COMPOSITION OF OPERATORS IN ORLICZ SPACES

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Abstract. In this work we find sharp conditions for boundedness on Orlicz spaces of the composition of j operators, each one being of restricted weak type (p, p) for some $p > 1$, and of strong type (∞, ∞). Particularly, we find necessary and sufficient conditions to obtain modular inequalities for the j-times composition of the Cesàro Maximal function of order α. With this approach we treat a kind of strong maximal function related to Cesàro averages over n-dimensional rectangles.

1. Introduction

Let (Ω, μ) be a measure space and $\mathfrak{M}(\Omega)$ the space of measurable functions. Let $j \in \mathbb{N}$ and $T_1, T_2, \dots, T_j$, be sublinear operators defined on $\mathfrak{M}(\Omega)$, so that all of them are of strong type (∞, ∞), and for a given $p \geq 1$, of restricted weak type (p, p), $1 \leq k \leq j$, that is, there exist constants A_k and B_k such that for any measurable function $f \in \mathfrak{M}(\Omega)$

$$\|T_k f\|_\infty \leq A_k \|f\|_\infty,$$

and

$$\mu_{T_k f}(s) \leq \left(\frac{A_k}{s} \int_0^\infty \mu_f^{1/p} \right)^p \qquad \text{for all } s > 0,$$

where μ_g denotes the distribution of a measurable function g.

It is well known that for a sublinear operator T these two conditions can be expressed in just one inequality, namely

$$\mu_{Tf}(t) \leq \left(\frac{C}{t} \int_{t/C}^\infty \mu_f(s)^{1/p} ds \right)^p \qquad \text{for all } t > 0, \tag{1}$$

where C is a constant independent of f and t.

In turn, the last inequality is equivalent to

$$(Tf)^*(t) \leq C \, \frac{1}{t^{1/p}} \int_0^t f^*(s) s^{\frac{1}{p}-1} ds \,, \tag{2}$$

where g^* denotes the rearrangement of a measurable function g (see for example [1]).

Observe that the right hand side of (2) is the modified Hardy operator, $\mathcal{H}_p$ defined in (22) below, acting on the rearrangement of f. Furthermore, for this operator, inequality (2) is known to be an equivalence.

As we shall see these and even more general Hardy type operators will play a central role in modelling our situation and in obtaining the main results.

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Now we remind the definition of Orlicz spaces. Let $\Psi : [0,\infty] \to [0,\infty]$ an increasing function such that $\lim_{t\to 0}\Psi(t) = 0$ and $\lim_{t\to\infty}\Psi(t) = \infty$. The set of functions

$$L^{\Psi}(\Omega) = \{ f \in \mathfrak{M}(\Omega) : \int_{\Omega} \Psi(\epsilon\,|f|)\,d\mu < \infty \text{ for some } \epsilon > 0 \ \}$$

is called *Orlicz space* associated to Ψ.

In this work we are interested in boundedness properties in Orlicz spaces for the composition operator

$$T_1 \circ T_2 \circ \cdots \circ T_j, \tag{3}$$

where T_k, $1 \le k \le j$, satisfies the above conditions.

In [6], Neuguebauer deals with modular inequalities for a finite iterated composition of a single operator that behaves like the Hardy-Littlewood maximal function, that is, of strong type (∞,∞) and of weak type $(1,1)$. Such composition may no longer be weak type $(1,1)$ and therefore classical interpolation results can not be applied directly. However, they satisfy more general *weak type inequalities* like those introduced in [3]. In fact, Theorem 2 in [6] might be derived from the results in [3] after using Lemma 1 in [6].

In this paper we will give answers to the problem of finding modular inequalities for the composition operator (3). Since for $p = 1$, weak type (p,p) and restricted weak type (p,p) are the same, we deal with the case $p > 1$. Other classical operators that satisfy the assumptions made on T_k are the *one sided Cesàro maximal function operators of order* $\alpha = \frac{1}{p}$, defined for $f \in \mathfrak{M}([0,1])$ and $x \in [0,1]$ by

$$M_{\alpha}^{-} f(x) = \sup_{0\le c<x} \frac{1}{(x-c)^{\alpha}} \int_c^x |f(s)|(s-c)^{\alpha-1}ds \tag{4}$$

taking averages to the left, and

$$M_{\alpha}^{+} f(x) = \sup_{x<c\le 1} \frac{1}{(c-x)^{\alpha}} \int_x^c |f(s)|(c-s)^{\alpha-1}ds \tag{5}$$

taking averages to the right (see [5]). Let us notice that the case $\alpha = 1$ gives the one sided Hardy-Littlewood maximal functions.

