DIFFERENCES OF ERGODIC AVERAGES FOR CESÀRO BOUNDED OPERATORS

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Abstract. We prove that the weighted differences of ergodic averages, induced by a Cesàro bounded, strongly continuous, one-parameter group of positive, invertible, linear operators on L_p, $1 < p < \infty$, converge a.e. and in the L^p norm. We obtain first the boundedness of the ergodic maximal operator and the convergence of the averages.

1. Introduction and Main Results

Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space and let $\Gamma = \{T^t : t \in \mathbb{R}\}$ be a one-parameter group of positive, invertible, linear operators on $L^p(\mu) = L^p(X, \mathcal{F}, \mu)$, for some fixed p in the range $1 < p < \infty$. One of the classical problems in ergodic theory is to study the convergence of the averages

$$\mathcal{A}_\varepsilon^+ f(x) = \frac{1}{\varepsilon} \int_0^\varepsilon T^t f(x) dt$$

as $\varepsilon \to 0^+$ and as $\varepsilon \to \infty$. If we know that this convergence holds in the almost everywhere sense or in the L^p-norm then it is reasonable to try to give some information about how the convergence occurs. In particular, given a lacunary sequence $\{\varepsilon_k\}_{k\in\mathbb{Z}}$, i.e., $\varepsilon_k > 0$ and $\frac{\varepsilon_{k+1}}{\varepsilon_k} \geq \rho > 1$ for all k, we may consider the series

$$\sum_{k=-\infty}^{\infty} \left(\frac{1}{\varepsilon_k} \int_0^{\varepsilon_k} T^t f(x) dt - \frac{1}{\varepsilon_{k-1}} \int_0^{\varepsilon_{k-1}} T^t f(x) dt \right)$$

which obviously converges. As the cancellation in this series is crucial (see [1]), it is natural to ask about the convergence properties of

$$\sum_{k=-\infty}^{\infty} v_k \left(\frac{1}{\varepsilon_k} \int_0^{\varepsilon_k} T^t f(x) dt - \frac{1}{\varepsilon_{k-1}} \int_0^{\varepsilon_{k-1}} T^t f(x) dt \right), \tag{1.1}$$

where v_k is a bounded sequence of real or complex numbers. Jones and Rosenblatt [4] studied this problem in the real line for $T^t f(x) = f(x+t)$ and when Γ is the group associated to an invertible, ergodic, measure preserving transformation. Our aim is to study the properties of convergence of (1.1) in a more

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general setting, that is, assuming that Γ is a strongly continuous, one-parameter group of positive, invertible, linear operators on $L^p(\mu) = L^p(X, \mathcal{F}, \mu)$. In order to prove a.e. convergence of the averages, the standard approach is to consider the maximal operator

$$\mathcal{M}^+ f(x) = \sup_{\varepsilon>0} |\mathcal{A}_\varepsilon^+ f(x)|$$

and to prove a dominated ergodic estimate, i.e.,

$$\int_X |\mathcal{M}^+ f|^p d\mu \leq C \int_X |f|^p d\mu.$$

It is clear that for such an inequality to hold, the averages $\mathcal{A}_\varepsilon^+$ must be uniformly bounded operators in $L^p(\mu)$, i.e.,

$$\sup_{\varepsilon>0} \|\mathcal{A}_\varepsilon f\|_p \leq C\|f\|_p.$$

In other words, the semigroup $\Gamma_+ = \{T^t : t > 0\}$, must be Cesàro bounded. (This is obviously the case if T^t is a measure preserving transformation for each t.) Our first result proves that this condition is sufficient for the boundedness of the maximal operator.

Theorem 1.1. *Let $1 < p < \infty$. Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space and let $\Gamma = \{T^t : t \in \mathbb{R}\}$ be a strongly continuous one-parameter group of positive, invertible, linear operators on $L^p(\mu)$. The following conditions are equivalent:*

(a) There exists $C > 0$ such that for all $f \in L^p(\mu)$,

$$\int_X |\mathcal{M}^+ f|^p d\mu \leq C \int_X |f|^p d\mu.$$

(b) The semigroup $\Gamma_+ = \{T^t : t > 0\}$ is Cesàro bounded in $L^p(\mu)$, i.e., there exists $C > 0$ such that for all $f \in L^p(\mu)$,

$$\sup_{\varepsilon>0} ||\mathcal{A}_\varepsilon^+ f||_{L^p(\mu)} \leq C||f||_{L^p(\mu)}.$$

As a consequence of this theorem we obtain that (b) in Theorem 1.1 implies the convergence of the averages.

Theorem 1.2. *Let $1 < p < \infty$. Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space and let $\Gamma = \{T^t : t \in \mathbb{R}\}$ be a strongly continuous one-parameter group of positive, invertible, linear operators on $L^p(\mu)$. Assume that the semigroup $\Gamma_+ = \{T^t : t > 0\}$ is Cesàro bounded in $L^p(\mu)$. Then the following statements hold for every $f \in L^p(\mu)$:*

(a) $\lim_{\varepsilon\to 0^+} \mathcal{A}_\varepsilon^+ f = f$ *a.e. and in $L^p(\mu)$.*

(b) The averages $\mathcal{A}_\varepsilon^+ f$ converge a.e. and in $L^p(\mu)$ as $\varepsilon \to +\infty$.

Once we have the convergence of the averages we may consider the series (1.1). In order to study it, we need to prove L^p inequalities for a suitable maximal operator. First, we introduce some definitions.

Definition 1.3. *Assume that* $\Gamma = \{T^t : t \in \mathbb{R}\}$ *is a strongly continuous one-parameter group of positive, invertible, linear operators on* $L^p(\mu)$, $1 < p < \infty$. *Given a lacunary sequence* $\{\varepsilon_k\}_{k\in\mathbb{Z}}$ *and a bounded sequence* $\{\upsilon_k\}_{k\in\mathbb{Z}}$ *of real numbers, we define, for each* $N = (N_1, N_2) \in \mathbb{Z}^2$, $N_1 \le N_2$, *the ergodic truncation operator by*

$$\mathcal{T}_N f(x) = \sum_{k=N_1}^{N_2} \upsilon_k \left(\frac{1}{\varepsilon_k} \int_0^{\varepsilon_k} T^t f(x) dt - \frac{1}{\varepsilon_{k-1}} \int_0^{\varepsilon_{k-1}} T^t f(x) dt \right)$$

and the corresponding maximal operator

$$\mathcal{T}^* f(x) = \sup_N |\mathcal{T}_N f(x)| .$$

The L^p estimate for $\mathcal{T}^*$ and the convergence of the series (1.1) are contained in the next theorem.

