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Differential Transforms in Weighted Spaces

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ABSTRACT. We extend the results by Jones and Rosenblatt about the series of the differences of differentiation operators along lacunary sequences to BMO and to the setting of weighted L^p-spaces. We use a different approach which allows to establish that the one-sided Sawyer A_p weights are the natural ones to study the boundedness and convergence of that series in weighted spaces.

1. Introduction

Jones and Rosenblatt studied [2] the behavior of the series of the differences of ergodic averages and the series of the differences of differentiation operators along lacunary sequences in the context of the L^p spaces. In this article we give a different approach to these questions and we extend their results about the series of the differences of differentiation operators to *BMO* and to the setting of weighted L^p spaces. We point out that the estimates in weighted spaces turn out to be a useful tool to establish the results for the series of the differences of ergodic averages associated to a strongly continuous one parameter group of positive operators acting on some $L^p(\mu)$; the details will appear in a forthcoming article.

Let $\rho > 1$ and let ε_k, $k \in \mathbb{Z}$, be a ρ-lacunary sequence of positive numbers, that is,

$$\varepsilon_{k+1}/\varepsilon_k \geq \rho \quad \text{for all } k .$$

This implies clearly that $\lim_{k\to-\infty} \varepsilon_k = 0$ and $\lim_{k\to\infty} \varepsilon_k = \infty$. Therefore if $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$, and

$$D_k f(x) = \frac{1}{\varepsilon_k} \int_0^{\varepsilon_k} f(x+t)\, dt \tag{1.1}$$

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then $\lim_{k\to-\infty} D_k f(x) = f(x)$ and $\lim_{k\to\infty} D_k f(x) = 0$ a.e. In order to give some information about how the convergence occurs, we may consider the series

$$\sum_{k=-\infty}^{\infty} (D_k f(x) - D_{k-1} f(x))$$

which obviously converges a.e. by the above remark. As it is explained in [2], the cancellation in this series is crucial. Therefore, it is natural to ask about the convergence properties of

$$\sum_{k=-\infty}^{\infty} v_k (D_k f(x) - D_{k-1} f(x)) \ , \tag{1.2}$$

where v_k is a bounded sequence of real or complex numbers. One of our aims is to study the convergence properties of that series in the setting of weighted spaces and not only in the Lebesgue measure case. This implies that our approach must be different from the one in [2]. In order to explain why our approach should be different, let us recall the argument in [2].

Jones and Rosenblatt obtained the results for the operator

$$\sum_{k=-\infty}^{0} v_k (D_k f(x) - D_{k-1} f(x)) \tag{1.3}$$

but this is probably a minor difference. They obtained the results by considering the following series:

$$\sum_{k=-\infty}^{0} v_k (D_k f(x) - E_k f(x)) \quad \text{and} \quad \sum_{k=-\infty}^{0} v_k (E_k f(x) - E_{k-1} f(x)) \ , \tag{1.4}$$

where $\varepsilon_k = 2^{-k}$ and E_k is the conditional expectation operator with respect to the dyadic σ-algebra with 2^{-k} atoms. Since the properties of convergence of the second series in (1.4) are well known, the problem is reduced to study the first series. Notice that the first series in (1.4) defines formally a two-sided operator in the sense that for fixed x the values of the series depend on the values of $f(y)$ for $y < x$ and $y > x$. In this way, if we study the weights for this series we are naturally led to consider Muckenhoupt A_p weights, i.e., the weights for the two-sided Hardy-Littlewood maximal function defined by

$$Mf(x) = \sup_{\eta,\varepsilon>0} \frac{1}{\eta+\varepsilon} \int_{-\eta}^{\varepsilon} |f(x+t)|\, dt \ .$$

[The same can be said for the second series in (1.4).] However, if we look at the original series (1.3), we realize that it is a one-sided operator, in the sense that for fixed x the values of the series depend only on the values of $f(y)$ for $y > x$. Therefore the natural weights to be considered are the good weights for the one-sided Hardy-Littlewood maximal function defined by

$$M^+ f(x) = \sup_{\varepsilon>0} \frac{1}{\varepsilon} \int_{0}^{\varepsilon} |f(x+t)|\, dt \ , \tag{1.5}$$

which are a class of weights wider than the Muckenhoupt A_p classes. So, we have to study directly the convergence of (1.2) without going through the conditional expectations. In what follows we fix the problems, the notations and state the results.

Throughout the article, we will consider a bounded sequence $\{v_k\}$, $k \in \mathbb{Z}$, of real or complex numbers. We shall say that $\{v_k\}$ is a multiplying sequence and we shall write $\|v_k\|_\infty = \sup_k |v_k|$. For a locally integrable function f we consider the averages $D_k f(x)$ as in (1.1), that is, $D_k f(x) = \varphi_k * f(x)$ where $\varphi_k(x) = \frac{1}{\varepsilon_k}\chi_{(-\varepsilon_k,0)}(x)$. For each $N \in \mathbb{Z}^2$, $N = (N_1, N_2)$ with $N_1 < N_2$ we define the sum

$$T_N f(x) = \sum_{k=N_1}^{N_2} v_k(D_k f(x) - D_{k-1} f(x)) = K_N * f(x) \,, \tag{1.6}$$

where

$$K_N(x) = \sum_{k=N_1}^{N_2} v_k(\varphi_k(x) - \varphi_{k-1}(x)) \,.$$

Our goal is to prove convergence results of $T_N f(x)$ as $N = (N_1, N_2)$ tends to $(-\infty, +\infty)$ which means that $N_1 \to -\infty$ and $N_2 \to +\infty$. As usual, to prove the a.e. convergence, we shall study the boundedness of the associated maximal operator

$$T^* f(x) = \sup_{N \in \mathbb{Z}^2} |T_N f(x)|$$

in the setting of the weighted spaces

$$L^p(w) = \left\{ f : f \text{ is measurable and } \|f\|_{L^p(w)} = \left(\int_{\mathbb{R}} |f|^p w \right)^{1/p} < \infty \right\} .$$

Since the operators T_N are convolution operators with kernels K_N supported in $(-\infty, 0)$, the study of T^* is related to the right-sided Hardy-Littlewood maximal operator M^+ defined in (1.5). We recall the well-known results about weights for M^+:

(1) The operator M^+ is of weak type $(1, 1)$ with respect to the measure $w(x)\,dx$ if and only if $w \in A_1^+$ [6, 3], i.e., there exists C such that $M^- w(x) \le C w(x)$ a.e. (M^- is the left-sided Hardy-Littlewood maximal function defined as $M^- f(x) = \sup_{\varepsilon>0} \frac{1}{\varepsilon} \int_0^\varepsilon |f(x-t)|\,dt$).

(2) The operator M^+ is bounded in $L^p(w)$, $1 < p < \infty$, if and only if $w \in A_p^+$ [6], i.e., if there exists C such that for any three points $a < b < c$

$$\left(\int_a^b w \right)^{\frac{1}{p}} \left(\int_b^c w^{1-p'} \right)^{\frac{1}{p'}} \le C(c-a) \,, \tag{1.7}$$

where $p + p' = pp'$.

Of course, there are similar results for the operator M^- reversing the orientation of $\mathbb{R}$.

Now we can state the main results.

Theorem 1. *Let $\{\varepsilon_k\}$ be a ρ-lacunary sequence. Let $\{v_k\}$ be a multiplying sequence. If $1 < p < \infty$ and $w \in A_p^+$ then there exists a constant C depending only on ρ, p, and $\|v_k\|_\infty$ such that*

$$\left\| T^* f \right\|_{L^p(w)} \le C \, \|f\|_{L^p(w)}$$

for all functions $f \in L^p(w)$.

Theorem 2. *Let* $\{\varepsilon_k\}$ *be a* ρ*-lacunary sequence. Let* $\{v_k\}$ *be a multiplying sequence. If* $w \in A_1^+$ *then there exists a constant* C *depending only on* ρ *and* $\|v_k\|_\infty$ *such that*

$$w\big(\big\{x \in \mathbb{R} : \big|T^* f(x)\big| > \lambda\big\}\big) \leq \frac{C}{\lambda} \|f\|_{L^1(w)} ,$$

for all $\lambda > 0$ *and all functions* $f \in L^1(w)$.

Using these theorems and proving the a.e. convergence in a suitable dense class of functions we obtain the following theorem.

