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Mathematical and Computer Modelling 46 (2007) 1236–1246

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Polynomial stability of a joint-leg-beam system with local damping

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Received 27 August 2006; accepted 30 November 2006

Abstract

Recent advances in the design and construction of large inflatable/rigidizable space structures and potential new applications of such structures have produced a demand for better analysis and computational tools to deal with the new class of structures. Understanding stability and damping properties of truss systems composed of these materials is central to the successful operation of future systems. In this paper, we consider a mathematical model for an assembly of two elastic beams connected to a joint through legs. The dynamic joint model is composed of two rigid bodies (the joint-legs) with an internal moment. In an ideal design all struts and joints will have identical material and geometric properties. In this case we previously established exponential stability of the beam-joint system. However, in order to apply theoretical stability estimates to realistic systems one must deal with the case where the individual truss components are not identical and still be able to analyze damping. We consider a problem of this type where one beam is assumed to have a small Kelvin–Voigt damping parameter and the second beam has no damping. In this case, we prove that the component system is only polynomially damped even if additional rotational damping is assumed in the joint.

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Keywords: Beam; Damping; Polynomial decay rate; Semigroup

1. Introduction

Inflatable and rigidizable space structures have been the subject of numerous scientific studies for the past fifty years and considerable progress has been made in the development of new materials and technologies for the design and manufacturing of these structures. Several proposed space antenna systems will require large ultra-light trusses to provide the “backbone” of the structure. In recent years, there has been renewed interest in inflatable/rigidizable space structures [6] because of the efficiency they offer in packaging during boost-to-orbit. Ground testing and in-orbit experiments on sub-scale models have provided limited data for model verification and validation. However, it has been recognized that practical precision requirements can only be achieved through the development of new high fidelity mathematical models and corresponding numerical tools. In particular, we need to better understand

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dynamic response characteristics, including inherent damping, of truss structures fabricated with these advanced material systems. In addition, several proposed designs make use of rigid joints with special attachment “legs” which lead to the joint-leg-beam system considered in the paper.

It has been proved in [4] that when both beams are subject to Kelvin–Voigt damping, the associated semigroup $S(t)$ is exponentially stable and analytic. Hence the energy of the system decays exponentially to zero, and the associated solution operator has smoothing properties. Numerical results are reported in [5].

Since beam damping may be achieved by additional processing and materials, it is of interest to know if treating only one of the beams is sufficient to ensure energy decay. Thus, our main interest here is to investigate the case in which only one of the beams is subject to Kelvin–Voigt damping. In this paper, we confirm that the decay rate is of polynomial type, with or without additional rotational damping in the joint.

Whereas polynomial stability has been investigated extensively in the literature, most of these are case studies. For example, spectral analysis methods were used in [7,10,12], and energy methods were used in [1,2,11,14,15]. Recently, sufficient conditions in frequency domain for polynomial stability of abstract first order linear evolution equations were given in [3,8], which added a new semigroup method for polynomial stability.

We will use the following result in [8]:

Theorem 1.1. *Let $T(t)$ be a bounded C_0 -semigroup on a Hilbert space H associated with the linear system $\dot{X}(t) = AX(t)$. If*

$$i\mathbb{R} \cap \sigma(A) = \emptyset \tag{1.1}$$

and for some positive constant ℓ

$$\lim_{\beta \rightarrow \infty} \left\| \frac{1}{\beta^\ell} (i\beta - A)^{-1} \right\| < \infty, \tag{1.2}$$

then for all $k \in \mathbb{N}$ there exists a finite positive constant c_k such that

$$\|T(t)X_0\|_{\mathcal{H}} \leq c_k \left(\frac{\ln t}{t}\right)^{\frac{k}{\ell}} \ln t \|X_0\|_{D(A^k)} \tag{1.3}$$

for all $X_0 \in D(A^k)$.