Theorem 1.4. *Let* $1 < p < \infty$, $\{\varepsilon_k\}_{k\in\mathbb{Z}}$ *a lacunary sequence and* $\{\upsilon_k\}_{k\in\mathbb{Z}}$ *a bounded sequence of real numbers. Let* $(X, \mathcal{F}, \mu)$ *be a* σ*-finite measure space and let* $\Gamma = \{T^t : t \in \mathbb{R}\}$ *be a strongly continuous one-parameter group of positive, invertible, linear operators on* $L^p(\mu)$. *Assume that the semigroup* $\Gamma_+ = \{T^t : t > 0\}$ *is Cesàro bounded in* $L^p(\mu)$. *Then the following statements hold:*

(a) There exists $C > 0$ *such that for every* $f \in L^p(\mu)$

$$\int_X (\mathcal{T}^* f(x))^p d\mu(x) \le C \int_X |f(x)|^p d\mu(x).$$

(b) For every $f \in L^p(\mu)$, $\lim_{N\to\infty} \mathcal{T}_N f(x)$ *exists a.e. and in* $L^p(\mu)$, *where* $\lim_{N\to\infty} \mathcal{T}_N f(x)$ *stands for* $\lim_{N_1\to-\infty, N_2\to\infty} \mathcal{T}_N f(x)$ *with* $N = (N_1, N_2)$.

Remark 1.5. *Theorem 1.1 is the continuous version of a result of F. J. Martín-Reyes and A. de la Torre* [6]*; the two-sided version of Theorem 1.1 is due to T.A. Gillespie and J.L. Torrea* [3]. *Theorem 1.4 in the case of a semigroup generated by an ergodic measure preserving transformation is due to R.L. Jones and J. Rosenblatt* [4].

2. Some previous results

Let $\Gamma = \{T^t : t \in \mathbb{R}\}$ be a strongly continuous one-parameter group of positive, invertible, linear operators on $L^p(\mu)$, $1 < p < \infty$, where μ is a σ-finite measure. The group structure of Γ and the positivity of each T^t assures that T^t is separation preserving. These properties and some technical facts as the meaning of $T^t f(x)$ as functions of (t, x) can be found in [3]. We list some of them that will be used in our proofs:

- For all t, there exists a function $H_t(x)$, such that for all $f \in L^p(\mu)$

$$\int_X |T^t f(x)|^p H_t(x) d\mu(x) = \int_X |f(x)|^p d\mu(x). \tag{2.1}$$

(See property (d), page 70, in [6] and (1.7)–(1.8) in [3]; in the notation in [3], $H_t(x) = h_t^{-p}(x)J_t(x)$.)

- Furthermore, if $(T^t)^*$ denotes de adjoint of T^t, then

$$H_t(x) = ((T^{-t})^* g^p)(x)(T^t g^{p'})^{1-p}(x) \tag{2.2}$$

for any function $g > 0$, $g \in L^{pp'}(\mu)$. (See property (e), page 70, in [6] or Remark 1.11 in [3]).

- For all $t \in \mathbb{R}$, for all $f \in L^p(\mu)$ and each compact subset of $\mathbb{R}$ we have

$$T^t\left(\int_K T^s f(x)ds\right) \le \int_K T^{t+s} f(x)ds. \tag{2.3}$$

The same property holds for the adjoint operators $(T^t)^*$. (See (1.9) in [3]).

- If $0 \le \gamma \le p$ the one-parameter group $\Gamma_\gamma = \{S^t : t \in \mathbb{R}\}$ defined by $S^t f = \left(T^t f^{1/\gamma}\right)^\gamma$ for all $f \ge 0$ is a strongly continuous group of positive invertible linear operators on $L^{p/\gamma}(\mu)$. We notice that if H_t is the function in (2.1) then

$$\int_X |S^t f(x)|^{p/\gamma} H_t(x)d\mu(x) = \int_X |f(x)|^{p/\gamma} d\mu(x). \tag{2.4}$$

for all $f \in L^{p/\gamma}(\mu)$.

We assume that the reader is familiar with the theory of weights for the one-sided Hardy-Littlewood maximal operator (see [8], [7] and [5]) defined by

$$M^+ f(x) = \sup_{h>0} \frac{1}{h}\int_x^{x+h} |f(t)|dt.$$

We recall that a weight w belongs to the class A_p^+, $p > 1$, if for any $a \in \mathbb{R}$ and any $h > 0$,

$$\int_a^{a+h} w(t)dt \left(\int_{a+h}^{a+2h} w^{1-p'}dt\right)^{p-1} \le Ch^p,$$

where $p + p' = pp'$. Condition A_p^+ is necessary and sufficient for the boundedness of M^+ from $L^p(w)$ into itself. A key fact is that if w belongs to the class A_p^+, $p > 1$, then there exists $\gamma > 1$, $1 < \gamma < p$, such that w belongs to the class $A_{p/\gamma}^+$, where γ depends only on the constant in the condition A_p^+. The following result from [2] will be used in the proof of Theorem 1.4.

