Spectral Duality for Finitely Generated Nilpotent Minimum Algebras, with Applications

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Abstract

We establish a categorical duality for the finitely generated Lindenbaum-Tarski algebras of propositional nilpotent minimum logic. The latter's conjunction is semantically interpreted by a left-continuous (but not continuous) triangular norm; implication is obtained through residuation. Our duality allows one to transfer to nilpotent minimum logic several known results about inutitionistic logic with the prelinearity axiom (also called Gödel-Dummett logic), *mutatis mutandis*. We give several such applications.

Keywords: Spectral duality, prime spectrum, amalgamation, interpolation, t-norm based logics.

1 Introduction

Hájek [15] introduced a family of non-classical propositional logics whose conjunction is semantically interpreted by *continuous triangular norms* [21], or *t-norms* for short. The semantics of implication remains thereby defined through *residuation* [see (1) in Section 2], a technique first investigated in algebraic contexts [20, and references therein]. Hájek's framework encompasses such time-honoured non-classical logics as Łukasiewicz infinite-valued logic, and intuitionistic logic with the prelinearity axiom [i.e. axiom (2) in Section 2], also known as *Gödel-Dummett logic*, or just *Gödel logic* in [15]. The literature on t-norm-based logics is nowadays extensive.

As it turned out [9], continuity is not necessary in order to perform residuation: continuity from the left alone suffices. Thus, one can extend Hájek's original scheme to include non-classical logics whose conjunction is semantically interpreted by *left-continuous t-norms*. The resulting framework, called *monoidal t-norm-based logic* (MTL, for short) is attracting growing interest (see for instance [9, 10, 14, 19, 24]). Note in particular that the system MTL is shown to have analytic hypersequent calculus in [2].

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In this article, we offer evidence that at least some logics based on left-continuous but discontinuous t-norms enjoy a reasonably rich duality theory. We carry out an in-depth case study of *nilpotent minimum logic* [9], from the point of view of algebraic and categorical logic. Even though the conjunction in nilpotent minimum is *not* idempotent, and thus, on the proof-theoretic side contraction fails (note that the weaker law $x * x * x = x * x$ holds), we shall nonetheless obtain tight connections with Gödel logic and its well-established duality theory. Let us outline the contents of the article.

The standard Lindenbaum-Tarski construction provides the algebraic counterpart of nilpotent minimum logic, namely, *nilpotent minimum algebras* [9], or *NM-algebras* for short (please see Section 2 for definitions). Throughout, we shall focus attention on finitely axiomatized theories in nilpotent minimum logic; their algebraic counterparts are finite NM-algebras.[1] Section 2 gathers the necessary preliminary notions on NM-algebras.

In Section 3, building on results in [3] and [6], we establish a spectral (or Stone-type) duality for finite NM-algebras. Theorem 3.5 and its Corollary 3.6 show that the dual of an NM-algebra can be described as a forest (i.e. a finite partially ordered set such that below any element there lies a subset that inherits a total order), such that each one of its trees (i.e. partially ordered subsets with a minimum) is enriched by one additional bit of information. Homomorphisms of NM-algebras dualize to order-preserving maps (between the corresponding forests) satisfying appropriate additional conditions. This seems to be the first instance of a (finite) spectral duality for a logic that is based on a discontinuous t-norm, and the rest of our article aims at showing that the construction is actually useful in obtaining further results.

Several applications of Corollary 3.6 are collected in Section 4.

- In 4.1, we build on the main result of [6] to obtain an explicit description of finite coproducts of finite NM-algebras;
- in 4.2, we derive a strong form of amalgamation for finite NM-algebras, along with the strongest possible form of the Deduction Theorem for nilpotent minimum logic;
- in 4.3, we establish a functional representation of free finitely generated NM-algebras that is the exact non-classical analogue of the folklore representation of free finitely generated Boolean algebras by Boolean functions on a Boolean cube and
- in 4.4, we give an exact recursive formula for the cardinality of free finitely generated NM-algebras,[2] that is, we solve the NM-algebraic analogue of Dedekind's problem on the cardinality of free finitely generated distributive lattices. (This should be compared with Horn's formula for the cardinality of free finitely generated Gödel algebras [17]; see also [1, 6] for different proofs of Horn's formula.)

For all our results we give essentially self-contained proofs, up to facts already proved in [3, 6].

2 Background on NM-algebras

An *integral commutative residuated lattice* is an algebraic structure $(A, *, \to, \wedge, \vee, \top)$ of type $(2, 2, 2, 2, 0)$ such that $(A, *, \top)$ is a commutative monoid, $(A, \wedge, \vee, \top)$ is a lattice with greatest element $\top$, and the following adjointness condition holds

$$x * y \leq z \ \text{ iff } \ x \leq y \to z \tag{1}$$

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[1] One easily checks that the variety of NM-algebras is locally finite; thus, finite, finitely generated and finitely presented NM-algebras coincide.

[2] compare with [1].

for any $x, y, z \in A$, where $\leq$ is the partial order induced on A by the lattice structure. (Throughout, 'iff' is short for 'if and only if'.) A *bounded* integral commutative residuated lattice is an integral residuated lattice with an extra constant $\perp$ that is the bottom of the induced lattice structure. In this case, a derived unary operation $\neg$ may be defined by $\neg x = x \to \perp$. For background on residuated lattices see [20]; for their relationship to non-classical logics, see [25].

An *NM-algebra* is a bounded integral commutative residuated lattice $(A, *, \to, \wedge, \vee, \perp, \top)$ that satisfies three extra equations:

$$(x \to y) \vee (y \to x) = \top \tag{2}$$

$$(x \to \perp) \to \perp = x \tag{3}$$

$$(x * y \to \perp) \vee (x \wedge y \to x * y) = \top. \tag{4}$$

For further background on NM-algebras and logics based on left-continuous t-norms, see [1, 3, 9, 14, 24].

As mentioned above, *negation* in NM-algebras is defined by

$$\neg x = x \to \perp. \tag{5}$$

Thus, (3) asserts that NM-algebras are a family of *involutive* residuated lattices, meaning that they satisfy the classical law of double negation. Following [14], we define the set of *positive* and *negative* elements of A by

$$A^+ = \{x \in A : x > \neg x\} \qquad A^- = \{x \in A : x < \neg x\},$$

respectively. In [16], it is proved that A can have at most one element x such that $x = \neg x$, which is then called the *negation fixpoint* (or simply *fixpoint*) of A.