We start by giving in Section 2 an estimate for the distribution function of the composition operator. This formula leads us to consider operators satisfying more general inequalities of this type (see (7)). These inequalities can be seen as the restricted weak type version of the ones appearing in [3]. In Section 3 we give modular inequalities for operators satisfying such general conditions and we apply the general result to obtain boundedness properties for the composition operator. Further, we show that the obtained modular inequalities are the best possible. As an application of these results, in Section 4 we deal with a kind of strong maximal function related with Cesàro averages over a family of rectangles on $\mathbb{R}^n$.

2. A distribution estimate

For clearness we will deal with the case when $T_1, T_2, \ldots, T_j$, are all the same operator, $T_k = T$ for $k = 1, 2, \ldots, j$. It is easy to modify the notation in the proofs of theorems and lemmas in this paper in order to obtain the same conclusions for the operator (3) when the $T_1, T_2, \ldots, T_j$, are not necessarily the same.

Let $T^{(j)} = \overbrace{T \circ \cdots \circ T}^{j \text{ times}}$ be the j-times composition of the operator T. Following [6], next Lemma gives us an estimate for the distribution of $T^{(j)} f$ for the case $p > 1$.

Lemma 1. *Let $p \geq 1$. If the operator T satisfies* (1) *then, $T^{(j)}$ satisfies*

$$\mu_{T^{(j)} f}(t) \leq \left[\frac{1}{(j-1)!t} \int_t^\infty \mu_f(s/C^j)^{1/p} \left[\log(s/t)\right]^{(j-1)} ds \right]^p \tag{6}$$

for all $t > 0$.

Proof. We proceed by induction on the number of iterations. For $j = 1$, inequalities (6) are (1) are the same. Now, suppose (6) is satisfied for some $j \geq 1$. If we call $g = T^{(j)} f$,

$$\begin{aligned} \mu(\{|T^{(j+1)} f| > t\}) &= \mu(\{|Tg| > t\}) \\ &\leq \left[\frac{C}{t} \int_{t/C}^\infty \mu_g(s)^{1/p} ds \right]^p \end{aligned}$$

and the last term is bounded by

$$\left[\frac{C}{t} \int_{t/C}^\infty \frac{1}{(j-1)!s} \int_s^\infty \mu_f(r/C^j)^{1/p} \left[\log(r/s)\right]^{(j-1)} dr\, ds \right]^p .$$

From Fubini-Tonelli's theorem, we have

$$\left[\frac{C}{(j-1)!t} \int_{t/C}^\infty \mu_f(r/C^j)^{1/p} \int_{t/C}^r \frac{1}{s} \left[\log(r/s)\right]^{(j-1)} ds\, dr \right]^p .$$

Performing the inner integral in the last expression we obtain

$$\left[\frac{C}{j!t} \int_{t/C}^\infty \mu_f(r/C^j)^{1/p} \left[\log(Cr/t)\right]^j dr \right]^p ,$$

which is the same as the second term of (6), after a change of variables.

□

3. Modular inequalities

Inequality (6) yields to consider operators T satisfying for some $p > 1$ and a constant C,

$$\mu_{Tf}(t) \leq \left[\frac{C}{t} \int_{t/C}^\infty \mu_f(s)^{1/p} w(s/t)\, ds \right]^p \qquad \text{for all } t > 0, \tag{7}$$

where $w : [1, \infty) \to [0, \infty)$ is continuous. Even tough we are interested in the case $w(s) = \log^{j-1}(s)$ there is no loss of clearness in working with a general w.

In what follows $a : [0, \infty) \to (0, \infty)$ and $b : [0, \infty) \to (0, \infty)$ will be continuous functions, with $\int_1^\infty a = \infty$ and b increasing. Let

$$\Phi(t) = \int_0^t a(s)\, ds \qquad \text{and} \qquad \Psi(t) = \int_0^t b(s)\, ds,$$

for all $t \geq 0$.

Since the more interesting applications concern the local behavior of Tf we shall work on a space Ω of finite measure. In this case the small values of t are irrelevant either in in condition (7) or in the definition of Ψ and Φ. Later, at the end of this section we present the corresponding results for Ω not necessarily of finite measure.

Theorem 1. *Let Ω be a finite measure space and T a sublinear operator satisfying (7). If there exists a constant C such that*

$$\sup_{t>1}\left(\int_1^t \frac{a(s)}{s^p}\, w^p(t/s)\, ds\right)^{1/p}\left(\int_t^\infty b(C\, s)^{-p'/p}\, ds\right)^{1/p'} < \infty \tag{8}$$

then, there exists a constant C' such that

$$\int_\Omega \Phi(|Tf|)\, d\mu \le C' + C' \int_\Omega \Psi(C'\, |f|)\, d\mu$$

for all $f \in \mathfrak{M}(\Omega)$.