Theorem 2.1. [2] *Let $\{\varepsilon_k\}$ be a lacunary sequence and let $\{\upsilon_k\}$ be a bounded sequence of real numbers. For each $N = (N_1, N_2) \in \mathbb{Z}^2$, $N_1 \le N_2$, we define* *the operator S_N acting on locally integrable functions $f:\mathbb{R}\to\mathbb{R}$, by*

$$S_N f(x)=\sum_{k=N_1}^{N_2} \upsilon_k \left(\frac{1}{\varepsilon_k}\int_0^{\varepsilon_k} f(x+t)dt-\frac{1}{\varepsilon_{k-1}}\int_0^{\varepsilon_{k-1}} f(x+t)dt\right),$$

and the corresponding maximal operator

$$S^* f(x)=\sup_N |S_N f(x)|.$$

If $1<p<\infty$ and $w\in A_p^+$ then there exists $C>0$, depending only only on p and on the constant in the definition of A_p^+, such that

$$\int_{\mathbb{R}} |S^* f(x)|^p w(x)\,dx \le C\int_{\mathbb{R}} |f(x)|^p w(x)\,dx.$$

3. Proof of Theorem 1.1

It is obvious that $(a)\Rightarrow(b)$. We shall prove that $(b)\Rightarrow(c)\Rightarrow(a)$ where (c) means the following:

- (c) For almost every $x\in X$, the function $t\to H_t(x)$ belongs to A_p^+, with a constant independent of x.

In order to prove $(b)\Rightarrow(c)$, we shall use the so called Rubio de Francia algorithm. Let us assume that (b) holds. By hypothesis we have that there exists a constant $C>0$ independent of $\varepsilon>0$ and f such that

$$\int_X |\mathcal{A}_{2\varepsilon}^+ f|^p d\mu \le C\int_X |f|^p d\mu \quad \text{for all } f\in L^p(\mu)$$

and

$$\int_X |(\mathcal{A}_{2\varepsilon}^+)^* f|^{p'} d\mu \le C\int_X |f|^{p'} d\mu \quad \text{for all } f\in L^{p'}(\mu),$$

where $(\mathcal{A}_{2\varepsilon}^+)^*$ is the adjoint operator of $\mathcal{A}_{2\varepsilon}^+ f$.

For $g\in L^{pp'}(\mu)$, we define

$$Q_\varepsilon g=\left(\mathcal{A}_{2\varepsilon}^+|g|^{p'}\right)^{1/p'} \quad \text{and} \quad P_\varepsilon g=\left((\mathcal{A}_{2\varepsilon}^+)^*|g|^p\right)^{1/p}.$$

Then Q_ε, P_ε and $R_\varepsilon=Q_\varepsilon+P_\varepsilon$ are bounded from $L^{pp'}(\mu)$ into $L^{pp'}(\mu)$ with constants independent of $\varepsilon>0$. Let us fix $C>0$ such that $||R_\varepsilon g||_{L^{pp'}(\mu)}\le C||g||_{L^{pp'}(\mu)}$, for all $g\in L^{pp'}(\mu)$ and all $\varepsilon>0$. Now, for fixed $g>0$, $g\in L^{pp'}(\mu)$ and $\varepsilon>0$, let

$$G(x)=\sum_{j=0}^{\infty}\frac{R_\varepsilon^{(j)}g(x)}{(2C)^j},$$

where $R_\varepsilon^{(j)}$ is the j-th iteration of R_ε. Then, $G\in L^{pp'}(\mu)$, $g\le G$ a.e., $R_\varepsilon G\le 2CG$ a.e. and, as a consequence, $P_\varepsilon G\le 2CG$ a.e. and $Q_\varepsilon G\le 2CG$ a.e., i.e., there exists $C>0$ such that,

$$\mathcal{A}_{2\varepsilon}^+ G^{p'}\le CG^{p'} \quad \text{a.e.} \tag{3.1}$$

and

$$(\mathcal{A}^+_{2\varepsilon})^* G^p \le C G^p \quad \text{a.e.} \tag{3.2}$$

Since the operators T^t are linear and positive, we get from (3.1) and (2.3) that for $s \le t \le s+\varepsilon$,

$$\begin{aligned} C T^t G^{p'}(x) &\ge T^t \left(\frac{1}{2\varepsilon} \int_0^{2\varepsilon} T^s G^{p'}(x) ds \right) = \frac{1}{2\varepsilon} \int_0^{2\varepsilon} T^{t+s} G^{p'}(x) ds \\ &= \frac{1}{2\varepsilon} \int_t^{2\varepsilon+t} T^u G^{p'}(x) du \ge \frac{1}{2\varepsilon} \int_{s+\varepsilon}^{s+2\varepsilon} T^u G^{p'}(x) du. \end{aligned}$$

Raising to $1-p<0$, multiplying by $(T^{-t})^* G^p(x)$, using (2.2) and integrating from s to $s+\varepsilon$, we get

$$C \int_s^{s+\varepsilon} H_t(x) dt \le \left(\frac{1}{2\varepsilon} \int_{s+\varepsilon}^{s+2\varepsilon} T^u G^{p'}(x) du \right)^{1-p} \int_s^{s+\varepsilon} (T^{-t})^* G^p(x) dt. \tag{3.3}$$

On the other hand, since the $(T^{-t})^*$ are also linear and positive, we get from (3.2) and (2.3) that for all $s+\varepsilon \le t \le s+2\varepsilon$,

$$\begin{aligned} C (T^{-t})^* G^p(x) &\ge \frac{1}{2\varepsilon} \int_0^{2\varepsilon} (T^{s-t})^* G^p(x) ds \\ &= \frac{1}{2\varepsilon} \int_{t-2\varepsilon}^{t} (T^{-u})^* G^p(x) du \ge \frac{1}{2\varepsilon} \int_s^{s+\varepsilon} (T^{-u})^* G^p(x) du. \end{aligned}$$

Raising to $1-p'<0$, multiplying by $T^t G^{p'}(x)$, using (2.2) and integrating from $s+\varepsilon$ to $s+2\varepsilon$ we get

$$C \int_{s+\varepsilon}^{s+2\varepsilon} (H_t(x))^{1-p'} dt \le \left(\frac{1}{2\varepsilon} \int_s^{s+\varepsilon} (T^{-u})^* G^p(x) du \right)^{1-p'} \int_{s+\varepsilon}^{s+2\varepsilon} T^t G^{p'}(x) dt. \tag{3.4}$$

From (3.4) and (3.3), using (3.1) and (3.2), we get

$$\int_s^{s+\varepsilon} H_t(x) dt \left(\int_{s+\varepsilon}^{s+2\varepsilon} (H_t(x))^{1-p'} dt \right)^{p-1} \le C \varepsilon^p,$$

which is (c).

