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Characterizations of postman sets

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Abstract

Using results by McKee and Woodall on binary matroids, we show that the set of postman sets has odd cardinality, generalizing a result by Toida on the cardinality of cycles in Eulerian graphs. We study the relationship between T-joins and blocks of the underlying graph, obtaining a decomposition of postman sets in terms of blocks. We conclude by giving several characterizations of T-joins which are postman sets.

Keywords: T-joins; Postman sets; Cardinality; Graph block; Decomposition

1. Basic notation and definitions

We will consider finite undirected graphs $G = (V, E)$, with loops and parallel edges allowed. The set of odd-degree vertices of G will be denoted by $O(G)$, or simply by O when it is clear what the underlying graph is. Most other notation and conventions for graphs are similar to those in West [5]. In particular, paths and cycles have no repeated vertices, and loops are cycles.

As it is defined for example in [1], given a subset T of vertices with $|T|$ even, a set of edges $J \subset E$ is a *T-join* if $O(G_J) = T$ where $G_J = (V, J)$. We will be interested in the family $\mathscr{J}(T)$ of minimal T-joins: an inclusion-wise minimal T-join is just a T-join such that G_J is acyclic (see Lemma 3.2 and Corollary 3.3 below). Of course, $\mathscr{J}(T)$ is a *clutter*, i.e. a family of subsets of some base set—here E—none of which is included in another.

When $T = \emptyset$, the empty set is the unique minimal $\emptyset$-join, and it is convenient to work instead with the clutter $\mathscr{C}$ of cycles (regarded as edge-sets), so that every non-empty $\emptyset$-join may be written as a union of disjoint cycles. When $T = O(G)$, the minimal T-joins are called *postman sets*, and we will indicate the corresponding clutter by $\mathscr{P}$.

We observe that although there are always postman sets, perhaps only the empty set (i.e. $\mathscr{P} = \{\emptyset\}$), we may have $\mathscr{J}(T) = \emptyset$ if some connected component of G contains an odd number of vertices of T. Similarly, $\mathscr{C}$ could be empty.

2. Introduction

In 1973, Toida [4] proved that in an Eulerian graph there is an odd number of cycles passing through any given edge. This can be shown by deleting the edge, say with endpoints u and v, from the graph and showing that there is an

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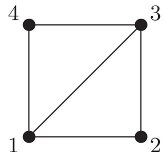

Fig. 1. Kite.

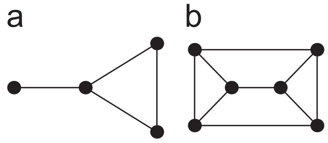

Fig. 2. Paw and triangular prism: (a) paw; (b) triangular prism ($K_2 \square C_3$).

odd number of (simple) u, v-paths in the resulting graph G'. In this case $O(G') = \{u, v\}$, and the u, v-paths in G' are precisely the postman sets in G'.

McKee [3] showed in 1984 that Toida's result actually characterizes Eulerian graphs: every edge is in an odd number of cycles if and only if $O(G) = \emptyset$. It is worth mentioning that in 1990, Woodall [6] gave an alternative proof of McKee's converse, and both McKee and Woodall obtained it as a consequence of more general results in the framework of binary matroids, which we reproduce here as Theorem 3.1.

We use McKee's and Woodall's results directly to show a characterization of the family of postman sets through a condition involving all minimal T-joins and (some) cycles, the precise statement being given in Corollary 3.4. As a consequence of this characterization, in Corollary 3.5 we generalize Toida's result to postman sets in any graph, obtaining that $\mathscr{P}$ has odd cardinality.

Certainly, for general T it is not true that $|\mathscr{J}(T)|$ is odd. For instance, in the kite of Fig. 1, there are four minimal T-joins when $T = \{2, 4\}$.