A *Gödel algebra* is a Heyting algebra $(G, \to, \wedge, \vee, \perp, \top)$ satisfying the prelinearity axiom (2). Equivalently, a Gödel algebra is a bounded integral commutative residuated lattice $(G, *, \to, \wedge, \vee, \perp, \top)$ in which *idempotence* $(x \to (x * x) = \top)$ and *prelinearity* (2) hold. Indeed, it follows at once that the monoidal operation $*$ and the meet operation $\wedge$ coincide for Gödel algebras. *Generalized Gödel algebras* (also known as *relative Stone algebras* [7] and *Gödel hoops* [10]) are integral commutative residuated lattices with prelinearity and idempotence, but not necessarily bounded.

As usual, an NM-algebra that does not split into the direct product of two non-trivial NM-algebras is called *directly indecomposable*. Let us recall a characterization of indecomposable NM-algebras that is crucial to the sequel. To this end, we need to introduce connected and disconnected rotations of generalized Gödel algebras.

Definition 2.1
Let $D = (D, *, \to, \wedge, \vee, \perp, \top)$ be a generalized Gödel algebra. We define its disconnected rotation[3]

$$DR(D) = (D \times \{1\} \cup D \times \{0\}, \otimes, \Rightarrow, \sqcap, \sqcup, \perp, \top)$$

[3]Readers familiar with the definitions of connected and disconnected rotations given in [18] should notice that our terminology differs somewhat from that adopted there.

as an algebra of type (2, 2, 2, 2, 0, 0) with operations defined as follows.

$$(x, i) \sqcap (y, j) = (y, j) \sqcap (x, i) = \begin{cases} (x \wedge y, 1) & \text{if } i = j = 1, \\ (x \vee y, 0) & \text{if } i = j = 0, \\ (x, 0) & \text{if } i < j. \end{cases}$$

$$(x, i) \sqcup (y, j) = (y, j) \sqcup (x, i) = \begin{cases} (x \vee y, 1) & \text{if } i = j = 1, \\ (x \wedge y, 0) & \text{if } i = j = 0, \\ (y, 1) & \text{if } i < j. \end{cases}$$

$$(x, i) \otimes (y, j) = (y, j) \otimes (x, i) = \begin{cases} (x * y, 1) & \text{if } i = j = 1, \\ (\top, 0) & \text{if } i = j = 0, \\ (y \to x, 0) & \text{if } i < j. \end{cases}$$

$$(x, i) \Rightarrow (y, j) = \begin{cases} (x \to y, 1) & \text{if } i = j = 1, \\ (y \to x, 1) & \text{if } i = j = 0, \\ (x * y, 0) & \text{if } i > j, \\ (\top, 1) & \text{if } i < j. \end{cases}$$

$$\top = (\top, 1) \qquad \bot = (\top, 0).$$

Note that, upon defining $\neg(x, i)$ as $(x, i) \Rightarrow \bot$,

$$\neg(x, i) = \begin{cases} (x, 1) & \text{if i} = 0, \\ (x, 0) & \text{if i} = 1. \end{cases}$$

Similarly, the connected rotation of D is

$$CR(D) = \left(D \times \{1\} \cup \left\{ \left(\frac{1}{2}, \frac{1}{2} \right) \right\} \cup D \times \{0\}, \otimes, \Rightarrow, \sqcap, \sqcup, \bot, \top \right),$$

which is an algebra of type (2, 2, 2, 2, 1, 0, 0) with the operations $\sqcap, \sqcup, \otimes, \Rightarrow, \bot, \top$ given as in the disconnected rotation over $D \times \{1\} \cup D \times \{0\}$, and extended by

$$(x, i) \sqcap \left(\frac{1}{2}, \frac{1}{2} \right) = \left(\frac{1}{2}, \frac{1}{2} \right) \sqcap (x, i) = \begin{cases} \left(\frac{1}{2}, \frac{1}{2} \right) & \text{if } i = 1, \\ (x, i) & \text{otherwise,} \end{cases}$$

$$(x, i) \sqcup \left(\frac{1}{2}, \frac{1}{2} \right) = \left(\frac{1}{2}, \frac{1}{2} \right) \sqcup (x, i) = \begin{cases} (x, i) & \text{if } i = 1, \\ \left(\frac{1}{2}, \frac{1}{2} \right) & \text{otherwise,} \end{cases}$$

$$(x, i) \otimes \left(\frac{1}{2}, \frac{1}{2} \right) = \left(\frac{1}{2}, \frac{1}{2} \right) \otimes (x, i) = \begin{cases} \left(\frac{1}{2}, \frac{1}{2} \right) & \text{if } i = 1, \\ (\top, 0) & \text{otherwise,} \end{cases}$$

$$(x,i)\Rightarrow\left(\frac{1}{2},\frac{1}{2}\right)=\begin{cases}\left(\frac{1}{2},\frac{1}{2}\right) & \text{if } i=1,\\ (\top,1) & \text{otherwise,}\end{cases}$$

$$\left(\frac{1}{2},\frac{1}{2}\right)\Rightarrow(x,i)=\begin{cases}\left(\frac{1}{2},\frac{1}{2}\right) & \text{if } i=0,\\ (\top,1) & \text{otherwise,}\end{cases}$$

Note that

$$\neg\left(\frac{1}{2},\frac{1}{2}\right)=\left(\frac{1}{2},\frac{1}{2}\right).$$

A routine verification shows that the disconnected rotation of a generalized Gödel algebra is an NM-algebra without negation fixpoint, while its connected rotation is an NM-algebra with fixpoint. Moreover, we have:

THEOREM 2.2 ([3])
An NM-algebra A is directly indecomposable if and only if it is isomorphic either to the connected or to the disconnected rotation of a generalized Gödel algebra $G(A)$.

Specifically, if A is a directly indecomposable NM-algebra with (respectively, without) a negation fixpoint, then A is isomorphic to the connected (respectively, disconnected) rotation of the generalized Gödel algebra $G(A)=(A^+,*,\to,\wedge,\vee,\top)$. Conversely, it is clear that each generalized Gödel algebra G uniquely determines one directly indecomposable NM-algebra without fixpoint and one directly indecomposable NM-algebra with fixpoint: namely, the disconnected rotation and the connected rotation of G, respectively.