Theorem 3. *Let* $\{\varepsilon_k\}$ *be a* ρ*-lacunary sequence. Let* $\{v_k\}$ *be a multiplying sequence. Assume that* w *is a weight in* A_p^+.
(i) *If* $1 < p < \infty$ *then* $T_N f$ *converges a.e. and in* $L^p(w)$ *norm for all* $f \in L^p(w)$ *as* $N = (N_1, N_2)$ *tends to* $(-\infty, +\infty)$.
(ii) *If* $p = 1$ *then* $T_N f$ *converges a.e. and in measure for all* $f \in L^1(w)$ *as* $N = (N_1, N_2)$ *tends to* $(-\infty, +\infty)$.

The above result includes the case of the Lebesgue measure ($w = 1$) which was obtained in [2]. In the case of the space *BMO* we can not expect to obtain a.e. convergence or norm convergence. However, a nice substitute can be found. We state the result in the next theorem.

Theorem 4. *Let* $\{\varepsilon_k\}$ *be a* ρ*-lacunary sequence. Let* $\{v_k\}$ *be a multiplying sequence. Then the operators* $T_N f$ *converge in the sense of the weak* $*$ *topology of BMO.*

The organization of the article is as follows. Sections 2 and 3 are devoted to state notations and properties of the lacunary sequences. In Section 4 we prove some properties of the kernels K_N. Section 5 is devoted to prove uniform boundedness of the operators T_N, while Theorems 1, 2, 3, and 4 are proved in Sections 6, 7, 8, and 9, respectively.

Throughout the article, the letter C means a positive constant not necessarily the same at each occurrence.

2. Some Notation and Previous Facts

We denote by $L^p = L^p(\mathbb{R})$, $1 \leq p < \infty$, the Lebesgue space consisting of all measurable functions f defined in $\mathbb{R}$ such that

$$||f||_{L^p} = \left(\int_{\mathbb{R}} |f(x)|^p \, dx \right)^{1/p} < \infty .$$

For $p = \infty$, we write L^∞ for the space of all measurable f such that

$$||f||_{L^\infty} = \text{ess sup}\, |f(x)| < \infty .$$

When Lebesgue measure is replaced by $w(x)\,dx$ (for some nonnegative measurable function w in $\mathbb{R}$) we denote the corresponding weighted spaces by $L^p(w)$.

For a locally integrable function f ($f \in L^1_{\text{loc}}$), we define the sharp maximal function

$$f^\sharp(x) = \sup_{x \in I} \frac{1}{|I|} \int_I |f(y) - f_I| \, dy ,$$

where I is an arbitrary interval in $\mathbb{R}$ and $f_I = |I|^{-1} \int_I f$. Now, we define the space

$$BMO = \left\{ f \in L^1_{\text{loc}} : \, ||f||_{BMO} = \left\| f^\sharp \right\|_\infty < \infty \right\} .$$

An atom is a function $a \in L^\infty$ supported in an interval I and such that

$$|a(x)| \leq 1/|I|, \qquad \int_I a(x)\, dx = 0$$

and the space H^1 is, as usual, the subspace of L^1 formed by all functions

$$f(x) = \sum_i \lambda_i a_i(x) ,$$

where $\sum_i |\lambda_i| < \infty$ and the functions a_i are atoms. The norm of f in H^1 is defined by $||f||_{H^1} = \inf \sum_i |\lambda_i|$, where the infimum is taken over the sequences $\{\lambda_i\}$ appearing in all the possible decompositions of f. It is known that if we identify in *BMO* the constant functions with 0, we have that *BMO* is the dual space of H^1.

The study of the operators T_N in weighted spaces requires the introduction of the following one-sided maximal functions:

$$M_s^+ f(x) = \sup_{h>0} \left(\frac{1}{h} \int_x^{x+h} |f(y)|^s \, dy \right)^{1/s} \quad (1 \leq s < \infty)$$

and

$$f^{+,\#}(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} \left(f(y) - \frac{1}{h} \int_{x+h}^{x+2h} f \right)^+ dy ,$$

where f is a locally integrable function taking values in $\mathbb{R}$ and z^+ stands for the positive part of the real number z. We shall also need the following theorem that was proved in [4].

Theorem 5. *For any $0 < p < \infty$ and $w \in A_\infty^+ = \cup_{p>1} A_p^+$ there exists C such that*

$$\int_{\mathbb{R}} \left| M^+ f(x) \right|^p w(x)\, dx \leq C \int_{\mathbb{R}} \left(f^{+,\#}(x) \right)^p w(x)\, dx$$

whenever the left-hand side is finite.

3. Lacunary Sequences

We prove in this section some properties of the ρ-lacunary sequence $\{\varepsilon_k\}$. The next proposition shows that, without loss of generality, we may assume that

$$1 < \rho \leq \frac{\varepsilon_{k+1}}{\varepsilon_k} \leq \rho^2 . \tag{3.1}$$

Proposition 1. *Given the ρ-lacunary sequence $\{\varepsilon_k\}$ and the multiplying sequence $\{v_k\}$, we can define a ρ-lacunary sequence $\{\eta_k\}$ and a multiplying sequence $\{w_k\}$ verifying the following properties:*

(i) $1 < \rho \leq \frac{\eta_{k+1}}{\eta_k} \leq \rho^2$ *and* $||v_k||_\infty = ||w_k||_\infty$.

(ii) *For all* $N = (N_1, N_2)$ *there exists* $M = (M_1, M_2)$ *with* $T_N = \tilde{T}_M$*, where* $\tilde{T}_M$ *is the operator defined in* (1.6) *for the new sequences* $\{\eta_k\}$ *and* $\{w_k\}$.

Proof. Let $\eta_0 = \varepsilon_0$, and let us construct η_l for positive l as follows (the argument for negative l is analogous). If $\rho^2 \geq \varepsilon_1/\varepsilon_0 \geq \rho$, define $\eta_1 = \varepsilon_1$. In the opposite case where $\varepsilon_1/\varepsilon_0 > \rho^2$, let $\eta_1 = \rho\varepsilon_0$. It verifies $\rho^2 \geq \eta_1/\eta_0 = \rho \geq \rho$. Further, $\varepsilon_1/\eta_1 \geq \rho^2\varepsilon_0/\rho\varepsilon_0 = \rho$. Again, if $\varepsilon_1/\eta_1 \leq \rho^2$, then $\eta_2 = \varepsilon_1$. If this is not the case, define $\eta_2 = \rho^2\varepsilon_0 \leq \varepsilon_1$. By the same calculations as before, η_0, η_1, η_2 are part of a lacunary sequence satisfying (3.1). To continue the sequence, either $\eta_3 = \varepsilon_1$ (if $\varepsilon_1/\eta_2 \leq \rho^2$) or $\eta_3 = \rho^3\eta_0$ (if $\varepsilon_1/\eta_2 > \rho^2$). Since $\rho > 1$, this process ends at some k_0 such that $\eta_{k_0} = \varepsilon_1$. The rest of the elements η_k are built in the same way, as the original ε_k plus the necessary terms put in between two consecutive ε_k to get (3.1).

Let $J(k) = \{j : \varepsilon_{k-1} < \eta_j \leq \varepsilon_k\}$, $\tilde{D}_j f(x) = \frac{1}{\eta_j}\int_0^{\eta_j} f(x+t)\,dt$ and $w_j = v_k$ if $j \in J(k)$. Then

$$v_k(D_k f(x) - D_{k-1} f(x)) = \sum_{j\in J(k)} w_j\big(\tilde{D}_j f(x) - \tilde{D}_{j-1} f(x)\big)\,.$$

If $M = (M_1, M_2)$ is such that $\eta_{M_2} = \varepsilon_{N_2}$ and $\eta_{M_1-1} = \varepsilon_{N_1-1}$ we get that

$$\begin{aligned} T_N f(x) &= \sum_{k=N_1}^{N_2} v_k(D_k f(x) - D_{k-1} f(x)) \\ &= \sum_{j=M_1}^{M_2} w_j\big(\tilde{D}_j f(x) - \tilde{D}_{j-1} f(x)\big) = \tilde{T}_M f(x)\,. \end{aligned}$$

□

It follows from this proposition that it is enough to prove all the results of this article in the case of a ρ-lacunary sequence satisfying (3.1). For this reason, in the rest of the article we assume that $\{\varepsilon_k\}$ satisfies (3.1) without saying it explicitly. Observe that the following properties hold:

$$\left(\frac{1}{\rho}\right)^{2(m-n)} \leq \frac{\varepsilon_n}{\varepsilon_m} \leq \left(\frac{1}{\rho}\right)^{m-n}, \quad \text{for all} \quad m > n\,. \tag{3.2}$$

In fact, these inequalities follow from (3.1) and the equality

$$\frac{\varepsilon_n}{\varepsilon_m} = \frac{\varepsilon_n}{\varepsilon_{n+1}}\frac{\varepsilon_{n+1}}{\varepsilon_{n+2}}\cdots\frac{\varepsilon_{m-1}}{\varepsilon_m}\,.$$

If we denote by α the smaller positive integer such that

$$1/\rho + (1/\rho)^\alpha \leq 1\,,$$

we get from (3.2) that

$$\varepsilon_i + \varepsilon_m \leq \varepsilon_{m+1} \quad \text{for all} \quad m \geq i + \alpha - 1\,. \tag{3.3}$$

4. Properties of the Kernels K_N

The next lemma allows us to prove the boundedness of the operators T_N uniformly on L^2.

