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Parabolic mean values and maximal estimates for gradients of temperatures ☆

Hugo Aimar [a,b], Ivana Gómez [a,c,*], Bibiana Iaffei [a,d]

[a] *Instituto de Matemática Aplicada del Litoral (CONICET–UNL), Güemes 3450, (S3000GLN) Santa Fe, Argentina*
[b] *Departamento de Matemática, FIQ (UNL), Santa Fe, Argentina*
[c] *Departamento de Matemática, FICH (UNL), Santa Fe, Argentina*
[d] *Departamento de Matemática, FHUC (UNL), Santa Fe, Argentina*

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Abstract

We aim to prove inequalities of the form $|\delta^{k-\lambda}(x,t)\nabla^k u(x,t)| \leqslant C M^{-}_{\mathbb{R}^+} M^{\#,\lambda,k}_D u(x,t)$ for solutions of $\frac{\partial u}{\partial t} = \Delta u$ on a domain $\Omega = D \times \mathbb{R}^+$, where $\delta(x,t)$ is the parabolic distance of (x,t) to parabolic boundary of Ω, $M^{-}_{\mathbb{R}^+}$ is the one-sided Hardy–Littlewood maximal operator in the time variable on $\mathbb{R}^+$, $M^{\#,\lambda,k}_D$ is a Calderón–Scott type d-dimensional elliptic maximal operator in the space variable on the domain D in $\mathbb{R}^d$, and $0 < \lambda < k < \lambda + d$. As a consequence, when D is a bounded Lipschitz domain, we obtain estimates for the $L^p(\Omega)$ norm of $\delta^{2n-\lambda}(\nabla^{2,1})^n u$ in terms of some mixed norm $\int_0^\infty \|u(\cdot,t)\|^p_{B^{\lambda,p}_p(D)}\,dt$ for the space $L^p(\mathbb{R}^+, B^{\lambda,p}_p(D))$ with $\|\cdot\|_{B^{\lambda,p}_p(D)}$ denotes the Besov norm in the space variable x and where $\nabla^{2,1} = (\nabla^2, \frac{\partial}{\partial t})$.

Keywords: Maximal operators; Gradient estimates; Mean value formula; Heat equation

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* Corresponding author at: Instituto de Matemática Aplicada del Litoral (CONICET–UNL), Güemes 3450, (S3000GLN) Santa Fe, Argentina. Fax: +54 342 4550944.

E-mail addresses: haimar@santafe-conicet.gov.ar (H. Aimar), ivanagomez@santafe-conicet.gov.ar (I. Gómez), biaffei@santafe-conicet.gov.ar (B. Iaffei).

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0. Introduction

The main result of this paper, which is contained in Theorem 5.4, is a pointwise estimate for the space time gradients of a temperature u on a cylindrical domain in terms of an iteration of two maximal operators. The result is an extension to the parabolic setting of the elliptic inequalities proved by S. Dahlke and R. DeVore in [2], see also D. Jerison and C. Kenig in [7]. After an improvement of the parabolic mean value formula and the analysis of the kernel and the operator that provides the space derivatives of temperatures, we obtain a pointwise estimate for space gradients weighted by powers of the distance to the parabolic boundary in terms of an iteration of two maximal operators which are well known in harmonic analysis: the one-sided maximal Hardy–Littlewood operator M^- in the time variable and the Calderón–Scott maximal operator $M^{\#,\lambda,k}$ in the space variable.

We would like to point out that these results are a part of a larger program which looks for a parabolic theory, similar to the elliptic one developed in [2], in order to obtain regularity improvements for temperatures in terms of adequate Besov type norms which could help in the analysis of the rate of convergence for nonlinear approximation methods for parabolic equations. In particular we mention that a time localized version of Corollary 6.2 can be used to obtain, following the interpolation technique used in [7], parabolic Besov type estimates in space and time variables in terms of mixed Lebesgue–Besov norms. These results shall be published elsewhere.

The paper is organized as follows. In Section 1 we prove a smooth mean value formula for temperatures and we introduce some basic notation that shall be used in the sequel. Section 2 is devoted to obtain a distributional representation for the space derivatives of the kernel obtained in Section 1. Here we also prove some basic but essential properties of that distribution. As a corollary we obtain a formula for derivatives of temperatures. In Section 3 we introduce the one-sided Hardy–Littlewood maximal operator M^- and the Calderón–Scott maximal operator $M^{\#,\lambda,k}$. We prove in this section a basic lemma which shall be used in Section 4 in order to get the pointwise estimate on $\mathbb{R}^{d+1}$ contained in Theorem 4.1. In Section 5 we obtain basic pointwise estimates for some parabolic gradients of temperatures on cylindrical domains. Section 6 is devoted to obtain L^p-estimates for space–time gradients of temperatures in terms of mixed Lebesgue–Besov norms for cylindrical domains of the form $D \times \mathbb{R}^+$ with D a Lipschitz domain in $\mathbb{R}^d$.

1. Smooth parabolic mean value formula

In 1966 W. Fulks proves in [6] a mean value property for caloric functions involving integration on the level surfaces of the fundamental solution $W_t(x) = (4\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}}$ for $t > 0$, for the heat equation $\frac{\partial u}{\partial t} = \Delta u$. In 1973, N.A. Watson gives in [12] a parabolic mean value formula in terms of $(d+1)$-volume integrals over the heat balls defined by

$$E(x,t;r) = \left\{(y,s) \in \mathbb{R}^{d+1}\colon\ s \leqslant t,\ \nu(x-y,t-s) \leqslant r\right\},$$

where $\nu(x,t) = (W_t(x))^{-\frac{1}{d}} = (4\pi t)^{\frac{1}{2}} e^{\frac{|x|^2}{4dt}}$. Precisely,

$$u(x,t) = \frac{1}{4r^d} \iint\limits_{E(x,t;r)} u(y,s) \frac{|x-y|^2}{(t-s)^2}\, dy\, ds \tag{1.1}$$

provided that $E(x,t;r)$ is contained in the domain of the temperature u. For a proof of (1.1) see [5]. From (1.1), by using spherical coordinates for the space variable, taking derivatives with respect to the radial variable, we get that

$$u(x,t) = -\frac{1}{2r^d} \int_{-\frac{r^2}{4\pi}}^{0} \left(\int_{S^{d-1}} u\big(x + R_r(s)w, t+s\big)\, dw \right) \frac{R_r(s)^d}{s}\, ds, \tag{1.2}$$

with $R_r(s) = |y| = \sqrt{-2sd \ln \frac{r^2}{-4\pi s}}$.

Even when smooth versions of (1.1) are considered elsewhere, see [11] for instance, for the sake of completeness and as a chance to write explicitly our notation we shall state and prove in the next lemma the formula that will be used throughout this paper.

Lemma 1.1. *Let η be a $\mathscr{C}^\infty(\mathbb{R})$, nonnegative function supported on $[0,1]$ satisfying $d\int_0^1 \eta(r) r^{d-1}\, dr = 1$. Then for every temperature u we have that*

$$u(x,t) = \iint_{\mathbb{R}^{d+1}} K_\delta(x-y, t-s) u(y,s)\, dy\, ds, \tag{1.3}$$

where

$$K(x,t) = \frac{1}{4} \frac{|x|^2}{t^2} \eta\big(\nu(x,t)\big)$$

and

$$K_\delta(x,t) = \frac{1}{\delta^{d+2}} K\left(\frac{x}{\delta}, \frac{t}{\delta^2}\right),$$

provided that the closure of $E(x,t;\delta)$ is contained in the domain of the temperature u.