If A and B are directly indecomposable NM-algebras and $f: A\to B$ is a homomorphism, then the restriction of f to A^+ is an order preserving map into B^+ that also preserves the implication. The following lemma is an easy consequence of the definitions of connected and disconnected rotation.

LEMMA 2.3
Let A and B be directly indecomposable NM-algebras such that if A has a negation fixpoint p, then B also has a negation fixpoint q. Let $f: A^+\to B^+$ be an order preserving map that also preserves implication. Then f can be uniquely extended to a morphism $f': A\to B$ by setting $f'(p)=q$ if A has a negation fixpoint, and $f'(x)=\neg f(\neg x)$ for each $x\in A^-$.

3 Prime spectra of finite NM-algebras

Given an integral commutative residuated lattice A, a *filter of A* is a nonempty subset $\rho\subseteq A$ closed under $*$, and upwards closed in the underlying lattice order. The filter ρ is *prime* if $\rho\neq A$ and $x\vee y\in\rho$ implies $x\in\rho$ or $y\in\rho$.

All partially ordered sets (*posets*, for short) considered in this article are finite. Following standard usage, by a *chain* we mean a totally ordered set. Building on [6], in this section we develop a categorical duality between finite NM-algebras, and a combinatorial category consisting of certain suitably enriched finite posets, along with an appropriate notion of order-preserving maps between them. We recall the needed notions and results from [6].

Definition 3.1
If P is a poset and $S \subseteq P$, the *down-set* of S is the set of elements of P smaller than or equal to some element of S. A *forest* is a poset F such that for every $x \in F$, the down-set of x (i.e. $\{y \in F \mid y \leq x\}$) is a chain when endowed with the order inherited from F. A *tree* is a forest with a minimum element, called the *root* of the tree. A function $f: F \to G$ between forests is *open*, or an *open map*, if it is order-preserving and carries down-sets to down-sets.

Clearly, open maps between forests (more generally, posets) compose, and the identity function on a poset is open. Thus, posets and open maps form a category. We denote by F the category of forests and open maps between them.

Theorem 3.2 ([6])
The category of finite Gödel algebras and F are dually equivalent via the functor *Spec* that sends a finite Gödel algebra to the poset of its prime filters (ordered by *reverse* set-theoretic inclusion), and a morphism $f: A \to B$ of Gödel algebras to the open map given by

$$\mathrm{Spec}\,(f): \rho \in \mathrm{Spec}\,(B) \mapsto \{a \in A \mid f(a) \in \rho\} \in \mathrm{Spec}\,(A).$$

While the preceding theorem is concerned with finite Gödel algebras, one easily checks that the variety of Gödel algebras is locally finite. Therefore, finitely generated, finitely presented and finite Gödel algebras coincide. Similarly, as already mentioned, the variety of NM-algebras is also locally finite. In the sequel, we shall work with finite NM-algebras only. It follows that the corresponding generalized Gödel algebras given by Theorem 2.2 are necessarily finite.

If A is a directly indecomposable NM-algebra and $G(A)$ is the generalized Gödel algebra whose underlying set is A^+, then it is immediately checked that the set of prime filters of A is given by the union of the set of proper prime filters of $G(A)$ and the maximal filter whose underlying set is A^+. We shall denote by $G(A)_\perp$ the Gödel algebra obtained adjoining a bottom element to $G(A)$. Observe that the identity bijection between $G(A)_\perp \setminus \{\perp\}$ and A^+ preserves implication and order.

We shall denote by Spec (A) the poset of prime filters of A, ordered by reverse inclusion, and call it the *prime spectrum* of A. By the foregoing, along with the Theorem 3.2, we infer:

Lemma 3.3
If A is a directly indecomposable NM-algebra, the prime spectrum of the Gödel algebra $G(A)_\perp$ is order-isomorphic to *Spec* (A).

Consider now directly indecomposable NM-algebras A and B such that $G(A)$ is isomorphic (as a generalized Gödel algebra) to $G(B)$. Assume further that A has a fixpoint, whereas B has not. Then Spec (A) is order-isomorphic to Spec (B). In other words, the prime spectrum does not carry enough information to tell A and B apart. To remedy this situation, we shall augment Spec (A) with one additional bit of information.

Definition 3.4
A *labelled tree* is a pair (T, j), where T is a tree and $j \in \{0, 1\}$. We say (T, j) is of *type* j. A *labelled forest* is a set of labelled trees $F = \{(T_i, j_i) \mid i \in I\}$, for I a finite index set. A *morphism* of labelled trees $\phi: (T, j) \to (T', j')$ is an open map from T to T' such that $j \leq j'$. A *morphism* of labelled forests $\phi: F_1 \to F_2$ is specified by morphisms of labelled trees $\phi_i: (T_i, j_i) \to (T'_{i'}, j'_{i'})$, $i \in I$, where $(T'_{i'}, j'_{i'}) \in F_2$, for each $(T_i, j_i) \in F_1$.

It is trivial to check that morphisms of labelled trees and forests compose, and that the identity morphisms are given by the underlying set-theoretical identities. Thus, we obtain a category LF of labelled forests and their morphisms. We let LT denote the full subcategory of labelled trees, and DNM the category of finite directly indecomposable NM-algebras and their homomorphisms.

We next define a contravariant augmented spectral functor Spec^+ from DNM to LT as follows: for each directly indecomposable NM-algebra A, let

$$\mathrm{Spec}^+(A) = (\mathrm{Spec}\,(A), j)\ ,$$

where $j=0$ if A has a negation fixpoint, and $j=1$ otherwise. For each homomorphism $f\colon A \to B$ in DNM, let $\mathrm{Spec}^+(f)\colon \mathrm{Spec}^+(B) \to \mathrm{Spec}^+(A)$ be given by

$$\mathrm{Spec}^+(f)(\rho) = \{x \in A \mid f(x) \in \rho\}\ .$$

Notice that since there are no homomorphisms from NM-algebras with negation fixpoint into algebras without fixpoint, the duals of the morphisms in DNM are well defined morphisms in LT. Now a routine verification shows that $\mathrm{Spec}^+\colon \mathsf{DNM} \to \mathsf{LT}$ indeed is a contravariant functor.

Theorem 3.5
The categories DNM and LT are dually equivalent via Spec^+.