Lemma 1. *There exists a constant C depending only on ρ and $\|v_k\|_\infty$ such that*

$$\sup_N \left|\hat{K}_N(\xi)\right| = \sup_N \left|\sum_{k=N_1}^{N_2} v_k\big(\hat{\varphi}(\varepsilon_k\xi) - \hat{\varphi}(\varepsilon_{k-1}\xi)\big)\right| \le C\ ,$$

for all $\xi \in \mathbb{R}$ and where $\hat{f}(\xi) = \int e^{-i\xi x} f(x)\,dx$ is the Fourier transform of f.

Proof. For fixed $\xi \in \mathbb{R}$, let k_0 be such that $\varepsilon_{k_0-1} < 1/|\xi| \le \varepsilon_{k_0}$. Then

$$\begin{aligned}\left|\hat{K}_N(\xi)\right| &\le \sum_{k=-\infty}^{\infty} |v_k|\left|\hat{\varphi}(\varepsilon_k\xi) - \hat{\varphi}(\varepsilon_{k-1}\xi)\right| \\ &\le \|v_k\|_\infty \sum_{k=-\infty}^{k_0-1} \cdots + \|v_k\|_\infty \sum_{k=k_0}^{\infty} \cdots = I + II\ .\end{aligned}$$

To estimate I, the mean value theorem and (3.1) yield

$$\begin{aligned}\left|\hat{\varphi}(\varepsilon_k\xi) - \hat{\varphi}(\varepsilon_{k-1}\xi)\right| &= \int_{-1}^{0} \left|e^{-i\xi\varepsilon_k x} - e^{-i\xi\varepsilon_{k-1}x}\right| dx \\ &\le C(\varepsilon_k - \varepsilon_{k-1})|\xi| \le C\rho^2\varepsilon_{k-1}|\xi|\ .\end{aligned}$$

Then, using (3.2) we get

$$\begin{aligned} I &\le C\|v_k\|_\infty|\xi|\rho^2 \sum_{k=-\infty}^{k_0-1} \varepsilon_{k-1} \le C\|v_k\|_\infty|\xi|\rho^2 \sum_{k=-\infty}^{k_0-1} \left(\frac{1}{\rho}\right)^{k_0+1-k} \varepsilon_{k_0} \\ &\le C\|v_k\|_\infty \frac{|\xi|\rho^2\varepsilon_{k_0}}{\rho^{k_0+1}} \sum_{k=-\infty}^{k_0-1} \rho^k = \frac{C\|v_k\|_\infty\rho|\xi|\varepsilon_{k_0}}{\rho-1} \\ &\le \frac{C\|v_k\|_\infty\rho^3}{\rho-1}\ ,\end{aligned}$$

where in the last inequality we have used that $\varepsilon_{k_0} \le \rho^2\varepsilon_{k_0-1} \le \frac{\rho^2}{|\xi|}$.

To estimate II we observe that $|\hat{\varphi}(\varepsilon_k\xi)| \le \frac{2}{|\xi|\varepsilon_k}$. Then, by (3.2) we get that

$$\begin{aligned} II &\le \|v_k\|_\infty \sum_{k=k_0}^{\infty} \left(\left|\hat{\varphi}_k(\xi)\right| + \left|\hat{\varphi}_{k-1}(\xi)\right|\right) \\ &\le \frac{4\|v_k\|_\infty}{|\xi|} \sum_{k=k_0}^{\infty} \frac{1}{\varepsilon_{k-1}} \\ &\le \frac{4\|v_k\|_\infty}{|\xi|\varepsilon_{k_0-1}} \sum_{k=k_0}^{\infty} \left(\frac{1}{\rho}\right)^{k-k_0} \le \frac{4\|v_k\|_\infty\rho^3}{\rho-1}\ ,\end{aligned}$$

where in the last inequality we have used that $|\xi|\varepsilon_{k_0-1} \ge |\xi|\frac{\varepsilon_{k_0}}{\rho^2} \ge \frac{1}{\rho^2}$. □

In what follows we shall prove that the kernels K_N verify a one-sided smoothness condition uniformly. The proofs are similar to the ones in [7]. First, we shall need the following lemma.

Lemma 2. *If $j \geq i+\alpha$, $0 < x \leq \varepsilon_i$ and $\varepsilon_j < y < \varepsilon_{j+1}$, then $\chi_{(x,x+\varepsilon_k)}(y)-\chi_{(0,\varepsilon_k)}(y) = 0$ unless $k = j$ in which case*

$$\chi_{(x,x+\varepsilon_k)}(y) - \chi_{(0,\varepsilon_k)}(y) = \chi_{(\varepsilon_j,x+\varepsilon_j)}(y) .$$

Proof. It is clear that for $k > j$,

$$0 < x \leq \varepsilon_i \leq \varepsilon_j < y < \varepsilon_{j+1} \leq \varepsilon_k < x + \varepsilon_k ,$$

and thus

$$[\chi_{(x,x+\varepsilon_k)}(y) - \chi_{(0,\varepsilon_k)}(y)]\chi_{(\varepsilon_j,\varepsilon_{j+1})}(y) = 0 .$$

For $k \leq j-1$, by (3.3), we get that

$$\varepsilon_k < x + \varepsilon_k \leq \varepsilon_i + \varepsilon_{j-1} \leq \varepsilon_j .$$

Then

$$\chi_{(x,x+\varepsilon_k)}(y)\chi_{(\varepsilon_j,\varepsilon_{j+1})}(y) = \chi_{(0,\varepsilon_k)}(y)\chi_{(\varepsilon_j,\varepsilon_{j+1})}(y) = 0 .$$

Finally, in the case $k = j$, since by (3.3) $x < \varepsilon_j < x + \varepsilon_j \leq \varepsilon_i + \varepsilon_j \leq \varepsilon_{j+1}$ we have that

$$\begin{aligned} [\chi_{(x,x+\varepsilon_j)}(y) - \chi_{(0,\varepsilon_j)}(y)]\chi_{(\varepsilon_j,\varepsilon_{j+1})}(y) &= \chi_{(x,x+\varepsilon_j)}(y)\chi_{(\varepsilon_j,\varepsilon_{j+1})}(y) \\ &= \chi_{(\varepsilon_j,x+\varepsilon_j)}(y) . \end{aligned}$$

□

Lemma 3. *Let $1 \leq r < \infty$, $j \geq i + \alpha$ and $0 < x \leq \varepsilon_i$. Then*

$$\left(\int_{\varepsilon_j}^{\varepsilon_{j+1}} |K_N(x-y) - K_N(-y)|^r \, dy \right)^{1/r} \leq C_j \, \varepsilon_j^{1/r-1} ,$$

where $C_j = \frac{2\|v_k\|_\infty}{\rho^{(j-i)/r}}$.