Let us prove $(c) \Rightarrow (a)$. Since $\mathcal{M}^+ f(x) \le \mathcal{M}^+(|f|)(x)$, we can assume that $f \ge 0$. For each $\eta > 0$, let us consider $\mathcal{M}^+_\eta f(x) = \sup_{0<\varepsilon\le\eta} \mathcal{A}^+_\varepsilon f(x)$. From the positivity of T^t and (2.3) we have that

$$T^t \mathcal{M}^+_\eta f(x) = T^t \mathcal{M}^+_\eta (T^{-t} T^t f)(x) \le \mathcal{M}^+_\eta (T^t f)(x).$$

If we define $g^x(t) = T^t g(x)$, we have that for all $R>0$ and all $t \le R$

$$\begin{aligned} \mathcal{M}^+_\eta (T^t f)(x) &= \sup_{0<\varepsilon\le\eta} \frac{1}{\varepsilon} \int_0^\varepsilon T^{s+t} f(x) ds = \sup_{0<\varepsilon\le\eta} \frac{1}{\varepsilon} \int_0^\varepsilon f^x(s+t) ds \\ &= \sup_{0<\varepsilon\le\eta} \frac{1}{\varepsilon} \int_0^\varepsilon f^x \chi_{[0,R+\eta]}(s+t) ds \le M^+(f^x \chi_{[0,R+\eta]})(t), \end{aligned} \tag{3.5}$$

where M^+ is the one-sided Hardy-Littlewood maximal operator in $\mathbb{R}$. Then, by (2.1), Fubini's theorem, (c) and the fact that A_p^+ implies boundedness of the one-sided Hardy-Littlewood maximal operator, we get that for each $R>0$,

$$
\begin{aligned}
\int_X (\mathcal{M}_\eta^+ f(x))^p d\mu(x) &= \frac{1}{R}\int_0^R \int_X |T^t \mathcal{M}_\eta^+ f(x)|^p H_t(x) d\mu(x) dt \\
&\le \int_X \frac{1}{R}\int_0^R |M^+(f^x \chi_{[0,R+\eta]})(t)|^p H_t(x) dt d\mu(x) \\
&\le C\int_X \frac{1}{R}\int_0^{R+\eta} |f^x(t)|^p H_t(x) dt d\mu(x) \\
&= C\frac{1}{R}\int_0^{R+\eta}\int_X |T^t f(x)|^p H_t(x) d\mu(x) dt \\
&= C\frac{1}{R}\int_0^{R+\eta}\int_X |f(x)|^p d\mu(x) dt \\
&= C\frac{R+\eta}{R}\int_X |f(x)|^p d\mu(x).
\end{aligned} \tag{3.6}
$$

Letting, first R, and then η, go to infinity we obtain

$$\int_X (\mathcal{M}^+ f(x))^p d\mu(x) \le C\int_X |f(x)|^p d\mu(x),$$

which is (a).

4. Proof of Theorem 1.2

Proof of (a) in Theorem 1.2. First we shall prove that

$$\lim_{\varepsilon\to 0^+} ||\mathcal{A}_\varepsilon^+ f - f||_{L^p(\mu)} = 0. \tag{4.1}$$

Using the strong continuity of the group Γ we have that for any $\gamma>0$, there exists $\delta>0$ such that if $|s|<\delta$ then $||T^s f - f||_{L^p(\mu)} < \gamma$. Then, by the Minkowski's integral inequality, for all $\varepsilon<\delta$,

$$
\begin{aligned}
&\left(\int_X \left|\frac{1}{\varepsilon}\int_0^\varepsilon T^s f(x) ds - f(x)\right|^p d\mu(x)\right)^{1/p} \\
&\le \left(\int_X \left(\frac{1}{\varepsilon}\int_0^\varepsilon |T^s f(x) - f(x)| ds\right)^p d\mu(x)\right)^{1/p} \\
&\le \frac{1}{\varepsilon}\int_0^\varepsilon \left(\int_X |T^s f(x) - f(x)|^p d\mu(x)\right)^{1/p} ds \le \gamma.
\end{aligned} \tag{4.2}
$$

Now we shall prove that

$$\lim_{\varepsilon\to 0^+} \mathcal{A}_\varepsilon^+ f = f \quad \text{a.e.}$$

We know by Theorem 1.1 that the ergodic maximal operator is of strong type (p,p). Therefore, we only need to check the convergence for a dense class. Since (4.1) holds, we have that

$$D = \{\mathcal{A}_\varepsilon^+ g : g \in L^p(\mu), \varepsilon > 0\}$$

is dense in $L^p(\mu)$. Let $f \in D$, $f = \mathcal{A}_\gamma^+ g$, $g \in L^p(\mu)$ and $\gamma > 0$. Then, for all $\varepsilon < \gamma$ we have

$$\begin{aligned}
|\mathcal{A}_\varepsilon^+(f)(x) - f(x)| &= \left|\frac{1}{\varepsilon}\int_0^\varepsilon (T^t(\mathcal{A}_\gamma^+ g)(x) - \mathcal{A}_\gamma^+ g(x))dt\right| \\
&\le \frac{1}{\varepsilon}\int_0^\varepsilon \left|T^t(\mathcal{A}_\gamma^+ g)(x) - \mathcal{A}_\gamma^+ g(x)\right| dt \\
&= \frac{1}{\varepsilon}\int_0^\varepsilon \left|\frac{1}{\gamma}\int_0^\gamma T^{t+s}g(x)ds \; - \frac{1}{\gamma}\int_0^\gamma T^s g(x)ds\right| dt \\
&= \frac{1}{\varepsilon}\int_0^\varepsilon \left|\frac{1}{\gamma}\int_t^{t+\gamma} T^s g(x)ds \; - \frac{1}{\gamma}\int_0^\gamma T^s g(x)ds\right| dt \\
&= \frac{1}{\varepsilon}\int_0^\varepsilon \left|-\frac{1}{\gamma}\int_0^t T^s g(x)ds + \frac{1}{\gamma}\int_\gamma^{t+\gamma} T^s g(x)ds\right| dt \\
&\le \frac{1}{\varepsilon}\int_0^\varepsilon \frac{1}{\gamma}\int_0^\varepsilon |T^s g(x)|ds + \frac{1}{\gamma}\int_\gamma^{\varepsilon+\gamma} |T^s g(x)|dsdt \\
&= \frac{\varepsilon}{\gamma}\frac{1}{\varepsilon}\int_0^\varepsilon |T^s g(x)|ds + \frac{\varepsilon}{\gamma}\frac{1}{\varepsilon}\int_0^\varepsilon |T^s(T^\gamma g(x))|ds \\
&\le \frac{\varepsilon}{\gamma}\mathcal{M}^+ g(x) + \frac{\varepsilon}{\gamma}\mathcal{M}^+(T^\gamma g(x)).
\end{aligned}$$

The last term tends to 0 as ε goes to 0^+ since, by Theorem 1.1, we have that $\mathcal{M}^+ g(x)$ and $\mathcal{M}^+(T^\gamma g(x))$ are finite a.e.