In view of McKee's result, it is natural to wonder whether $|\mathscr{J}(T)|$ odd implies $T = O$. However, this is not true. For example, in the kite of Fig. 1 with $T = \{1, 2\}$ we have $|\mathscr{J}(T)| = 3$, but $T \not\subset O$ and $O \not\subset T$.

A simple way of looking at McKee's converse of Toida's result is to consider the symmetric difference of all cycles. Similarly, E will be itself a T-join (see Lemma 3.2 below) and therefore $T = O$ if every edge is in an odd number of minimal T-joins and the number of minimal T-joins is odd. However, even for postman sets we do not always have the latter property. For example, in the paw of Fig. 2(a), the edges in the cycle do not belong to any postman set. And in the triangular prism (the Cartesian product $K_2 \square C_3$) shown in Fig. 2(b), each edge in the triangular bases belong to an even number (6) of postman sets.

Most of our results depend on Lemma 4.1 on the intersection of two clutters. We use it in Section 4 to show how T-joins and postman sets relate to cycles. Thus, if for general T we define

$$E_T = \{e \in E : e \in J \text{ for some } J \in \mathscr{J}(T)\},$$
$$H_T = \{e \in E : e \notin J \text{ for all } J \in \mathscr{J}(T)\},$$

we see in Lemma 4.2 that a cycle C is contained either in E_T or in H_T, in Lemma 4.3 that if $C \subset E_T$ then it is the symmetric difference of two T-joins, and in Theorem 4.4 that E_O may be written as the symmetric difference of some postman sets (and no cycles).

The example of the paw also suggests that the blocks of the graph play an important role in the structure of T-joins and postman sets, and we study this interplay in Section 5. In Theorem 5.4 we see that E_T and H_T are unions of blocks of G, necessarily disjoint. This is strengthened for postman sets in Lemmas 5.5 and 5.6, and Theorem 5.7, which gives a block decomposition of postman sets.

In Section 6 we combine our findings to give several characterizations of postman sets.

3. Toida and McKee's results for postman sets

Following Woodall [6], a *binary matroid* is a pair (S, W) where S is a finite set and W is a subspace of 2^S (with scalar operations modulo 2). Also, a *circuit* in a binary matroid (S, W) is a minimal non-empty set in W.

One of the main results in McKee [3] and Woodall [6] is:

3.1 Theorem (*McKee [3], Woodall [6]*). *Suppose (S, W) is a binary matroid. Then $S \in W$ if and only if each element of S lies in an odd number of circuits. Equivalently, S is the Boolean sum of some set of circuits if and only if S is the Boolean sum of the set of all circuits.*

Denoting by $A \triangle Z$ the symmetric difference of the sets A and Z, we will make frequent use of the following well known result (see e.g. [1, p. 168]):

3.2 Lemma. *If J' is a T'-join, then J is a T-join if and only if $J \triangle J'$ is a $(T \triangle T')$-join.*

One immediate consequence of this Lemma is that the symmetric difference of T-joins is a $\emptyset$-join which, if not empty, is the disjoint union of cycles. Also, a T-join J containing a cycle C cannot be minimal, since $J \triangle C = J \setminus C$ is a T-join strictly inside J. We state these results formally for future reference:

3.3 Corollary. *Suppose J is a T-join, and $\mathscr{J}(T)$ is the clutter of minimal T-joins. If J contains a cycle, then $J \notin \mathscr{J}(T)$. On the other hand, if $J \notin \mathscr{J}(T)$, then it is the disjoint union of a minimal T-join and cycles.*

We now consider the matroid (S, W) where $S = E$ and W is the linear subspace spanned by minimal T-joins and cycles. By the previous corollary, in this matroid every minimal T-join is a circuit, but there may be some cycles that are not circuits, namely cycles that contain all of the vertices in T. (Since $|T|$ is even, such a cycle can be split into two—necessarily minimal—T-joins. All other cycles are circuits of the matroid.) Thus we have:

3.4 Corollary. *E is the symmetric difference of all the postman sets and some of the cycles (namely, the circuits of the binary matroid (E, W) just mentioned).*

Conversely, if $O \neq \emptyset$ and E is the symmetric difference of all minimal T-joins and some cycles, then $T = O$.