Proof. According to [22, Chapter IV], it is enough to prove that Spec^+ is full, faithful and essentially surjective, meaning that for every $(T,j) \in \mathsf{LT}$ there is $A \in \mathsf{DNM}$ such that $\mathrm{Spec}^+(A)$ is isomorphic to (T,j) in LT.

Claim 1. Spec^+ is essentially surjective.

Indeed, let (T,j) be given. Consider the Gödel algebra G_T whose dual (in the sense of Theorem 3.2) is order-isomorphic to T. If $j=0$, then (T,j) is isomorphic in LT to the image under Spec^+ of the connected rotation of the generalized Gödel algebra whose underlying set is $G_T \backslash \{\bot\}$.

If $j=1$, then (T,j) is isomorphic in LT to the image under Spec^+ of the disconnected rotation of that same Gödel algebra.

Claim 2. Spec^+ is full.

Let A, $B \in \mathsf{DNM}$, and let $\mathrm{Spec}^+(A) = (T_A, j)$ and $\mathrm{Spec}^+(B) = (T_B, i)$. Consider a morphism $\phi\colon (T_B, i) \to (T_A, j)$ of labelled trees. By definition, ϕ is an open map from T_B into T_A and $i \le j$. Consider the Gödel algebra G_{T_A} whose dual (in the sense of Theorem 3.2) is order-isomorphic to T_A. Define G_{T_B} analogously. Then there is a unique Gödel algebra homomorphism $f\colon G_{T_A} \to G_{T_B}$ such that ϕ is the dual of f. Note that G_{T_A} and G_{T_B} are Gödel algebras with a unique maximal filter, corresponding to the roots of T_A and T_B, respectively. Therefore, f restricts to a map

$$f'\colon G_{T_A} \backslash \{\bot\} \to G_{T_B} \backslash \{\bot\}$$

of generalized Gödel algebras that is order preserving, and preserves implication. By Lemma 2.3 there is a unique NM-algebra homomorphism $h\colon A \to B$ extending f'. If ρ is a prime filter of G_{T_B}, then

$$\mathrm{Spec}^+(h)(\rho) = \{x \in A \mid h(x) \in \rho\} = \{x \in A^+ \mid f'(x) \in \rho\} = \phi(\rho),$$

with the last equality granted by Theorem 3.2.

Claim 3. Spec^+ is faithful.

If $f\colon A \to B$ and $g\colon A \to B$ are distinct homomorphisms of NM-algebras in DNM, by Lemma 2.3 the order-preserving maps $A^+ \to B^+$ they induce are distinct. By Theorem 3.2, $\mathrm{Spec}^+(f)$ and $\mathrm{Spec}^+(g)$ are distinct morphisms in LT, and the proof is complete. ■

We now let FNM denote the category of all finite NM-algebras and their homomorphisms.

Corollary 3.6
The categories FNM and LF are dually equivalent via Spec^+.

Proof. By [3, Lemma 2.5], a finite NM-algebra A is isomorphic to a direct product of a finite family $(A_i, i \in I)$ of directly indecomposable NM-algebras, and this decomposition is unique up to isomorphism. Thus, $\mathrm{Spec}^+(A)$ is isomorphic to the labelled forest $\{(\mathrm{Spec}^+(A_i)) \mid i \in I\}$. A routine verification using Theorem 3.5 shows that $\mathrm{Spec}^+\colon \mathsf{FNM} \to \mathsf{LF}$ is full, faithful and essentially surjective. ■

The proof of Theorem 3.5 does not explicitly construct the (contravariant) adjoint to Spec^+, which we shall call NMFun for *NM-functions*. For our applications in the sequel, however, it is important to have an explicit description of this functor, and we provide one here.

Note that NMFun can be regarded as the finite analogue for NM-logic of the functor that associates to a Stone space X the Boolean algebra of all continuous $\{0, 1\}$-valued functions on X, where $\{0, 1\}$ is endowed with the discrete topology. Here the role of $\{0, 1\}$ is taken over by the augmented prime spectrum of the free NM-algebra on one generator, and that of finite Stone spaces (i.e. finite sets) by labelled forests. The continuity required in the Boolean case (that becomes immaterial for finite sets) is here replaced by the requirement that functions are open maps. As is well known, while the order-preserving requirement dually corresponds to preservation of the lattice operations (see e.g. [8]), the stronger openness requirement dually corresponds to preservation of the Heyting implication (see e.g. [13]).

For the rest of this article, for each integer $n \geq 0$, let C_n denote a (fixed, but otherwise arbitrary) chain of $n+1$ elements. Further, let $\mathcal{S}$ denote the labelled forest consisting of the labelled tree $(C_0, 0)$, along with two copies of the labelled tree $(C_1, 1)$ (Figure 1). As we shall see in Subsection 4.3, $\mathcal{S}$ is the augmented spectrum of the free singly generated NM-algebra.

For a labelled forest F, let

$$\mathrm{NMFun}(F) = \{f\colon F \to \mathcal{S} \mid f \text{ is a morphism in } \mathsf{LF}\} \tag{6}$$

denote the set of all morphisms in LF from F to $\mathcal{S}$. We shall now endow $\mathrm{NMFun}(F)$ with operations $*, \to, \wedge, \vee, \bot, \top$ so as to make it an NM-algebra.

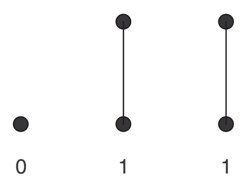

Figure 1. A picture of $\mathrm{Spec}^+(\mathcal{F}_1)$ — where $\mathcal{F}_1$ denotes the free singly generated NM-algebra — consisting of the chain C_0 of type 0, and two copies of the chain C_1 of type 1

First we need to distinguish the two copies of $(C_1, 1)$ in $\mathcal{S}$. To this purpose, let us denote the bottom elements of the first and second copy by $\perp$ and $\top$ and the top elements by $\perp^+$ and $\top^-$, respectively. We denote by 0 the only element of $(C_0, 0)$.

Since morphisms in LF are open maps, each $f \in \mathrm{NMFun}\,(F)$ carries trees to trees. It follows that each $f \in \mathrm{NMFun}\,(F)$ is completely determined by the preimages of 0, $\perp$ and $\top$.

