 - r_0^2 + \alpha r_0^2}$$

and observing that

$$du = r_0^4 \frac{2|y|^2 - 2r_0^2 + \alpha r_0^2}{|y|^2 - r_0^2 + \alpha r_0^2} \frac{\alpha d\alpha}{|y|^2 - r_0^2 + \alpha r_0^2} \geq r_0^4 \frac{\alpha d\alpha}{|y|^2 - r_0^2 + \alpha r_0^2},$$

we see that the expression (27) is bounded by $\int_0^\infty e^{-cu}u^{(n-3)/2}du \leq C$, since $n \geq 2$.

To prove (26), a similar argument as the one above may be applied. In order to take care of the integral restricted to the angles θ for which $\sin\theta \geq 1/2$ we use again $|t|^{n-1}e^{-ct^2} \leq C$, and for the remaining integral the same argument applies in order to make the change of variables $\alpha = \sin\theta$ and therefore we get

$$r_1^{n-1}\int_{\{\alpha r_1 \geq c\}} e^{-cr_1^2\alpha^2}\alpha^{n-2}d\alpha \leq C\int_0^\infty e^{-cu^2}u^{n-2}du \leq C,$$

since again $n \geq 2$. Having proved (25) and (26), by changing the order of integration in (22) and (23) we get also the desired estimate for the terms involving I_0 and I_1 respectively and at the same time we end the proof of Theorem 2.

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