Proof. Let us first notice that is enough to prove formula (1.3) for $(x,t) = (0,0)$. Take $\delta > 0$ small enough in such a way that the closure of $E(0,0;\delta)$ is contained in the domain of the temperature u. Multiplying both sides of (1.2) by $2d\eta(\frac{r}{\delta}) r^{d-1}$ and integrating with respect to r on the interval $(0,\delta)$, we get

$$\begin{aligned} u(0,0) \int_0^\delta 2d\eta\left(\frac{r}{\delta}\right) r^{d-1}\, dr \\ = -d \int_0^\delta \int_{-\frac{r^2}{4\pi}}^{0} \left(\int_{S^{d-1}} u\big(R_r(s)w, s\big)\, dS(w) \right) \frac{R_r(s)^d}{s}\, ds\, \eta\left(\frac{r}{\delta}\right) \frac{dr}{r}. \end{aligned} \tag{1.4}$$

Notice now that the choice of the support of η allows us to apply Fubini's theorem to interchange orders of integration on (1.4). Then for s fixed, performing the change of variables $r \mapsto$

$\tau = R_r(s)$ and taking into account that $\frac{dR_r(s)}{dr} = -\frac{1}{2}(R_r(s))^{-1}2s\,d\frac{(-4\pi s)}{r^2}\frac{2r}{(-4\pi s)} = -2s\,d\tau^{-1}r^{-1}$ from which $\frac{dr}{r} = -\frac{\tau}{2sd}\,d\tau$, we get the desired formula

$$\begin{aligned}
2\delta^d u(0,0) &= -d \int_{-\frac{\delta^2}{4\pi}}^{0} \frac{1}{s} \int_{\sqrt{-4\pi s}}^{\delta} \int_{S^{d-1}} u\big(R_r(s)w, s\big)\, dS(w)\, R_r(s)^d \eta\left(\frac{r}{\delta}\right)\frac{dr}{r}\, ds \\
&= \frac{1}{2} \int_{-\frac{\delta^2}{4\pi}}^{0} \frac{1}{s} \int_{R_{\sqrt{-4\pi s}}(s)}^{R_\delta(s)} \int_{S^{d-1}} u(\tau w, s)\, dS(w) \frac{\tau^{d+1}}{s} \eta\left(\frac{R_\tau^{-1}(s)}{\delta}\right) d\tau\, ds \\
&= \frac{1}{2} \int_{-\frac{\delta^2}{4\pi}}^{0} \frac{1}{s^2} \int_{0}^{R_\delta(s)} \tau^{d+1} \int_{S^{d-1}} u(\tau w, s)\, dS(w)\, \eta\left(\frac{1}{\delta}(-4\pi s)^{\frac{1}{2}} e^{\frac{-\tau^2}{4ds}}\right) d\tau\, ds \\
&= \frac{1}{2} \int_{-\frac{\delta^2}{4\pi}}^{0} \int_{B(0;R_\delta(s))} u(y,s)\, \eta\left(\frac{1}{\delta}(-4\pi s)^{\frac{1}{2}} e^{\frac{-|y|^2}{4ds}}\right) \frac{|y|^2}{s^2}\, dy\, ds,
\end{aligned}$$

in the third equality we have used that $R_{\sqrt{-4\pi s}}(s) = 0$. Here $B(0; R_\delta(s))$ denotes the d-dimensional Euclidean ball centered at the origin with radius $R_\delta(s)$. $\square$

2. Spatial derivatives of the caloric mean value kernel

The aim of this section is to obtain an explicit formula for spatial derivatives of the kernel K_δ introduced in Section 1. We also prove here some useful structural properties of the family of kernels which represent those derivatives.

Let us first observe that for fixed $\delta > 0$ and $t < 0$ the kernel $K_\delta(x,t)$ is a smooth function of x, actually $\mathscr{C}^\infty$. Let us write $N^\alpha(x,t)$ to denote the classical derivative $\partial^\alpha K = \frac{\partial^{|\alpha|}}{\partial x_d^{\alpha_d} \ldots \partial x_1^{\alpha_1}} K$ of K for fixed t and $\alpha = (\alpha_1, \ldots, \alpha_d)$ a multi-index of nonnegative integers ($\alpha \in \mathbb{N}_0^d$). Let us observe that N^α needs not generally be an integrable function on $\mathbb{R}^{d+1}$. In fact, for $\alpha = 1$ and $d = 1$ for example, with $\eta_1(s) = s\eta'(s)$ we have that

$$N^\alpha(x,t) = \frac{1}{2}\frac{x^3}{t^3}\eta_1\big(\nu(x,t)\big) + 2\frac{x}{t^2}\eta\big(\nu(x,t)\big).$$

It is easy to see that if for example η is a positive constant on the interval $(\frac{1}{4}, \frac{3}{4})$, we have that

$$\iint_{\{(x,t):\ \frac{1}{4} < \nu(x,t) < \frac{3}{4},\ x>0\}} \frac{x}{t^2}\eta\big(\nu(x,t)\big)\, dx\, dt = +\infty.$$

On the other hand, since $\eta_1(s)$ vanishes for $s \in (\frac{1}{4}, \frac{3}{4})$, we see that

$$\iint\limits_{\{(x,t):\ \frac{1}{4}<\nu(x,t)<\frac{3}{4},\ x>0\}} \left|N^\alpha(x,t)\right| dx\,dt = \iint\limits_{\{(x,t):\ \frac{1}{4}<\nu(x,t)<\frac{3}{4},\ x>0\}} 2\frac{x}{t^2}\eta\big(\nu(x,t)\big)\,dx\,dt = +\infty.$$

Since K is an $L^1(\mathbb{R}^{d+1})$ function with compact support, the derivatives of order α of K, with $\alpha = (\alpha_1, \dots, \alpha_d)$, make also sense as distributions. Let us denote by $D^\alpha K$ the distributional derivatives of K. Even when these derivatives are generally not functions themselves, we shall prove some integral representation formulas. The precise result is contained in the next theorem. The angle brackets $\langle\cdot,\cdot\rangle$ are used for the distributional duality of $\mathcal{E} = \mathscr{C}^\infty(\mathbb{R}^{d+1})$ and $\mathcal{E}'$. For a given $\mathscr{C}^\infty(\mathbb{R}^{d+1})$ function φ we write $P_k\varphi(x,t)$ to denote the Taylor polynomial at $x_0 = 0$ (MacLaurin) of degree k for the function defined on $\mathbb{R}^d$ by $x \mapsto \varphi(x,t)$ for t fixed.

Theorem 2.1. *For $\delta > 0$ and $\alpha = (\alpha_1, \dots, \alpha_d)$ a multi-index of nonnegative integers and for every $\varphi \in \mathscr{C}^\infty(\mathbb{R}^{d+1})$ we have that*

$$\big\langle D^\alpha(K_\delta), \varphi\big\rangle = \delta^{-|\alpha|} \iint\limits_{\mathbb{R}^{d+1}} \big(N^\alpha\big)_\delta(x,t)\big[\varphi(x,t) - P_{|\alpha|-1}\varphi(x,t)\big]\,dx\,dt, \tag{2.1}$$

where the integral on the right-hand side converges absolutely.

The proof of Theorem 2.1 will be a consequence of the following basic properties of the derivatives of K. We shall use the following notation. For $m \in \mathbb{N}_0$, $h(m)$ equals $\frac{m}{2}$ if m is even and $\frac{m-1}{2}$ if m is odd. For a given multi-index $\alpha \in \mathbb{N}_0^d$ we write $h(\alpha)$ to denote the d-vector of integers given by $h(\alpha) = (h(\alpha_1), h(\alpha_2), \dots, h(\alpha_d))$.

Lemma 2.2. *For $\delta > 0$, a fixed multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$ with $|\alpha| > 0$, we have:*

(2.2.1) *For every t real $K(x,t)$ is a $\mathscr{C}^\infty$ function of $x \in \mathbb{R}^d$ and*

$$N^\alpha(x,t) = \sum_{i=1}^{d}\sum_{j=1}^{3} \sum_{0 \leqslant \beta \leqslant h(\alpha-(j-1)e_i)} \frac{x^{\alpha+(4-2j)e_i+2\beta}}{t^{|\alpha|-|\beta|+3-j}} \eta_\beta^{ij}\big(\nu(x,t)\big), \tag{2.2}$$

where η_β^{ij} are $\mathscr{C}^\infty$ functions of a real variable with support contained in $\operatorname{supp}\eta$.