Proof. First, we have

$$\begin{aligned} &|K_N(x-y) - K_N(-y)| \\ &\quad = \left| \sum_{k=N_1}^{N_2} v_k \left(\frac{1}{\varepsilon_k} \chi_{(-\varepsilon_k,0)}(x-y) - \frac{1}{\varepsilon_{k-1}} \chi_{(-\varepsilon_{k-1},0)}(x-y) \right) \right. \end{aligned}$$

$$
\begin{aligned}
&\quad - \sum_{k=N_1}^{N_2} v_k \left(\frac{1}{\varepsilon_k} \chi_{(-\varepsilon_k,0)}(-y) - \frac{1}{\varepsilon_{k-1}} \chi_{(-\varepsilon_{k-1},0)}(-y) \right) \Bigg| \\
&\le \left| \sum_{k=N_1}^{N_2} \frac{v_k}{\varepsilon_k} (\chi_{(-\varepsilon_k,0)}(x-y) - \chi_{(-\varepsilon_k,0)}(-y)) \right| \\
&\quad + \left| \sum_{k=N_1}^{N_2} \frac{v_k}{\varepsilon_{k-1}} (\chi_{(-\varepsilon_{k-1},0)}(x-y) - \chi_{(-\varepsilon_{k-1},0)}(-y) \right| \\
&\le 2\|v_k\|_\infty \sum_{k=N_1-1}^{N_2} \frac{1}{\varepsilon_k} |\chi_{(-\varepsilon_k,0)}(x-y) - \chi_{(-\varepsilon_k,0)}(-y)| \\
&= 2\|v_k\|_\infty \sum_{k=N_1-1}^{N_2} \frac{1}{\varepsilon_k} |\chi_{(x,x+\varepsilon_k)}(y) - \chi_{(0,\varepsilon_k)}(y)| \,.
\end{aligned}
$$

Now, applying Lemma 2 we get that

$$
\begin{aligned}
\left(\int_{\varepsilon_j}^{\varepsilon_{j+1}} |K_N(x-y) - K_N(-y)|^r \, dy \right)^{1/r} &\le \frac{2\|v_k\|_\infty x^{1/r}}{\varepsilon_j} \\
&\le \frac{2\|v_k\|_\infty}{\rho^{(j-i)/r}} \varepsilon_j^{1/r-1} \,,
\end{aligned}
$$

where in the last inequality we have used that, by (3.2), $x \le \varepsilon_i \le \varepsilon_j/\rho^{j-i}$. □

Remark 1. It follows from the proof of the above lemma that if $N = (N_1, N_2)$ with $N_2 < i + \alpha \le j$ or $i + \alpha \le j < N_1$ then

$$
|K_N(x-y) - K_N(-y)| = 0
$$

for all $x \in (0, \varepsilon_i]$ and $y \in (\varepsilon_j, \varepsilon_{j+1})$.

The condition in Lemma 3 is called the one-sided D_r condition, that is, we have proved that for every $r \in [1, \infty)$ the kernels K_N satisfy the one-sided condition D_r uniformly on N. This condition implies Hörmander's condition as we prove in the next corollary.

Corollary 1. *The kernels K_N satisfy the following Hörmander's condition uniformly: There exists a constant C depending only on ρ and $\|v_k\|_\infty$ such that*

$$
\int_{|y|>C_\rho|x|} |K_N(x-y) - K_N(-y)| \, dy \le C, \quad x \in \mathbb{R} \,,
$$

where $C_\rho = \rho^{2(\alpha+1)}$ and $x \neq 0$.

Proof. Given $x \in \mathbb{R}\setminus\{0\}$ let $i \in \mathbb{Z}$ such that $\varepsilon_{i-1} < |x| \le \varepsilon_i$. It is clear by the lacunarity of the sequence $\{\varepsilon_j\}$ that

$$
\{y : |y| > C_\rho|x|\} \subset \{y : |y| > C_\rho\varepsilon_{i-1}\} \subset \{y : |y| > \varepsilon_{i+\alpha}\} \,.
$$

Now, let us observe that if $\varepsilon_{i-1} < |x| \le \varepsilon_i$ and $-\varepsilon_{j+1} < y \le -\varepsilon_j$ with $j \ge i + \alpha$, then $K_N(x-y) = K_N(-y) = 0$. So that we only need to consider $y > 0$. Thus,

$$
\int_{|y|>C_\rho|x|} |K_N(x-y) - K_N(-y)| \, dy \le \int_{y>\varepsilon_{i+\alpha}} |K_N(x-y) - K_N(-y)| \, dy \,.
$$

Now, if $x > 0$, the lemma follows simply from Lemma 3 with $r = 1$. If $x < 0$, by a change of variables we can obtain the same boundedness. □

5. Boundedness of the Operators T_N

This section is devoted to prove the uniform boundedness of the operators T_N. More precisely, we shall prove the following theorem.

Theorem 6. *The operators T_N verify the following inequalities with constants independent of N and f.*
(i) *For all* $\lambda > 0$, $|\{x \in \mathbb{R} : |T_N f(x)| > \lambda\}| \le \frac{C}{\lambda} \|f\|_{L^1}$.
(ii) *For* $1 < p < \infty$, $\|T_N f\|_{L^p} \le C_p \|f\|_{L^p}$.
(iii) $\|T_N f\|_{BMO} \le C \|f\|_{BMO}$.
(iv) *For* $1 < s < \infty$, $[|T_N f|]^{+,\#}(x) \le C\, M_s^+ f(x)$.

Proof. It follows from Lemma II.6.1 in [1] that (i) and (ii) hold since the kernels K_N satisfy Hörmander's condition uniformly and the Fourier transforms are uniformly bounded in L^∞.

Before proving (iii) we state and prove the following technical lemma.

Lemma 4. *Given $f \in BMO$, $x_0 \in \mathbb{R}$, $1 < p < \infty$, $h > 0$, $i \in \mathbb{Z}$ such that $\varepsilon_{i-1} < h \le \varepsilon_i$, $j > i$ and $I = (x_0, x_0 + h)$, we have that*

$$\left(\frac{1}{\varepsilon_{j+1}} \int_{x_0}^{x_0+\varepsilon_{j+1}} |f(x) - f_I|^p\, dx\right)^{1/p} \le C(1 + j - i)\|f\|_{BMO}\,.$$

Proof. Let $I_j = (x_0, x_0 + \varepsilon_j)$. Then,

$$\left(\frac{1}{\varepsilon_{j+1}} \int_{x_0}^{x_0+\varepsilon_{j+1}} |f(x) - f_I|^p\, dx\right)^{1/p} \le \left(\frac{1}{\varepsilon_{j+1}} \int_{x_0}^{x_0+\varepsilon_{j+1}} |f(x) - f_{I_{j+1}}|^p\, dx\right)^{1/p} + \sum_{l=i+1}^{j+1} |f_{I_l} - f_{I_{l-1}}| + |f_{I_i} - f_I| = I + II + III\,.$$

The first term is bounded by a constant times $\|f\|_{BMO}$ by John-Nirenberg's theorem. On the other hand, by (3.1), we have that

$$\begin{aligned} II + III &\le \sum_{l=i+1}^{j+1} \frac{1}{\varepsilon_{l-1}} \int_{x_0}^{x_0+\varepsilon_{l-1}} |f(x) - f_{I_l}|\, dx + \frac{1}{|I|} \int_I |f(x) - f_{I_i}|\, dx \\ &\le \sum_{l=i+1}^{j+1} \frac{\rho^2}{\varepsilon_l} \int_{x_0}^{x_0+\varepsilon_l} |f(x) - f_{I_l}|\, dx + \frac{\rho^2}{\varepsilon_i} \int_{x_0}^{x_0+\varepsilon_i} |f(x) - f_{I_i}|\, dx \\ &\le \rho^2(j - i + 2)\|f\|_{BMO}\,. \end{aligned}$$

□

Now, we shall prove (iii). Fix $x_0 \in \mathbb{R}$ and $h > 0$. Consider the interval $I = (x_0, x_0 + h)$. We split f according to the interval I as $f = f_0 + f_1 + f_2 + f_I$, where $f_0 = (f - f_I)\chi_{(-\infty, x_0)}$, $f_1 = (f - f_I)\chi_{(x_0, x_0+\rho^{2(\alpha+1)}h)}$ and $f_2 = (f - f_I)\chi_{(x_0+\rho^{2(\alpha+1)}h, \infty)}$.

