WEIGHTED WEAK TYPE INEQUALITIES FOR GENERALIZED HARDY OPERATORS

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Abstract. We characterize the pairs of weights (v, w) for which the Hardy-Steklov type operator $Tf(x) = g(x)\int_{s(x)}^{h(x)} K(x,y)f(y)\,dy$ applies $L^p(v)$ into weak$-L^q(w)$, $q < p$, assuming certain monotonicity conditions on g, s, h and K.

1. Introduction

Let us consider the Hardy-Steklov type operator defined by

$$Tf(x) = g(x)\int_{s(x)}^{h(x)} K(x,y)f(y)\,dy, \qquad f \geq 0,$$

where g is a nonnegative measurable function, s and h are continuous and increasing functions ($x < y \Rightarrow s(x) \leq s(y), h(x) \leq h(y)$) defined on an interval (a, b) such that $s(x) \leq h(x)$ for all $x \in (a, b)$ and the kernel $K(x, y)$ defined on $\{(x, y) : x \in (a, b) \text{ and } s(x) \leq y \leq h(x)\}$ satisfies

(i) $K(x, y) \geq 0$;
(ii) it is increasing and continuous in x and decreasing in y;
(iii) $K(x, z) \leq D[K(x, h(y)) + K(y, z)]$ for $y \leq x$ and $s(x) \leq z \leq h(y)$, where the constant $D > 1$ is independent of x, y and z.

Gogatishvili and Lang [3] characterized the pairs of weights for the strong and weak type (p, q) inequalities for the operator T in the case $p \leq q$. Actually, in [3] the authors deal with Banach functions spaces with some extra condition. On the other hand, Chen and Sinnamon [2] have characterized the weighted strong-type inequality for $1 < p, q < \infty$ in terms of a normalizing measure. In both papers they work with more general functions s, h and K.

The goal of this paper is to characterize the weighted weak type inequalities in the case $q < p$. It is well known that strong type inequalities for the operator T can be deduced directly from the corresponding ones for $g(x) = 1$, but this is not the case when we work with weak type inequalities. In [5] it was characterized the weighted weak type inequality in the case $q < p$ for the operator T when $s \equiv 0$, $h(x) = x$ and $K \equiv 1$. The result was obtained for monotone functions g. In fact, in the proof of the result the authors used the condition

$$\inf_{x\in E} g(x) = \inf_{x\in(\alpha,\beta)} g(x), \tag{1.1}$$

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for any bounded set E, where $\alpha = \inf E$ and $\beta = \sup E$. This property clearly holds if g is monotone or if there exists x_0 such that g is increasing in $(a, x_0]$ and decreasing in $[x_0, b)$. In our result we shall assume (1.1) and the same condition for the function $g(x)K(x,y)$, that is, for all y and every bounded set $E_y \subset \{x : s(x) \le y \le h(x)\}$,

$$\inf_{x\in E_y} [g(x)K(x,y)] = \inf_{x\in(\alpha_y,\beta_y)} [g(x)K(x,y)], \tag{1.2}$$

where $\alpha_y = \inf E_y$ and $\beta_y = \sup E_y$.

Examples of Hardy-Steklov type operators are the modified Riemann-Liouville operators defined for $\alpha > 0$ and $\eta \in \mathbb{R}$ as $x^\eta \int_0^x (x-y)^\alpha f(y)\,dy$ or the more general version $x^\eta \int_{Ax}^{Bx} (x-y)^\alpha f(y)\,dy$, with $0 < A < B \le 1$ and $x > 0$; the modified logarithmic kernel operators $x^\eta \int_0^x \log^\beta(x/y) f(y)\,dy$, with $\beta > 0$ and $\eta \in \mathbb{R}$; the Steklov operator $Tf(x) = \int_{x-1}^{x+1} f$; the Riemann-Liouville operators with general variable limits $\int_{s(x)}^{h(x)} (x-y)^\alpha f(y)\,dy$, with $s(x) \le h(x) \le x$. This last operator was studied in [6] in the case $-1 < \alpha < 0$.

As far as we know, our result is new even for the particular cases $Tf(x) = g(x)\int_0^x K(x,y)f(y)\,dy$ and $Tf(x) = \int_{s(x)}^{h(x)} K(x,y)f(y)\,dy$. For this last operator, conditions (1.1) and (1.2) holds trivially because $K(x,y)$ is increasing in x.