Proof of (b) in Theorem 1.2. We shall need some results which are interesting by itself.

Lemma 4.1. *Assume that we are in the conditions of Theorem 1.2. If* $1 \le \gamma < p$ *let* $\Gamma_\gamma = \{S^t : t \in \mathbb{R}\}$ *be the strongly continuous, one-parameter group of positive operators on* $L^{p/\gamma}(\mu)$ *such that* $S^t f = \left(T^t f^{1/\gamma}\right)^\gamma$ *for all* $f \ge 0$. *Then there exists* γ, $1 < \gamma < p$ *such that the semigroup* $\Gamma_{\gamma,+} = \{S^t : t > 0\}$ *is Cesàro bounded*

Proof. We have seen in the proof of Theorem 1.1 that Γ_+ is Cesàro bounded if and only if for almost every x the functions $t \to H_t(x)$ belong to A_p^+ with a constant independent of x. Then by the properties of A_p^+ classes, we have that there exists γ, $1 < \gamma < p$, such that $t \to H_t(x)$ belongs to $A_{p/\gamma}^+$ with a constant independent of x (see [5]). Again, by the proof of Theorem 1.1, we obtain that $\Gamma_{\gamma,+}$ is Cesàro bounded in $L^{p/\gamma}(\mu)$. □

Lemma 4.2. *Assume that we are in the conditions of Theorem 1.2. Then, for all $f \in L^p(\mu)$ and all $s > 0$,*

(a) $\lim_{\varepsilon\to\infty}[\mathcal{A}^+_\varepsilon f(x) - T^s(\mathcal{A}^+_\varepsilon f)(x)] = 0$, *a.e. x.*

(b) $\lim_{\varepsilon\to\infty}||\mathcal{A}^+_\varepsilon f - T^s(\mathcal{A}^+_\varepsilon f)||_{L^p(\mu)} = 0$.

Proof. Let us fix $s > 0$. For any $\varepsilon > s > 0$ we have

$$\begin{aligned}\mathcal{A}^+_\varepsilon f(x) - T^s(\mathcal{A}^+_\varepsilon f)(x) &= \frac{1}{\varepsilon}\int_0^\varepsilon T^t f(x)dt - \frac{1}{\varepsilon}\int_s^{s+\varepsilon} T^t f(x)dt \\ &= \frac{1}{\varepsilon}\int_0^s T^t f(x)dt - \frac{1}{\varepsilon}\int_\varepsilon^{s+\varepsilon} T^t f(x)dt.\end{aligned}$$

It is clear that

$$\lim_{\varepsilon\to\infty}\frac{1}{\varepsilon}\int_0^s T^t f(x)dt = 0.$$

To control the other term we use Lemma 4.1. Let $p > \gamma > 1$ and let $\Gamma_\gamma = \{S^t : t \in \mathbb{R}\}$ be as in that lemma and let $\tilde{\mathcal{M}}^+$ be the maximal operator associated to $\Gamma_{\gamma,+} = \{S^t : t > 0\}$. By Lemma 4.1 and Theorem 1.1, $\tilde{\mathcal{M}}^+$ is bounded from $L^{p/\gamma}(\mu)$ into $L^{p/\gamma}(\mu)$. Consequently, $\tilde{\mathcal{M}}^+(|f|^\gamma)(x) < \infty$ a.e. for $f \in L^p(\mu)$. It follows that $\frac{1}{\varepsilon}\int_\varepsilon^{s+\varepsilon} T^t f(x)dt$ tends to 0 a.e. as ε goes to infinity since

$$\begin{aligned}\frac{1}{\varepsilon}\int_\varepsilon^{s+\varepsilon} T^t f(x)dt &\le \frac{1}{\varepsilon}\left(\int_\varepsilon^{s+\varepsilon}(T^t f(x))^\gamma dt\right)^{1/\gamma} s^{1/\gamma'} \\ &\le \frac{(s+\varepsilon)^{1/\gamma}}{\varepsilon}\left(\frac{1}{s+\varepsilon}\int_0^{s+\varepsilon} S^t(|f|^\gamma)(x)dt\right)^{1/\gamma} s^{1/\gamma'} \\ &\le \frac{(s+\varepsilon)^{1/\gamma}s^{1/\gamma'}}{\varepsilon}[\tilde{\mathcal{M}}^+(|f|^\gamma)(x)]^{1/\gamma},\end{aligned}$$

This proves (a).

The proof of (b) follows from (a), the dominated convergence theorem, Theorem 1.1 and the fact that

$$|\mathcal{A}^+_\varepsilon f(x) - T^s(\mathcal{A}^+_\varepsilon f)(x)| \le \mathcal{M}^+ f(x) + \mathcal{M}^+(T^s f)(x) \in L^p(\mu).$$

□

The next theorem follows from Lemma 4.2 using a standard argument. We include it for the sake of completeness.

Theorem 4.3. *Assume that we are in the conditions of Theorem 1.2. Let*

$$A = \{f \in L^p(\mu) : T^s f = f \text{ for all } s > 0\}$$

and let B be the linear manifold generated by

$$\{f - T^s f : f \in L^p(\mu), s > 0\}.$$

Then, $A \oplus \bar{B} = L^p(\mu)$, where $\bar{B}$ stands for the closure of B and $A \oplus \bar{B} = \{f + g : f \in A, g \in \bar{B}\}$. In particular $A \oplus B$ is dense in $L^p(\mu)$.