(2.2.2) *For $\varphi \in \mathscr{C}^\infty(\mathbb{R}^{d+1})$ and each $s \in \mathbb{R}$ the function of y given by $N^\alpha(y,s)\varphi(y,s)$ belongs to $L^1(\mathbb{R}^d)$. Moreover the function of s $\int_{\mathbb{R}^d} N^\alpha(y,s)\varphi(y,s)\,dy$ belongs to $L^1(\mathbb{R})$ and the distribution $D^\alpha K$ is given as the iterated integral*

$$\big\langle D^\alpha K, \varphi\big\rangle = \int\limits_{\mathbb{R}} \Bigg\{ \int\limits_{\mathbb{R}^d} N^\alpha(y,s)\varphi(y,s)\,dy \Bigg\}\,ds. \tag{2.3}$$

(2.2.3) *For each t, the function $N^\alpha(x,t)x^\beta$ belongs to $L^1(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d} N^\alpha(x,t)x^\beta\,dx = 0$ for every $0 \leqslant |\beta| < |\alpha|$.*

(2.2.4) $N^\alpha[\varphi - P_{|\alpha|-1}\varphi] \in L^1(\mathbb{R}^{d+1})$ *for every $\varphi \in \mathscr{C}^\infty(\mathbb{R}^{d+1})$.*

Let us give the proof of Theorem 2.1 assuming that Lemma 2.2 holds.

Proof of Theorem 2.1. We shall first show that the integral on the right-hand side of (2.1) is absolutely convergent. From (2.2.4) we know that this property is true when $\delta = 1$. On the other hand, since for any positive δ we have that $P_{|\alpha|-1}\varphi_\delta = (P_{|\alpha|-1}\varphi)_\delta$, the convergence of the integral for general δ follows from (2.2.4) by changing variables. From (2.2.2) and (2.2.3) we have, for each $\varphi \in \mathscr{C}^\infty(\mathbb{R}^{d+1})$, that

$$\langle D^\alpha K, \varphi\rangle = \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}^d} N^\alpha(x,t)\big[\varphi(x,t) - P_{|\alpha|-1}\varphi(x,t)\big]\,dx \right\} dt.$$

Now from (2.2.4) and from Fubini–Tonelli theorem we have (2.1) for $\delta = 1$. Take now any $\delta > 0$. By the change of variables $y = \frac{x}{\delta}$ and $s = \frac{t}{\delta^2}$, we have that

$$\begin{aligned}
&\iint_{\mathbb{R}^{d+1}} \delta^{-|\alpha|}\big(N^\alpha\big)_\delta(x,t)\big[\varphi(x,t) - (P_{|\alpha|-1}\varphi)(x,t)\big]\,dx\,dt \\
&\quad = \delta^{-|\alpha|} \iint_{\mathbb{R}^{d+1}} N^\alpha(y,s)\big[\varphi\big(\delta y, \delta^2 s\big) - (P_{|\alpha|-1}\varphi)\big(\delta y, \delta^2 s\big)\big]\,dy\,ds \\
&\quad = \delta^{-|\alpha|-d-2} \iint_{\mathbb{R}^{d+1}} N^\alpha(y,s)\big[\varphi_{\frac{1}{\delta}}(y,s) - (P_{|\alpha|-1}\varphi_{\frac{1}{\delta}})(y,s)\big]\,dy\,ds.
\end{aligned}$$

Now, applying the case of $\delta = 1$ already considered with $\varphi_{\frac{1}{\delta}}$ instead of φ, we readily see that the right-hand side in (2.1) is given by

$$\delta^{-|\alpha|-d-2}\big\langle D^\alpha K, \varphi_{\frac{1}{\delta}}\big\rangle = \delta^{-|\alpha|-d-2}(-1)^{|\alpha|}\big\langle K, \partial^\alpha\big(\varphi_{\frac{1}{\delta}}\big)\big\rangle = \delta^{-d-2}(-1)^{|\alpha|}\big\langle K, (\partial^\alpha\varphi)_{\frac{1}{\delta}}\big\rangle.$$

Finally, since $K \in L^1(\mathbb{R}^{d+1})$,

$$\begin{aligned}
&\iint_{\mathbb{R}^{d+1}} \delta^{-|\alpha|}\big(N^\alpha\big)_\delta(x,t)\big[\varphi(x,t) - (P_{|\alpha|-1}\varphi)(x,t)\big]\,dx\,dt \\
&\quad = (-1)^{|\alpha|} \iint_{\mathbb{R}^{d+1}} K(x,t)\partial^\alpha\varphi\big(\delta x, \delta^2 t\big)\,dx\,dt \\
&\quad = \delta^{-d-2}(-1)^{|\alpha|} \iint_{\mathbb{R}^{d+1}} K\left(\frac{y}{\delta}, \frac{s}{\delta^2}\right)\partial^\alpha\varphi(y,s)\,dy\,ds \\
&\quad = (-1)^{|\alpha|} \iint_{\mathbb{R}^{d+1}} K_\delta(y,s)\partial^\alpha\varphi(y,s)\,dy\,ds \\
&\quad = \big\langle D^\alpha(K_\delta), \varphi\big\rangle. \qquad \square
\end{aligned}$$

For the proof of (2.2.1) in Lemma 2.2 we shall make use of a somehow explicit expression for the space derivatives of $\phi(\nu(x,t))$ where ϕ is any $\mathscr{C}^\infty$ function of a real variable.

Lemma 2.3. *Let ϕ be a $\mathscr{C}^\infty$ function of a positive real variable. Then for any multi-index $\gamma \in \mathbb{N}_0^d$ we have*

$$\partial^\gamma\big(\phi\big(\nu(x,t)\big)\big) = \sum_{0\leqslant\beta\leqslant h(\gamma)} \frac{x^{\gamma-2\beta}}{t^{|\gamma|-|\beta|}}\phi_\beta^\gamma\big(\nu(x,t)\big),$$

where each ϕ_β^γ is a $\mathscr{C}^\infty$ function of a positive real variable with support contained in the support of ϕ.

Proof. Let us start by showing by induction on m that for each $i=1,\dots,d$ the formula

$$\frac{\partial^m}{\partial x_i^m}\big(\phi\big(\nu(x,t)\big)\big) = \sum_{0\leqslant n\leqslant h(m)} \frac{x_i^{m-2n}}{t^{m-n}}\phi_n^m\big(\nu(x,t)\big) \tag{2.4}$$

holds for some smooth functions ϕ_n^m supported in the support of ϕ. When $m=1$ (2.4) reads $\frac{\partial}{\partial x_i}(\phi(\nu(x,t))) = \frac{x_i}{t}\phi_0^1(\nu(x,t))$ where $\phi_0^1(s) = s\phi'(s)$. Let us assume that (2.4) holds as stated for derivatives of order m. Let us assume that m is even. The case m odd can be handled in a similar way. Hence

$$\begin{aligned}
&\frac{\partial^{m+1}}{\partial x_i^{m+1}}\big(\phi\big(\nu(x,t)\big)\big)\\
&\quad= \sum_{0\leqslant n\leqslant\frac{m}{2}} \frac{1}{t^{m-n}}\frac{\partial}{\partial x_i}\big(x_i^{m-2n}\phi_n^m\big(\nu(x,t)\big)\big)\\
&\quad= \frac{1}{t^{\frac{m}{2}}}\frac{d\phi_{m/2}^m}{ds}\big(\nu(x,t)\big)\nu(x,t)\frac{x_i}{t}\\
&\qquad+ \sum_{0\leqslant n<\frac{m}{2}} \frac{1}{t^{m-n}}\left[(m-2n)x_i^{m-2n-1}\phi_n^m\big(\nu(x,t)\big) + x_i^{m-2n}\frac{d\phi_n^m}{ds}\big(\nu(x,t)\big)\nu(x,t)\frac{x_i}{t}\right]\\
&\quad= \sum_{0\leqslant n<\frac{m}{2}} \frac{x_i^{(m+1)-2(n+1)}}{t^{(m+1)-(n+1)}}\phi_n^m\big(\nu(x,t)\big) + \sum_{0\leqslant n\leqslant\frac{m}{2}} \frac{x_i^{(m+1)-2n}}{t^{(m+1)-n}}\widetilde{\phi}_n^m\big(\nu(x,t)\big).
\end{aligned}$$