Proof. Let f be a function in the domain of T. From (7),

$$\begin{aligned}\int_\Omega \Phi(|Tf|) d\mu &= \int_0^\infty a(s)\mu_{Tf}(s) ds\\ &\le \left(\int_0^1 + \int_1^\infty\right) a(s)\mu_{Tf}(s) ds\end{aligned}$$

is bounded by

$$\Phi(1)\mu(\Omega) + \int_1^\infty \frac{a(s)}{s^p}\left[\int_s^\infty \mu_f(t/C)^{1/p} w(t/s)\, dt\right]^p ds,$$

for some constant C. Now, if we call

$$h(t) = \left[\int_t^\infty b(Cr)^{-p'/p} dr\right]^{1/pp'} \qquad \text{and} \qquad g(t) = [\mu_f(t/C) b(Ct)]^{1/p},$$

from Hölder's inequality,

$$\begin{aligned}&\int_1^\infty \frac{a(s)}{s^p}\left[\int_s^\infty \mu_f(t/C)^{1/p}\, w(t/s) dt\right]^p ds =\\ &\qquad = \int_1^\infty \frac{a(s)}{s^p}\left[\int_s^\infty g(t)\, w(t/s) h(t) \frac{1}{h(t) b(Ct)^{1/p}} dt\right]^p ds\end{aligned}$$

can be bounded by

$$\int_1^\infty \frac{a(s)}{s^p}\left[\int_s^\infty [g(t)\, w(t/s)\, h(t)]^p\, dt\right]\left[\int_s^\infty b(Cr)^{-p'/p} h(r)^{-p'} dr\right]^{p/p'} ds\ .$$

If we apply Fubini-Tonelli's Theorem, the last expression is

$$\int_1^\infty [g(t)\, h(t)]^p \int_1^t \frac{a(s)}{s^p}\, w^p(t/s)\, \left[\int_s^\infty b(Cr)^{-p'/p} h(r)^{-p'} dr\right]^{p/p'} ds\, dt.$$

Integrating by parts, we have

$$\int_s^\infty b(Cr)^{-p'/p} h(r)^{-p'} dr = p' \left[\int_s^\infty b(Cr)^{-p'/p} dr\right]^{1/p'} \tag{9}$$

and

$$\int_1^t \frac{a(s)}{s^p}\, w^p(t/s) \left[\int_1^s \frac{a(r)}{r^p}\, w^p(s/r)\, dr\right]^{-1/p'} ds = \tag{10}$$

$$= p \left[\int_1^t \frac{a(r)}{r^p}\, w^p(t/r)\, dr\right]^{1/p}.$$

Now, identity (9) and inequality (8) gives

$$\begin{aligned}
&\int_1^t \frac{a(s)}{s^p}\, w^p(t/s) \left[\int_s^\infty b(Cr)^{-p'/p} h(r)^{-p'} dr\right]^{p/p'} ds = \\
&\quad = (p')^{p/p'} \int_1^t \frac{a(s)}{s^p}\, w^p(t/s) \left[\int_s^\infty b(Cr)^{-p'/p} dr\right]^{p/(p')^2} ds \\
&\quad \le (p')^{p/p'} C^{p/p'} \int_1^t \frac{a(s)}{s^p}\, w^p(t/s) \left[\int_1^s \frac{a(r)}{r^p}\, w^p(s/r)\, dr\right]^{-1/p'} ds,
\end{aligned}$$

and from identity (10) the last expression is a constant times

$$\begin{aligned}
\left[\int_1^t \frac{a(r)}{r^p}\, w^p(t/r)\, dr\right]^{1/p} &\le C \left[\int_t^\infty b(Cr)^{-p'/p} dr\right]^{-1/p'} \\
&= C h(t)^{-p},
\end{aligned}$$

where the last inequality is due to (8). Therefore,

$$\begin{aligned}
\int_\Omega \Phi(|T^{(j)} f|) d\mu &\le \Phi(1)\mu(\Omega) + p(p')^{p/p'} C^p \int_1^\infty \mu_f(t/C) b(Ct) dt \\
&\le \Phi(1)\mu(\Omega) + p(p')^{p/p'} C^{p-1} \int_\Omega \Psi(C^2 f) d\mu,
\end{aligned}$$

and the proof is finished.

□

As a consequence of Lemma 1 and Theorem 1 for the case $w(s) = \log^{(j-1)}(s)$, we obtain modular inequalities for the j-times composition of an operator of restricted weak type (p,p) and strong type (∞,∞).

Theorem 2. *Let Ω be a finite measure space, $p > 1$, and T an operator that satisfies inequality* (1)*. If for some constant C, a and b satisfy*

$$\sup_{t>1} \left(\int_1^t \frac{a(s)}{s^p}\, \log^{p(j-1)}(t/s)\, ds\right)^{1/p} \left(\int_t^\infty b(C\, s)^{-p'/p}\, ds\right)^{1/p'} < \infty \tag{11}$$

then, there exists a constant C' such that

$$\int_\Omega \Phi(|T^{(j)} f|)\, d\mu \le C' + C' \int_\Omega \Psi(C'\, |f|)\, d\mu$$

for all $f \in \mathfrak{M}(\Omega)$.