Proof. We first prove that $\{\mathcal{A}_\varepsilon^+ f\}$ is weakly convergent as ε goes to infinity for all $f \in L^p(\mu)$.

Let $f \in L^p(\mu)$. By hypothesis, $\sup_{\varepsilon>0} ||\mathcal{A}_\varepsilon^+ f||_{L^p(\mu)} \leq C||f||_{L^p(\mu)}$. This gives that the set $\{\mathcal{A}_\varepsilon^+ f : \varepsilon > 0\}$ is bounded in $L^p(\mu)$. Therefore there exists a sequence $\{\varepsilon_k\} \to \infty$ such that $\{\mathcal{A}_{\varepsilon_k}^+ f\}$ is weakly convergent. If we suppose that $\{\mathcal{A}_\varepsilon^+ f\}$ is not weakly convergent as ε goes to infinity, then there exist another sequence $\{\eta_k\} \to \infty$ and $g_1, g_2 \in L^p(\mu)$, $g_1 \neq g_2$, such that $\{\mathcal{A}_{\varepsilon_k}^+ f\}$ converges weakly to g_1 and $\{\mathcal{A}_{\eta_k}^+ f\}$ converges weakly to g_2. The continuity of T^s gives that $\{\mathcal{A}_{\varepsilon_k}^+ f - T^s(\mathcal{A}_{\varepsilon_k}^+ f)\}$ converges weakly to $g_1 - T^s g_1$. On the other hand, by part (*b*) of Lemma 4.2, $\{\mathcal{A}_{\varepsilon_k}^+ f - T^s(\mathcal{A}_{\varepsilon_k}^+ f)\}$ converges to 0 in $L^p(\mu)$. Therefore, $g_1 \in A$. The same argument gives that $g_2 \in A$ and, as a consequence, $0 \neq g_1 - g_2 \in A$.

We shall prove now that $g_1 - g_2 \in \bar{B}$. If $g_1 - g_2 \notin \bar{B}$, then there exists a linear functional $\Lambda : L^p(\mu) \to \mathbb{R}$, such that $\Lambda(\bar{B}) = 0$ and $\Lambda(g_1 - g_2) = 1$. It follows that $\Lambda g = \Lambda(T^s g)$ for all $g \in L^p(\mu)$ and all $s > 0$. Furthermore, there exists $h \in L^{p'}(\mu)$ such that $\Lambda g = \int_X gh\, d\mu$. Therefore,

$$\begin{aligned}\Lambda(\mathcal{A}_{\varepsilon_k}^+ f) &= \int_X \mathcal{A}_{\varepsilon_k}^+ f(x)h(x)d\mu(x) = \int_X \left(\frac{1}{\varepsilon_k}\int_0^{\varepsilon_k} T^t f(x)\, dt\right) h(x)d\mu(x)\\ &= \frac{1}{\varepsilon_k}\int_0^{\varepsilon_k}\int_X T^t f(x)h(x)d\mu(x)dt = \frac{1}{\varepsilon_k}\int_0^{\varepsilon_k} \Lambda(T^t f)dt = \Lambda f.\end{aligned}$$

On the other hand, $\{\Lambda(\mathcal{A}_{\varepsilon_k}^+ f)\}$ converges to Λg_1 in $\mathbb{R}$. Then $\Lambda g_1 = \Lambda f$. In analogous way we get that $\Lambda g_2 = \Lambda f$. It follows that $1 = \Lambda(g_1 - g_2) = \Lambda g_1 - \Lambda g_2 = 0$, which is a contradiction. This proves that $g_1 - g_2 \in \bar{B}$.

Let us prove now that $||\mathcal{A}_\varepsilon^+ g||_{L^p(\mu)} \to 0$ as ε tends to infinity, for all $g \in \bar{B}$. If $g = g_0 - T^s g_0$ for some $g_0 \in L^p(\mu)$ and $s > 0$, this follows from part (*b*) of Lemma 4.2, and therefore it holds for any $g \in B$. Let now fix $g \in \bar{B}$. For any $\gamma > 0$, there exists $g_0 \in B$ such that $||g - g_0||_{L^p(\mu)} < \gamma$. As a consequence,

$$\begin{aligned}||\mathcal{A}_\varepsilon^+ g||_{L^p(\mu)} &\leq ||\mathcal{A}_\varepsilon^+ g - \mathcal{A}_\varepsilon^+ g_0||_{L^p(\mu)} + ||\mathcal{A}_\varepsilon^+ g_0||_{L^p(\mu)}\\ &= ||\mathcal{A}_\varepsilon^+ (g - g_0)||_{L^p(\mu)} + ||\mathcal{A}_\varepsilon^+ g_0||_{L^p(\mu)} \leq C\gamma + ||\mathcal{A}_\varepsilon^+ g_0||_{L^p(\mu)},\end{aligned}$$

and $||\mathcal{A}_\varepsilon^+ g_0||_{L^p(\mu)} \to 0$ as ε tends to infinity, since $g_0 \in B$.

As we have seen, $g_1 - g_2 \in \bar{B}$, and then $\{\mathcal{A}_\varepsilon^+(g_1 - g_2)\}$ converges to 0 in $L^p(\mu)$. On the other hand, $g_1 - g_2 \in A$ which gives that $\mathcal{A}_\varepsilon^+(g_1 - g_2) = g_1 - g_2$. Then $g_1 - g_2 = 0$, against $g_1 \neq g_2$. Therefore, $\{\mathcal{A}_\varepsilon^+ f\}$ is weakly convergent as ε goes to infinity. (Observe that the preceding argument also proves that $A \cap \bar{B} = \{0\}$.)