Since in the case m even we have that $h(m+1) = \frac{m}{2}$, we only have to observe that each term in the above two sums can be identified with some of the terms of the following

$$\sum_{0\leqslant k\leqslant\frac{m}{2}} \frac{x_i^{(m+1)-2k}}{t^{(m+1)-k}}\psi_k^{m+1}\big(\nu(x,t)\big)$$

for adequate ψ_k^{m+1} with support contained in the support of ϕ. The desired result for any arbitrary multi-index γ follows by iteration of (2.4). $\square$

Proof of (2.2.1). From Leibniz rule, we have that

$$\begin{aligned} N^\alpha(x,t) &= \sum_{i=1}^d \partial^\alpha\left(\frac{x_i^2}{t^2}\eta\big(\nu(x,t)\big)\right) \\ &= \frac{1}{t^2}\sum_{i=1}^d \sum_{0\leqslant\beta\leqslant\alpha}\binom{\alpha}{\beta}\partial^\beta\big(x^{2e_i}\big)\partial^{\alpha-\beta}\big(\eta\big(\nu(x,t)\big)\big) \\ &= \sum_{i=1}^d\binom{\alpha}{0}\frac{x_i^2}{t^2}\partial^\alpha\eta\big(\nu(x,t)\big)+\sum_{i=1}^d 2\binom{\alpha}{e_i}\frac{x_i}{t^2}\partial^{\alpha-e_i}\eta\big(\nu(x,t)\big) \\ &\quad+\sum_{i=1}^d 2\binom{\alpha}{2e_i}\frac{1}{t^2}\partial^{\alpha-2e_i}\eta\big(\nu(x,t)\big). \end{aligned}$$

For each one of the three derivatives in the last term above, we apply Lemma 2.3. The first one gives the terms in (2.2) corresponding to $j=1$. The second to $j=2$ and the third to $j=3$. $\square$

Proof of (2.2.2). Take $\varphi\in\mathscr{C}^\infty(\mathbb{R}^{d+1})$, and $s\in\mathbb{R}$. If $s\geqslant 0$, then $N^\alpha(y,s)\varphi(y,s)\equiv 0$ on $\mathbb{R}^d$. For $s<0$, the function of y defined by $N^\alpha(y,s)$ is bounded and has bounded support. So that $N^\alpha(y,s)\varphi(y,s)$ is in $L^1(\mathbb{R}^d)$ as a function of $y\in\mathbb{R}^d$. On the other hand, since for s fixed $K(y,s)$ is $\mathscr{C}^\infty(\mathbb{R}^d)$ of y, integrating by parts, we see that

$$\int_{\mathbb{R}^d} N^\alpha(y,s)\varphi(y,s)\,dy = (-1)^{|\alpha|}\int_{\mathbb{R}^d} K(y,s)\partial^\alpha\varphi(y,s)\,dy.$$

Notice now that $K(y,s)\partial^\alpha\varphi(y,s)$ is absolutely integrable on $\mathbb{R}^{d+1}$ since K belongs to $L^1(\mathbb{R}^{d+1})$, K has compact support and $\partial^\alpha\varphi$ is bounded on the support of K. Hence from Fubini' theorem the function of s given by $\int_{\mathbb{R}^d}K(y,s)\partial^\alpha\varphi(y,s)\,dy$ belongs to $L^1(\mathbb{R})$. So does $\int_{\mathbb{R}^d}N^\alpha(y,s)\varphi(y,s)\,dy$. Let us finally check (2.3). By the same integration by parts in the integral with respect to y performed before,

$$\begin{aligned} \int_{\mathbb{R}}\left\{\int_{\mathbb{R}^d} N^\alpha(y,s)\varphi(y,s)\,dy\right\}ds &= (-1)^{|\alpha|}\int_{\mathbb{R}}\left\{\int_{\mathbb{R}^d}K(y,s)\partial^\alpha\varphi(y,s)\,dy\right\}ds \\ &= (-1)^{|\alpha|}\iint_{\mathbb{R}^{d+1}}K(y,s)\partial^\alpha\varphi(y,s)\,dy\,ds \\ &= (-1)^{|\alpha|}\big\langle K,\partial^\alpha\varphi\big\rangle \\ &= \big\langle D^\alpha K,\varphi\big\rangle. \qquad\square \end{aligned}$$

Proof of (2.2.3). The integrability of $\partial^\alpha K(x,t)x^\beta$ as functions of x are easy to check. In fact, for $t\geqslant 0$ the function of x given by $\partial^\alpha K(x,t)$ is identically zero. For $t<0$, $\partial^\alpha K(\cdot,t)$ has compact support and is bounded as a function of x. Integrating by parts, with $0\leqslant|\beta|<|\alpha|$, we have then $\int_{\mathbb{R}^d}\partial^\alpha K(x,t)x^\beta\,dx=(-1)^{|\alpha|}\int_{\mathbb{R}^d}K(x,t)\partial^\alpha(x^\beta)\,dx=0$. $\square$

Proof of (2.2.4). Let us write φ_t to denote the function of the variable $x \in \mathbb{R}^d$ defined by $\varphi_t(x) = \varphi(x,t)$ when $\varphi \in \mathscr{C}^\infty(\mathbb{R}^{d+1})$. The MacLaurin polynomial for φ_t of order $|\alpha| - 1$ is given by

$$P_{|\alpha|-1}\varphi_t(x) = \sum_{k=0}^{|\alpha|-1} \frac{1}{k!} \sum_{|\gamma|=k} \partial^\gamma \varphi_t(0) x^\gamma,$$

and

$$\varphi_t(x) - P_{|\alpha|-1}\varphi_t(x) = \frac{1}{(|\alpha|-1)!} \sum_{|\gamma|=|\alpha|} x^\gamma \int_0^1 \partial^\gamma \varphi_t(\sigma x)(1-\sigma)^{|\alpha|-1}\, d\sigma.$$

In order to prove that $N^\alpha[\varphi - P_{|\alpha|-1}\varphi]$ as a function of (x,t) belongs to $L^1(\mathbb{R}^{d+1})$ we only have to check that each function of the form $x^\gamma N^\alpha(x,t)$, with $|\gamma| = |\alpha|$ belongs to $L^1(\mathbb{R}^{d+1})$. On the other hand, using formula (2.2) for N^α, it will be enough to show that each function of the form $\widetilde{\eta}(\nu(x,t)) \frac{x^{\alpha+(4-2j)e_i+2\beta}}{t^{|\alpha|-|\beta|+3-j}} x^\gamma$ belongs to $L^1(\mathbb{R}^{d+1})$ when $|\gamma| = |\alpha|$; $i = 1, \dots, d$; $j = 1, 2, 3$; $0 \leqslant \beta \leqslant h(\alpha - (j-1)e_i)$ and $\widetilde{\eta}$ is a bounded function of real variable with support contained in that of η. So that it shall be enough to show that

$$\iint_{E^*(0,0;1)} \frac{|x^{\alpha+(4-2j)e_i+2\beta+\gamma}|}{t^{|\alpha|-|\beta|+3-j}}\, dx\, dt < \infty$$

