Algebra univers. **58** (2008) 335–340
0002-5240/08/030335 – 06, *published online* May 14, 2008
DOI 10.1007/s00012-008-2074-0

**Algebra Universalis**

# Free $\mathrm{MV_n}$-algebras

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Abstract. In this note we characterize free algebras in varieties of MV-algebras generated by a finite chain $\mathbf{L}_n$ as algebras of continuous functions from the spectrum of the Boolean skeleton of the free algebra into $\mathbf{L}_n$.

## 1. Introduction

$\mathrm{MV}_n$-algebras, for $n$ an integer $\geq 2$, were introduced in [10] as the algebraic counterpart of Łukasiewicz $n$-valued logic. Of course, $\mathrm{MV}_2$-algebras coincide with Boolean algebras.

Since the variety of $\mathrm{MV}_n$-algebras is generated by a semi-primal algebra, it follows that each $\mathrm{MV}_n$-algebra can be represented by $n$-valued continuous functions over the Stone space of its Boolean skeleton [5, 11, 12].

A characterization of the Boolean skeleton of a free MVn-algebra as a free Boolean algebra over a certain poset was given in [2]. The aim of this paper is to use this fact to give a description of free $\mathrm{MV}_n$-algebras over an arbitrary cardinal in terms of $n$-valued continuous functions over the Stone space of a free Boolean algebra over a certain poset. When the number of free generators is finite, our description coincides with the one given by A. Monteiro in 1969 (see [6, Theorem 8.6.1] and the corresponding Bibliographical Remarks).

The description of free $\mathrm{MV}_n$-algebras given in this paper should be compared with the one given in [12]. Alternative descriptions are in [9] and [8].

## 2. Preliminaries

We assume familiarity of the reader with the theory of MV-algebras. We refer to [6] for all unexplained notions.

Presented by J. Berman.
Received May 9, 2006; accepted in final form May 15, 2007.
2000 *Mathematics Subject Classification*: 03D30, 03D35.
*Key words and phrases*: $\mathrm{MV}_n$-algebras, free algebras, Stone space, Moisil operators.

Let $n \in \mathbb{N}$. $\mathbb{MV}_n$ denotes the variety of MV-algebras generated by the $n$-element MV-chain $\mathbf{L}_n$. The universe $L_n$ of $\mathbf{L}_n$ is the finite set

$$\left\{\frac{0}{n-1}, \frac{1}{n-1}, \ldots, \frac{n-1}{n-1}\right\}.$$

Recall that $\mathbf{L}_{d+1}$ is a subalgebra of $\mathbf{L}_n$ iff $d$ is a divisor of $n-1$.

As it is well known, the set of idempotent elements of each algebra $\mathbf{A} \in \mathbb{MV}_n$ forms the universe of a subalgebra of $\mathbf{A}$ which is a Boolean algebra. This algebra will be denoted by $\mathbf{B}(\mathbf{A})$.

Given an integer $n \geq 1$, $\mathrm{Div}(n)$ will denote the set of divisors of $n$, and $\mathrm{Div}^*(n)$ the set of proper divisors of $n$, i.e., $\mathrm{Div}^*(n) = \{d \in \mathrm{Div}(n) \colon d < n\}$. Both sets become distributive lattices under the divisibility order.

For each $n \geq 2$, an *$n$-valued Boolean space* is a pair $\langle X, \rho\rangle$, such that $X$ is a Boolean space and $\rho$ is a meet-homomorphism from the lattice of divisors of $n-1$ into the lattice of closed subsets of $X$, such that $\rho(n-1) = X$.

If the set $L_n$ is equipped with the discrete topology, and $(X, \rho)$ is an $n$-valued Boolean space, then $\mathcal{C}_n(X, \rho)$ denotes the $\mathrm{MV}_n$-algebra formed by the continuous functions $f$ from $X$ into $L_n$ such that $f(\rho(d)) \subseteq L_{d+1}$ for each $d \in \mathrm{Div}^*(n-1)$, with the algebraic operations defined pointwise. Clearly, the correspondence that assigns to each clopen $U$ its characteristic function defines an isomorphism from $\mathrm{Clop}(X)$ onto $\mathbf{B}(\mathcal{C}_n(X, \rho))$.

Since the variety $\mathbb{MV}_n$ is generated by the algebra $L_n$, which is a semiprimal algebra (a discriminator term for $L_n$ is $\tau(x, y, z) = ((\sigma_1^n((x \to y) \wedge (y \to x)) \wedge z) \vee (\neg\sigma_1^n((x \to y) \wedge (z \to x)) \wedge x)$, where $\to$ denotes Łukasiewicz implication), the next theorem follows at once from [11, Theorem 6.5] (cf [5, 12]). An elementary direct proof is given in [7, Theorem 1.5].