To equip $\mathrm{NMFun}\,(F)$ with a structure of NM-algebra, we first pick a labelled tree (T, j) in the labelled forest F and consider the subset G_T of the set of morphisms $\mathrm{NMFun}\,(T, j)$ defined as

$$G_T = \{f \in \mathrm{NMFun}\,(T, j) : f^{-1}(\{\perp, \top, \top^-\}) = T\}.$$

Observe that the subforest $\{\perp, \top, \top^-\}$ of $\mathcal{S}$ is isomorphic to the prime spectrum of the free Gödel algebra over one free generator, and hence, by the duality of Theorem 3.2 and by [6, Remark 2], $(G_T, \to, \wedge, \vee, \perp, \top)$ is a Gödel algebra, with operations completely determined by specifying the preimage of $\top$, as follows: for any subset A of T, set

$$A\uparrow = \{x \in T \mid y \leq x \text{ for some } y \in A\}.$$

We then have:

$$\begin{aligned}(f \wedge g)^{-1}(\{\top\}) &= f^{-1}(\{\top\}) \cap g^{-1}(\{\top\})\\ \perp^{-1}(\{\top\}) &= \emptyset\\ (f \to g)^{-1}(\{\top\}) &= f^{-1}(\{\perp\}) \cup [(f^{-1}(\{\top, \top^-\}) \cap g^{-1}(\{\top, \top^-\}))\backslash \alpha\uparrow],\end{aligned}$$

where

$$\alpha = f^{-1}(\{\top\}) \cap g^{-1}(\{\top^-\}).$$

Clearly, the $\perp$-free reduct of $G_T\backslash\{\perp\}$ is the generalized Gödel algebra $(G_T^+, \to, \wedge, \vee, \top)$, for

$$G_T^+ = \{f \in \mathrm{NMFun}\,(T, j) \mid f^{-1}(\{\top, \top^-\}) = T\}.$$

Given $f \in G_T^+$, let $f' \in \mathrm{NMFun}\,(T, j)$ be given by $f'(x) = \perp$ if $f(x) = \top$ and $f'(x) = \perp^+$ if $f(x) = \top^-$. If the labelled tree (T, j) is such that $j = 1$ then there are no morphisms from the labelled tree (T, j) into $(C_0, 0)$. Consider the NM-algebra $DR(G_T^+)$. It is easy to see that the correspondence ε from $DR(G_T^+)$ into $\mathrm{NMFun}\,(T, j)$ given by

$$\varepsilon(f) = \begin{cases} f & \text{if } f \in G_T^+ \\ g' & \text{if } f = \neg g \quad \text{for some } g \in G_T^+ \end{cases}$$

is bijective. Thus we can give $\mathrm{NMFun}\,(T, j)$ a structure of NM-algebra isomorphic to $DR(G_T^+)$. Similarly, if (T, j) is such that $j = 0$, the correspondence that assigns to the fixpoint of $CR(G_T^+)$ the only morphism $f : (T, j) \to (C_0, 0)$ and behaves like ε over the remaining elements of $CR(G_T^+)$ defines a bijection between $CR(G_T^+)$ and $\mathrm{NMFun}\,(T, j)$. In this case we can equip $\mathrm{NMFun}\,(T, j)$ with a structure of NM-algebra isomorphic to $CR(G_T^+)$.

We conclude from Theorem 2.2 that $(\mathrm{NMFun}\,(T,j), *, \to, \wedge, \vee, \top, \bot)$ is a finite directly indecomposable NM-algebra. For the general case of the labelled forest F, $(\mathrm{NMFun}\,(F), *, \to, \wedge, \vee, \top, \bot)$ simply is the NM-algebra arising as the direct product of the family $\{\mathrm{NMFun}\,(T,j) \mid (T,j) \text{ a labelled tree in } F\}$.

Let us now turn to morphisms. If F and G are labelled forests, consider a morphism $\phi\colon F \to G$ in LF. We define a function $\mathrm{NMFun}\,(\phi)\colon \mathrm{NMFun}\,(G) \to \mathrm{NMFun}\,(F)$ as follows. If $g\colon G \to \mathcal{S}$ is an element of $\mathrm{NMFun}\,(G)$, we set

$$(\mathrm{NMFun}\,(\phi))(g) = g \circ \phi\ .$$

A routine verification shows that the ensuing function $\mathrm{NMFun}\,(\phi)$ preserves all operations $*, \to, \wedge, \vee, \bot, \top$, and thus is a homomorphism of NM-algebras. By definition, NMFun preserves composition and identities; hence, it is a contravariant functor from labelled forests to finite NM-algebras. One now straightforwardly verifies that $\mathrm{NMFun}: \mathsf{LF} \to \mathsf{FNM}$ is adjoint (in the contravariant sense) to $\mathrm{Spec}^+\colon \mathsf{FNM} \to \mathsf{LF}$.

4 Applications of Spectral Duality

4.1 Co-products

Recall that any variety is complete and co-complete (see e.g. [23]). Products of NM-algebras are just direct products with pointwise operations. Co-products in varieties of algebras are usually much harder to describe explicitly. As it turns out, an efficient Stone-type duality often provides the key to such a description, cf. [6, Introduction]. Using the main results in the latter paper, we shall show how to explicitly compute finite co-products of NM-algebras. As before, F is the category of forests and their open maps; now, we let T be the full subcategory of trees. Given trees S and T, the product $S \times T$ with its projection maps $\pi_S\colon S \times T \to S$, $\pi_T\colon S \times T \to T$ is constructed in [6, Theorem 3.6], to which the reader is referred for further information.

Proposition 4.1
Let (S, i) and (T, j) be labelled trees, and let $S \times T$, together with its projections π_S, π_T, be the product of S and T in T. The labelled tree

$$(S \times T, i \times j),$$

where $i \times j \in \{0, 1\}$ denotes multiplication of binary digits, endowed with the projection maps

$$\pi_{(S,i)}\colon (S \times T, i \times j) \to (S, i)\ , \pi_{(T,j)}\colon (S \times T, i \times j) \to (T, j)$$

induced by π_S and π_T, respectively, is the product of (S, i) and (T, j) in LT.