for $E^*(0,0;1) = \{(x,t)\colon (x,-t) \in E(0,0;1)\}$ and those values of α, β, γ, i and j. The above integral can be estimated after an application of Fubini's theorem and the introduction of spherical coordinates in $\mathbb{R}^d$ by an integral of the form

$$\begin{aligned}
&\int_0^1 t^{-|\alpha|+|\beta|-3+j} \left(\int_0^{R_1(-t)} \rho^{|\alpha|+4-2j+2|\beta|+|\gamma|+d-1}\, d\rho \right) dt \\
&\quad = c \int_0^1 t^{-|\alpha|+|\beta|-3+j} \big(R_1(-t)\big)^{2|\alpha|+4-2j+2|\beta|+d}\, dt \\
&\quad = c \int_0^1 t^{-|\alpha|+|\beta|-3+j}\, t^{|\alpha|+2-j+|\beta|+\frac{d}{2}} \left(\ln \frac{1}{4\pi t} \right)^{|\alpha|+2-j+|\beta|+\frac{d}{2}} dt \\
&\quad = c \int_0^1 t^{2|\beta|+\frac{d}{2}-1} \left(\ln \frac{1}{4\pi t} \right)^{\varepsilon} dt
\end{aligned}$$

for some positive number ε. Since the last integral is finite, we are done. $\square$

Since from Theorem 2.1 $D^\alpha K_\delta$ is a compactly supported Schwartz distribution on $\mathbb{R}^{d+1}$ given by (2.1) its convolution with a $\mathscr{C}^\infty(\mathbb{R}^{d+1})$ function $v(x,t)$ is well defined and is nothing but the weak derivative of $K_\delta * v$.

Corollary 2.4. *For any $v \in \mathscr{C}^\infty(\mathbb{R}^{d+1})$ we have that*

$$D^\alpha(K_\delta * v)(x,t) = \big(D^\alpha K_\delta\big) * v(x,t)$$
$$= \delta^{-|\alpha|} \iint\limits_{\mathbb{R}^{d+1}} \big(N^\alpha\big)_\delta (x-y, t-s)\big[v(y,s) - P_{|\alpha|-1}v(y,s)\big]\,dy\,ds.$$

3. Hardy–Littlewood and Calderón–Scott maximal operators. A technical lemma

In this section we introduce the maximal operators that we shall use in the proof of Theorem 4.1 of Section 4. We shall also review the basic boundedness properties of those operators and we shall state and prove some technical lemmas that will be used in Section 4.

The one-sided character in the time variable of the kernel $K(x,t)$ introduced in Section 1 leads us to handle the one-sided Hardy–Littlewood maximal operator

$$M^-(g)(t) = \sup_{h>0} \frac{1}{h} \int_{t-h}^{t} \big|g(s)\big|\,ds$$

defined for any $g \in L^1_{\mathrm{loc}}(\mathbb{R})$.

On the other hand, since the projection on the space of the space–time heat ball $E(x,t;r)$ is an Euclidean ball centered at x and since in the space variable we shall look for regularity properties of temperatures, the natural operator is given by the sharp maximal function of order λ. For a given positive number λ and a given $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ function f, define

$$M^{\#,\lambda}(f)(x) = \sup_{r>0} \inf_{\pi \in \mathcal{P}_{[\lambda]}} \frac{1}{|B(x;r)|^{1+\frac{\lambda}{d}}} \int_{B(x;r)} \big|f(y) - \pi(y)\big|\,dy,$$

where $[\lambda]$ is the largest integer less than or equal to λ and $\mathcal{P}_m$ is the space of all polynomials of degree at most m.

The boundedness properties of these two types of operators have been extensively studied. Regarding the one-sided operator M^-, let us only point out that since $M^-(g) \leqslant 2M(g)$ where M is the centered Hardy–Littlewood maximal operator, then the boundedness of M^- on $L^p(dt)$ $(1 < p \leqslant \infty)$ follow from the same property for M. On the other hand the boundedness of M^- as an operator on weighted Lebesgue spaces has been obtained by E. Sawyer [10], see also K. Martín-Reyes [8]. Actually these results provide a class of weights which is strictly larger than the usual Muckenhoupt A_p weights which is associated to M.

Regarding the sharp maximal operator, let us mention the comprehensive treatment of maximal functions measuring regularity by R. DeVore and R. Sharpley [4]. From [4] we shall borrow the following estimate which holds for any $\lambda > 0$ and any $1 \leqslant p \leqslant \infty$,

$$\big\|M^{\#,\lambda}(f)\big\|_{L^p(\mathbb{R}^d)} \leqslant c\|f\|_{B_p^{\lambda,p}(\mathbb{R}^d)},$$

where $\|f\|_{B_p^{\lambda,p}(\mathbb{R}^d)}$ denotes the Besov norm with regularity exponent λ and $p = q$. See [9].

We shall actually deal with the maximal operator defined for any smooth function and $k > \lambda > 0$, with $k \in \mathbb{N}$ by

$$M^{\#,\lambda,k}(f)(x) = \sup_{r>0} \frac{1}{|B(x;r)|^{1+\frac{\lambda}{d}}} \int_{B(x;r)} \left| f(y) - P_x(y) \right| dy, \tag{3.1}$$

where $P_x(y)$ is the Taylor polynomial of degree $k-1$ for f at x. Notice that when λ is noninteger and $k = [\lambda] + 1$, this operator is the Calderón–Scott maximal operator (see [1]).

The next lemma provides pointwise estimates in terms of the one-sided maximal function for convolution operators which naturally appear in the proof of Theorem 4.1.

For a given $L^1(\mathbb{R})$ kernel κ, a given $L^1(\mathbb{R})$ function g and a given positive number δ, let us define

$$\kappa^*(g)(t) = \sup_{\delta>0} \left| \frac{1}{\delta} \int_{\mathbb{R}} \kappa\left(\frac{s}{\delta}\right) g(t-s)\, ds \right|.$$

Lemma 3.1. *Set $\kappa(t) = t^{\vartheta}(\ln \frac{1}{t})^{\theta} \mathcal{X}_{(0,1)}(t)$ with $-1 < \vartheta < 0$ and $\theta > 0$. Then, there exists a constant C depending only on ϑ and θ such that $\kappa^*(g)(t) \leqslant C M^-(g)(t)$ for every integrable function g defined on $\mathbb{R}$.*

Proof. Let us first show following the lines of [3, Chapter 10], that if κ is a nonnegative integrable kernel supported on $\mathbb{R}^+$ and nonincreasing on $\mathbb{R}^+$, then for every $\delta > 0$

$$\left| \frac{1}{\delta} \int_{\mathbb{R}} \kappa\left(\frac{s}{\delta}\right) g(t-s)\, ds \right| \leqslant 4 \left(\int_{\mathbb{R}} \kappa \right) M^-(g)(t). \tag{3.2}$$

By dyadic decomposition of $\mathbb{R}^+$ and since κ is nonincreasing, we get

$$\begin{aligned}
\left| \frac{1}{\delta} \int_{\mathbb{R}} \kappa\left(\frac{s}{\delta}\right) g(t-s)\, ds \right| &\leqslant \sum_{j\in\mathbb{Z}} \frac{1}{\delta} \int_{\delta 2^j \leqslant s < \delta 2^{j+1}} \kappa\left(\frac{s}{\delta}\right) \left| g(t-s) \right| ds \\
&\leqslant \sum_{j\in\mathbb{Z}} \frac{1}{\delta} \kappa\left(2^j\right) \int_{0 \leqslant s \leqslant \delta 2^{j+1}} \left| g(t-s) \right| ds \\
&= 2 \sum_{j\in\mathbb{Z}} 2^j \kappa\left(2^j\right) \left(\frac{1}{\delta 2^{j+1}} \int_{0 \leqslant s \leqslant \delta 2^{j+1}} \left| g(t-s) \right| ds \right) \\
&\leqslant 2 \left(\sum_{j\in\mathbb{Z}} 2^j \kappa\left(2^j\right) \right) M^-(g)(t)
\end{aligned}$$

and we also have that

$$\int_{\mathbb{R}^+} \kappa(s)\, ds = \sum_{j\in\mathbb{Z}} \int_{2^{j-1}}^{2^j} \kappa(s)\, ds \geqslant \sum_{j\in\mathbb{Z}} \kappa\left(2^j\right)\left(2^j - 2^{j-1}\right) = \frac{1}{2} \sum_{j\in\mathbb{Z}} 2^j \kappa\left(2^j\right).$$

Hence we obtain (3.2).