Since $|T_N f(x)| < \infty$, it is enough to prove that there exists a constant C depending only on ρ and $\|v_k\|_\infty$, but not on N, such that

$$\frac{1}{|I|}\int_I |T_N f(x) - T_N f_2(x_0)|\, dx \le C\|f\|_{BMO} \, .$$

Observe that $T_N f_0(x) = 0$ for all $x \in I$. Also, since the operators D_k are averages, we have that $T_N f_I(x) = 0$. Therefore,

$$|T_N f(x) - T_N f_2(x_0)| \le |T_N f_1(x)| + |T_N f_2(x) - T_N f_2(x_0)| \, .$$

Now, since the operators T_N are uniformly bounded in L^p for $1 < p < \infty$ we have that

$$\begin{aligned} \frac{1}{|I|}\int_I |T_N f_1(x)|\, dx &\le \left(\frac{1}{|I|}\int_I |T_N f_1(x)|^p\, dx\right)^{1/p} \\ &\le C\left(\frac{1}{h}\int_{x_0}^{x_0+\rho^{2(\alpha+1)}h} |f(x) - f_I|^p\, dx\right)^{1/p} \le C\|f\|_{BMO} \, , \end{aligned}$$

where the last inequality follows by arguing as in the proof of Lemma 4. In order to handle II, let us fix i such that $\varepsilon_{i-1} \le h \le \varepsilon_i$. Thus, for $x \in I$, we obtain

$$\begin{aligned} |T_N f_2(x) - T_N f_2(x_0)| &\le \int_{\rho^{2(\alpha+1)}h}^{\infty} |K_N((x-x_0)-y) - K_N(-y)||f(y+x_0) - f_I|\, dy \\ &\le \sum_{j=i+\alpha}^{\infty}\int_{\varepsilon_j}^{\varepsilon_{j+1}} |K_N((x-x_0)-y) - K_N(-y)||f(y+x_0) - f_I|\, dy \, , \end{aligned}$$

where in the last inequality we have used that by (3.2), $\rho^{2(\alpha+1)}h \ge \rho^{2(\alpha+1)}\varepsilon_{i-1} \ge \varepsilon_{i+\alpha}$. Then, by Hölder inequality, Lemma 3 and Lemma 4 we get that

$$\begin{aligned} |T_N f_2(x) - T_N f_2(x_0)| &\le C\sum_{j=i+\alpha}^{\infty} \varepsilon_j^{1/r'}\left(\int_{\varepsilon_j}^{\varepsilon_{j+1}} |K_N((x-x_0)-y) - K_N(-y)|^r\, dy\right)^{1/r} \\ &\quad \times \left(\frac{1}{\varepsilon_{j+1}}\int_{x_0}^{x_0+\varepsilon_{j+1}} |f(y) - f_I|^{r'}\, dy\right)^{1/r'} \\ &\le C\sum_{j=i+\alpha}^{\infty} \frac{1}{\rho^{(j-i)/r}}(1+j-i)||f||_{BMO} \le C\,||f||_{BMO} \, . \end{aligned}$$

Finally, we shall prove (iv). It is proved in [4] that

$$f^{+,\#}(x) \le \sup_{h>0}\inf_{a\in\mathbb{R}}\left(\frac{1}{h}\int_x^{x+h}(f(y)-a)^+dy + \frac{1}{h}\int_{x+h}^{x+2h}(a-f(y))^+dy\right) .$$

If we use the above inequality we get that

$$\begin{aligned} f^{+,\#}(x) &\le \sup_{h>0}\inf_{a\in\mathbb{R}}\left(\frac{1}{h}\int_x^{x+h}|f(y)-a|\, dy + \frac{1}{h}\int_{x+h}^{x+2h}|a-f(y)|\, dy\right) \\ &\le C\sup_{h>0}\inf_{a\in\mathbb{R}}\frac{1}{h}\int_x^{x+h}|f(y)-a|\, dy \, . \end{aligned}$$

Thus, it is enough to prove that for fixed x_0 there is, for each positive h, a real number a that may depend on x_0 and h such that

$$\frac{1}{h}\int_{x_0}^{x_0+h}\big||T_N f|-a\big|\,dx \le C\, M_s^+ f(x_0)\ .$$

As in the proof of (iii), we split f as $f = f_0 + f_1 + f_2$, where $f_0 = f\chi_{(-\infty,x_0)}$, $f_1 = f\chi_{(x_0,x_0+\rho^{2(\alpha+1)}h)}$ and $f_2 = f\chi_{(x_0+\rho^{2(\alpha+1)}h,\infty)}$ and choose $a = |T_N f_2(x_0)|$. It is clear that, since $|T_N f(x)| < \infty$ and $|T_N f_2(x_0)| < \infty$,

$$\big||T_N f(x)| - |T_N f_2(x_0)|\big| \le |T_N f_1(x)| + |T_N f_2(x) - T_N f_2(x_0)|\ .$$

Now, again as in the proof of (iii), for $x \in (x_0, x_0+h)$ and $\varepsilon_{i-1} < h \le \varepsilon_i$, we get that

$$\begin{aligned}|T_N f_2(x) - T_N f_2(x_0)| &\le C\sum_{j=i+\alpha}^{\infty}\varepsilon_j^{1/s}\left(\int_{\varepsilon_j}^{\varepsilon_{j+1}}|K((x-x_0)-y)-K(-y)|^{s'}\,dy\right)^{1/s'}\\ &\quad\times\left(\frac{1}{\varepsilon_{j+1}}\int_{x_0}^{x_0+\varepsilon_{j+1}}|f(y)|^s\,dy\right)^{1/s} \le C\, M_s^+ f(x_0)\ ,\end{aligned}$$

and

$$\begin{aligned}\frac{1}{h}\int_{x_0}^{x_0+h}|T_N f_1(x)|\,dx &\le \left(\frac{1}{h}\int_{x_0}^{x_0+h}|T_N f_1(x)|^s\,dx\right)^{1/s}\\ &\le \left(\frac{1}{h}\int_{x_0}^{x_0+\rho^{2(\alpha+1)}h}|f(x)|^s\,dx\right)^{1/s} \le C\, M_s^+ f(x_0)\ .\end{aligned}$$

Thus, the theorem follows. □

6. Proof of Theorem 1

First, we shall prove that the operators T_N are uniformly bounded in the weighted L^p spaces.

Theorem 7. *Let $1 < p < \infty$ and $w \in A_p^+$. There exists a constant C depending on p, ρ, and $\|v_k\|_\infty$ such that*

$$\int |T_N f(x)|^p w(x)\,dx \le C\int |f(x)|^p w(x)\,dx\ .$$

Proof. We know that $A_p^+ \subset A_\infty^+$ and that for each $w \in A_p^+$ there is an s bigger than one such that $p/s > 1$ and $w \in A_{p/s}^+$ (see [6]). On the other hand, since

$$|T_N f(x)| \le 2\|v_k\|_\infty(N_2 - N_1 + 1)M^+ f(x)\ ,$$

we get that

$$\int_{\mathbb{R}}\big[M^+(|T_N f|)(x)\big]^p w(x)\,dx < \infty\ .$$

Thus, applying Theorem 5 and Theorem 6 (iv) we have

$$\begin{aligned}\int |T_N f(x)|^p w(x)\,dx &\le C\int \left[M^+(|T_N f|)(x)\right]^p w(x)\,dx\\ &\le C\int \left[(|T_N f|)^{+,\#}(x)\right]^p w(x)\,dx\\ &\le C\int \left[M_s^+ f(x)\right]^p w(x)\,dx\\ &= C\int \left[M^+\big(|f|^s\big)(x)\right]^{p/s} w(x)\,dx\\ &\le C\int |f(x)|^p w(x)\,dx\ ,\end{aligned}$$

where in the last inequality we have applied that M^+ is bounded in $L^{p/s}(w)$ since $w \in A^+_{p/s}$.
□

Now, we prove the L^p boundedness of the operator T^*. For $N = (N_1, N_2)$, we shall use T_{N_1,N_2} to denote the operator T_N. We start proving a pointwise estimate for the operators

$$T^*_M f(x) = \sup_{|N_1|,|N_2|\le M} |T_{N_1,N_2} f(x)|\ .$$

Theorem 8. *For each $s \in (1,\infty)$ there exists a constant C depending on s, ρ, and $\|v_k\|_\infty$ such that for every $x \in \mathbb{R}$ and every $M > 0$*

$$T^*_M f(x) \le C\left[M^+(|T_{-M,M} f|)(x) + M_s^+ f(x)\right].$$

Proof. Since the operators T_{N_1,N_2} are given by convolutions, they are invariant under translations, and therefore it is enough to prove the theorem for $x = 0$. Observe that

$$T_{N_1,N_2} f(x) = T_{N_1,M} f(x) - T_{N_2+1,M} f(x)\ .$$

Then, it will suffice to estimate $|T_{m,M} f(0)|$ for $|m| \le M$ with constants independent of m and M. Let us split f as $f = f_1 + f_2 + f_3$, where $f_1 = f\chi_{(0,\varepsilon_{m-1})}$, $f_2 = f\chi_{(\varepsilon_{m-1},\infty)}$, and $f_3 = f\chi_{(-\infty,0)}$. First, notice that $T_{m,M}(f_3)(0) = 0$. Then

$$\begin{aligned}|T_{m,M} f(0)| &\le |T_{m,M} f_1(0)| + |T_{m,M} f_2(0)|\\ &= I + II\ .\end{aligned}$$