**Theorem 2.1.** *For each $\mathbf{A} \in \mathbb{MV}_n$ , there is a Boolean space $X(\mathbf{A})$ and a meet homomorphism $\rho_{\mathbf{A}}$ from $\mathrm{Div}(n-1)$ into the lattice of closed subsets of $X(\mathbf{A})$, satisfying $\rho_{\mathbf{A}}(n-1) = X(\mathbf{A})$, such that $\mathbf{A} \cong \mathcal{C}_n(X(\mathbf{A}), \rho_{\mathbf{A}})$. Moreover, $X(\mathbf{A})$ is isomorphic to the Stone space of the Boolean algebra $\mathbf{B}(\mathbf{A})$.*

For every $n \geq 2$ and for every $\mathbf{A} \in \mathbb{MV}_n$, we can define the *Moisil operators* in $A$. These operators are one-variable terms $\sigma_1^n(x), \ldots, \sigma_{n-1}^n(x)$ in the language $(\neg, \to, \top)$ such that for every $j = 1, \ldots, n-1$,

$$\sigma_j^n \colon \mathbf{A} \to \mathbf{B}(\mathbf{A}).$$

In particular, when evaluated on the algebras $\mathbf{L}_n$ we have

$$\sigma_i^n\left(\frac{j}{(n-1)}\right) = \begin{cases} 1 & \text{if } i + j \geq n, \\ 0 & \text{if } i + j < n, \end{cases} \tag{2.1}$$

for $i = 1, \dots, n-1$ (see [1], [4] or [13]). The most important property of these operators is the following one (see [3]).

**Theorem 2.2.** *Let $\mathbf{A} \in \mathbb{MV}_n$. Then $x \in B(\mathbf{A})$ if and only if $\sigma^n_{n-1}(x) = x$. Furthermore,*

$$\sigma^n_{n-1}(x) = \min\{b \in B(\mathbf{A}) : x \leq b\} \text{ and } \sigma^n_1(x) = \max\{a \in B(\mathbf{A}) : a \leq x\}.$$

Recall that a Boolean algebra $\mathbf{B}$ is said to be *free over a poset* $Y$ if for each Boolean algebra $\mathbf{C}$ and for each non-decreasing function $f\colon Y \to \mathbf{C}$, $f$ can be uniquely extended to a homomorphism from $\mathbf{B}$ into $\mathbf{C}$. We shall denote by $\mathbf{Free}_{\mathbb{MV}_n}(X)$ the free algebra in the variety $\mathbb{MV}_n$ over a set $X$ of generators. The next theorem is proved in [2, 2.12].

**Theorem 2.3.** $\mathbf{B}(\mathbf{Free}_{\mathbb{MV}_n}(X))$ *is the free Boolean algebra over the poset $Y := \{\sigma^n_i(x) : x \in X, i = 1, \dots, n-1\}$.*

The proof of [2, Lemma 3.6] can be easily adapted to give the following.

**Lemma 2.4.** *Consider the poset $Y = \{\sigma^n_i(x) : x \in X, i = 1, \dots, n-1\}$. The correspondence that assigns to each upwards closed subset $S \subseteq Y$ the Boolean filter $U_S$ generated by the set $S \cup \{\neg y : y \in Y \setminus S\}$, defines a bijection from the set of upwards closed subsets of $Y$ onto the ultrafilters of $\mathbf{B}(\mathbf{Free}_{\mathbb{MV}_n}(X))$.*

## 3. Main result

Using the information in Theorem 2.1 and Theorem 2.3, we shall give a characterization of the free algebras in $\mathbb{MV}_n$ as algebras of continuous functions.

Given $X$ and the poset $Y = \{\sigma^n_i(x) : x \in X, i = 1, \dots, n-1\}$, each chain $K^n_j(x) := \sigma^n_j(x) < \dots < \sigma^n_{n-1}(x)$ for $1 \leq j \leq n-1$ and $x \in X$ will be called an *x-chain* of $Y$. Clearly, $K^n_i(x) \cap K^n_j(y) = \emptyset$ whenever $x \neq y$, $1 \leq i, j \leq n-1$, and

$$Y = \bigcup_{x \in X} K^n_1(x). \tag{3.2}$$

Let $R_{n-1}$ be the set of upwards closed subsets of $Y$. Notice that a subset $S$ of $Y$ belongs to $R_{n-1}$ if and only if it is a union of $x$-chains. Let

$$Ch_x = \{K^n_j(x) \mid j = 1, \dots, n-1\} \cup \{\emptyset\}.$$

For each $d \in \mathrm{Div}^*(n-1)$ we define the set

$$H^x_d = \{C \in Ch_x \mid \frac{\#C}{n-1} \in L_{d+1}\},$$

where $\#C$ denotes the cardinal of the set $C$.