Remark. For $k = 1$ in the above theorem, we see that the solution decays at a rate of $\left(\frac{1}{t}\right)^{\frac{1}{\ell}} (\ln t)^{1+\frac{1}{\ell}}$ for all initial states in the domain of \mathcal{A} . This rate is faster than $\left(\frac{1}{t}\right)^{\frac{1}{\ell}-\epsilon}$ for any $\epsilon > 0$. In this case we say that $T(t)$ is polynomially stable with order $\frac{1}{\ell}$. There are examples of specific systems whose decay rate is equal to $\frac{1}{\ell}$. Therefore, the polynomial decay rate obtained by the semigroup method is almost optimal.

2. System equations and semigroup setting

The joint-leg-beam system is shown in Fig. 2.1. The equations of motion of this system, which have been derived in [4], are:

$$\rho_i A_i \frac{\partial^2 w^i(t, s_i)}{\partial t^2} + \frac{\partial^2}{\partial s_i^2} \left[E_i I_i \frac{\partial^2 w^i(t, s_i)}{\partial s_i^2} + \gamma_i \frac{\partial^3 w^i(t, s_i)}{\partial s_i^2 \partial t} \right] = 0, \tag{2.1}$$

$$\rho_i A_i \frac{\partial^2 u^i(t, s_i)}{\partial t^2} - \frac{\partial}{\partial s_i} \left[E_i A_i \frac{\partial u^i(t, s_i)}{\partial s_i} + \mu_i \frac{\partial^2 u^i(t, s_i)}{\partial s_i \partial t} \right] = 0 \tag{2.2}$$

and

$$m\ddot{x}(t) - m_1 d_1 \cos \varphi_1 \ddot{\theta}_1(t) + m_2 d_2 \cos \varphi_2 \ddot{\theta}_2(t) = F_1(t) \sin \varphi_1 - N_1(t) \cos \varphi_1 + F_2(t) \sin \varphi_2 + N_2(t) \cos \varphi_2, \tag{2.3}$$

$$m\ddot{y}(t) + m_1d_1 \sin \varphi_1 \ddot{\theta}_1(t) + m_2d_2 \sin \varphi_2 \ddot{\theta}_2(t) = F_1(t) \cos \varphi_1 + N_1(t) \sin \varphi_1 - F_2(t) \cos \varphi_2 + N_2(t) \sin \varphi_2, \tag{2.4}$$

$$I_Q^1 \ddot{\theta}_1(t) = M_Q(t) + M_1(t) + l_1 N_1(t) + m_1 d_1 [\ddot{x}(t) \cos \varphi_1 - \ddot{y}(t) \sin \varphi_1], \tag{2.5}$$

$$I_Q^2 \ddot{\theta}_2(t) = -M_Q(t) + M_2(t) + l_2 N_2(t) - m_2 d_2 [\ddot{x}(t) \cos \varphi_2 + \ddot{y}(t) \sin \varphi_2], \tag{2.6}$$

where $w^i(t, s_i), u^i(t, s_i)$ are the transversal and longitudinal displacements of the beam, $0 \leq s_i \leq L_i, t \geq 0, i = 1, 2$; $x(t), y(t)$ are the planar, Cartesian displacements of the pivot joint; $\theta_i(t)$ is the perturbation of the angle between leg i and the positive x axis, and the “dots” represent derivatives with respect to t . The physical parameters in the above equations are given by

- $L_i, A_i, I_i, E_i, \rho_i$: length, cross-section area, area moment of inertia, Young’s modulus and mass density of beam $i, i = 1, 2$ (with $L_i, A_i, I_i, E_i, \rho_i > 0$).
- $\ell_i, m_i, I_\ell^i, d_i$: length, mass, mass moment of inertia about center of mass and distance from pivot to center of mass of joint-leg $i, i = 1, 2$.
- $I_Q^i = I_\ell^i + m_i d_i^2 > 0$: mass moment of inertia of joint-leg i about pivot, $i = 1, 2$.
- μ_i, γ_i, b, k : nonnegative constants representing the Kelvin–Voigt damping parameters in the axial motions, in the transverse bending, viscous joint damping, and joint stiffness parameters, respectively.
- m_p : mass of the pivot.