In order to show that condition (8) is the best possible we introduce a Hardy type operator for which (7) holds with equivalence. If $p > 1$, $w:\ [1,\infty) \to [0,\infty)$ is non-decreasing and f is an integrable function of $[0,1]$, we define the operator

$$\mathcal{T}_p^w f(x) = \frac{1}{p\, x^{1/p}} \int_0^x f(s)\, s^{\frac{1}{p}-1}\, w(x/s)\, ds \qquad \text{for all } x \in [0,1].$$

We will need the following lemma.

Lemma 2. *If f is non-increasing, then $\mathcal{T}_p^w f$ is also non-increasing.*

Proof. Let $x, y \in [0,1]$ and $x < y$. By a change of variables, since f is non-increasing, we have

$$\begin{aligned} \mathcal{T}_p^w f(x) &= \frac{1}{x^{1/p}} \int_0^x f(s)\, s^{1/p-1}\, w(x/s)\, ds \\ &= \frac{1}{y^{1/p}} \int_0^y f\left(\frac{x}{y}\, t\right) t^{1/p-1} w(y/t)\, dt \\ &\geq \frac{1}{y^{1/p}} \int_0^y f(t)\, t^{1/p-1} w(y/t)\, dt \;=\; \mathcal{T}_p^w f(y). \end{aligned}$$

□

Theorem 3. *If b is monotone and for some constant C,*

$$\int_0^1 \Phi(|\mathcal{T}_p^w f(x)|)\, dx \leq C + C \int_0^1 \Psi(C\,|f(x)|)\, dx \qquad \textit{for all } f \in \mathfrak{M}([0,1]), \tag{12}$$

then the functions a and b satisfy (8).

Proof. We deal first with the case when b has the property that there exists a constant C_1 such that

$$w(t) \left(\int_t^\infty b(Cs)^{-p'/p} ds \right)^{1/p'} \leq C_1 \qquad \text{for all } t > 1. \tag{13}$$

Fix $t > 1$ and let

$$h_t(s) = \frac{1}{A_t}\, b(Cs)^{-p'} \qquad s > 0,$$

with $A_t = t\, b(Ct)^{-p'/p} + \int_t^\infty b(Cs)^{-p'/p} ds$. The fact that b is monotone and (13), implies b is non-increasing and $\lim\limits_{s\to\infty} b(s) = \infty$. Thus, h_t is non-decreasing, and also $\lim_{s\to\infty} h_t(s) = 0$ and $h_t^{-1}(s)$ is well defined for $s > 0$.

Now, consider the function $f_t \in \mathfrak{M}([0,1])$ defined by

$$f_t = h_t^{-1} \chi_{(0, h_t(t))}.$$

Observe that, if (12) is satisfied with a constant C, then it is also satisfied with any constant greater than C. Hence, we can assume $b(C) \geq 1$, and

$$h_t(t) = \frac{1}{A_t}\, b(Ct)^{-p'} \leq \frac{1}{t b(Ct)} \leq 1.$$

Then, $h_t(t)$ is in the interval $[0,1]$.

The distribution of f_t is

$$\mu_{f_t}(s) = \begin{cases} h_t(t) & \text{for } 0 < s \leq t \\ h_t(s) & \text{for } s > t. \end{cases} \tag{14}$$

Then, as b is non-decreasing, we have

$$\begin{aligned}\frac{1}{C}\int_0^1 \Psi(C|f_t(x)|)\,dx &= \int_0^\infty b(Cs)\mu_{f_t}(s)\,ds \\ &\le tb(Ct)h_t(t) + \int_t^\infty b(Cs)h_t(s)\,ds \\ &\le \frac{1}{A_t}\left[tb(Ct)^{-p'/p} + \int_t^\infty b(Cs)^{-p'/p}\,ds\right] \\ &\le 1,\end{aligned}$$

and thus

$$C + C\int_{[0,1]} \Psi(C|f_t|) \le C + C^2. \tag{15}$$

On the other hand,

$$\begin{aligned}\int_0^1 \Phi(|\mathcal{T}_p^w f_t(x)|)dx &= \int_0^\infty a(s)\,\mu_{\mathcal{T}_p^w f_t}(s)\,ds \\ &\ge \int_1^t a(s)\,\mu_{\mathcal{T}_p^w f_t}(s)\,ds.\end{aligned} \tag{16}$$

If $s > \mathcal{T}_p^w f_t(1)$, by Lemma 2, the function $\mathcal{T}_p^w f_t$ non-increases, then there exists $x_s \in [0,1]$, so that

$$\{x:\ \mathcal{T}_p^w f_t(x) > s\} = [0, x_s),$$

and since $\mathcal{T}_p^w$ is continuous,

$$s = \mathcal{T}_p^w f_t(x_s) = \frac{1}{{x_s}^{1/p}}\int_0^{x_s} f_t(y)\,y^{\frac{1}{p}-1}\,w(x_s/y)\,dy. \tag{17}$$