To show the integrability of the special kernel $\kappa(t) = t^{\vartheta}(\ln \frac{1}{t})^{\theta}\mathcal{X}_{(0,1)}(t)$ notice that with $\vartheta - \varepsilon > -1$ and $\varepsilon > 0$ we have

$$\kappa(t) = t^{\vartheta-\varepsilon} \cdot t^{\varepsilon}\left(\ln \frac{1}{t}\right)^{\theta}\mathcal{X}_{(0,1)}(t)$$

and that $t^{\varepsilon}(\ln \frac{1}{t})^{\theta}$ is bounded on $(0, 1]$. It is easy to see for $t \in (0, 1)$ that κ is nonincreasing. □

4. Maximal function estimates for the convolution operator induced by the distribution $D^{\alpha}K_{\delta}$ on $\mathbb{R}^{d+1}$

We know, see Corollary 2.4, that the derivatives of the convolution of K_{δ} ($\delta > 0$) with a $\mathscr{C}^{\infty}(\mathbb{R}^{d+1})$ function v is given by

$$D^{\alpha}(K_{\delta} * v)(x,t) = \delta^{-|\alpha|} \iint\limits_{\mathbb{R}^{d+1}} \left(N^{\alpha}\right)_{\delta}(x-y, t-s)\left[v(y,s) - P_{|\alpha|-1}v(y,s)\right] dy\, ds. \quad (4.1)$$

For a given positive real number λ, and k any integer larger than λ, let us define the maximal operator

$$\mathcal{M}^{\lambda,k}(v)(x,t) = \sup_{\delta>0} \delta^{k-\lambda}\left|\nabla^{k}(K_{\delta} * v)(x,t)\right|,$$

where ∇^{k} is the vector of all the space derivatives of order k. The main result of this section is the following pointwise estimate for $\mathcal{M}^{\lambda,k}$.

Theorem 4.1. *For $0 < \lambda < k < \lambda + d$ and $k \in \mathbb{N}$ there exists a constant $C = C(\lambda, k, d)$ such that the inequality*

$$\mathcal{M}^{\lambda,k}(v)(x,t) \leqslant C M^{-}\left[M^{\#,\lambda,k}(v)\right](x,t) \quad (4.2)$$

holds for every $\mathscr{C}^{\infty}$ function v defined on $\mathbb{R}^{d+1}$.

Let us point out that the right-hand side in (4.2) is the iteration of the operators $M^{\#,\lambda,k}$ acting on x and M^{-} acting on the time variable, precisely,

$$M^{-}\left[M^{\#,\lambda,k}(v)\right](x,t) = \sup_{h>0} \frac{1}{h}\int\limits_{t-h}^{t} M^{\#,\lambda,k}\left(v(\cdot,s)\right)(x)\, ds.$$

Proof of Theorem 4.1. Take $v \in \mathscr{C}^{\infty}(\mathbb{R}^{d+1})$ and fix $\delta > 0$. In order to estimate $\nabla^{k}(K_{\delta} * v)$ we shall consider a fixed multi-index α of length k and we shall estimate $D^{\alpha}(K_{\delta} * v)$ using the representation formula (4.1). Now, from (2.2.1), the kernel N^{α} splits as a finite sum $\sum_{i\in I} N_i^{\alpha}$ of kernels each of them bounded above in absolute value by kernels with the following basic shape $\widetilde{N}_i^{\alpha}(x,t) = \mathcal{Q}_i^{\alpha}(|x|,t)\eta_i^{\alpha}(\nu(x,t))$ with η_i^{α} a $\mathscr{C}^{\infty}$ function of a real variable with compact support and $\mathcal{Q}_i^{\alpha}(|x|,t)$ is $\mathscr{C}^{\infty}$ on $\mathbb{R}_{+}^{d+1} = \{(x,t) \in \mathbb{R}^{d+1}\colon t > 0\}$, and increasing as a function of $|x|$ for t

fixed. Also the kernel $\mathcal{Q}_i^\alpha$ is parabolically homogeneous of degree $-|\alpha|-2$, in other words, $\mathcal{Q}_i^\alpha(\mu|x|,\mu^2 t)=\mu^{-|\alpha|-2}\mathcal{Q}_i^\alpha(|x|,t)$, $\mu>0$. Since in the proof of (2.2.4) we actually show that $N_i^\alpha[\varphi-P_{|\alpha|-1}\varphi]$ is integrable for each $i\in I$, each integral

$$\iint\limits_{\mathbb{R}^{d+1}}\big(N_i^\alpha\big)_\delta(x-y,t-s)\big[v(y,s)-P_{|\alpha|-1}v(y,s)\big]\,dy\,ds$$

is absolutely convergent and its sum for $i\in I$ gives us a representation of $D^\alpha(K_\delta*v)(x,t)$. Hence in order to show (4.2), we have to estimate the maximal operator induced by anyone of these terms with kernel $(N_i^\alpha)_\delta$. From the absolute convergence of the integral, and from Fubini's theorem we have that

$$\begin{aligned}\mathcal{M}_{i,\delta}^\lambda(v)(x,t)&:=\delta^{k-\lambda}\left|\delta^{-|\alpha|}\iint\limits_{\mathbb{R}^{d+1}}\big(N_i^\alpha\big)_\delta(x-y,t-s)\big[v(y,s)-P_{|\alpha|-1}v(y,s)\big]\,dy\,ds\right|\\&\leqslant\delta^{-\lambda}\int\limits_{t-\frac{\delta^2}{4\pi}}^{t}\int\limits_{y\in B}\big(\widetilde{N}_i^\alpha\big)_\delta(x-y,t-s)\big|v(y,s)-P_{|\alpha|-1}v(y,s)\big|\,dy\,ds,\end{aligned}$$

where B is the Euclidean ball $B(x,R_\delta(t-s))$ with

$$R_\delta(t-s)=\sqrt{2d(t-s)\ln\frac{\delta^2}{4\pi(t-s)}}.$$

Next we multiply and divide the inner space integral by the Lebesgue measure of the ball B raised to the power of $1+\frac{\lambda}{d}$ and we use the boundedness of η_i^α in order to obtain the upper estimate

$$\begin{aligned}&\mathcal{M}_{i,\delta}^\lambda(v)(x,t)\\&\quad\leqslant c\delta^{-\lambda}\int\limits_{t-\delta^2}^{t}|B|^{1+\frac{\lambda}{d}}\left\{\frac{1}{|B|^{1+\frac{\lambda}{d}}}\int\limits_{B}\big(\mathcal{Q}_i^\alpha\big)_\delta\big(|x-y|,t-s\big)\big|v(y,s)-P_{|\alpha|-1}v(y,s)\big|\,dy\right\}ds\\&\quad\leqslant\frac{c}{\delta^2}\int\limits_{t-\delta^2}^{t}\delta^{2-\lambda+k-d}\big(R_\delta(t-s)\big)^{d+\lambda}\mathcal{Q}_i^\alpha\big(R_\delta(t-s),t-s\big)M^{\#,\lambda,k}\big(v(\cdot,s)\big)(x)\,ds,\end{aligned}$$

where the monotonicity property of $\mathcal{Q}_i^\alpha$ in its first variable together with the fact $|x-y|<R_\delta(t-s)$, the homogeneity of $\mathcal{Q}_i^\alpha$ and the definition of $M^{\#,\lambda,k}$ have been used. Notice that from the definition of $R_\delta(t-s)$ and the homogeneity of $\mathcal{Q}_i^\alpha$, we have

$$\mathcal{Q}_i^\alpha\big(R_\delta(t-s),t-s\big)=\mathcal{Q}_i^\alpha\left((t-s)^{\frac{1}{2}}\sqrt{2d\ln\frac{\delta^2}{4\pi(t-s)}},t-s\right)$$

$$= \frac{1}{(t-s)^{\frac{k+2}{2}}} \mathcal{Q}_i^\alpha\left(\sqrt{2d\ln\frac{\delta^2}{4\pi(t-s)}}, 1\right).$$