Proof. Since π_S and π_T are open maps, their labelled counterparts $\pi_{(S,i)}$ and $\pi_{(T,j)}$ are morphisms in LT, because $i \times j \le i, j$ for each $i, j \in \{0, 1\}$. To verify the universal property of products, let (P, k) be a labelled tree, and let $p_{(S,i)}\colon (P, k) \to (S, i)$, $p_{(T,j)}\colon (P, k) \to (T, j)$ be two morphisms of labelled trees. This implies $k \le i, j$. By Definition 3.4, there are unique corresponding open maps $p_S\colon P \to S$, $p_T\colon P \to T$ in T. Since $S \times T$, with π_S and π_T, is the product of S and T in T, there exists a unique open map $f\colon P \to S \times T$ such that

$$p_S = \pi_S \circ f \text{ and } p_T = \pi_T \circ f. \tag{7}$$

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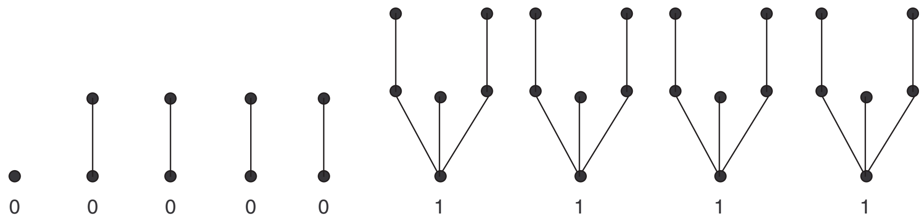

FIGURE 2. The augmented prime spectrum $\mathcal{S}^2$ of the free 2-generated NM-algebra

Then f induces a unique morphism $g: (P,k) \to (S \times T, i \times j)$: indeed, since $k \leq i, j$, we have $k \leq i \times j$, whence g exists. From (7) we immediately infer

$$p_{(S,i)} = \pi_{(S,i)} \circ g \text{ and } p_{(T,j)} = \pi_{(T,j)} \circ g\ . \tag{8}$$

To see that g is the unique morphism in LT satisfying (9), it is enough to note that its underlying open map of trees f is unique, by the universal property of products in T.
The extension of the preceding result to labelled forests is straightforward, cf. [6, p. 204], and shall be omitted for the sake of brevity. See Figure 2 in Subsection 4.3 below for a picture of $\mathcal{S}^2 = \mathcal{S} \times \mathcal{S}$, where $\mathcal{S}$ is as in Figure 1.

4.2 *Amalgamation, interpolation and the deduction theorem*

For background, and for an extensive analysis of interpolation in algebraic logic, see [11]. Recall that a variety has the *amalgamation property* if whenever S, A and B are given algebras with injective homomorphisms $S \to A$ and $S \to B$, there exists an algebra C with injective homomorphisms $A \to C$ and $B \to C$ such that the diagram

$$\begin{array}{ccc} S & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & C \end{array} \tag{9}$$

commutes.

Several possible stronger forms of the amalgamation property, or variations thereof, have been considered in the literature. (Please, see again [11] and references therein). Observe that a natural condition to ask of the commutative square (9) is that it be a push-out square—in other words, that C with the given maps be a co-product of A and B fibred over S. Explicitly, this means that for any algebra $\tilde{C}$ and homomorphisms $A \to \tilde{C}$, $B \to \tilde{C}$ commuting with the given $S \to A$, $S \to B$ as in (9), there is a unique homomorphism $C \to \tilde{C}$ making the triangles

$$\begin{array}{ccc} B & \longrightarrow & C \\ \| & & \downarrow \\ B & \longrightarrow & \tilde{C} \end{array} \qquad \begin{array}{ccc} A & \longrightarrow & C \\ \| & & \downarrow \\ A & \longrightarrow & \tilde{C} \end{array}$$

commute. When this holds, the amalgam C is canonical, because fibred co-products are of course unique up to isomorphism. Since fibred co-products exist in any variety, the point is then to show that the push-out of monomorphisms (that is, in the context of varieties, injective homomorphisms) are again monomorphisms. In traditional algebraic (especially group-theoretic) language, the same construction is known as a *free product with amalgamated subalgebra*, even though what is really involved is a *co*-product. To sum up, following tradition we shall say that a variety of algebras has *free products with amalgamation* if injectivity of homomorphisms is stable under push-outs.

In [6] it is proved by duality that finite Gödel algebras have free products with amalgamation. Here we shall prove the analogous result for finite NM-algebras.

PROPOSITION 4.2
The category of finite NM-algebras has free products with amalgamation.

PROOF. We first prove the dual property that epimorphisms are stable under pull-backs in LF. The question is at once reduced to LT. We shall use the following easy observation: a morphism in LT is an epimorphism if and only if its underlying open map of trees is an epimorphism in T if and only if its underlying set-theoretic function is surjective.

Now let (A, i), (B, j), and (S, k) be labelled trees, and let $s_{(A,i)}: (A, i) \to (S, k)$, $s_{(B,j)}: (B, j) \to (S, k)$ be epimorphisms. Form the pull-back

$$\begin{tikzcd} (C,h) \arrow[r, "p_{(A,i)}"] \arrow[d, "p_{(B,j)}"'] & (A,i) \arrow[d, "s_{(A,i)}"'] \\ (B,j) \arrow[r, "s_{(B,j)}"] & (S,k) \end{tikzcd} \tag{10}$$

Let us denote by p_A, p_B, s_A, s_B the underlying open maps of $p_{(A,i)}$, $p_{(B,j)}$, $s_{(A,i)}$, $s_{(B,j)}$, respectively. Note that s_A and s_B are epimorphisms, because they come from epimorphisms in LT. It is immediate to verify that

$$\begin{tikzcd} C \arrow[r, "p_A"] \arrow[d, "p_B"'] & A \arrow[d, "s_A"'] \\ B \arrow[r, "s_B"] & S \end{tikzcd}$$

is a pull-back square in T, because (10) is a pull-back square in LT. By [6, Corollary 4.1], if p_A and p_B are epimorphisms, so are s_A and s_B, and thus so are $s_{(A,i)}$ and $s_{(B,j)}$.

Upon applying Theorem 3.5, we conclude that monomorphisms are stable under push-outs in FNM. Since monomorphisms are the same thing as injective homomorphisms of NM-algebras, we are done. ■

The logical relevance of amalgams is that, for a variety of algebras, the amalgamation property together with the congruence extension property is equivalent to strong deductive interpolation (see [12]). In the presence of the deduction theorem for the corresponding propositional logic, deductive interpolation is equivalent to Craig (implicative) interpolation (cf. [5, 3.1.1]). Recall that Craig interpolation holds for a logic if, given two formulas ϕ, ψ such that $\phi \to \psi$ is provable in the logic, then there exists a formula θ, the *interpolant*, whose variables occur both in ϕ and ψ, such that $\phi \to \theta$ and $\theta \to \psi$ are provable.