The notation is standard: $w(E)$ denotes the integral $\int_E w$, if $1 < p < \infty$, then p' denotes the conjugate exponent of p defined by $1/p + 1/p' = 1$ and $L^{q,\infty}(w)$ will denote the space of measurable functions f such that

$$\|f\|_{q,\infty;w} = \sup_{\lambda>0} \lambda \left(w(\{x : |f(x)| > \lambda\})\right)^{\frac{1}{q}} < \infty.$$

2. Statement and proof of the result

In the next theorem we state the result of this article.

Theorem *Let s and h be increasing continuous functions defined on an interval (a,b) satisfying $s(x) \le h(x)$ for $x \in (a,b)$. Let $K(x,y)$ defined on $\{(x,y) : x \in (a,b)$ and $s(x) \le y \le h(x)\}$ satisfying (i), (ii), (iii) and let g be a nonnegative function defined on (a,b) satisfying (1.1) and (1.2). Let q, p and r be such that $0 < q < p$, $1 < p < \infty$ and $1/r = 1/q - 1/p$. Let w and v be nonnegative measurable functions defined on (a,b) and $(s(a), h(b))$, respectively. The following statements are equivalent.*

(i) There exists a positive constant C such that

$$[w(\{x \in (a,b) : Tf(x) > \lambda\})]^{1/q} \le \frac{C}{\lambda} \left(\int_{s(a)}^{h(b)} f^p v \right)^{1/p}$$

for all $f \ge 0$ and all positive real number λ.

(ii) The functions

$$\Phi_1(x) = \sup \left\{ \inf_{t\in(c,d)} [g(t)K(t, h(\bar c))] \left(\int_c^d w \right)^{1/p} \left(\int_{s(d)}^{h(\bar c)} v^{1-p'} \right)^{1/p'} \right\},$$

where the supremum is taken over all the numbers $\bar{c}$, c *and* d *such that* $a \le \bar{c} \le c < x < d \le b$ *and* $s(d) \le h(\bar{c})$ *and*

$$\Phi_2(x) = \sup \left\{ (\inf_{t\in(c,d)} g(t)) \left(\int_c^d w \right)^{1/p} \left(\int_{s(d)}^{h(c)} K^{p'}(c,y) v^{1-p'}(y)\, dy \right)^{1/p'} \right\},$$

where the supremum is taken over all the numbers c *and* d *such that* $a \le c < x < d \le b$ *and* $s(d) \le h(c)$*, belong to* $L^{r,\infty}(w)$

Let us observe that if $g \equiv 1$ we get that $\Phi_1 \le \Phi_2$. Then, in this case, the weighted weak type inequality (i) is equivalent to $\Phi_2 \in L^{r,\infty}(w)$. On the other hand, if $K \equiv 1$ then $\Phi_1 = \Phi_2$ and we recover Theorem 1.9 in [1].

To prove the theorem we shall use the following lemma (see Lemma 1.4 in [1] for the proof).

Lemma *Let* a *and* b *be real numbers such that* $a < b$*. Let* $s, h : (a,b) \to \mathbb{R}$ *be increasing and continuous functions such that* $s(x) \le h(x)$ *for all* $x \in (a,b)$*. Let* $\{(a_j, b_j)\}_j$ *be the connected components of the open set* $\Omega = \{x \in (a,b) : s(x) < h(x)\}$*. Then*
(a) $(s(a_j), h(b_j)) \cap (s(a_i), h(b_i)) = \emptyset$ *for all* $j \ne i$*.*
(b) For every j *there exists a (finite or infinite) sequence* $\{m_k^j\}$ *of real numbers such that:*

(i) $a_j \le m_k^j < m_{k+1}^j \le b_j$ *for all* k *and* j*;*
(ii) $(a_j, b_j) = \cup_k (m_k^j, m_{k+1}^j)$ *a. e. for all* j*;*
(iii) $s(m_{k+1}^j) \le h(m_k^j)$ *for all* k *and* j *and* $s(m_{k+1}^j) = h(m_k^j)$ *if* $a_j < m_k^j < m_{k+1}^j < b_j$*.*

Proof of the Theorem: $(i) \Rightarrow (ii)$: First, we shall prove that $\Phi_1 \in L^{r,\infty}(w)$, i.e. we shall prove that

$$\sup_{\lambda>0} \lambda \left[w(\{x \in (a,b) : \Phi_1(x) > \lambda\}) \right]^{1/r} < \infty. \tag{2.1}$$