By inspection of the terms in the expansion for N^α given in (2.2.1) we see that for each $i \in I$, we have

$$\mathcal{Q}_i^\alpha\left(\sqrt{2d\ln\frac{\delta^2}{4\pi(t-s)}}, 1\right) \leqslant c\left(\ln\frac{1}{4\pi(\frac{t-s}{\delta^2})}\right)^{\theta_i} \quad \text{with } \theta_i > 0.$$

Hence

$$\mathcal{M}_{i,\delta}^\lambda(v)(x,t) \leqslant \frac{c}{\delta^2}\int_{t-\delta^2}^{t} \delta^{2-\lambda+k-d}\frac{(t-s)^{\frac{d+\lambda}{2}}}{(t-s)^{1+\frac{k}{2}}}\left(\ln\frac{1}{4\pi(\frac{t-s}{\delta^2})}\right)^{\theta_i+\frac{d+\lambda}{2}} M^{\#,\lambda,k}\big(v(\cdot,s)\big)(x)\,ds$$

$$= \frac{c}{\delta^2}\int_{t-\delta^2}^{t}\left(\frac{t-s}{\delta^2}\right)^{\frac{d+\lambda-2-k}{2}}\left(\ln\frac{1}{4\pi(\frac{t-s}{\delta^2})}\right)^{\theta_i+\frac{d+\lambda}{2}} M^{\#,\lambda,k}\big(v(\cdot,s)\big)(x)\,ds.$$

Since from our choice of k and λ, $\frac{d+\lambda-2-k}{2} > -1$, from Lemma 3.1 the last integral is bounded by the one-sided Hardy–Littlewood maximal operator M^- for each $\delta > 0$ and each $i \in I$, hence $\mathcal{M}^{\lambda,k}(v) \leqslant CM^-[M^{\#,\lambda,k}(v)]$. $\square$

5. Estimates for space and time derivatives of temperatures on cylindrical domains

In this section we shall write Ω to denote a cylindrical domain on $\mathbb{R}^{d+1}$ of the form $D \times \mathbb{R}^+$, where D is an open set in $\mathbb{R}^d$.

We shall use the standard parabolic distance function defined on $\mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$ as

$$\rho\big((x,t);(y,s)\big) = \max\big\{|x-y|, \sqrt{|t-s|}\big\}.$$

Let us define the function $\delta(x,t)$ on Ω as the parabolic distance of $(x,t) \in \Omega$ to the parabolic boundary of Ω. Precisely,

$$\delta(x,t) = \rho\big((x,t), \partial_{\text{par}}\Omega\big) = \inf\big\{\rho\big((x,t);(y,s)\big)\colon (y,s) \in \partial_{\text{par}}\Omega\big\},$$

where $\partial_{\text{par}}\Omega = (D \times \{0\}) \cup (\partial D \times \mathbb{R}^+)$ is the parabolic boundary of Ω.

Notice that for any temperature u in Ω, any $(x,t) \in \Omega$ and any $0 < \delta < \delta(x,t)$, the mean value formula (1.3) holds true since for those values of δ the support of $K_\delta(x-y,t-s)$ as a function of (y,s) is contained in Ω. Moreover, the same is true for the support of the kernel $(N^\alpha)_\delta(x-y,t-s)$ when $0 < \delta < \delta(x,t)$. In particular the formula for the space derivatives of $K_\delta * v$ given in Corollary 2.4 remains true for $v \in \mathscr{C}^\infty(\Omega)$ for these values of δ.

From the above observations and the results of the previous section, we readily realize that we shall be able to obtain estimates for a local version of the maximal function $\mathcal{M}^{\lambda,k}$ in terms of local versions of M^- and $M^{\#,\lambda,k}$.

For a given $\lambda > 0$, k any integer larger than λ and $v \in \mathscr{C}^\infty(\Omega)$, we define

$$\mathcal{M}_\Omega^{\lambda,k}(v)(x,t) = \sup_{0<\delta<\delta(x,t)} \delta^{k-\lambda} \left|\nabla^k (K_\delta * v)(x,t)\right|.$$

For a given $L^1_{\mathrm{loc}}(\mathbb{R})$ function g supported on $\mathbb{R}^+$ and a given $t > 0$, let us write

$$M^-_{\mathbb{R}^+}(g)(t) = \sup_{0<h<t} \frac{1}{h} \int_{t-h}^{t} \left|g(s)\right| ds,$$

to denote the local version of the one-sided maximal operator restricted to $\mathbb{R}^+$. On the other hand, for a given f smooth function on D we define the local version of the Calderón–Scott maximal function of order λ by

$$M_D^{\#,\lambda,k}(f)(x) = \sup_{0<\delta<\delta(x)} \frac{1}{|B(x;\delta)|^{1+\frac{\lambda}{d}}} \int_{B(x;\delta)} \left|f(y) - P_x(y)\right| dy,$$

where $\delta(x) = \inf\{|x-y|\colon\ y \in \partial D\}$ and P_x is the Taylor polynomial of degree $k-1$ for f at x.

By simple inspection of the proofs of Theorem 4.1 and Lemma 3.1 and the above remarks, we have the following result.

Theorem 5.1. *For $0 < \lambda < k < \lambda + d$ and $k \in \mathbb{N}$, there exists a constant $C = C(\lambda, k, d)$ such that for every $\Omega = D \times \mathbb{R}^+$ with D open in $\mathbb{R}^d$ the inequality*

$$\mathcal{M}_\Omega^{\lambda,k}(v)(x,t) \leqslant C M^-_{\mathbb{R}^+}\left[M_D^{\#,\lambda,k}(v)\right](x,t)$$

holds for every function v in $\mathscr{C}^\infty(\Omega)$ and every $(x,t) \in \Omega$.

When the above result is applied to a temperature $u = u(x,t)$ in Ω we have the following statement.

Corollary 5.2. *If u is a temperature in Ω, then*

$$\delta^{k-\lambda}(x,t)\left|\nabla^k u(x,t)\right| \leqslant C M^-_{\mathbb{R}^+}\left[M_D^{\#,\lambda,k}(u)\right](x,t)$$

for every $(x,t) \in \Omega$, when $0 < \lambda < k < \lambda + d$, $k \in \mathbb{N}$.

In order to obtain estimates for mixed space–time derivatives of temperatures, let us introduce some notation. Given a smooth function v on Ω let us write $\nabla^{2,1} v(x,t)$ to denote the (d^2+1)-vector given by the d^2 second-order purely spatial derivatives of v and the first derivative of v with respect to time, i.e., $\nabla^{2,1} v = (\nabla^2 v, \frac{\partial v}{\partial t})$. We shall also say that given a multi-index $\widetilde{\alpha} = (\alpha_1, \ldots, \alpha_d; \alpha_{d+1})$ in $\mathbb{N}_0^{d+1}$ the derivative $\partial^{\widetilde{\alpha}} v$ is of parabolic order $|\alpha| + 2\alpha_{d+1}$. By $(\nabla^{2,1})^n(v)$, $n \in \mathbb{N}$, we mean the vector of all the derivatives of parabolic order $2n$ of the smooth function v. Explicitly, each component of $(\nabla^{2,1})^n(v)$ has the form $\partial^{\widetilde{\alpha}} v$ with $|\alpha| + 2\alpha_{d+1} = 2n$. So that we always have, in each one of these derivatives, an even number of space derivatives. We shall use the notation $|(\nabla^{2,1})^n u|$ for the Euclidean length of $(\nabla^{2,1})^n u$. The next elementary lemma shall allow us to transfer the spatial estimates in Corollary 5.2 to mixed space–time derivatives for temperatures.