It should be noted that nilpotent minimum logic does not enjoy Craig interpolation, as the formula $\psi = (x \wedge \neg x) \to (y \vee \neg y)$ shows. For, ψ is clearly a tautology of the logic, and an interpolant would be a variable-free formula θ such that both $(x \wedge \neg x) \rightarrow \theta$ and $\theta \rightarrow (y \vee \neg y)$ are tautologies. Needless to say, such a θ does not exist.

For nilpotent minimum logic the Deduction Theorem in its classical form would state that for all pair of formulas ϕ and ψ over the set of propositional variables $\{x_1, \ldots, x_n\}$, it holds that $\phi \models \psi$ iff $\models \phi \rightarrow \psi$, where $\phi \models \psi$ iff in any NM-algebra A and for all $a_1, \ldots, a_n \in A$, $\phi(a_1, \ldots, a_n) = \top$ implies $\psi(a_1, \ldots, a_n) = \top$, while $\models \psi$ means that $\psi(a_1, \ldots, a_n) = \top$ in each NM-algebra A and for all $a_1, \ldots, a_n \in A$.

To show that the classical Deduction Theorem does not hold for nilpotent minimum logic, it suffices to check the following counterexample. Consider the NM-algebra $\mathrm{NMFun}(\mathcal{S})$ and let $i \in \mathrm{NMFun}(\mathcal{S})$ denote the identity function $i(x) = x$ for all $x \in \mathcal{S}$. Notice that $(i * i)(x) = \bot$ if $x \in \{0, \bot, \bot^+\}$, while $(i * i)(\top) = \top$ and $(i * i)(\top^-) = \top^-$. Hence $i^{-1}(\{\top\}) = (i * i)^{-1}(\{\top\})$, but $(i \rightarrow (i * i))^{-1}(\{\top\}) \neq \mathcal{S}$, as $(i \rightarrow (i * i))(0) = 0$ and $(i \rightarrow (i * i))(\bot^+) = \top^-$.

On the other hand, the following Local Deduction Theorem (see [4, Thm. 1]) is stated (for the larger class of monoidal t-norm-based logics) in [9]: $\phi \models \psi$ iff there is an integer $n > 0$ such that

$$\models \underbrace{\phi * \cdots * \phi}_{n \text{ times}} \rightarrow \psi$$

holds. Nilpotent minimum logic enjoys the following form of Global Deduction Theorem.

Proposition 4.3
For each pair of formulas ϕ, ψ,

$$\phi \models \psi \qquad \text{iff} \qquad \models (\phi * \phi) \rightarrow \psi.$$

Proof. Immediate, since the equation $x * x = x * x * x$ holds in every NM-algebra. ∎

4.3 Representation of free NM-algebras

The description of co-products provided in Subsection 4.1, along with [6, Theorem 3.6], affords a representation theorem for finite NM-algebras, to which we now turn. For related work based on different techniques, we refer the reader to [1, 3, 26].

For each integer $n \geq 0$, let us write $\mathcal{F}_n$ to denote the free NM-algebra over n generators. Let us further write $\mathcal{S}^n$ for the product in LF of $\mathcal{S}$ with itself n times.

Lemma 4.4
For each integer $n \geq 0$, $\mathrm{Spec}^+(\mathcal{F}_n)$ is isomorphic, as a labelled forest, to $\mathcal{S}^n$.

Proof. The main point is:

Claim. $\mathrm{Spec}^+(\mathcal{F}_1)$ is isomorphic to $\mathcal{S}$.

By the duality between FNM and LF, the claim is equivalent in showing that $\mathrm{NMFun}(\mathcal{S})$ is isomorphic to $\mathcal{F}_1$ as an NM-algebra.

One checks that the subdirectly irreducible finite NM-algebras are precisely the finite NM-chains. Direct inspection shows that there are precisely three singly generated NM-chains that are not proper quotients of some other singly generated NM-chain. Specifically, they are $N_0 = \{\bot < x = \neg x < \top\}$, $N_1 = \{\bot < x < \neg x < \top\}$, $N_2 = \{\bot < \neg x < x < \top\}$, where x denotes the generator. Clearly, $\mathcal{F}_1$ is the largest singly generated NM-algebra, and it embeds into the direct product $N_0 \times N_1 \times N_2$. Hence $|\mathcal{F}_1| \leq 48$.

NMFun$(\mathcal{S})$ is generated by the identity function $i{:}\, x \mapsto x$ for all $x \in \mathcal{S}$: it is an easy (if somewhat lengthy) exercise to verify that each one of the 48 elements $f \in$ NMFun$(\mathcal{S})$ can be written as $\phi_f(i)$ for some NM-term $\phi_f(x)$.

In details (compare with [1, Sec. V, Case C]), let $\delta_0 := (x \to \neg x) * (\neg x \to x) * (x \to \neg x) * (\neg x \to x)$, $\delta_\top := \neg(\neg(x * x) * \neg(x * x))$ and $\delta_\bot := \neg(\neg(\neg x * \neg x) * \neg(\neg x * \neg x))$. Moreover, let ϕ_0 be the term $\bot$ if $f(0) = \bot$, the term x if $f(0) = 0$, the term $\top$ if $f(0) = \top$. Similarly, let $\phi_\bot$ (respectively $\phi_\top$) be the term $\bot$ if $f(\bot^+) = \bot$ (resp. $f(\top^-) = \bot$), the term $\top$ if $f(\bot^+) = \top$ (resp. $f(\top^-) = \bot$), the term x if $f(\bot^+) = \bot^+$ (resp. $f(\top^-) = \top^-$), the term $\neg x$ if $f(\bot^+) = \top^-$ (resp. $f(\top^-) = \bot^+$). One checks that $\phi_f := (\delta_0 \wedge \phi_0) \vee (\delta_\bot \wedge \phi_\bot) \vee (\delta_\top \wedge \phi_\top)$ is such that $\phi_f(i) = f$. Hence NMFun$(\mathcal{S}) \cong N_0 \times N_1 \times N_2 \cong \mathcal{F}_1$. Take an NM-algebra B and let $b(i)$ be an element of B. The function φ: NMFun$(\mathcal{S}) \to B$ defined as $\varphi(f) := \phi_f(b(i))$ is the (necessarily unique) NM-homomorphism extending b. This settles our claim.[4]

To complete the proof, note that, as in any variety, $\mathcal{F}_n$ is a co-product of n copies of $\mathcal{F}_1$. By Corollary 3.6, an application of Spec$^+$ yields the lemma. ■

For the sake of clarity, we observe that $\mathcal{F}_0$ is the NM-algebra $\{\bot, \top\}$, whose augmented spectrum Spec$^+(\mathcal{F}_0)$ is isomorphic to $\{(C_0, 1)\}$. Dually, then, $\mathcal{S}^0 = \{(C_0, 1)\}$.