Proof. Since E is an O-join, by Corollary 3.3 we may write E as the symmetric difference of a postman set and cycles. This implies $E = S \in W$ and Theorem 3.1 gives the first part of the result.

For the remaining part, we notice that a symmetric difference of T-joins and cycles is either a T-join or a $\emptyset$-join, according to the number of T-joins is odd or even. If $O \neq \emptyset$, E is not a $\emptyset$-join and therefore it must be both a T-*join* and an O-join, i.e. $T = O$. □

By Lemma 3.2, the symmetric difference of some postman sets and some cycles is either an O-join or a $\emptyset$-join, depending on whether or not an odd number of postman sets is considered in the symmetric difference. Since E is an O-join, the previous Corollary shows that the total number of postman sets must be odd. This is true even if $O = \emptyset$, where $\mathscr{P} = \{\emptyset\}$. Thus, we obtain the generalization of Toida's result to postman sets:

3.5 Corollary. *The family of postman sets of G has odd cardinality.*

Remark. Although to prove this result we relied on McKee's and Woodall's results, it could also be proved directly inside graph theory (without explicit mention of binary matroids), for example by induction on the number of edges.

Remark. If in Corollary 3.4 we have $O = \emptyset$, E may be written as a disjoint union of cycles, and by Theorem 3.1, E is the symmetric difference of all minimal T-joins and some cycles. But we may have $T \neq O$, e.g. if G is a triangle and $|T| = 2$. When $O = \emptyset \neq T$, $|\mathscr{J}(T)|$ must be even.

4. T-joins and cycles

We will need the following result on the intersection of two clutters. Notice that although it has a matroid—and even binary matroid—flavor, we are not asking directly for any matroid property.

4.1 Lemma. *Let $\mathscr{Y}$ and $\mathscr{Z}$ be clutters on the same base set X, and suppose $Y \in \mathscr{Y}$ is such that for every $Z \in \mathscr{Z}$ there exist $Y' \in \mathscr{Y}$ and $Z' \in \mathscr{Z}$ with $Y' \cap Z' = \emptyset$ and $Y' \cup Z' \subset Y \triangle Z$. Then*

$$Y \cap Z = \emptyset \quad \textit{for all } Z \in \mathscr{Z}.$$

Proof. Suppose there exists $Z \in \mathscr{Z}$ such that $Y \cap Z \neq \emptyset$ and consider $Z_0 \in \mathscr{Z}$ such that $Y \cap Z_0 \neq \emptyset$ and

$$|Y \cup Z_0| = \min\{|Y \cup Z|: Z \in \mathscr{Z} \text{ and } Y \cap Z \neq \emptyset\}.$$

By hypothesis, there exist $Y' \in \mathscr{Y}$ and $Z' \in \mathscr{Z}$ such that Y' and Z' are disjoint and $Y' \cup Z' \subset Y \triangle Z_0$.

Since $Z' \subset Y \triangle Z_0$ but $Y \cap Z_0 \neq \emptyset$, there are elements in Z_0 which are not in Z', i.e. $Z' \neq Z_0$ and since $\mathscr{Z}$ is a clutter, we must have $Z' \not\subset Z_0$. Therefore, there exist elements of Z' in Y, that is,

$$Y \cap Z' \neq \emptyset. \tag{4.1}$$

Since $Z' \subset Y \triangle Z_0 \subset Y \cup Z_0$, we have $Y \cup Z' \subset Y \cup Z_0$. But since $Z' \in \mathscr{Z}$ and $Y \cap Z' \neq \emptyset$, by our choice of Z_0 we must have $|Y \cup Z'| = |Y \cup Z_0|$, and (given the inclusion),

$$Y \cup Z' = Y \cup Z_0.$$

Furthermore, $Y' \setminus Y \subset (Y \cup Z_0) \setminus Y = (Y \cup Z') \setminus Y = Z' \setminus Y$, which implies $Y' \setminus Y = \emptyset$ since Z' and Y' are disjoint. Since Y and Y' are members of the clutter $\mathscr{Y}$ and $Y' \subset Y$, we have $Y' = Y$, but then $Y \cap Z' = Y' \cap Z' = \emptyset$, contradicting (4.1). □