For each $d \in \mathrm{Div}^*(n-1)$, we define $R_d \subseteq R_{n-1}$ such that $S \in R_d$ iff $S = \bigcup_{x \in X} C_x$ for $C_x \in H^x_d$. Then we have the following:

**Theorem 3.1.** $\mathbf{Free}_{\mathbb{MV}_n}(X)$ *is isomorphic to the algebra of continuous functions $f$ from the Stone space of the free Boolean algebra over the poset $Y = \{\sigma_i^n(x) : x \in X, i = 1, \ldots, n-1\}$ into $\mathbf{L}_n$ such that for each $d \in \mathrm{Div}^*(n-1)$ and each $S \in R_d$, $f(U_S) \in L_{d+1}$.*

*Proof.* From Theorem 2.1 and Theorem 2.3 we can assert that $\mathbf{Free}_{\mathbb{MV}_n}(X)$ is isomorphic to the algebra of continuous functions from the Stone space of the free Boolean algebra over the poset $Y = \{\sigma_i^n(x) : x \in X, i = 1, \ldots, n-1\}$ into $\mathbf{L}_n$ such that $f(\rho(d)) \subseteq L_{d+1}$ for each $d \in \mathrm{Div}^*(n-1)$, where $\rho$ is the corresponding meet homomorphism. Then we only need to prove the following claim.

*Claim* For each $d \in \mathrm{Div}^*(n-1)$, $U_S \in \rho(d)$ iff $S \in R_d$.

*Proof of the claim.* If $U_S \in \rho(d)$, from the proof of [7, Theorem 1.5], we know that

$$U_S = \chi^{-1}(\{1\}) \cap B(\mathbf{Free}_{\mathbb{MV}_n}(X))$$

for some homomorphism $\chi\colon \mathbf{Free}_{\mathbb{MV}_n}(X) \to L_{d+1}$. Fix $x \in X$ and let $\chi(x) = \frac{j}{n-1} \in L_{d+1}$. Thus

$$\sigma_i^n(x) \in S \text{ iff } \chi(\sigma_i^n(x)) = 1 \text{ iff } \sigma_i^n(\chi(x)) = 1,$$

and from equation (2.1) this happens iff $i + j \geq n$. This implies that the $x$-chain $C_x$ which is included in $S$ has cardinality $j$. Since $\frac{j}{n-1} \in L_{d+1}$ we obtain that $C_x \in H_d^x$. Hence if $U_S \in \rho(d)$, then $S \in R_d$.

Now let $S \in R_d$ and $x \in X$. If we denote by $C_x$ the only $x$-chain included in $S$, our hypothesis asserts that $\frac{\#C_x}{n-1} \in L_{d+1}$. Let

$$j_x = \begin{cases} 0 & \text{if } \sigma_1^n(x) \in S, \\ \max\{i \in \{1, \ldots, n-1\} : \sigma_i^n(x) \notin S\} & \text{otherwise.} \end{cases}$$

Clearly $\#C_x = n - j_x - 1$. We define $\chi\colon \mathbf{Free}_{\mathbb{MV}_n}(X) \to \mathbf{L}_n$ by $\chi(x) = \frac{n-j_x-1}{n-1}$, for each $x \in X$. Since $X$ is a set of generators of $\mathbf{Free}_{\mathbb{MV}_n}(X)$, $\chi(\mathbf{Free}_{\mathbb{MV}_n}(X)) \subseteq L_{d+1}$. We also have

$$U_S = \chi^{-1}(\{1\}) \cap B(\mathbf{Free}_{\mathbb{MV}_n}(X)).$$

Therefore $U_S \in \rho(d)$. □

Taking into account Lemma 2.4, we obtain that for each $x \in X$, the function $\hat{x}\colon \mathrm{Spec}(\mathbf{B}(\mathbf{Free}_{\mathbb{MV}_n}(X))) \to L_n$ given by Theorem 2.1 is defined as follows, for each upwards closed subset $S$ of $Y$:

$$\hat{x}(U_S) = \begin{cases} 1 & \text{if } \sigma_1^n(x) \in U_S, \\ \dfrac{n-j_x-1}{n-1} & \text{otherwise.} \end{cases} \tag{3.3}$$

## 4. Free $\mathrm{MV}_{\mathbf{n}}$-algebras over a finite set of generators

We shall consider now the algebra $\mathbf{Free}_{\mathbb{MV}_n}(X)$ where $X$ is a finite set of cardinality $r \in \mathbb{N}$. According with Theorem 2.3 the Stone space of $\mathbf{B}(\mathbf{Free}_{\mathbb{MV}_n}(X))$ is a discrete space with $n^r$ elements (i.e, the cardinality of $R_{n-1}$). For each $S \in R_{n-1}$ let

$$r_S = \{d : S \in R_d \text{ and } S \notin R_j \text{ for any } j \in \mathrm{Div}^*(d)\}.$$