As an immediate consequence of the fact that NMFun is (contravariant) adjoint to Spec$^+$, together with the fact that, by Lemma 4.4, Spec$^+(\mathcal{F}_n)$ is isomorphic to $\mathcal{S}^n$, we obtain the following representation result.

Proposition 4.5
For each integer $n \geq 0$, NMFun$(\mathcal{S}^n)$ is isomorphic to $\mathcal{F}_n$, the free NM-algebra over n generators.

We emphasize once more that the preceding proposition only becomes fully effective when coupled with the explicit description of products of labelled forests provided in Subsection 4.1. By way of example, Figure 2 displays a picture of $\mathcal{S}^2$.

4.4 Determining the size of free NM-algebras

As a final application of our duality, we obtain an exact recursive formula for the cardinality of the free finitely generated NM-algebra.

Throughout this subsection, fix an integer $n \geq 1$. Let us display $\mathcal{S}$ as

$$\left\{ \binom{1}{0}(C_0, 0),\ 2\binom{1}{1}(C_1, 1) \right\} \tag{11}$$

meaning that it consists of $\binom{1}{0} = 1$ copies of $(C_0, 0)$, and $2\binom{1}{1} = 2$ copies of $(C_1, 1)$. Consider Newton's binomial expansion

$$(1 + 2x)^n = 1 + \sum_{k=1}^{n} \binom{n}{k} 2^k x^k. \tag{12}$$

[4]For a less computational proof of the same claim, one can verify the dual of the universal property of free objects for $\mathcal{S}$ in the category of labelled forests; that is, one verifies that $\mathcal{S}$ is cofree in LF.

Observe that C_0 acts as neutral element under product in T, whereas C_1 acts as an element of infinite period (i.e. $C_1^h \neq C_1^k$ unless $h=k$). Thus, by (11) and (12), we can display $\mathcal{S}^n$ as

$$\mathcal{S}^n = \left\{(C_0,0), 2\binom{n}{1}(C_1^1,0), \ldots, 2^k\binom{n}{k}(C_1^k,0), \ldots, 2^n(C_1^n,1)\right\} \tag{13}$$

Note that here the type of C_1^k is 1 if and only if $k=n$, because otherwise by a trivial induction (C_1^k, j) has at least one factor of type 0, and thus has type $j=0$. By Proposition 4.5, we know

$$|\mathcal{F}_n| = |\,\mathsf{Mor}\ (\mathcal{S}^n, \mathcal{S})| \tag{14}$$

where $|\cdot|$ denotes cardinality, and $\mathsf{Mor}\ (A,B)$ is the number of morphisms in LF from A to B. By direct inspection,

$$|\mathsf{Mor}\ ((C_1^n,0),\ \mathcal{S})| = 1 + 2|\,\mathsf{TMor}\ (C_1^n, C_1)| \tag{15}$$

and, similarly,

$$|\mathsf{Mor}\ ((C_1^n,1),\ \mathcal{S})| = 2|\,\mathsf{TMor}\ (C_1^n, C_1)| \tag{16}$$

where $\mathsf{TMor}\ (A,B)$ denotes the number of morphisms from A to B in the category of trees T. By (14) we have

$$|\mathcal{F}_n| = 3 \times \left(\prod_{k=1}^{n-1} |\mathsf{Mor}\ ((C_1^k,0),\ \mathcal{S})|^{2^k\binom{n}{k}}\right) \times |\mathsf{Mor}\ ((C_1^n,1),\ \mathcal{S})|^{2^n} \tag{17}$$

where the first factor 3 accounts for the number of morphisms $(C_0,0) \to \mathcal{S}$; note that the second factor evaluates to 1 when $n=1$. By (15), (16) and (17), we obtain

$$|\mathcal{F}_n| = 3 \times \left(\prod_{k=1}^{n-1} (1 + 2|\mathsf{TMor}\ (C_1^k, C_1)|)^{2^k\binom{n}{k}}\right) \times 2^{2^n}|\mathsf{TMor}\ (C_1^n, C_1)|^{2^n} \tag{18}$$

Thus it remains to determine $|\,\mathsf{TMor}\ (C_1^k, C_1)|$. But by [6, Corollary 4.2], these numbers satisfy the recurrence

$$|\mathsf{TMor}\ (C_1^k, C_1)| = \prod_{m=1}^{k} (1 + |\mathsf{TMor}\ (C_1^{k-m}, C_1)|)^{\binom{k}{m}} \tag{19}$$

for each integer $k \geq 0$, where $C_1^0 = C_0$, and, again, the product evaluates to 1 when $k=0$. We have thus proved (compare with [1, Theorem 4.12]):

Proposition 4.6
For each integer $n \geq 1$, the cardinality of the free NM-algebra $\mathcal{F}_n$ satisfies the recurrence relations (18) and (19), subject to the initial condition $|\,\mathsf{TMor}\ (C_1^0, C_1)| = 1$.

See Figure 3 for a table of the first few values of $|\mathcal{F}_n|$.

n	$\lvert\mathcal{F}_n\rvert$
0	2
1	48
2	$3 \times 5^4 \times 2^4 \times 18^4 = 3.14928 \times 10^9$
3	$2.79794 \times 10^{70} < \lvert\mathcal{F}_3\rvert < 2.79795 \times 10^{70}$
4	$7.22237 \times 10^{750} < \lvert\mathcal{F}_4\rvert < 7.22238 \times 10^{750}$

FIGURE 3. The first few values of the cardinality $|\mathcal{F}_n|$ of the free NM-algebra on n generators

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