A first consequence of Lemma 4.1 is the following:

4.2 Lemma. *Suppose $\mathscr{J}(T) \neq \emptyset$ and let $e \in E$ be such that $e \notin J$ for every minimal T-join J, i.e. $e \in H_T$. Then*

$$C \subset H_T \quad \textit{for every cycle } C \textit{ such that } e \in C.$$

Consequently for any cycle C either $C \subset E_T$ or $C \subset H_T$.

Also, every edge on a cycle which intersects some $J \in \mathscr{J}(T)$ must be in some minimal T-join.

Proof. For fixed $J \in \mathscr{J}(T)$, we use Lemma 4.1 with

$$\mathscr{Y} = \mathscr{J}(T), \quad \mathscr{Z} = \{C \in \mathscr{C}: e \in C\} \quad \text{and} \quad Y = J.$$

We need to show that given $C \in \mathscr{Z}$, there exist $J' \in \mathscr{J}(T)$ and $C' \in \mathscr{Z}$ such that $J' \cap C' = \emptyset$ and $J' \cup C' \subset J \triangle C$.

But $J_0 = J \triangle C$ is a T-join, which cannot be minimal since by hypothesis e is not in any minimal T-join and $e \in C \setminus J \subset J_0$. Therefore, there exist $J' \in \mathscr{J}(T)$ with $J' \subset J_0$ and a cycle C' with $e \in C' \subset J_0 \triangle J' = J_0 \setminus J'$, so that $J' \in \mathscr{J}(T)$ and $C' \in \mathscr{Z}$ are disjoint and $J' \triangle C' \subset J \triangle C$.

It follows from Lemma 4.1, that $J \cap C = \emptyset$ for every cycle C with $e \in C$. Since this holds for every $J \in \mathscr{J}(T)$, it follows that $C \subset H_T$ for every such cycle C. □

Similarly, we also have:

4.3 Lemma. *Suppose C_0 is a cycle such that $C_0 \cap J_0 \neq \emptyset$ for some $J_0 \in \mathscr{J}(T)$. Then there exist $J_1, J_2 \in \mathscr{J}(T)$ such that*

$$C_0 = J_1 \triangle J_2.$$

Remark. This is another way of showing that $C_0 \cap E_T \neq \emptyset$ implies $C_0 \subset E_T$.

Proof. It will be enough to show that

$$J_1 \triangle C_0 \in \mathscr{J}(T) \quad \text{for some } J_1 \in \mathscr{J}(T), \tag{4.2}$$

and then take $J_2 = J_1 \triangle C_0$.

Assume (4.2) does not hold, so that for each $J \in \mathscr{J}(T)$ the T-join $J \triangle C_0$ is not minimal.

In this case, by Corollary 3.3, for each $J \in \mathscr{J}(T)$ we may find a cycle C' and $J' \in \mathscr{J}(T)$ such that $J' \cap C' = \emptyset$ and $J \triangle C_0 \supset J' \triangle C' = J' \cup C'$.

Applying Lemma 4.1 with

$$\mathscr{Y} = \mathscr{C}, \mathscr{Z} = \mathscr{J}(T) \quad \text{and} \quad Y = C_0,$$

we have

$$J \cap C_0 = \emptyset \quad \text{for every } J \in \mathscr{J}(T),$$

contradicting the hypothesis of the lemma. □

We have seen in Corollary 3.4 that E may be written as the symmetric difference of (all) postman sets and (some) cycles. We now show that we need not consider cycles if $H_O = \emptyset$.