Furthermore, $m = m_1 + m_2 + m_p > 0$ is the total mass of the joint-leg system, while the angles φ_i describe the equilibrium orientation of beam i (see Fig. 2.1). Finally,

$$\begin{cases} M_i(t) = E_i I_i \frac{\partial^2 w^i}{\partial s_i^2}(t, L_i) + \gamma_i \frac{\partial^3 w^i}{\partial s_i^2 \partial t}(t, L_i), \\ N_i(t) = \frac{\partial}{\partial s_i} \left(E_i I_i \frac{\partial^2 w^i}{\partial s_i^2} + \gamma_i \frac{\partial^3 w^i}{\partial s_i^2 \partial t} \right) (t, L_i), \\ F_i(t) = \frac{\partial}{\partial s_i} \left(E_i A_i u^i + \mu_i \frac{\partial u^i}{\partial t} \right) (t, L_i) \end{cases} \tag{2.7}$$

represent the bending moment, shear force and axial force at the end $s_i = L_i$ of beam i , and

$$M_Q(t) = k (\theta_2(t) - \theta_1(t)) + b (\dot{\theta}_2(t) - \dot{\theta}_1(t)) \tag{2.8}$$

is the internal torque exerted on joint-leg 1 by joint-leg 2.

Geometric compatibility between the joint-leg and the $s_i = L_i$ end of each beam requires that, for beam 1 — leg 1:

$$\begin{cases} x(t) - l_1 \theta_1(t) \cos \varphi_1 + w^1(t, L_1) \cos \varphi_1 + u^1(t, L_1) \sin \varphi_1 = 0 \\ y(t) + l_1 \theta_1(t) \sin \varphi_1 - w^1(t, L_1) \sin \varphi_1 + u^1(t, L_1) \cos \varphi_1 = 0 \\ \theta_1(t) + w_s^1(t, L_1) = 0, \end{cases} \tag{2.9}$$

while, for beam 2 — leg 2:

$$\begin{cases} x(t) + l_2 \theta_2(t) \cos \varphi_2 - w^2(t, L_2) \cos \varphi_2 + u^2(t, L_2) \sin \varphi_2 = 0 \\ y(t) + l_2 \theta_2(t) \sin \varphi_2 - w^2(t, L_2) \sin \varphi_2 - u^2(t, L_2) \cos \varphi_2 = 0 \\ \theta_2(t) + w_s^2(t, L_2) = 0. \end{cases} \tag{2.10}$$

These conditions require that the Cartesian position of each beam’s tip and the corresponding leg’s tip remain the same, and that the end slope of each beam remains equal to that of the corresponding leg.

At the end $s_i = 0$ of each beam, we have clamped boundary conditions:

$$u^i(t, 0) = w^i(t, 0) = \frac{\partial w^i}{\partial s_i}(t, 0) = 0. \tag{2.11}$$

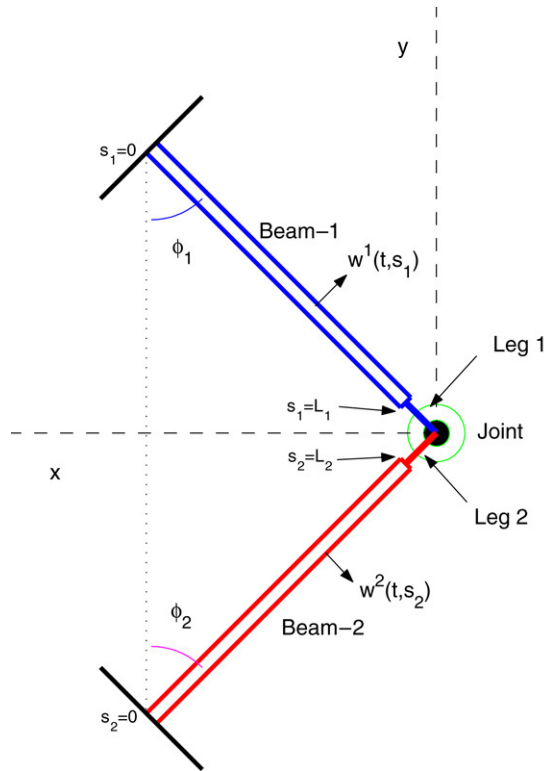


Fig. 2.1. Joint beam system.