If $s < t$, and as $f_t \le \mathcal{T}_p^w f_t$, we have

$$x_s \ge x_t \ge \mu_{f_t}(t) = h_t(t).$$

Then, since w increases, from (17) we have

$$x_s \ge \left(\frac{w(x_s/x_t)}{s}\int_0^{h_t(t)} f_t(y)\,y^{\frac{1}{p}-1}\,dy\right)^p. \tag{18}$$

If $\mathcal{T}_p^w f_t(1) < s < t$,

$$\frac{x_s}{x_t} = \left(\frac{t\int_0^{x_s} f(y)\,y^{\frac{1}{p}-1}\,w(x_s/y)\,dy}{s\int_0^{x_t} f(y)\,y^{\frac{1}{p}-1}\,w(x_t/y)\,dy}\right)^p \ge \left(\frac{t}{s}\right)^p > \frac{t}{s}.$$

Hence, from (18) it follows

$$x_s \ge \left(\frac{w(t/s)}{s}\int_0^{h_t(t)} f_t(y)\,y^{\frac{1}{p}-1}\,dy\right)^p. \tag{19}$$

From the definition of f_t,

$$\int_0^{h_t(t)} f_t(y)\, y^{\frac{1}{p}-1}\, dy = t\, h_t(t)^{1/p} + \int_t^\infty h_t(r)^{1/p}\, dr$$
$$= A_t^{1/p'}$$
$$\geq \left(\int_t^\infty b^{-p'/p}(Cr)dr\right)^{1/p'}.$$

Then, by inequality (19),

$$x_s \geq \frac{w^p(t/s)}{s^p}\left(\int_t^\infty b^{-p'/p}(Cr)dr\right)^{p/p'} \qquad \text{for all } s \in (\mathcal{T}_p^w f_t(1), t).$$

If $s \in (1, \mathcal{T}_p^w f_t(1))$, $\mu_{\mathcal{T}_p^w f_t}(s) = 1$, by (13) and w non-decreasing,

$$1 \geq \frac{1}{C_1^p} w^p(t)\left(\int_t^\infty b^{-p'/p}(Cr)dr\right)^{p/p'}$$
$$\geq \frac{1}{C_1^p}\frac{w^p(t/s)}{s^p}\left(\int_t^\infty b^{-p'/p}(Cr)dr\right)^{p/p'},$$

and thus, there exists a constant C_2 such that

$$\mu_{\mathcal{T}_p^w f_t}(s) \geq \frac{1}{C_2}\frac{w^p(t/s)}{s^p}\left(\int_t^\infty b(Cr)^{-p'/p}\, dr\right)^{p/p'} \quad \text{for all } 1 < s < t. \tag{20}$$

Therefore, from (12), (15), (16) and (20) we get (8).

In order to finish the proof of this theorem, it remains to consider the case when the function b does not have the property (13). In the case

$$\int_1^\infty b^{-p'/p} = \infty,$$

following the example in [2], we consider

$$f = h^{-1}\chi_{[0,1]}$$

where

$$h(x) = \frac{b(x)^{-p'}}{\left(\int_{1/2}^x b^{-p'/p} ds\right)^p} \qquad x \geq 1,$$

and, as it was shown there,

$$\int_0^1 f(r) r^{1/p-1} dr = \infty.$$

Since $\int_0^1 f(r) r^{1/p-1} dr \leq \mathcal{T}_p^w f(1) \leq \mathcal{T}_p^w f(x)$ for all $x \in [0,1]$, the left side of (12) is infinite.

The remaining case is when

$$\int_1^\infty b^{-p'/p} < \infty. \tag{21}$$

Thus, we can choose an increasing sequence of numbers t_n for $n = 1, 2, \ldots$, so that

$$w(t_n)\left(\int_{t_n}^\infty b(s)^{-p'/p} ds\right)^{1/p'} \geq n.$$

Also, as b is monotone and satisfies (21) it must be non-decreasing, and so we can choose $t_1 \geq 1$ and $b(t_1) \geq 1$.

We will see that the operator $\mathcal{T}_p^w$ can not satisfy (12) for any Φ increasing.

For $n = 1, 2, \ldots$ let

$$h_n(s) = \frac{b(s)^{-p'}}{A_n} \qquad \text{for all } s \geq 0,$$

with $A_n = t_n\, b(t_n)^{-p'/p} + \int_{t_n}^{\infty} b(s)^{-p'/p} ds$, and consider the function

$$f_n(x) = h_n^{-1}(x)\chi_{[0,h_n(t_n))}(x)$$

for all x in the interval $[0,1]$. Since $b(t_1) \geq 1$,

$$h_n(t_n) = \frac{b(t_n)^{-p'}}{A_n} \leq \frac{1}{t_n\, b(t_n)} \leq \frac{1}{t_n}$$

and due to $t_1 \geq 1$ and $t_n \geq t_1$, the number $h_n(t_n)$ is a point of $[0,1]$.