It is clear that

$$\begin{aligned} I &= \left|\int_0^{\varepsilon_{m-1}} \sum_{k=m}^{M} v_k\left(\frac{1}{\varepsilon_k} - \frac{1}{\varepsilon_{k-1}}\right) f(y)\,dy\right|\\ &\le \|v_k\|_\infty \frac{1}{\varepsilon_{m-1}} \int_0^{\varepsilon_{m-1}} |f(y)|\,dy \le \|v_k\|_\infty M^+ f(0)\ .\end{aligned}$$

On the other hand,

$$\begin{aligned} II &= \frac{1}{\varepsilon_{m-1-\alpha}} \int_0^{\varepsilon_{m-1-\alpha}} |T_{m,M} f_2(0)|\, dx \\ &\le \frac{1}{\varepsilon_{m-1-\alpha}} \int_0^{\varepsilon_{m-1-\alpha}} |T_{-M,M} f(x)|\, dx \\ &\quad + \frac{1}{\varepsilon_{m-1-\alpha}} \int_0^{\varepsilon_{m-1-\alpha}} |T_{-M,M} f_1(x)|\, dx \\ &\quad + \frac{1}{\varepsilon_{m-1-\alpha}} \int_0^{\varepsilon_{m-1-\alpha}} |T_{m,M} f_2(0) - T_{m,M} f_2(x)|\, dx \\ &\quad + \frac{1}{\varepsilon_{m-1-\alpha}} \int_0^{\varepsilon_{m-1-\alpha}} |T_{-M,m-1} f_2(x)|\, dx \\ &= A_1 + A_2 + A_3 + A_4 \,. \end{aligned}$$

(If $m = -M$ we understand that $A_4 = 0$). It is obvious that

$$A_1 \le M^+(|T_{-M,M} f|)(0) \,.$$

For the second term, we use the uniform boundedness on L^s of the operators $T_{-M,M}$ given in Theorem 6 (ii). Thus,

$$\begin{aligned} A_2 &\le \left(\frac{1}{\varepsilon_{m-1-\alpha}} \int_0^{\varepsilon_{m-1-\alpha}} |T_{-M,M} f_1(x)|^s\, dx \right)^{1/s} \\ &= C \left(\frac{1}{\varepsilon_{m-1-\alpha}} \int_0^{\varepsilon_{m-1}} |f(x)|^s\, dx \right)^{1/s} \le \rho^{2\alpha/s} M_s^+ f(0) \,, \end{aligned}$$

where in the last inequality we have used condition (3.2). To estimate A_3, we proceed as in Theorem 6 (iv) and thus

$$|T_{m,M} f_2(0) - T_{m,M} f_2(x)| \le C\, M_s^+ f(0) \,.$$

Finally, we estimate A_4. First, it is clear that

$$A_4 \le \frac{C}{\varepsilon_{m-1-\alpha}} \int_0^{\varepsilon_{m-1-\alpha}} \sum_{-M-1}^{m-1} \frac{1}{\varepsilon_k} \int_{\mathbb{R}} \chi_{(\varepsilon_{m-1},\infty)}(y) \chi_{(x,x+\varepsilon_k)}(y) |f(y)|\, dy\, dx \,.$$

By using (3.3) we get that for $x \in (0, \varepsilon_{m-1-\alpha})$ and $k \le m-2$, $x + \varepsilon_k \le \varepsilon_{m-1-\alpha} + \varepsilon_{m-2} \le \varepsilon_{m-1}$. Therefore, the sum in the above inequality reduces to the term $k = m-1$. Thus, by using (3.3) again we get

$$A_4 \le \frac{C}{\varepsilon_{m-1}} \int_0^{\varepsilon_{m-1}+\varepsilon_{m-1-\alpha}} |f(y)|\, dy \le C M^+ f(0) \,.$$

Putting all these inequalities together and using that $M^+ f(x) \le M_s^+ f(x)$ for all $s \in (1,\infty)$, we are done. □

Proof of Theorem 1. Since if $w \in A_p^+$ there is an s bigger than one and such that $p/s > 1$ and $w \in A_{p/s}^+$ [6] then

$$\int \left[M_s^+ f(x) \right]^p w(x)\, dx \le C \int |f(x)|^p w(x)\, dx \,.$$

Now, from Theorems 7 and 8 we get that

$$\int \left[T_M^* f(x)\right]^p w(x)\, dx \leq C \int |f(x)|^p w(x)\, dx \ ,$$

where the constant C does not depend on M. Consequently, letting M increase to infinity, we see that the same holds for the operator T^*, and we are done. □

7. Proof of Theorem 2

We begin studying the behavior of T^* on the functions of compact support and zero average. For this, we shall need the following remark.

Remark 2. It is clear that A_1^+ implies the following condition: There exists C such that for any $M > 1$ and every interval $I = (a, a+h)$

$$\int_{a-Mh}^{a} w \leq CMh \operatorname{ess\,inf}\{w(x) : x \in I\} \ .$$

It follows from this property that if w satisfies A_1^+ then the following one-sided doubling property holds: There exists C such that for any $M > 1$

$$\int_{a-Mh}^{a+h} w \leq (CM+1) \int_a^{a+h} w \ .$$

Lemma 5. *Let a be supported on $I = (0, \varepsilon_i)$ and such that $\int_I a = 0$ and let $w \in A_1^+$. There exists C such that*

$$\int_{z<-\varepsilon_{i+\alpha}} T^* a(z) w(z)\, dz \leq C \int_I |a(z)| w(z)\, dz \ .$$

Proof. Let us write

$$\int_{z<-\varepsilon_{i+\alpha}} T^* a(z) w(z)\, dz = \sum_{m=i+\alpha}^{\infty} \int_{-\varepsilon_{m+1}}^{-\varepsilon_m} T^* a(z) w(z)\, dz \ .$$

If $z \in (-\varepsilon_{m+1}, -\varepsilon_m)$ and $u \in I$, then by (3.3) $z - u \in (-\varepsilon_{m+2}, -\varepsilon_m)$. Thus, $\chi_{(-\varepsilon_k,0)}(z - u) = 1$ for all $k \geq m+2$ and $\chi_{(-\varepsilon_k,0)}(z-u) = 0$ for all $k \leq m$. Then, for fixed $N \in \mathbb{Z}^2$ and by using that $\int_I a = 0$ we get that

$$\begin{aligned} |T_N a(z)| &\leq \sum_{k=-\infty}^{\infty} \left| \int_I v_k \left(\frac{1}{\varepsilon_k} \chi_{(-\varepsilon_k,0)}(z-u) - \frac{1}{\varepsilon_{k-1}} \chi_{(-\varepsilon_{k-1},0)}(z-u) \right) a(u)\, du \right| \\ &= \left| \int_I v_{m+1} \frac{1}{\varepsilon_{m+1}} \chi_{(-\varepsilon_{m+1},0)}(z-u) a(u)\, du \right| \\ &\quad + \left| \int_I v_{m+2} \frac{1}{\varepsilon_{m+1}} \chi_{(-\varepsilon_{m+1},0)}(z-u) a(u)\, du \right| . \end{aligned}$$

On one hand, we can see that if $z \geq -\varepsilon_{m+1} + \varepsilon_i$, then $z - u > z - \varepsilon_i \geq -\varepsilon_{m+1}$ and as a consequence $\chi_{(-\varepsilon_{m+1},0)}(z-u) = 1$. Thus, applying again that $\int_I a = 0$, we get that both terms in the above inequalities are zero for $z \geq -\varepsilon_{m+1} + \varepsilon_i$. On the other hand, both terms are dominated by $C \frac{1}{\varepsilon_{m+1}} \int_I |a|$. Therefore

$$\int_{z<-\varepsilon_{i+\alpha}} T^* a(z) w(z)\, dz \leq C \sum_{m=i+\alpha}^{\infty} \frac{1}{\varepsilon_{m+1}} \left(\int_I |a(u)|\, du \right) \int_{-\varepsilon_{m+1}}^{-\varepsilon_{m+1}+\varepsilon_i} w(z)\, dz\ .$$