We shall prove now that $A \oplus \bar{B} = L^p(\mu)$. Let Pf be the weak limit of $\{\mathcal{A}_\varepsilon^+ f\}$ as ε tends to infinity. Then $f = Pf + (f - Pf)$. From the continuity of T^s and part (*b*) of Lemma 4.2, it follows that $Pf \in A$. If we suppose that $f - Pf \notin \bar{B}$, then there exists a linear functional $\Lambda : L^p(\mu) \to \mathbb{R}$, such that $\Lambda(\bar{B}) = 0$ and $\Lambda(f - Pf) = 1$. But Pf is the weak limit of $\mathcal{A}_\varepsilon^+ f$ and therefore $\Lambda(Pf) = \lim_{\varepsilon\to\infty} \Lambda(\mathcal{A}_\varepsilon^+ f)$. However, we have seen above that $\Lambda(\mathcal{A}_\varepsilon^+ f) = \Lambda f$ for any Λ such that $\Lambda(\bar{B}) = 0$. Therefore $\Lambda(Pf) = \Lambda f$, i.e., $\Lambda(f - Pf) = 0$, which is a contradiction. □

Now we can conclude the proof of Theorem 1.2. Since the maximal operator is bounded in $L^p(\mu)$ it is enough to prove the a.e. convergence in the dense class $D_1 = A \oplus B$. If $f \in A$ it is obvious. For $f \in B$, part (a) of Lemma 4.2 proves that $\{\mathcal{A}_\varepsilon^+ f\}$ converges to 0 a.e. as ε tends to infinity.

5. Proof of Theorem 1.4

Proof of (a) in Theorem 1.4. For each natural N, we consider the set

$$Q_N = \{M \in \mathbb{Z}^2 : M = (M_1, M_2), M_1 \le M_2, |M_1| \le N, |M_2| \le N\}$$

and the operator

$$\mathcal{T}_N^* f(x) = \sup_{M \in Q_N} |\mathcal{T}_M f(x)|\,.$$

For each $R > 0$ we have by (2.1)

$$\begin{aligned} \int_X (\mathcal{T}_N^* f(x))^p d\mu(x) &= \frac{1}{R}\int_0^R \int_X |T^t(\mathcal{T}_N^* f(x))|^p H_t(x) d\mu(x) dt \\ &= \int_X \frac{1}{R}\int_0^R |T^t(\mathcal{T}_N^* f(x))|^p H_t(x) dt d\mu(x). \end{aligned} \tag{5.1}$$

Observe that since T^t is positive we have for each $M \in Q_N$,

$$|\mathcal{T}_M(T^t f)(x)| = |T^t \mathcal{T}_M f(x)| \le T^t \mathcal{T}_N^* f(x)$$

and therefore

$$\mathcal{T}_N^*(T^t f(x)) \le T^t \mathcal{T}_N^* f(x).$$

Consequently,

$$T^t \mathcal{T}_N^* f(x) = T^t \mathcal{T}_N^*(T^{-t} T^t) f(x) \le \mathcal{T}_N^*(T^t f)(x).$$

Now for any t, $0 < t < R$,

$$\begin{aligned} \mathcal{T}_N^*(T^t f)(x) &= \sup_{M \in Q_N} \left| \sum_{k=M_1}^{M_2} \upsilon_k \left(\frac{1}{\varepsilon_k} \int_0^{\varepsilon_k} T^{t+s} f(x) ds - \frac{1}{\varepsilon_{k-1}} \int_0^{\varepsilon_{k-1}} T^{t+s} f(x) ds \right) \right| \\ &= \sup_{M \in Q_N} \left| \sum_{k=M_1}^{M_2} \upsilon_k \left(\frac{1}{\varepsilon_k} \int_0^{\varepsilon_k} f^x(t+s) ds - \frac{1}{\varepsilon_{k-1}} \int_0^{\varepsilon_{k-1}} f^x(t+s) ds \right) \right| \\ &\le S^* \left(f^x \chi_{(0, \varepsilon_N + R)} \right)(t). \end{aligned}$$

Therefore, using that for almost every x the functions $t \to H_t(x)$ belong to A_p^+ with a constant independent of x and Theorem 2.1 we can dominate the last term in inequality (5.1) by

$$\begin{aligned}&\int_X \frac{1}{R}\int_0^R \left|S^*\left(f^x\chi_{(0,\varepsilon_N+R)}\right)(t)\right|^p H_t(x)dtd\mu(x)\\&\leq C\int_X \frac{1}{R}\int_0^{\varepsilon_N+R} |f^x(t)|^p H_t(x)dtd\mu(x)\\&= C\frac{1}{R}\int_0^{\varepsilon_N+R}\int_X |T^t f(x)|^p H_t(x)d\mu(x)dt\\&= C\frac{\varepsilon_N+R}{R}\int_X |f(x)|^p d\mu(x).\end{aligned}$$

Letting R go to infinity, we obtain

$$\int_X (\mathcal{T}_N^* f(x))^p d\mu(x) \leq C\int_X |f(x)|^p d\mu(x),$$

with constant independent of N. Letting $N\to\infty$ we are done.

Proof of (b) in Theorem 1.4. It suffices to prove that there exist the limits $\lim_{N\to\infty}\mathcal{T}_N^1 f(x)$ and $\lim_{N\to\infty}\mathcal{T}_N^2 f(x)$ a.e., where

$$\mathcal{T}_N^1 f(x) = \sum_{k=-N}^{0} \upsilon_k \left(\mathcal{A}^+_{\varepsilon_k} f(x) - \mathcal{A}^+_{\varepsilon_{k-1}} f(x)\right)$$

and

$$\mathcal{T}_N^2 f(x) = \sum_{k=1}^{N} \upsilon_k \left(\mathcal{A}^+_{\varepsilon_k} f(x) - \mathcal{A}^+_{\varepsilon_{k-1}} f(x)\right).$$

(Here N stands for a natural number.) We start with the convergence of $\mathcal{T}_N^1 f(x)$. Since $\mathcal{T}^*$ is of strong type (p,p) (Theorem 1.4) it suffices to prove the a.e. convergence for f in the set $D=\{\mathcal{A}^+_\varepsilon g : g\in L^p(\mu), \varepsilon>0\}$ which is dense in $L^p(\mu)$ by (a) in Theorem 1.2. Assume that $f\in D$, i.e., $f=\mathcal{A}^+_\gamma g$, for some $g\in L^p(\mu)$ and some $\gamma>0$. In this case,

$$\begin{aligned}|\upsilon_k|\left|\mathcal{A}^+_{\varepsilon_k} f(x) - \mathcal{A}^+_{\varepsilon_{k-1}} f(x)\right| &\leq C\left|\mathcal{A}^+_{\varepsilon_k}(\mathcal{A}^+_\gamma g)(x) - \mathcal{A}^+_\gamma g(x)\right|\\&\quad + C\left|\mathcal{A}^+_\gamma g(x) - \mathcal{A}^+_{\varepsilon_{k-1}}(\mathcal{A}^+_\gamma g)(x)\right|.\end{aligned}$$

We can deal with both terms in the same way. We only write the details for the first one.