Let $\lambda > 0$ and $S_\lambda = \{x \in (a,b) : \Phi_1(x) > \lambda\}$. For every $z \in S_\lambda$ there exist $\bar{c}_z$, c_z and d_z with $a \le \bar{c}_z \le c_z < z < d_z \le b$ such that $s(d_z) \le h(\bar{c}_z)$ and

$$\lambda < \inf_{t\in(c_z,d_z)} [g(t) K(t, h(\bar{c}_z))] \left(\int_{c_z}^{d_z} w \right)^{1/p} \left(\int_{s(d_z)}^{h(\bar{c}_z)} v^{1-p'} \right)^{1/p'}. \tag{2.2}$$

Let $\mathcal{K} \subset S_\lambda$ be a compact set. Then there exist $(c_{z_1}, d_{z_1}), \ldots, (c_{z_k}, d_{z_k})$ which cover $\mathcal{K}$. We may assume without loss of generality that $\sum_{j=1}^k \chi_{(c_{z_j}, d_{z_j})} \le 2\chi_{\cup_{j=1}^k (c_{z_j}, d_{z_j})}$. Let $f : (s(a), h(b)) \to \mathbb{R}$ defined by

$$f(y) = \left(\sum_{j=1}^k \frac{v^{-p'}(y) \chi_{(s(d_{z_j}), h(\bar{c}_{z_j}))}(y)}{\left(\inf_{t\in(c_{z_j}, d_{z_j})} [g(t) K(t, h(\bar{c}_{z_j}))] \int_{s(d_{z_j})}^{h(\bar{c}_{z_j})} v^{1-p'} \right)^p} \right)^{1/p}.$$

If $z \in (c_{z_j}, d_{z_j})$ then we have $Tf(z) = g(z)\int_{s(z)}^{h(z)} K(z,y)f(y)\,dy \geq 1$. Therefore, $\cup_{j=1}^k (c_{z_j}, d_{z_j}) \subset \{x \in (a,b) : Tf(x) \geq 1\}$. Applying the weighted weak type inequality and (2.2) we obtain

$$\begin{aligned}
\int_{\cup_{j=1}^k (c_{z_j},d_{z_j})} w &\leq C\left(\sum_{j=1}^k \frac{\int_{s(d_{z_j})}^{h(\bar{c}_{z_j})} v^{1-p'}}{\left(\inf_{t\in(c_{z_j},d_{z_j})}[g(t)K(t,h(\bar{c}_{z_j}))]\int_{s(d_{z_j})}^{h(\bar{c}_{z_j})} v^{1-p'}\right)^p}\right)^{q/p} \\
&= C\left(\sum_{j=1}^k \frac{1}{(\inf_{t\in(c_{z_j},d_{z_j})}[g(t)K(t,h(\bar{c}_{z_j}))]^p(\int_{s(d_{z_j})}^{h(\bar{c}_{z_j})} v^{1-p'})^{p-1}}\right)^{q/p} \\
&\leq \frac{C}{\lambda^q}\left(\sum_{j=1}^k \int_{c_{z_j}}^{d_{z_j}} w\right)^{q/p} \\
&\leq \frac{C}{\lambda^q}\left(\int_{\cup_{j=1}^k (c_{z_j},d_{z_j})} w\right)^{q/p}.
\end{aligned}$$

The last inequality implies that $\lambda\left(\int_{\mathcal{K}} w\right)^{1/r} \leq C$ for any compact set $\mathcal{K} \subset S_\lambda$ which implies (2.1). The proof of (2.1) for the function Φ_2 follows in a similar way applying (i) to the function

$$f(y) = \left(\sum_{j=1}^k \frac{K^{p'}(c_{z_j},y)v^{-p'}(y)\chi_{(s(d_{z_j}),h(c_{z_j}))}(y)}{\left(\inf_{t\in(c_{z_j},d_{z_j})} g(t)\int_{s(d_{z_j})}^{h(c_{z_j})} K^{p'}(c_{z_j},t)v^{1-p'}(t)\,dt\right)^p}\right)^{1/p}.$$

$(ii) \Rightarrow (i)$: Let $\{a^N\}_{N=1}^\infty$ and $\{b^N\}_{N=1}^\infty$ be sequences in (a,b) such that

$$\lim_{N\to\infty} a^N = a \quad \text{and} \quad \lim_{N\to\infty} b^N = b.$$

In order to prove (i) it will suffice to show that

$$w(\{x \in (a^N, b^N) : Tf(x) > \lambda\}) \leq \frac{C}{\lambda^q},$$

for all nonnegative function f bounded with compact support such that $\int_{s(a)}^{h(b)} f^p v = 1$ and with a constant C independent of N, λ and f.