Following Theorem 3.1, to every $f \in \mathbf{Free}_{\mathbb{MV}_n}(X)$ we can assign an element $\overline{x} \in \prod_{S\in R_{n-1}} L_{r_S+1}$ such that $x_S = f(U_S)$. It is not hard to check that the assignment is an injective homomorphism. On the other hand, if we consider an element $\overline{x} \in \prod_{S\in R_{n-1}} L_{r_S+1}$ and define $f(U_S) = x_S$ for each $U_S \in \mathrm{Spec}\,\mathbf{B}(\mathbf{Free}_{\mathbb{MV}_n}(X))$, it is obvious that $f$ is continuous and that $f(U_S) \in L_{d+1}$ for each $S \in R_d$. Therefore the algebra $\mathbf{Free}_{\mathbb{MV}_n}(X)$ is isomorphic to the direct product $\prod_{S\in R_{n-1}} L_{r_S+1}$.

Therefore we can conclude the following:

**Theorem 4.1.**

$$\mathbf{Free}_{\mathbb{MV}_n}(X) \cong \prod_{d\in \mathrm{Div}(n-1)} \mathbf{L}_{d+1}^{\alpha_d},$$

*where for each $d \in \mathrm{Div}(n-1)$, $\alpha_d$ is the cardinality of the set $(R_d \setminus \bigcup_{k\in \mathrm{Div}^*(d)} R_k)$.*

**Example 4.2.** Let $p$ be a prime number. Consider $\mathbf{Free}_{\mathbb{MV}_{p+1}}(X)$ where $X$ is a finite set of cardinality $r$. Then $R_p$ has $p^r$ elements while $R_1$ has $2^r$ elements. According to Theorem 4.1 we have

$$\mathbf{Free}_{\mathbb{MV}_{p+1}}(X) \cong \mathbf{L}_{p+1}^{p^r-2^r} \times \mathbf{L}_2^{2^r}.$$

Notice that our description coincides with the one given in [6, Theorem 8.6.1].

## References


[1] V. Boicescu, A. Filipoiu, G. Georgescu, and S. Rudeanu, *Lukasiewicz-Moisil Algebras*, Elsevier, Amsterdam, 1991.

[2] M. Busaniche and R. Cignoli, *Free algebras in varieties of BL-algebras generated by a $BL_n$-chain*, Journal of the Australian Mathematical Society **80** (2006), 419–439.

[3] R. Cignoli, *Moisil Algebras*, Notas de Lógica Matemática, Universidad Nac. del Sur, Bahía Blanca, Argentina **27** (1970).

[4] R. Cignoli, *Proper n-Valued Łukasiewicz Algebras as S-algebras of Łukasiewicz n-Valued Propositional Calculi*, Stud. Log. **41** (1982), 3–16.

[5] R. Cignoli, *Natural dualities for the algebras of Lukasiewicz finitely-valued logics*, Bull. Symb. Logic. **2** (1996), 218.

[6] R. Cignoli, M. I. D'Ottaviano, and D. Mundici, *Algebraic foundations of many-valued reasoning*, Kluwer Academic Pub., Dordrecht, 2000.

[7] R. Cignoli and L. Monteiro, *Maximal subalgebras of $MV_{\mathbf{n}}$-algebras. A proof of a conjecture of A. Monteiro*, Stud. Log. **84** (2006), 393–405.

[8] A. Di Nola, R. S. Grigolia, and G. Panti, *Finitely generated free MV-algebras and their automorphism groups*, Stud. Log. **61** (1998), 65–78.

[9] A. Di Nola and R. S. Grigolia, *Projective MV-algebras and their automorphism groups*, J of Mult.-Valued Logic and Soft Computing, 9 (2003), 291–317.

[10] R. S. Grigolia, *An algebraic analysis of Łukasiewicz - Tarski n-valued logical systems*. In: R. Wójcicki, G. Malinowski (Eds.), Selected papers on Łukasiewicz sentential calculus, Ossolineum, Wrocław (1977), 81–92.

[11] K. Keimel and H. Werner, *Stone duality for varieties generated by a quasi-primal algebra*, Mem. Amer. Math. Soc. No. **148** (1974), 59–85.

[12] P. Niederkorn, *Natural dualities for varieties of MV-algebras*, J. Math. Anal. Appl. **225** (2001), 58–73.

[13] A. J. Rodríguez and A. Torrens, *Wajsberg Algebras and Post Algebras*, Stud. Log. **53** (1994), 1–19.

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