We denote by $H^n(0, L)$ the usual Sobolev space of functions in $L^2(0, L)$ with derivatives up to order n in $L^2(0, L)$. With $H_\ell^n(0, L)$ and $H_0^n(0, L)$ we denote the spaces of functions in $H^n(0, L)$ that vanish, together with all derivatives up to the order $n - 1$, at the left end and at both ends, respectively.

Define the Hilbert space

$$\mathcal{H}_z \doteq H_\ell^2(0, L_1) \times H_\ell^2(0, L_2) \times H_\ell^1(0, L_1) \times H_\ell^1(0, L_2)$$

with the inner product

$$\langle z, \tilde{z} \rangle_{\mathcal{H}_z} \doteq \sum_{i=1}^2 [E_i I_i \langle D^2 w_i, D^2 \tilde{w}_i \rangle + E_i A_i \langle Du_i, D\tilde{u}_i \rangle]$$

where $z \doteq (w_1, w_2, u_1, u_2)^T$, $\tilde{z} \doteq (\tilde{w}_1, \tilde{w}_2, \tilde{u}_1, \tilde{u}_2)^T$, $D^n \doteq \frac{d^n}{ds^n}$ and $\langle \cdot, \cdot \rangle$ denotes the usual L^2 -inner product.

We also define

$$\mathcal{V}_z \doteq L^2(0, L_1) \times L^2(0, L_2) \times L^2(0, L_1) \times L^2(0, L_2)$$

with the inner product

$$\langle z, \tilde{z} \rangle_{\mathcal{V}_z} \doteq \sum_{i=1}^2 \rho_i A_i [\langle w_i, \tilde{w}_i \rangle + \langle u_i, \tilde{u}_i \rangle].$$

Furthermore, define

$$\mathcal{H}_J \doteq [\ker(C)]^\perp = \text{range}(C^T) \subset \mathbb{R}^6$$

with the inner product

$$\langle \xi, \tilde{\xi} \rangle_{\mathcal{H}_J} \doteq \xi^T (C^T M^{-1} C)^\dagger \tilde{\xi} = \langle \xi, (C^T M^{-1} C)^\dagger \tilde{\xi} \rangle_{\mathbb{R}^6}$$

where the matrices M and C are given by:

$$M \doteq \begin{pmatrix} mI_2 & P \\ P^T & \text{diag}(I_Q^1, I_Q^2) \end{pmatrix}, \quad P \doteq \begin{pmatrix} -m_1d_1 \cos \varphi_1 & m_2d_2 \cos \varphi_2 \\ m_1d_1 \sin \varphi_1 & m_2d_2 \sin \varphi_2 \end{pmatrix}, \tag{2.12}$$

$$C \doteq \begin{pmatrix} 0 & -\cos \varphi_1 & 0 & \cos \varphi_2 & \sin \varphi_1 & \sin \varphi_2 \\ 0 & \sin \varphi_1 & 0 & \sin \varphi_2 & \cos \varphi_1 & -\cos \varphi_2 \\ 1 & \ell_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \ell_2 & 0 & 0 \end{pmatrix}, \tag{2.13}$$

and $(C^T M^{-1} C)^\dagger$ denotes the Moore–Penrose generalized inverse of the matrix $C^T M^{-1} C$. An immediate calculation using properties of the Moore–Penrose generalized inverse shows that if $\xi \in \mathcal{H}_J, \xi = C^T \zeta, \zeta \in \mathbb{R}^4$, then $\|\xi\|_{\mathcal{H}_J}^2 = \zeta^T M \zeta$.