Let us fix $N \in \mathbb{N}$. Observe that if $O_\lambda = \{x \in (a^N, b^N) : Tf(x) > \lambda\}$ and $U = \{x \in (a,b) : \Phi_1(x) \leq \lambda^{q/r}, \Phi_2(x) \leq \lambda^{q/r}\}$, then

$$\begin{aligned}
w(O_\lambda) \leq & w(O_\lambda \cap U) + 2w(\{x \in (a,b) : \Phi_1(x) > \lambda^{q/r}\}) \\
& + w(\{x \in (a,b) : \Phi_2(x) > \lambda^{q/r}\}) \\
\leq & w(O_\lambda \cap U) + 2\frac{\|\Phi_1\|_{r,\infty;w}^r}{\lambda^q} + \frac{\|\Phi_2\|_{r,\infty,;w}^r}{\lambda^q}.
\end{aligned}$$

Therefore the implication will be proved if we establish that $w(O_\lambda \cap U) \leq \frac{C}{\lambda^q}$. Let (a_j, b_j) and $\{m_k^j\}$ be the sequences given by the lemma for the set $\Omega_N = \{x \in$

$(a^N, b^N) : s(x) < h(x)\}$. Then, for fixed j,

$$w(O_\lambda \cap U \cap (a_j, b_j)) = \sum_k w(O_\lambda \cap U \cap (m_k^j, m_{k+1}^j)). \tag{2.3}$$

If $x \in (m_k^j, m_{k+1}^j)$, since $s(m_{k+1}^j) \le h(m_k^j)$, we get that

$$\begin{aligned} Tf(x) =& g(x) \int_{s(x)}^{s(m_{k+1}^j)} K(x,y)f(y)dy + g(x) \int_{s(m_{k+1}^j)}^{h(m_k^j)} K(x,y)f(y)dy \\ &+ g(x) \int_{h(m_k^j)}^{h(x)} K(x,y)f(y)dy = T_{j,k}^1 f(x) + T_{j,k}^2 f(x) + T_{j,k}^3 f(x) \end{aligned}$$

It is clear that

$$w(O_\lambda \cap U \cap (m_k^j, m_{k+1}^j)) \le w(E^1) + w(E^2) + w(E^3),$$

where $E^\ell = \{x \in (m_k^j, m_{k+1}^j) \cap U : \ T_{j,k}^\ell f(x) > \lambda/3\}$, $\ell = 1, 2, 3$.

First, notice that the property (iii) of the kernel K implies

$$K(x, y) \le D[K(x, h(m_k^j)) + K(m_k^j, y)] \tag{2.4}$$

for $x \in (m_k^j, m_{k+1}^j)$ and $y \in (s(m_{k+1}^j), h(m_k^j))$.
In order to estimate $w(E^1)$ let us observe that

$$\begin{aligned} T_{j,k}^1 f(x) \le& D g(x) K(x, h(m_k^j)) \int_{s(x)}^{s(m_{k+1}^j)} f(y)\, dy \\ &+ D g(x) \int_{s(x)}^{s(m_{k+1}^j)} K(m_k^j, y) f(y)\, dy = D T_{j,k}^{1,1} f(x) + D T_{j,k}^{1,2} f(x). \end{aligned}$$

Then, $w(E^1) \le w(E^{1,1}) + w(E^{1,2})$, where

$$E^{1,\ell} = \{x \in (m_k^j, m_{k+1}^j) \cap U : \ T_{j,k}^{1,\ell} f(x) > \lambda/6D\}, \quad \ell = 1, 2.$$