In the same way as before, from the expression of the distribution of f_n,

$$\begin{aligned}
\int_0^1 \Psi(|f_n(x)|)dx &= \int_0^{\infty} b(s)\mu_{f_n}(s)ds \\
&\leq h_n(t_n) \int_0^{t_n} b(s)ds + \int_{t_n}^{\infty} b(s)h_n(s)ds \\
&\leq \frac{1}{A_n}\left[t_n b(t_n)^{-p'/p} + \int_{t_n}^{\infty} b(s)^{-p'/p} ds\right] \\
&\leq 1.
\end{aligned}$$

On the other hand, due to $h_n(t_n) \leq 1/t_n$,

$$\begin{aligned}
\mathcal{T}_p^w f_n(1) &\geq w(t_n) \int_0^{h_n(t_n)} f(y)\, y^{\frac{1}{p}-1}\, dy \\
&= w(t_n)\, A_n^{1/p'} \\
&\geq w(t_n) \left(\int_{t_n}^{\infty} b^{-p'/p}(r)\, dr\right)^{1/p'} \\
&\geq n.
\end{aligned}$$

and since $\mathcal{T}_p^w f_n$ is non-increasing,

$$\mathcal{T}_p^w f_n(x) \geq n \qquad \text{for all } x \in [0,1].$$

Therefore, if we have (12), as $\mathcal{T}_p^w$ is lineal, we would have

$$\begin{aligned}
\Phi\left(\frac{n}{C\,(j-1)!\,p^j}\right) &\leq \int_0^1 \Phi(\mathcal{T}_p^w(f_n(x)/C))\, dx \\
&\leq C + C\int_0^1 \Psi(f_n(x))dx \\
&\leq 2C
\end{aligned}$$

for all $n \in \mathbb{N}$, and this is a contradiction because Φ is unbounded.

$\square$

A particular case of the operator $\mathcal{T}_p^w$, when $w \equiv \frac{1}{p}$, is the operator

$$\mathcal{H}_p f(x) = \frac{1}{p\, x^{1/p}} \int_0^x f(y)\, y^{\frac{1}{p}-1}\, dy. \tag{22}$$

It is easy to check that $\mathcal{H}_p$ is of restricted weak type (p,p) (but not of weak type (p,p)) and strong type (∞,∞), then it satisfies inequality (1).

An other example of $\mathcal{T}_p^w$ is the j-times composition of $\mathcal{H}_p$ as following lemma shows.

Lemma 3. *Let $j \in \mathbb{N}$ and f be an integrable function on $[0,1]$ with respect to the Lebesgue measure. Then,*

$$\mathcal{H}_p^{(j)} f(x) = \frac{1}{(j-1)!p^j\, x^{1/p}} \int_0^x f(y)\, y^{\frac{1}{p}-1} \log^{j-1}(x/y)\, dy, \tag{23}$$

for all x in $[0,1]$.

Proof. We proceed by induction. For $j=1$, (23) is the definition of $\mathcal{H}_p$. Suppose that (23) is true for some $j \geq 1$. Then, from Fubini-Tonelli's Theorem, if x belongs to $[0,1]$,

$$\begin{aligned}
\mathcal{H}_p^{(j+1)} f(x) &= \frac{1}{p\, x^{1/p}} \int_0^x \mathcal{H}_p^{(j)} f(y)\, y^{\frac{1}{p}-1}\, dy \\
&= \frac{1}{(j-1)!p^{j+1} x^{1/p}} \int_0^x \frac{1}{y} \int_0^y f(r)\, r^{\frac{1}{p}-1} \log^{j-1}(y/r)\, dr\, dy \\
&= \frac{1}{(j-1)!p^{j+1} x^{1/p}} \int_0^x f(r)\, r^{\frac{1}{p}-1} \int_r^x \frac{1}{y} \log^{j-1}(y/r)\, dy\, dr \\
&= \frac{1}{j!p^{j+1} x^{1/p}} \int_0^x f(r)\, r^{\frac{1}{p}-1} \log^{j-1}(x/r)\, dr.
\end{aligned}$$

□

From the previous lemma and Theorem 3 we obtain *sharp* modular inequalities for j-times composition of $\mathcal{H}_p$.

Theorem 4. *Let $p > 1$ and b monotone. There exists a constant C, such that*

$$\int_0^1 \Phi(|\mathcal{H}_p^{(j)} f(x)|)\, dx \leq C + C \int_0^1 \Psi(C\,|f(x)|)\, dx$$

for all $f \in \mathfrak{M}([0,1])$ if, and only if, the functions a and b satisfy (11).

Now we present a result for the Cesàro operators.