Let $r > 1$ such that $w^r \in A_1^+$ [6]. By Hölder's inequality and Remark 2 we have that

$$\begin{aligned} \int_{-\varepsilon_{m+1}}^{-\varepsilon_{m+1}+\varepsilon_i} w(z)\, dz &\leq \left(\int_{-\varepsilon_{m+1}}^{-\varepsilon_{m+1}+\varepsilon_i} w^r(z)\, dz \right)^{\frac{1}{r}} \varepsilon_i^{\frac{1}{r'}} \\ &\leq C\, \varepsilon_i^{\frac{1}{r'}} (\varepsilon_{m+1})^{\frac{1}{r}} \operatorname{ess\,inf}\{w(x) : x \in I\}\ . \end{aligned}$$

Hence,

$$\int_{z<-\varepsilon_{i+\alpha}} T^* a(z) w(z)\, dz \leq C\, \varepsilon_i^{\frac{1}{r'}} \sum_{m=i+\alpha}^{\infty} \frac{1}{(\varepsilon_m)^{\frac{1}{r'}}} \int_I |a| w \leq C \int_I |a| w\ . \qquad \square$$

Corollary 2. *Let a be supported on $I = (x^*, x^* + h)$ and such that $\int_I a = 0$ and let $w \in A_1^+$. If $A = \rho^{2(\alpha+1)}$ there exists C independent of x^*, h, and a, such that*

$$\int_{z<x^*-Ah} T^* a(z) w(z)\, dz \leq C \int_I |a(z)| w(z)\, dz\ .$$

Proof. Observe that it is sufficient to prove the corollary for $x^* = 0$. Choose i such that $\varepsilon_{i-1} \leq h < \varepsilon_i$. Then a is supported on $(0, \varepsilon_i)$ and has integral zero. Furthermore, by (3.2), $-Ah < -\varepsilon_{i+\alpha}$ and

$$\int_{z<-Ah} T^* a(z) w(z)\, dz \leq \int_{z<-\varepsilon_{i+\alpha}} T^* a(z) w(z)\, dz \leq C \int_I |a(z)| w(z)\, dz\ . \qquad \square$$

Theorem 9. *Let $w \in A_1^+$. Then there exists C, depending only on w, so that for any $\lambda > 0$ and all $f \in L^1(w)$*

$$w\big(\{x \in \mathbb{R} :\ T^* f(x) > \lambda\}\big) \leq \frac{C}{\lambda} \int |f(x)| w(x)\, dx\ .$$

Proof. Let $O_\lambda = \{x : M^+ f(x) > \lambda\}$. It is well known that if $\{I_i\}$ are the connected components of O_λ, then $\lambda = \frac{1}{|I_i|} \int_{I_i} f = f_{I_i}$. We decompose f as

$$f = f \chi_{\mathbb{R}\setminus O_\lambda} + \sum f_{I_i} \chi_{I_i} + \sum (f - f_{I_i}) \chi_{I_i}\ .$$

As usual, $f \chi_{\mathbb{R}\setminus O_\lambda} + \sum f_{I_i} \chi_{I_i}$ will be denoted by g and $\sum (f - f_{I_i}) \chi_{I_i} = \sum b_i$ by b. Observe that each b_i has support on I_i and average zero. Now,

$$\begin{aligned} \int_{\mathbb{R}} |g(y)| w(y)\, dy &\leq \int_{\mathbb{R}\setminus O_\lambda} |f(y)| w(y)\, dy + \sum w(I_i) f_{I_i} \qquad (7.1) \\ &= \int_{\mathbb{R}\setminus O_\lambda} |f(y)| w(y)\, dy + \lambda \sum w(I_i) \\ &= \int_{\mathbb{R}\setminus O_\lambda} |f(y)| w(y)\, dy + \lambda w(O_\lambda) \\ &\leq C \int_{\mathbb{R}} |f(y)| w(y)\, dy\ , \end{aligned}$$

because the operator $M^+ f$ is of weak type $(1, 1)$ with respect to w.

For each interval $I = (a, a + h)$, let us denote by I^* the interval $(a - Ah, a + h)$, where $A = \rho^{2(\alpha+1)}$. We also denote $\tilde{O}_\lambda$ the union of all the intervals I_i^*. Observe that

$$\begin{aligned} w\big(\{x : T^* f(x) > \lambda\}\big) \quad &\le \quad w\big(\{x : T^* g(x) > \lambda/2\}\big) + w\big(\tilde{O}_\lambda\big) \\ &\quad + w\big(\{x \notin \tilde{O}_\lambda : T^* b(x) > \lambda/2\}\big) = I + II + III \ . \end{aligned}$$

The one-sided doubling property of the weight and the weak type $(1, 1)$ inequality for M^+ give

$$II = w\big(\cup_i I_i^*\big) \le C w(O_\lambda) \le \frac{C}{\lambda} \int |f(y)| w(y)\, dy \ .$$

On the other hand, since A_1^+ implies condition A_p^+ for any $p > 1$, Theorem 1 implies that T^* is a bounded operator in $L^p(w)$. Then we have

$$\begin{aligned} w\big(\{x : T^* g(x) > \lambda/2\}\big) \quad &\le \quad \frac{C}{\lambda^p} \int \big(T^* g(y)\big)^p w(y)\, dy \le \frac{C}{\lambda^p} \int |g(y)|^p w(y)\, dy \\ &\le \quad \frac{C}{\lambda} \int |g(y)| w(y)\, dy \le \frac{C}{\lambda} \int |f(y)| w(y)\, dy \ . \end{aligned}$$

Observe that in the last two inequalities we have used $|g| \le \lambda$ and (7.1). Finally, by using Corollary 2 and the one-sided nature of the operator T^*, we have

$$\begin{aligned} III \quad &\le \quad \frac{C}{\lambda} \int_{\mathbb{R}\setminus \tilde{O}_\lambda} T^* b(x) w(x)\, dx \le \frac{C}{\lambda} \sum_i \int_{\mathbb{R}\setminus I_i^*} T^* b_i(x) w(x)\, dx \\ &\le \quad \frac{C}{\lambda} \sum_i \int_{I_i} |b_i(x)| w(x)\, dx \ . \end{aligned}$$

Since the I_i's are disjoint and $b(x) = b_i(x)$ on each I_i the last term is bounded by

$$\frac{C}{\lambda} \int |b(x)| w(x)\, dx = \frac{C}{\lambda} \int |f(x) - g(x)| w(x)\, dx \le \frac{C}{\lambda} \int |f(x)| w(x)\, dx \ . \qquad \square$$

8. Proof of Theorem 3

We start proving the convergence in the Schwartz's class $\mathcal{S}$.

Theorem 10. *The functions $T_N \psi(x)$ converge for all $\psi \in \mathcal{S}$ and for every $x \in \mathbb{R}$ as $N = (N_1, N_2)$ tends to $(-\infty, +\infty)$.*

Proof. It will suffice to show that $T_{0,M}\psi(x)$ and $T_{-M,0}\psi(x)$ converge as $M \to +\infty$. We shall prove that

$$\begin{aligned} &|T_{-M,0}\psi(x) - T_{-N,0}\psi(x)| + |T_{0,M}\psi(x) - T_{0,N}\psi(x)| \\ &\quad = |T_{-M,-N-1}\psi(x)| + |T_{N+1,M}\psi(x)| = I + II \end{aligned}$$

is small for $N < M$ and N big enough. First, let us observe that from (3.2) we have that for each $m \le n$

$$\sum_{k=m}^{n} \varepsilon_k \le C\, \varepsilon_n \qquad \text{and} \qquad \sum_{k=m}^{n} \frac{1}{\varepsilon_k} \le C\, \frac{1}{\varepsilon_m} \ . \tag{8.1}$$

Since $\int K_N = 0$ for every $N \in \mathbb{Z}^2$, by using the mean value theorem and (8.1) we get that

$$\begin{aligned} I &= \left| \int K_{-M,-N-1}(x-y)[\psi(y)-\psi(x)]\,dy \right| \\ &\le 2\left\|\psi'\right\|_{L^\infty} ||v_k||_\infty \int \sum_{k=-M-1}^{-N-1} \varphi_k(u)|u|\,du \\ &\le C \sum_{k=-M-1}^{-N-1} \frac{1}{\varepsilon_k} \int_{-\varepsilon_k}^{0} |u|\,du \le C \sum_{k=-M-1}^{-N-1} \varepsilon_k \le C\varepsilon_{-N-1}\,, \end{aligned}$$

which is small when N is big enough. On the other hand, by using (8.1) again we have that

$$\begin{aligned} II &\le 2||v_k||_\infty \int \sum_{k=N}^{M} \varphi_k(x-y)|\psi(y)|\,dy \\ &\le C \sum_{k=N}^{M} \frac{1}{\varepsilon_k} ||\psi||_{L^1} \le \frac{C}{\varepsilon_N} ||\psi||_{L^1}\,, \end{aligned}$$

which is small taking for N big enough. □

The above theorem and Theorems 1 and 2 with $w = 1$ allow us to prove that there exists the limit of $T_N f(x)$ a.e. for all $f \in L^p(dx)$ with $1 \le p < \infty$. Now, since $L^p(dx) \cap L^p(w)$ is a dense subset of $L^p(w)$, this fact and Theorem 1 allow us to prove that there exists the limit of $T_N f(x)$ a.e. for all $f \in L^p(w)$ with $1 \le p < \infty$. On the other hand, by using the dominated convergence theorem, we can prove the convergence in the $L^p(w)$ norm for $1 < p < \infty$. For $p = 1$, convergence in measure follows by standard arguments.