Let us select an increasing sequence $\{x_i\}_i$, $x_i \in (m_k^j, m_{k+1}^j)$, such that $x_0 = m_k^j$ and

$$\int_{s(x_i)}^{s(m_{k+1}^j)} f = \int_{s(x_{i-1})}^{s(x_i)} f.$$

Let $E_i^{1,1} = E^{1,1} \cap (x_i, x_{i+1})$, $\alpha_i^1 = \inf E_i^{1,1}$ and $\beta_i^1 = \sup E_i^{1,1}$. If $E_i^{1,1} \neq \emptyset$, let $t \in E_i^{1,1}$. Using the property of the sequence $\{x_i\}_i$ we have that

$$\frac{\lambda}{6D} \le 4 g(t) K(t, h(m_k^j)) \int_{s(x_{i+1})}^{s(x_{i+2})} f.$$

Now, by using (1.2) and Hölder inequality we get

$$\frac{\lambda}{6D} \le 4 \inf_{t \in (\alpha_i^1, \beta_i^1)} [g(t) K(t, h(m_k^j))] \left(\int_{s(x_{i+1})}^{s(x_{i+2})} v^{1-p'} \right)^{1/p'} \left(\int_{s(x_{i+1})}^{s(x_{i+2})} f^p v \right)^{1/p}.$$

Now, multiplying by $\left(\int_{\alpha_i^1}^{\beta_i^1} w\right)^{1/p}$ and using the inequalities $s(\beta_i^1) \le s(x_{i+1})$ and $s(x_{i+2}) \le s(m_{k+1}^j) \le h(m_k^j)$ we get that

$$\begin{aligned}\frac{\lambda}{6D}\left(\int_{\alpha_i^1}^{\beta_i^1} w\right)^{1/p} &\le 4\Phi_1(x)\left(\int_{s(x_{i+1})}^{s(x_{i+2})} f^p v\right)^{1/p} \\ &\le 4\lambda^{q/r}\left(\int_{s(x_{i+1})}^{s(x_{i+2})} f^p v\right)^{1/p},\end{aligned}$$

where x is any element of $E_i^{1,1}$ and summing up in i we obtain

$$w(E^{1,1}) \le \frac{C}{\lambda^q}\int_{s(m_k^j)}^{s(m_{k+1}^j)} f^p v. \tag{2.5}$$

To estimate $w(E^{1,2})$, we select an increasing sequence $\{z_i\}_i$, $z_i \in (m_k^j, m_{k+1}^j)$, such that $z_0 = m_k^j$ and

$$\int_{s(z_i)}^{s(m_{k+1}^j)} K(m_k^j, y) f(y)\, dy = \int_{s(z_{i-1})}^{s(z_i)} K(m_k^j, y) f(y)\, dy.$$

As before, let $E_i^{1,2} = E^{1,2} \cap (z_i, z_{i+1})$, $\alpha_i^2 = \inf E_i^{1,2}$ and $\beta_i^2 = \sup E_i^{1,2}$. If $E_i^{1,2} \ne \emptyset$ then Hölder inequality and (1.1) give

$$\frac{\lambda}{6D} \le 4 \inf_{t\in(\alpha_i^2,\beta_i^2)} g(t)\left(\int_{s(z_{i+1})}^{s(z_{i+2})} K^{p'}(m_k^j, t) v^{1-p'}(t)\, dt\right)^{1/p'}\left(\int_{s(z_{i+1})}^{s(z_{i+2})} f^p v\right)^{1/p}.$$

Notice that $s(\beta_i^2) \le s(z_{i+1})$, $m_k^j \le \alpha_i^2$ and $s(z_{i+2}) \le s(m_{k+1}^j) \le h(m_k^j) \le h(\alpha_i^2)$. Then multiplying by $\left(\int_{\alpha_i^2}^{\beta_i^2} w\right)^{1/p}$ both members of the above inequality we get

$$\begin{aligned}\frac{\lambda}{6D}\left(\int_{\alpha_i^2}^{\beta_i^2} w\right)^{1/p} &\le 4\Phi_2(x)\left(\int_{s(z_{i+1})}^{s(z_{i+2})} f^p v\right)^{1/p} \\ &\le 4\lambda^{q/r}\left(\int_{s(z_{i+1})}^{s(z_{i+2})} f^p v\right)^{1/p},\end{aligned}$$

where x is any element of $E_i^{1,2}$. Now, summing up in i and putting together with (2.5) we obtain

$$w(E^1) \le \frac{C}{\lambda^q}\int_{s(m_k^j)}^{s(m_{k+1}^j)} f^p v.$$

To estimate $w(E^2)$ we proceed in a similar way. In fact, by using (2.4) we get that