4.4 Theorem. *Suppose $O \neq \emptyset$ and let $E_O = \{e \in E : e \in P \text{ for some } P \in \mathscr{P}\}$. Then there exists $\{P_1, P_2, \ldots, P_s\} \subset \mathscr{P}$ with s odd and*

$$E_O = P_1 \triangle P_2 \triangle \cdots \triangle P_s.$$

If $H_O \neq \emptyset$, there also exist disjoint cycles $C_1, C_2, \ldots, C_t$ contained in H_O such that

$$H_O = C_1 \triangle \cdots \triangle C_t,$$

that is, H_O is a union of disjoint cycles.

Therefore,

$$E = P_1 \triangle \cdots \triangle P_s \triangle C_1 \triangle \cdots C_t,$$

where the cycles are pairwise disjoint and the postman sets do not intersect the cycles.

Proof. Since $O \neq \emptyset$, there exists $P_1 \in \mathscr{P}$, $P_1 \neq \emptyset$, and $E \backslash P_1$ is a (possibly empty) disjoint union of cycles. By Lemma 4.2 each of these cycles is either contained in E_O or in H_O, and so we may write

$$E \backslash P_1 = C'_1 \triangle \cdots \triangle C'_r \triangle C_1 \triangle \cdots C_t,$$

where $C'_i \subset E_O$ for $i = 1, \ldots, r$ and $C_i \subset H_O$ for $i = 1 \ldots t$.

By Lemma 4.3, for every $i = 1, \ldots, r$ we may find P_{2i} and P_{2i+1} in $\mathscr{P}$ such that $C'_i = P_{2i} \triangle P_{2i+1}$, hence

$$E_O = P_1 \triangle \cdots \triangle P_s,$$

with $s = 2r + 1$, and

$$H_O = C_1 \triangle \cdots \triangle C_t,$$

proving the result. □

Remark. The previous proof does not rely on the results by McKee and Woodall.

5. T-joins and blocks

In this section we study the connection between the block structure of the graph $G = (V, E)$ and the structure of minimal T-joins of G, using the following definition for blocks, which differs from that given by West [5, p. 155]:

5.1 Definition. Given a graph $G = (V, E)$, a set $B \subset E$ is a *block* of G if B consists of either a single cut-edge, a single loop, or the set of edges of a maximal 2-connected loopless subgraph of G.

Lemma 4.2 gives information about 2-connected subgraphs induced by blocks of G: either for any edge e in such a block there exists $J \in \mathscr{J}(T)$ with $e \in J$, or else the edges of the block do not intersect any minimal T-join. Since loops are in no minimal T-join, the other interesting blocks to us are the cut-edges (bridges), and these are taken care of by Lemma 5.3. To prove it, we will use the following well known result (see e.g. [1, p. 180]):

5.2 Lemma. *If $S \subset V$ and J is a T-join, then*

$$|S \cap T| \equiv |\delta(S) \cap J| \pmod 2,$$

where $\delta(S)$ is the set of edges having exactly one endpoint in S.

In particular, if $|S \cap T|$ is odd then $\delta(S) \cap J \neq \emptyset$.

5.3 Lemma. *Suppose $\mathscr{J}(T) \neq \emptyset$.*

(a) *If $e \in E$ is a cut-edge of G, then either $e \in J$ for every T-join J or $e \notin J$ for every $J \in \mathscr{J}(T)$.*
(b) *If e is not a cut-edge, then there exists a T-join J with $e \notin J$.*

Proof. Suppose e is a cut-edge such that $e \in J$ for some $J \in \mathscr{J}(T)$, and let u and v be its endpoints. If $G_u = (V_u, E_u)$ is the connected component of $G' = (V, E \setminus \{e\})$ containing u, then $\delta(V_u) \cap J = \{e\}$, and therefore, by Lemma 5.2, $|V_u \cap T|$ must be odd and $e \in J'$ for every T-join J'.