We denote with $z(t) \doteq (w^1(t, \cdot), w^2(t, \cdot), u^1(t, \cdot), u^2(t, \cdot))^T, \eta(t) \doteq (x(t), y(t), \theta_1(t), \theta_2(t))^T, v(t) \doteq \dot{z}(t) \doteq (y^1(t, \cdot), y^2(t, \cdot), v^1(t, \cdot), v^2(t, \cdot))^T, \xi(t) \doteq \dot{\eta}(t) = (p(t), q(t), \tau_1(t), \tau_2(t))^T$, and define two “boundary projection operators” P_1^B and P_2^B from \mathcal{V}_z to \mathbb{R}^6 by

$$\begin{aligned} \text{dom}(P_1^B) &\doteq H^2(0, L_1) \times H^2(0, L_2) \times H^1(0, L_1) \times H^1(0, L_2), \\ \text{dom}(P_2^B) &\doteq H^4(0, L_1) \times H^4(0, L_2) \times H^2(0, L_1) \times H^2(0, L_2), \\ P_1^B (w^1, w^2, u^1, u^2)^T &\doteq (-Dw^1(L_1), w^1(L_1), -Dw^2(L_2), w^2(L_2), -u^1(L_1), -u^2(L_2))^T, \\ P_2^B (w^1, w^2, u^1, u^2)^T &\doteq (D^2w^1(L_1), D^3w^1(L_1), D^2w^2(L_2), D^3w^2(L_2), Du^1(L_1), Du^2(L_2))^T. \end{aligned}$$

Then, the geometric compatibility conditions (2.9) and (2.10) can be written in the form

$$P_1^B z(t) = C^T \eta(t), \tag{2.14}$$

whereas the dynamic compatibility conditions (2.7), written in terms of the boundary projection operator P_2^B defined above, take the form

$$EP_2^B (z(t) + S_1 \dot{z}(t)) = F(t) \tag{2.15}$$

where $E = \text{diag}(E_1 I_1, E_1 I_1, E_2 I_2, E_2 I_2, E_1 A_1, E_1 A_2)$ and $S_1 = \text{diag}(\frac{\gamma_1}{E_1 I_1}, \frac{\gamma_2}{E_2 I_2}, \frac{\mu_1}{E_1 A_1}, \frac{\mu_2}{E_2 A_2})$ and $F(t) = (M_1(t), N_1(t), M_2(t), N_2(t), F_1(t), F_2(t))^T$.

By defining $\xi(t) \doteq \frac{d}{dt} P_1^B z(t)$ it follows by (2.14) that $\xi(t) = P_1^B v(t) = C^T \zeta(t)$ and, using (2.15) and (2.8) with $k = 0$, the joint-leg system (2.3)–(2.6) can be written in the form

$$\dot{\xi}(t) + C^T M^{-1} B (C C^T)^{-1} C \xi(t) = C^T M^{-1} C E P_2^B (z(t) + S_1 v(t)) \tag{2.16}$$

where $B \doteq \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b & -b \\ 0 & 0 & -b & b \end{pmatrix}$.

We also define on \mathcal{H}_z the operator \mathcal{A}_1 by

$$\begin{aligned} \text{dom}(\mathcal{A}_1) &\doteq H_\ell^2 \cap H^4(0, L_1) \times H_\ell^2 \cap H^4(0, L_2) \times H_\ell^1 \cap H^2(0, L_1) \times H_\ell^1 \cap H^2(0, L_2), \\ \mathcal{A}_1 &\doteq \text{diag} \left(\frac{E_1 I_1}{\rho_1 A_1} D^4, \frac{E_2 I_2}{\rho_2 A_2} D^4, -\frac{E_1}{\rho_1} D^2, -\frac{E_2}{\rho_2} D^2 \right). \end{aligned}$$

Note that $\text{dom}(\mathcal{A}_1)$ is contained in both $\text{dom}(P_1^B)$ and $\text{dom}(P_2^B)$, so both projections are well defined on $\text{dom}(\mathcal{A}_1)$.