$$\begin{aligned}T_{j,k}^2 f(x) &\le Dg(x)K(x, h(m_k^j))\int_{s(m_{k+1}^j)}^{h(m_k^j)} f(y)\, dy + Dg(x)\int_{s(m_{k+1}^j)}^{h(m_k^j)} K(m_k^j, y) f(y)\, dy \\ &= DT_{j,k}^{2,1} f(x) + DT_{j,k}^{2,2} f(x),\end{aligned}$$

which implies that $w(E^2) \le w(E^{2,1}) + w(E^{2,2})$, where the sets $E^{2,\ell}$, $\ell = 1, 2$ are defined as the sets $E^{1,\ell}$ with $T_{j,k}^{2,\ell} f$ instead of $T_{j,k}^{1,\ell} f$. Now, the estimates of $w(E^{2,1})$

and $w(E^{2,2})$ follow as in the previous cases obtaining

$$w(E^2) \leq \frac{C}{\lambda^q} \int_{s(m_{k+1}^j)}^{h(m_k^j)} f^p v.$$

Actually, the estimations are easier because we do not need to split the sets $E^{2,\ell}$. For the estimation of $w(E^3)$ let us define the function

$$H(x) = \int_{h(m_k^j)}^{h(x)} K(x,y) f(y)\, dy.$$

Since h is continuous and K is continuous in the first variable, we may select a decreasing sequence $\{x_i\}_i$ in (m_k^j, m_{k+1}^j) such that $x_0 = m_{k+1}^j$ and $H(x_i) = \int_{h(m_k^j)}^{h(x_i)} K(x_i, y) f(y)\, dy = (D+1)^{-i} H(m_{k+1}^j)$. We claim that

$$H(x_i) \leq (D+1)^4 \left(K(x_{i+2}, h(x_{i+3})) \int_{h(m_k^j)}^{h(x_{i+3})} f + \int_{h(x_{i+3})}^{h(x_{i+2})} K(x_{i+2}, y) f(y) dy \right).$$

In fact, first notice that

$$\begin{aligned} H(x_i) &= (D+1)^2 \int_{h(m_k^j)}^{h(x_{i+2})} K(x_{i+2}, y) f(y)\, dy \\ &= (D+1)^2 \left[\int_{h(m_k^j)}^{h(x_{i+3})} K(x_{i+2}, y) f(y)\, dy + \int_{h(x_{i+3})}^{h(x_{i+2})} K(x_{i+2}, y) f(y)\, dy \right]. \end{aligned}$$

Now, applying property (iii) of K we get that

$$\begin{aligned} H(x_i) \leq & D(D+1)^2 \left[K(x_{i+2}, h(x_{i+3})) \int_{h(m_k^j)}^{h(x_{i+3})} f + \int_{h(m_k^j)}^{h(x_{i+3})} K(x_{i+3}, y) f(y)\, dy \right] \\ & + (D+1)^2 \int_{h(x_{i+3})}^{h(x_{i+2})} K(x_{i+2}, y) f(y)\, dy \\ \leq & (D+1)^3 \left[K(x_{i+2}, h(x_{i+3})) \int_{h(m_k^j)}^{h(x_{i+3})} f + \int_{h(x_{i+3})}^{h(x_{i+2})} K(x_{i+2}, y) f(y)\, dy \right] \\ & + \frac{D}{D+1} H(x_i), \end{aligned}$$

and the claim follows. Now, we have that

$$w(E^3) \leq \sum_{i \geq 0} [w(E_i^{3,1}) + w(E_i^{3,2})],$$

where

$$E_i^{3,1} = \left\{ x \in (x_{i+1}, x_i) \cap U : \ g(x) K(x_{i+2}, h(x_{i+3})) \int_{h(m_k^j)}^{h(x_{i+3})} f > \frac{\lambda}{6(D+1)^4} \right\}$$

and

$$E_i^{3,2} = \left\{ x \in (x_{i+1}, x_i) \cap U : \ g(x) \int_{h(x_{i+3})}^{h(x_{i+2})} K(x_{i+2}, y) f(y) dy > \frac{\lambda}{6(D+1)^4} \right\}.$$

Working as in previous cases we have that

$$\sum_{i\geq 0} w(E_i^{3,2}) \leq \frac{C}{\lambda^q}\int_{h(m_k^j)}^{h(m_{k+1}^j)} f^p v.$$