For the second part, if e is not a cut-edge, then there exists a cycle C with $e \in C$. If $J \in \mathscr{J}(T)$ is such that $e \in J$, then $J \triangle C$ contains a minimal T-join J' with $e \notin J'$. □

Combining Lemmas 4.2 and 5.3 we have:

5.4 Theorem. *E_T is the union of some blocks of G, and H_T is the union of the remaining blocks.*

Since the blocks of the underlying graph are shared by T-joins for different T's, it is not surprising that we may use them to relate minimal T-joins and postman sets:

5.5 Lemma. *Let $H_O = \{e \in E: e \notin P$ for every $P \in \mathscr{P}\}$. Then for arbitrary T, either no minimal T-join intersects H_O or else every T-join does.*

Proof. If $O = \emptyset$, then $H_O = E$, and the result is obvious. So let us consider the case $O \neq \emptyset$ and suppose there exist $J \in \mathscr{J}(T)$ and $e \in H_O \cap J$. We will show that every $J' \in \mathscr{J}(T)$ must intersect H_O.

If $e \in J'$, then obviously J' intersects H_O. Otherwise $J \triangle J'$ contains a cycle C such that $e \in C \subset J \triangle J'$. But C is inside H_O, and therefore J' has an edge in H_O ($C \not\subset J$ since J contains no cycles). □

5.6 Lemma. *$e \in E$ is a cut-edge if and only if $e \in P$ for every $P \in \mathscr{P}$.*

Proof. Suppose e is a cut-edge. Then, since H_O is a union of cycles or empty (Lemma 4.4), $e \notin H_O$. This implies that e is in some postman set, and by the first part of Lemma 5.3, that $e \in P$ for every $P \in \mathscr{P}$.

The converse is covered by the second part of Lemma 5.3. □

Let $B_1, B_2, \ldots, B_r$ be the blocks of $G = (V, E)$, and for $i = 1, \ldots, r$ let O_i be the set of odd-degree vertices of $G_i = (V_i, B_i)$, where V_i is the set of endpoints of the edges in B_i.

Since the B_i's are pairwise disjoint, we may write $E = B_1 \triangle \cdots \triangle B_r$ and therefore $O(G) = O_1 \triangle \cdots \triangle O_r$. Hence, if $\mathscr{P}_i$ is the family of postman sets in G_i and for each $i = 1, \ldots, r$ we choose $P_i \in \mathscr{P}_i$,

$$P_1 \triangle \cdots \triangle P_r = P_1 \cup \cdots \cup P_r$$

will be a postman set in G since it is an O-join having no cycles (if it contained one, it would be inside a block and hence contained in one of the P_i's).

Suppose now $P \in \mathscr{P}$ and the block B_i consists of a single edge e. If e is a loop then $\mathscr{P}_i = \{\emptyset\}$ and $e \notin P$, i.e. $P_i = \emptyset = P \cap B_i$ is the only postman set in G_i. On the other hand, if e is a cut-edge in G, $P_i = \{e\}$ is the unique postman set in $\mathscr{P}_i$ by Lemma 5.6, and since $e \in P$ we have $P_i = P \cap B_i$.

If G_i is 2-connected, we see that $E \backslash P$ is a union of disjoint cycles in G ($P \neq E$, since B_i contains a cycle), so that $B_i \cap (E \backslash P) = B_i \backslash P$ is also a union of disjoint cycles in G_i, and therefore $B_i \cap P$ is an O_i-join which must contain a postman set $P_i \in \mathscr{P}_i$.

Taking the union (which equals the symmetric difference) of all P_i's so constructed, we obtain a postman set in G which is contained in P and therefore must be precisely P. Thus $P_i = P \cap B_i$ for all $i = 1, \ldots, r$.