Theorem 5. *Let $0 < \alpha < 1$ and b monotone. There exists a constant C such that*

$$\int_0^1 \Phi(|(M_\alpha^-)^{(j)} f(x)|)\, dx \leq C + C \int_0^1 \Psi(C\,|f(x)|)\, dx \tag{24}$$

for all $f \in \mathfrak{M}([0,1])$ if, and only if, for some constant C'

$$\sup_{t>1} \left(\int_1^t \frac{a(s)}{s^{1/\alpha}} \log^{\frac{j-1}{\alpha}}(t/s)\, ds \right)^{\alpha} \left(\int_t^\infty b(C'\,s)^{-\frac{\alpha}{1-\alpha}}\, ds \right)^{1-\alpha} < \infty. \tag{25}$$

The same holds for M_α^+.

Proof. As we have mentioned in the introduction, the operator M_α^- is of restricted weak type $(1/\alpha, 1/\alpha)$ and strong type (∞,∞) and so it satisfies inequality (1). Then, as consequence of Theorem 2 condition (25) implies (24). To see that condition (25) is necessary we just observe that for $x \in [0,1]$, $\mathcal{H}_p f(x) \le M_\alpha^- f(x)$ and then $\mathcal{H}_p^{(j)} f(x) \le (M_\alpha^-)^{(j)} f(x)$ for any j and then we can use Theorem 4.

The result for M_α^+ follows from the identity $M_\alpha^+ f(x) = M_\alpha^- g(1-x)$, with $g(x) = f(1-x)$, for all $x \in [0,1]$; since f and g have the same distribution. □

The results contained in this section have a corresponding version when Ω is not of finite measure. Conditions about a and b like (8) and (11) changes because the values of t near 0 may be important when Ω has not finite measure. Also the wanted modular inequality is different. The results are presented without proofs in the following theorems.

Theorem 6. *Let Ω be a finite measure space, $w : [1,\infty) \to [0,\infty)$ continuous and $T : \mathfrak{M}(\Omega) \to \mathfrak{M}(\Omega)$ an operator satisfying for some $p > 1$,*

$$\mu_{Tf}(t) \le \left[\frac{1}{t}\int_t^\infty \mu_f(s)^{1/p} w(s/t) ds\right]^p \qquad \text{for all } t > 0.$$

If there exists a constant C such that

$$\sup_{t>0}\left(\int_0^t \frac{a(s)}{s^p}\, w^p(t/s)\, ds\right)^{1/p}\left(\int_t^\infty b(C\, s)^{-p'/p}\, ds\right)^{1/p'} < \infty$$

then, there exists a constant C' such that

$$\int_\Omega \Phi(|Tf|)\, d\mu \le C' \int_\Omega \Psi(C'\,|f|)\, d\mu$$

for all $f \in \mathfrak{M}(\Omega)$.

If we consider the operators M_α^- and M_α^+ defined over the whole real line (see [5]) we have the analogous of Theorem 5.

Theorem 7. *Let $0 < \alpha < 1$. There exists a constant C such that*

$$\int_{\mathbb{R}} \Phi(|(M_\alpha^-)^{(j)} f(x)|)\, dx \le C \int_{\mathbb{R}} \Psi(C\,|f(x)|)\, dx$$

for all $f \in \mathfrak{M}(\mathbb{R})$ if, and only if, for some constant C'

$$\sup_{t>0}\left(\int_0^t \frac{a(s)}{s^{1/\alpha}} \log^{\frac{j-1}{\alpha}}(t/s)\, ds\right)^\alpha \left(\int_t^\infty b(C'\, s)^{-\frac{\alpha}{1-\alpha}}\, ds\right)^{1-\alpha} < \infty.$$

The same holds for M_α^+.

4. The strong Cesàro maximal function on $\mathbb{R}^n$

For a locally integrable function f and $x \in \mathbb{R}^n$, the *Hardy-Littlewood maximal function* operator over cubes is defined as

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)|\, dy$$

where the supremum is taken over all cubes Q with sides parallel to the coordinates axes and containing x.

If we take rectangles instead of cubes, the resulting operator is known as the *strong maximal function* operator defined by

$$M^S f(x) = \sup_{x\in R} \frac{1}{|R|} \int_R |f(y)|\,dy$$

where the supremum is taken over all rectangles R with sides parallel to the coordinates axes and containing x.

We may write this maximal function as

$$M^S f(x) = \sup_{a_i<x_i<b_i} \frac{1}{\prod_{i=1}^n (b_i - a_i)} \int_{a_1}^{b_1}\int_{a_2}^{b_2} \cdots \int_{a_n}^{b_n} |f(y)|\,dy_n \dots dy_1$$

for all $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$.

It is well known that this operator is strong type (p, p) for all $p > 1$, nevertheless it is not of weak type $(1, 1)$. In [4] modular inequalities in Orlicz spaces are treated for this operator.