9. Proof of Theorem 4

We shall see that $T_{0,M} f$ and $T_{-M,0} f$ converges in the sense of the weak $*$ topology. Since the operators T_N, $N = (N_1, N_2)$, are uniformly bounded in *BMO* [Theorem 6 (iii)], it will be enough to show that for each sequence of atoms $\{a_i\}$ and each sequence $\{\lambda_i\} \in \ell^1$

$$\begin{aligned} a_{N,M} &= \left| \sum_{i=1}^{\infty} \lambda_i \int_{I_i} a_i(x)(T_{-M,0} f(x) - T_{-N,0} f(x))\,dx \right| \\ &\quad + \left| \sum_{i=1}^{\infty} \lambda_i \int_{I_i} a_i(x)(T_{0,M} f(x) - T_{0,N} f(x))\,dx \right| \end{aligned}$$

tends to 0 for $N, M \to +\infty$. Let us take $N < M$. Then

$$\begin{aligned} a_{N.M} &\le \sum_{i=1}^{\infty} |\lambda_i| \left| \int_{I_i} a_i(x) T_{N+1,M} f(x)\,dx \right| \\ &\quad + \sum_{i=1}^{\infty} |\lambda_i| \left| \int_{I_i} a_i(x) T_{-M,-N-1} f(x)\,dx \right| = b_{N,M} + c_{N,M}\,. \end{aligned}$$

For each $i \in \mathbb{N}$, let $I_i = [x_i, x_i + h_i]$ and $k_i \in \mathbb{Z}$ such that $\varepsilon_{k_i-1} < h_i \leq \varepsilon_{k_i}$. As in the proof of Theorem 6 (iii), we split $f = f_0^i + f_1^i + f_2^i + f_{I_i}$, where $f_0^i = (f - f_{I_i})\chi_{(-\infty,x_i)}$, $f_1^i = (f - f_{I_i})\chi_{(x_i,x_i+\rho^{2(\alpha+1)}h_i)}$ and $f_2^i = (f - f_{I_i})\chi_{(x_i+\rho^{2(\alpha+1)}h_i,\infty)}$. Observe that $T_N f_0^i(x) = 0$ for all $x \in I_i$ and all $N \in \mathbb{Z}^2$. Also, since the operators D_k are averages, we have that $T_N f_{I_i}(x) = 0$, for all $N \in \mathbb{Z}^2$. Then, since $|T_N f(x)| < \infty$ for all $x \in \mathbb{R}$ and each $N \in \mathbb{Z}^2$ and $\int_{I_i} a_i = 0$ we have that

$$\begin{aligned} b_{N,M} &\leq \sum_{i=1}^{L} |\lambda_i| \int_{I_i} |a_i(x)| \left|T_{N+1,M} f_1^i(x)\right| dx \\ &\quad + \sum_{i=1}^{L} |\lambda_i| \int_{I_i} |a_i(x)| \left|T_{N+1,M} f_2^i(x) - T_{N+1,M} f_2^i(x_i)\right| dx \\ &\quad + \sum_{i=L+1}^{\infty} |\lambda_i| \int_{I_i} |a_i(x)| |T_{N+1,M} f(x) - (T_{N+1,M} f)_{I_i}| \, dy = A_1 + A_2 + A_3 \,. \end{aligned}$$

On one hand, for some $1 < p < \infty$ we get that

$$A_1 \leq \sum_{i=1}^{L} |\lambda_i| \|a_i\|_{L^{p'}} \left\|T_{N+1,M} f_1^i\right\|_{L^p}$$

and, since $f_1^i \in L^p$ for all $1 < p < \infty$, by applying Theorem 3 (i) with $w = 1$ we get that $A_1 \to 0$ for $N \to \infty$. On the other hand, by (3.2) we have that $\rho^{2(\alpha+1)} h_i > \rho^{2(\alpha+1)} \varepsilon_{k_i-1} \geq \varepsilon_{k_i+\alpha}$. Then, by Hölder's inequality, Remark 1, Lemma 3, and Lemma 4 we get that

$$\begin{aligned} &\left|T_{N+1,M} f_2^i(x) - T_{N+1,M} f_2^i(x_i)\right| \\ &\quad \leq \int_{y>\rho^{2(\alpha+1)}h_i} |K_{N+1,M}((x - x_i) - y) - K_{N+1,M}(-y)| |f(y + x_i) - f_{I_i}| \, dy \\ &\quad \leq \int_{y>\varepsilon_{k_i+\alpha}} |K_{N+1,M}((x - x_i) - y) - K_{N+1,M}(-y)| |f(y + x_i) - f_{I_i}| \, dy \\ &\quad \leq \sum_{j=N+1}^{\infty} \left(\int_{\varepsilon_j}^{\varepsilon_{j+1}} |K_{N+1,M}((x - x_i) - y) - K_{N+1,M}(-y)|^r \, dy \right)^{1/r} \\ &\qquad \times \left(\int_0^{\varepsilon_{j+1}} |f(y + x_i) - f_{I_i}|^{r'} \, dy \right)^{1/r'} \\ &\quad \leq C ||f||_{BMO} \sum_{j=N+1}^{\infty} \frac{1 + j - k_i}{\rho^{(j-k_i)/r}} \,. \end{aligned}$$

Then, taking N big enough we get that $A_2 \to 0$. Finally, since the operators T_N are uniformly bounded on *BMO* we get that

$$A_3 \leq \sum_{i=L+1}^{\infty} |\lambda_i| ||T_{N+1,M} f||_{BMO} \leq C ||f||_{BMO} \sum_{i=L+1}^{\infty} |\lambda_i| \,,$$

which is small if we choose L big enough. In the same way we can prove that $c_{N,M} \to 0$.

10. Final Remark: The Vector Valued Setting

If we work in a vector valued setting we obtain results of boundedness and convergence for more general operators. In order to see this, assume that v_k is a bounded sequence in a Hilbert space $\mathcal{H}$. Then the operators T_N are well defined for locally integrable functions $f : \mathbb{R} \to \mathbb{R}$. We notice that $T_N f(x) \in \mathcal{H}$ for all $x \in \mathbb{R}$. All the results in this article hold in this setting for the appropriate Bochner-Lebesgue spaces $L^p_{\mathcal{H}}$ (see [5] for definitions and notations). The proofs are the same. We only may notice that we require a Hilbert space and not a Banach space because in the proof of the boundedness of T_N in $L^2_{\mathcal{H}}$ we need to use that the Fourier transform is bounded in $L^2_{\mathcal{H}}$.

One of the results in this vector valued setting says that if $w \in A^+_p$, $1 < p < \infty$, then there exists C such that

$$\int \left|T^* f\right|^p w \le C \int |f|^p w \tag{10.1}$$

for all functions $f \in L^p(w)$, where T^* is the maximal operator defined by

$$T^* f(x) = \sup_N ||T_N f(x)||_{\mathcal{H}} \, ,$$

and $|| \quad ||_{\mathcal{H}}$ stands for the norm in the Hilbert space. If $\mathcal{H}$ is the space ℓ^2 and v_k is the canonical basis of ℓ^2 then $T^* f$ is equal to the square function

$$Sf(x) = \left(\sum_{k=-\infty}^{\infty} |D_k f(x) - D_{k-1} f(x)|^2 \right)^{1/2}$$

and, therefore, the above result means in this special case that if $w \in A^+_p$, $1 < p < \infty$, then S applies $L^p(w)$ into $L^p(w)$. This result was previously obtained in [7].

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