In order to estimate $\sum_{i\geq 0} w(E_i^{3,1})$ we shall apply Lemma 1 in [4]. Let $\{u'_s\}$ be the decreasing sequence in (m_k^j, m_{k+1}^j) defined by $u'_0 = m_{k+1}^j$ and

$$\int_{h(m_k^j)}^{h(u'_s)} f = 2^{-s}\int_{h(m_k^j)}^{h(m_{k+1}^j)} f$$

and let $\{u_n\}$ be the subsequence of $\{u'_s\}$ defined by $u_0 = u'_0$ and if $[u'_{s+1}, u'_s)\cap\{x_i\} = \emptyset$, then we delete the term u'_{s+1} of $\{u'_s\}$. Let $\tilde{E}_n^{3,1} = \cup_{\{i\geq 0:\ u_{n+1}\leq x_{i+3}<u_n\}} E_i^{3,1}$, $\tilde{\alpha}_n = \inf \tilde{E}_n^{3,1}$ and $\tilde{\beta}_n = \sup \tilde{E}_n^{3,1}$. If $u'_{s+1} = u_{n+1} \leq x_{i+3} < u_n$, by the construction of the sequences we get that $x_{i+3}\leq u'_s$ and $u_{n+2}\leq u'_{s+2}$, then

$$\int_{h(m_k^j)}^{h(x_{i+3})} f \leq \int_{h(m_k^j)}^{h(u'_s)} f = 4\int_{h(u'_{s+2})}^{h(u'_{s+1})} f \leq 4\int_{h(u_{n+2})}^{h(u_{n+1})} f.$$

Let us assume that $\tilde{E}_n^{3,1}\neq\emptyset$. By the above inequalities and the monotonicity of K we have for all $t\in\tilde{E}_n^{3,1}$

$$\begin{aligned}\frac{\lambda}{6(D+1)^4} &\leq 4g(t)K(t,h(x_{i+3}))\int_{h(u'_{s+2})}^{h(u'_{s+1})} f\\ &\leq 4g(t)K(t,h(u_{n+1}))\int_{h(u_{n+2})}^{h(u_{n+1})} f.\end{aligned}$$

Now, multiplying by $\left(\int_{\tilde{\alpha}_n}^{\tilde{\beta}_n} w\right)^{1/p}$, applying Hölder inequality and using that $s(\tilde{\beta}_n)\leq h(u_{n+2})$ we get that

$$\begin{aligned}\frac{\lambda}{6(D+1)^4}\left(\int_{\tilde{\alpha}_n}^{\tilde{\beta}_n} w\right)^{1/p} &\leq 4\Phi_1(x)\left(\int_{h(u_{n+2})}^{h(u_{n+1})} f^p v\right)^{1/p}\\ &\leq 4\lambda^{q/r}\left(\int_{h(u_{n+2})}^{h(u_{n+1})} f^p v\right)^{1/p},\end{aligned}$$

where x is any point in $\tilde{E}_n^{3,1}$. Then

$$\begin{aligned}\sum_{i\geq 0} w(E_i^{3,1}) &= \sum_n \sum_{\{i\geq 0: u_{n+1}\leq x_{i+3}<u_n\}} w(E_i^{3,1}) \leq \sum_n w(\tilde{E}_n^{3,1})\\ &\leq \sum_n \int_{\tilde{\alpha}_n}^{\tilde{\beta}_n} w \leq \frac{C}{\lambda^q}\sum_n \int_{h(u_{n+2})}^{h(u_{n+1})} f^p v \leq \frac{C}{\lambda^q}\int_{h(m_k^j)}^{h(m_{k+1}^j)} f^p v.\end{aligned}$$

Putting together the estimations of $w(E^1)$, $w(E^2)$ and $w(E^3)$ we have

$$w(O_\lambda\cap U\cap(m_k^j, m_{k+1}^j)) \leq \frac{C}{\lambda^q}\int_{s(m_k^j)}^{h(m_{k+1}^j)} f^p v.$$

Summing up in k in the above inequality and by (2.3) we get that

$$w(O_\lambda \cap U \cap (a_j, b_j)) \leq \frac{C}{\lambda^q} \int_{s(a_j)}^{h(b_j)} f^p v.$$

Keeping in mind the lemma and summing up in j we obtain the desired inequality. $\square$

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