Since $\varepsilon_k \to 0$ as $k \to -\infty$, there exists k_0 such that $\varepsilon_k < \gamma$ for all $k \le k_0$. Therefore, for almost every x,

$$\begin{aligned}
&\sum_{k=-\infty}^{k_0} \left|\mathcal{A}^+_{\varepsilon_k}(\mathcal{A}^+_\gamma g)(x) - \mathcal{A}^+_\gamma g(x)\right| \\
&\qquad \le \sum_{k=-\infty}^{k_0} \frac{1}{\gamma}\left(\int_0^{\varepsilon_k} |T^s g(x)|ds + \int_\gamma^{\varepsilon_k+\gamma} |T^s g(x)|ds\right) \\
&\qquad \le \frac{1}{\gamma}\sum_{k=-\infty}^{k_0} \varepsilon_k(\mathcal{M}^+ g(x) + \mathcal{M}^+(T^\gamma g)(x)) \\
&\qquad \le \frac{\varepsilon_{k_0}}{\gamma}(\mathcal{M}^+ g(x) + \mathcal{M}^+(T^\gamma g)(x)) \sum_{k=-k_0}^{\infty} \frac{1}{\rho^k} < \infty.
\end{aligned}$$

To prove the convergence of $\mathcal{T}^2_N f(x)$, it is enough to establish it for functions $f \in A \oplus B$, where A and B are the sets in Theorem 4.3. If $f \in A$ there is nothing to prove. Suppose $f = g - T^s g$, for some $g \in L^p(\mu)$ and some $s > 0$. Then

$$\begin{aligned}
|\upsilon_k|\left|\mathcal{A}^+_{\varepsilon_k} f(x) - \mathcal{A}^+_{\varepsilon_{k-1}} f(x)\right| &\le C\left|\mathcal{A}^+_{\varepsilon_k}(g - T^s g)(x)\right| \\
&\quad + C\left|\mathcal{A}^+_{\varepsilon_{k-1}}(g - T^s g)(x)\right|.
\end{aligned}$$

Again, we can deal with both terms in the same way. Since $\varepsilon_k \to \infty$ as $k \to \infty$, there exists k_0 such that for all $k \ge k_0$ we have that $\varepsilon_k > s$. Therefore, for $k \ge k_0$,

$$\begin{aligned}
\left|\mathcal{A}^+_{\varepsilon_k}(g - T^s g)(x)\right| &= \left|\frac{1}{\varepsilon_k}\int_0^{\varepsilon_k} T^t g(x)dt - \frac{1}{\varepsilon_k}\int_s^{s+\varepsilon_k} T^t g(x)dt\right| \\
&\le \frac{1}{\varepsilon_k}\int_0^s |T^t g(x)|dt + \frac{1}{\varepsilon_k}\int_{\varepsilon_k}^{s+\varepsilon_k} |T^t g(x)|dt.
\end{aligned}$$

It is clear that

$$\sum_{k=k_0}^{\infty} \frac{1}{\varepsilon_k}\int_0^s |T^t g(x)|dt \le \varepsilon_{k_0}\int_0^s |T^t g(x)|dt \sum_{k=k_0}^{\infty}\frac{1}{\rho^k} < \infty.$$

On the other hand, if $p > \gamma > 1$, $\Gamma_\gamma = \{S^t : t \in \mathbb{R}\}$ is the group in Lemma 4.1 and $\tilde{\mathcal{M}}^+$ is the maximal operator associated to $\Gamma_{\gamma,+} = \{S^t : t > 0\}$ we have, by Lemma 4.1 and Theorem 1.1, that $\tilde{\mathcal{M}}^+$ is bounded from $L^{p/\gamma}(\mu)$ into $L^{p/\gamma}(\mu)$

and

$$\begin{aligned}\frac{1}{\varepsilon_k}\int_{\varepsilon_k}^{s+\varepsilon_k}|T^t g(x)|dt &\le \frac{1}{\varepsilon_k}\left(\int_{\varepsilon_k}^{s+\varepsilon_k}|T^t g(x)|^\gamma dt\right)^{1/\gamma} s^{1/\gamma'}\\ &\le \frac{1}{\varepsilon_k}\left(\int_{\varepsilon_k}^{s+\varepsilon_k}|S^t(|g|^\gamma)(x)|dt\right)^{1/\gamma} s^{1/\gamma'}\\ &\le \frac{(s+\varepsilon_k)^{1/\gamma}s^{1/\gamma'}}{\varepsilon_k}\left(\tilde{\mathcal{M}}^+(|g|^\gamma)(x)\right)^{1/\gamma}.\end{aligned}$$

Therefore,

$$\sum_{k=k_0}^{\infty}\frac{1}{\varepsilon_k}\int_{\varepsilon_k}^{s+\varepsilon_k}|T^t g(x)|dt \le \left(\tilde{\mathcal{M}}^+(|g|^\gamma)(x)\right)^{1/\gamma}\sum_{k=k_0}^{\infty}\frac{(s+\varepsilon_k)^{1/\gamma}s^{1/\gamma'}}{\varepsilon_k}.$$

The last term is finite a.e. since $\tilde{\mathcal{M}}^+$ is bounded in $L^{p/\gamma}(\mu)$ and the sum

$$\sum_{k=k_0}^{\infty}\frac{(s+\varepsilon_k)^{1/\gamma}s^{1/\gamma'}}{\varepsilon_k}$$

is essentially dominated by $\sum_{k=k_0}^{\infty}\frac{1}{(\rho^{1-\frac{1}{\gamma}})^k}$.

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