We sum up these findings in a theorem:

5.7 Theorem. *There is a one to one correspondence between* $\mathscr{P}$ *and* $\mathscr{P}_1 \times \cdots \times \mathscr{P}_r$, *given by*

$$P \to (P \cap B_1, \ldots, P \cap B_r) \quad \text{and} \quad (P_1, \ldots, P_r) \to P_1 \triangle \cdots \triangle P_r.$$

Consequently, if E' *is the union of some of the blocks of* G *and* $G' = (V, E')$, *then* $P \cap E'$ *is a minimal* $O(G')$*-join for every* $P \in \mathscr{P}$.

Remark. We may have $\mathscr{P}_i = \{\emptyset\}$—when the corresponding G_i is Eulerian—and this is the case if and only if B_i is one of the blocks forming H_O.

6. Postman sets

Let us denote by R the symmetric difference $O \triangle T$ (which may be empty), and by $\mathscr{J}(R)$ the corresponding clutter of minimal R-joins (with possibly $\mathscr{J}(R) = \{\emptyset\}$). We have:

6.1 Theorem. *If* $\mathscr{J}(T) \neq \emptyset$ *and* $G_T = (V, E_T)$, *then the following conditions are equivalent*:

(i) T *is the set of odd-degree vertices of* (V, E_T), *i.e.* $\mathscr{J}(T)$ *is the set of postman sets of* G_T.
(ii) $E_T = J_1 \triangle J_2 \triangle \cdots \triangle J_s$, *for some* $\{J_1, J_2, \ldots, J_s\} \subset \mathscr{J}(T)$ *and odd* s.
(iii) $|\mathscr{J}(T)|$ *is odd, and* $E_T = J_1 \triangle J_2 \triangle \cdots \triangle J_s$ *for some* $\{J_1, J_2, \ldots, J_s\} \subset \mathscr{J}(T)$.
(iv) *For every* $P \in \mathscr{P}$ *there exists* $J \in \mathscr{J}(T)$ *such that* $J \subset P$.
(v) *For every* $P \in \mathscr{P}$ *there exist* $J_P \in \mathscr{J}(T)$ *and* $D_P \in \mathscr{J}(R)$ *such that* J_P *and* D_P *are disjoint and* $P = J_P \cup D_P$.
(iv) E *is the disjoint union of* E_T, E_R *and* H_O.
(vii) E_T *and* E_R *are disjoint*.

Proof. (ii) Follows from (i) by Theorem 4.4, and if (ii) holds, then E_T is a T-join, which implies $O(G_T) = T$ and (i). Thus, (i) and (ii) are equivalent.

By Corollary 3.5, (iii) follows from (i) and (ii). Conversely, (iii) implies (i): by Theorem 3.1, E_T is the symmetric difference of all minimal T-joins and some cycles in G_T, and since $|\mathscr{J}(T)|$ is odd, E_T is both an $O(G_T)$-join and a T-join, which implies (i).

Let us now show the implications (i) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) $\Rightarrow$ (vii) $\Rightarrow$ (i).

The block decomposition given in Theorem 5.4 tells us that each of the blocks $B_1, \ldots, B_r$ of G is contained in either E_T (intersecting every T-join) or H_T (intersecting no minimal T-join). By Theorem 5.7, we know that for given $P \in \mathscr{P}$ and i ($1 \leqslant i \leqslant r$), $P \cap B_i \in \mathscr{P}_i$. By the same Theorem applied now to the graph $G_T = (V, E_T)$, we see that

$$J = \bigcup_{i: B_i \subset E_T} P \cap B_i$$

(that is, $J = \triangle_{i:B_i \subset E_T} P \cap B_i$) is a postman set in G_T. Thus, if $T = O(G_T)$, then $J = P \cap E_T$ is a T-join in G_T, so that (i) implies (iv).

Suppose now (iv) holds and for given $P \in \mathscr{P}$, let J be a T-join contained in P. Since P has no cycles, J is minimal and—even more—is the unique T-join contained in P, since if there were two, their symmetric difference would contain a cycle inside P. Let us denote by J_P this unique (minimal) T-join.