In order to study α-Cesàro continuity in $\mathbb{R}^n$ we may deal with the *Cesàro maximal function operator of order* α in $\mathbb{R}^n$ defined for $f \in \mathfrak{M}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, as

$$M_\alpha f(x) = \sup_{x\in Q} \frac{\alpha}{|Q|^{(n-1+\alpha)/n}} \int_Q |f(y)| d(y, Q^c)^{\alpha-1} dy. \tag{26}$$

Here $d(y, Q^c)$ denotes the distance form the point y to the complement of Q, and the supremum is taken over all cubes Q containing the point x and with sides parallel to the coordinate axes.

When $n = 1$, for $f \in \mathfrak{M}(\mathbb{R})$ and $x \in \mathbb{R}$, (26) can be written as

$$M_\alpha f(x) = \sup_{c<x<d} \frac{\alpha}{d-c} \int_c^d |f(s)| \left(1 - \frac{|2s - d - c|}{d-c}\right)^{\alpha-1} ds,$$

which is the bilateral version of the Cesàro maximal function operators (5) and (4).

Let $f \in \mathfrak{M}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we define the *strong Cesàro maximal function operator of order* α as

$$\begin{aligned} M^S_\alpha f(x) = \sup_{c_i<x_i<d_i} & \frac{\alpha^n}{\prod_{i=1}^n (d_i - c_i)} \int_{c_1}^{d_1}\int_{c_2}^{d_2} \cdots \\ & \cdots \int_{c_n}^{d_n} |f(y)| \prod_{i=1}^n \left(1 - \frac{|2\,y_i - d_i - c_i|}{d_i - c_i}\right)^{\alpha-1} dy_n \dots dy_2\, dy_1 \end{aligned}$$

for all $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$.

Before dealing with the strong Cesàro Maximal function, we would like to point out that for the Cesàro maximal function operator over cubes (26), we may obtain the same results as the one dimensional case contained in [2]. This sublinear operator, is also of restricted weak type $(1/\alpha, 1/\alpha)$ and strong type (∞, ∞). In fact, the same arguments of [2] can be easily adapted, giving an estimate for the rearrangement of the function $d(x, Q^c)$ for a cube Q contained in $\mathbb{R}^n$.

The rest of this section is devoted to apply Theorem 2 in order to obtain modular inequalities for the strong Cesàro maximal function operator.

Theorem 8. *Let $0 < \alpha < 1$ and Ω be a bounded set of $\mathbb{R}^n$, and suppose the functions a and b satisfy for some constant C*

$$\sup_{t>1} \left(\int_1^t \frac{a(s)}{s^{1/\alpha}} \log^{\frac{n-1}{\alpha}}(t/s)\, ds\right)^\alpha \left(\int_t^\infty b(C\,s)^{-\frac{\alpha}{1-\alpha}}\, ds\right)^{1-\alpha} < \infty, \tag{27}$$

then there exists a constant C' such that

$$\int_\Omega \Phi(|M_\alpha^S f(x)|)\, dx \ \leq \ C' + C' \int_\Omega \Psi(C'\,|f(x)|)\, dx \tag{28}$$

for all f supported in Ω.

Proof. For $f \in \mathfrak{M}(\mathbb{R}^n)$ and $i = 1, \ldots, n$, let us consider the operator

$$T_i f(x) = M_\alpha(f(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n))(x_i) \qquad \text{for all } x \text{ in } \mathbb{R}^n.$$

Then, we have

$$M_\alpha^S \leq T_1 \circ T_2 \circ \cdots \circ T_n.$$

For $i = 1, \ldots, n$, we check that the operator T_i satisfies (1). For clearness in the notation, we suppose $i = 1$. Let $x' = (x_2, \ldots, x_n)$, then $x = (x_1, x')$. Since M_α satisfies (1), for $t > 0$ we have

$$\begin{aligned}
\mu_{T_1 f}(t) &= \int_{R^n} \chi_{\{y\in\mathbb{R}^n:\ T_1 f(y)>t\}}(x)\, dx \\
&= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \chi_{\{u\in\mathbb{R}:\ (M_\alpha f(\cdot,x'))(u)>t\}}(x_1) dx_1 dx' \\
&\leq \int_{R^{n-1}} \left(\frac{C}{t} \int_{t/C}^\infty |\{x_1 \in \mathbb{R}:\ f(x_1, x') > s\}|^\alpha ds \right)^{1/\alpha} dx',
\end{aligned}$$

and due to Minkowski integral inequality, the last integral is bounded by

$$\left[\frac{C}{t} \int_{t/C}^\infty \left(\int_{R^{n-1}} |\{x_1 \in \mathbb{R}:\ f(x_1, x') > s\}|\, dx' \right)^\alpha ds \right]^{1/\alpha} =$$

$$= \left[\frac{C}{t} \int_{t/C}^\infty \mu_f(s)^\alpha ds \right]^{1/\alpha}.$$

Therefore the operator M_α^S is bounded by a composition of operators which satisfies the hypotheses of Theorem 2, then condition (27) implies modular inequality (28).

□

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