Lemma 5.3. *Let u be a temperature in Ω and $\widetilde{\alpha} = (\alpha; \alpha_{d+1}) \in \mathbb{N}_0^{d+1}$ with $2n = |\alpha| + 2\alpha_{d+1}$, $n \in \mathbb{N}$. Then the derivative $\partial^{\widetilde{\alpha}} u$ belongs to the linear span of $\nabla^{2n} u$.*

Proof. Induction in α_{d+1}. When $\alpha_{d+1} = 1$, we have that

$$\partial^{\widetilde{\alpha}} u = \partial^{(\alpha;0)}\big(\partial^{(0;1)} u\big) = \partial^{(\alpha;0)}(\Delta u) \in \operatorname{span} \nabla^{2n} u.$$

Assuming that result holds for $\alpha_{d+1} = j \in \mathbb{N}$ and let us prove it for $\alpha_{d+1} = j+1$,

$$\partial^{\widetilde{\alpha}} u = \partial^{(\alpha;0)} \partial^{(0;j+1)} u = \partial^{(\alpha;0)} \partial^{(0;j)} \Delta u = \partial^{(\alpha;0)} \Delta \partial^{(0;j)} u.$$

Notice now that the last term is a linear combination of derivatives of the form $\partial^{(\beta;0)}\partial^{(0;j)} u$, with $|\beta| = |\alpha| + 2$. So that we can apply the induction hypothesis to each one of this terms to obtain that $\partial^{\widetilde{\alpha}} u$ is in the linear span of $\nabla^{2n} u$. $\square$

As a corollary of Theorem 5.1, its Corollary 5.2 and the above considerations we readily have that the next statement holds true.

Theorem 5.4. *For $0 < \lambda < 2n < \lambda + d$ and $n \in \mathbb{N}$ there exists a constant C such that the inequality*

$$\delta^{2n-\lambda}(x,t)\big|\big(\nabla^{2,1}\big)^n u(x,t)\big| \leqslant C M^{-}_{\mathbb{R}^+}\big[M_D^{\#,\lambda,2n}(u)\big](x,t)$$

holds for every temperature u in Ω and every $(x,t) \in \Omega$.

6. L^p-estimates for space–time gradients of temperatures in terms of mixed Lebesgue–Besov norms

In analogy with the definition of $C_p^\lambda(D)$ (see [4, Section 6]) let us write $\mathscr{C}_p^{\lambda,m}(D)$ to denote the space of all those $L^p(D)$ functions f for which $M_D^{\#,\lambda,m}(f)$ belongs to $L^p(D)$ equipped with the norm $\|f\|_{\mathscr{C}_p^{\lambda,m}(D)} = \|f\|_{L^p(D)} + \|M_D^{\#,\lambda,m}(f)\|_{L^p(D)}$. From Corollary 5.4 in [4], when λ is a non-integer positive number, $M_D^{\#,\lambda} = \sup_{0<r<\delta(x)} \inf_{\pi\in\mathcal{P}_{[\lambda]}} |B(x;r)|^{-1-\frac{\lambda}{d}} \int_{B(x;r)} |f - \pi|$ is pointwise equivalent to the Calderón–Scott maximal operator, so that in this case $\mathscr{C}_p^{\lambda,[\lambda]+1}(D) = C_p^\lambda(D)$. Hence, from the immersion of Besov $B_p^{\lambda,p}(D)$ spaces into the space $C_p^\lambda(D)$ ([4, Corollary 11.6] and the footnote [2, p. 7]) when D is a Lipschitz domain, we conclude that $B_p^{\lambda,p}(D) \hookrightarrow \mathscr{C}_p^{\lambda,[\lambda]+1}(D)$ which, for a bounded D is continuously immersed in $\mathscr{C}_p^{\lambda,m}(D)$ for $m \geqslant [\lambda]+1$. Let us introduce the mixed norm spaces defined by $L^q(\mathbb{R}^+, \mathscr{C}_p^{\lambda,m}(D))$ and by $L^q(\mathbb{R}^+, B_p^{\lambda,p}(D))$ with $1 \leqslant q \leqslant \infty$, $1 \leqslant p \leqslant \infty$ and $\lambda > 0$. The corresponding norms for a $d+1$ variables function v are

$$\|v\|_{L^q(\mathbb{R}^+,\mathscr{C}_p^{\lambda,m}(D))} = \bigg(\int_{\mathbb{R}^+} \|v(\cdot,t)\|^q_{\mathscr{C}_p^{\lambda,m}(D)}\, dt\bigg)^{\frac{1}{q}}$$

and

$$\|v\|_{L^q(\mathbb{R}^+, B_p^{\lambda,p}(D))} = \left(\int_{\mathbb{R}^+} \|v(\cdot,t)\|^q_{B_p^{\lambda,p}(D)}\, dt \right)^{\frac{1}{q}}.$$

As a corollary of Theorem 5.4 we obtain the next basic result.

Theorem 6.1. *Let $\Omega = D \times \mathbb{R}^+$ with D a bounded domain in $\mathbb{R}^d$. Let $1 < p \leqslant \infty$ be given. For $0 < \lambda < 2n < \lambda + d$ and $n \in \mathbb{N}$ there exists a constant C_1 depending on p, λ and n such that for every temperature u in Ω we have the inequalities*

$$\left\| \delta^{2n-\lambda} \left| \left(\nabla^{2,1}\right)^n u \right| \right\|_{L^p(\Omega)} \leqslant C_1 \|u\|_{L^p(\mathbb{R}^+, \mathscr{C}_p^{\lambda,2n}(D))}.$$

Proof. Follows from Theorem 5.4 and the boundedness of $M^-_{\mathbb{R}^+}$ on $L^p(\mathbb{R})$ for $p > 1$. □

Notice also that for $p = 1$ a weak type inequality of the form

$$\begin{aligned} &\int_D \left| \left\{ t \in \mathbb{R}^+ \colon \delta^{2n-\lambda}(x,t) \left| \left(\nabla^{2,1}\right)^n u(x,t) \right| > \mu \right\} \right| dx \\ &\quad \leqslant \frac{C}{\mu} \int_D \left\| M_D^{\#,\lambda,2n} u(x,\cdot) \right\|_{L^1(\mathbb{R}^+)} dx \\ &\quad \leqslant \frac{C_1}{\mu} \|u\|_{L^1(\mathbb{R}^+, \mathscr{C}_1^{\lambda,2n}(D))} \end{aligned}$$

follows from the weak type $(1,1)$ of the time operator $M^-_{\mathbb{R}^+}$.

Since the weight functions $\omega(t)$ for which the one-sided maximal operator M^- is bounded on $L^p(\omega\, dt)$ $(p > 1)$ are characterized by the one-sided Muckenhoupt condition A_p^- (see [8,10]), the above results extend to mixed norms with weighted L^p-norms in the time variable.

Corollary 6.2. *If $\Omega = D \times \mathbb{R}^+$ with D a bounded Lipschitz domain in $\mathbb{R}^d$, $1 < p \leqslant \infty$, λ a non-integer positive number and n an integer such that $[\lambda] + 1 \leqslant 2n < \lambda + d$, then*

$$\left\| \delta^{2n-\lambda} \left| \left(\nabla^{2,1}\right)^n u \right| \right\|_{L^p(\Omega)} \leqslant C_2 \|u\|_{L^p(\mathbb{R}^+, B_p^{\lambda,p}(D))}$$

for some constant C_2 and every temperature u in Ω.

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