If $R = O \triangle T$, we observe that $D_P = P \triangle J_P$ is an R-join contained in P, and by the argument just used for J_P, we obtain that $D_P \in \mathscr{J}(R)$ and that D_P is the only R-join contained in P, so that (iv) implies (v).

We will now show that if (v) holds, then $J \cap D = \emptyset$ for every $J \in \mathscr{J}(T)$ and $D \in \mathscr{J}(R)$.[1] This follows from Lemma 4.1 by considering for fixed $J \in \mathscr{J}(T)$:

$$\mathscr{Y} = \mathscr{J}(T), \quad \mathscr{X} = \mathscr{J}(R) \quad \text{and} \quad Y = J,$$

and observing that given $D \in \mathscr{X}$, the O-join $J \triangle D$ contains a postman set P, which may be written as $J_P \cup D_P$ (by (v)), with $J_P \in \mathscr{J}(T)$, $D_P \in \mathscr{J}(R)$ and $J_P \cap D_P = \emptyset$. Therefore

$$E_T \cap E_R = \emptyset, \tag{6.1}$$

since no $e \in E$ may be simultaneously in a minimal T-join and a minimal R-join.

Moreover, by Lemma 5.5, we know that either H_O intersects every minimal T-join or it intersects none. Since $J_P \cap H_O = \emptyset$, we must have $J \cap H_O = \emptyset$ for all $J \in \mathscr{J}(T)$, and thus

$$E_T \cap H_O = \emptyset. \tag{6.2}$$

Similarly, since in Lemma 5.5 T is arbitrary, we may apply the same reasoning with T replaced by R (by the symmetry in condition (v)), so that

$$E_R \cap H_O = \emptyset. \tag{6.3}$$

Observing that every postman set is the disjoint union of a minimal T-join and a minimal R-join, we see that E_O is the disjoint union of E_T and E_R. Finally, since by definition $E_O \cup H_O = E$, we must have

$$E = E_T \cup E_R \cup H_O. \tag{6.4}$$

From Eqs. (6.1)–(6.4), we see that (v) implies (vi).

(vi) clearly implies (vii).

If (vii) holds then for $J \in \mathscr{J}(T)$ and $D \in \mathscr{J}(R)$ it must be that $J \cap D = \emptyset$. Hence $P = J \triangle D = J \cup D$ is an O-join which has no cycles since the blocks forming E_T and E_R are disjoint. So $P \in \mathscr{P}$. Also since $J \subset E_T$, $D \subset E_R$ and $E_T \cap E_R = \emptyset$, we have $J = P \cap E_T$, that is, J is the restriction of P to the blocks forming E_T. So by the last part of Theorem 5.7, $O(G_T) = T$ which is (i). □

Remark. Guenin in [2] gives another characterization of postman sets. By defining T-*cuts* to be sets of the form $\delta(U)$ with $|U \cap T|$ odd, when $T \neq \emptyset$ we have $T = O$ if and only if every T-cut (minimal or not) has an odd number of edges.

References

[1] W.J. Cook, W.H. Cunningham, W.R. Pulleyblank, A. Schrijver, Combinatorial Optimization, Wiley, New York, 1998.
[2] B. Guenin, Packing T-joins and edge colouring in planar graphs, Math. Oper. Res., to appear.
[3] T.A. McKee, Recharacterizing Eulerian: intimations of new duality, Discrete Math. 51 (1984) 237–242.
[4] S. Toida, Properties of a Euler graph, J. Franklin Inst. 295 (1973) 343–345.
[5] D. West, Introduction to Graph Theory, second ed., Prentice-Hall, Englewood Cliffs, NJ, 2001.
[6] D.R. Woodall, A proof of McKee's Eulerian-bipartite characterization, Discrete Math. 84 (1990) 217–220.

[1] Notice that this is actually condition (vii).