Let $X(t) \doteq (z(t), v(t), \xi(t))^T$. Then the joint-leg-beam system (2.1)–(2.6) can be written as an abstract first order evolution equation on the Hilbert space

$$\mathcal{H} \doteq \mathcal{H}_z \times \mathcal{V}_z \times \mathcal{H}_J$$

in the form

$$\dot{X}(t) = \mathcal{A}X(t) = \begin{pmatrix} v(t) \\ -\mathcal{A}_1(z(t) + S_1v(t)) \\ -C^T M^{-1} B(C C^T)^{-1} C \xi(t) + C^T M^{-1} C E P_2^B (z(t) + S_1v(t)) \end{pmatrix} \tag{2.17}$$

with

$$\text{dom}(\mathcal{A}) = \left\{ X = \begin{pmatrix} z \\ v \\ \xi \end{pmatrix} \in \mathcal{H} \mid v \in \mathcal{H}_z, z + S_1v \in \text{dom}(\mathcal{A}_1), P_1^B v = \xi \right\}. \tag{2.18}$$

It was proved in [4] that the operator \mathcal{A} is dissipative and it generates a C_0 -semigroup of contractions $S(t)$ which is exponentially stable if $\mu_1, \mu_2, \gamma_1, \gamma_2 > 0$ [9,13].

3. Polynomial stability for $b > 0$

With damping in only one beam we have $\mu_1, \gamma_1 > 0$ and $\mu_2, \gamma_2 = 0$. We first consider the case of $b > 0$, i.e., there is rotational damping in the joint.

Theorem 3.1. *The semigroup $S(t)$ for the case of $b > 0$ is polynomially stable with order $\frac{1}{2}$.*

Proof. We will first check condition (1.2) for $\ell = 2$. If condition (1.2) is false, then there exists a sequence $\{\beta_n\}_{n=1}^\infty \subset \mathbb{R}^+$ with $\beta_n \rightarrow \infty$, and a sequence $\{X_n\}_{n=1}^\infty \subset D(\mathcal{A})$, $X_n = \begin{pmatrix} z_n \\ v_n \\ \xi_n \end{pmatrix}$, $z_n = \begin{pmatrix} w_n^1 \\ w_n^2 \\ u_n^1 \\ u_n^2 \end{pmatrix}$, $v_n = \begin{pmatrix} y_n^1 \\ y_n^2 \\ v_n^1 \\ v_n^2 \end{pmatrix}$, $\xi_n =$

$$C^T \zeta_n, \zeta_n = \begin{pmatrix} p_n \\ q_n \\ r_n^1 \\ r_n^2 \end{pmatrix}, \text{ with}$$

$$\|X_n\|_{\mathcal{H}}^2 = \sum_{i=1}^2 \left[E_i A_i \|D u_n^i\|^2 + E_i I_i \|D^2 w_n^i\|^2 + \rho_i A_i (\|v_n^i\|^2 + \|y_n^i\|^2) \right] + \zeta_n^T M \zeta_n = 1 \tag{3.1}$$

such that

$$\lim_{n \rightarrow \infty} \|\beta_n^\ell (i\beta_n - \mathcal{A})X_n\|_{\mathcal{H}} = 0. \tag{3.2}$$

Our goal is to show that (3.2) will yield the contradiction $\|X_n\|_{\mathcal{H}} \rightarrow 0$. For simplicity of notation, we shall hereafter omit the subscript n .

By a straightforward calculation, we have

$$\beta^\ell \text{Re} \langle \mathcal{A}X, X \rangle_{\mathcal{H}} = -\beta^\ell \left(\mu_1 \|Dv^1\|^2 + \gamma_1 \|D^2 y^1\|^2 + b |\tau_1 - \tau_2|^2 \right). \tag{3.3}$$

Since $|\text{Re} \langle \mathcal{A}X, X \rangle_{\mathcal{H}}| = |\text{Re} \langle (i\beta - \mathcal{A})X, X \rangle_{\mathcal{H}}| \leq \|(i\beta - \mathcal{A})X\|_{\mathcal{H}}$, (3.2) and (3.3) lead to

$$\beta^{\frac{\ell}{2}} \|Dv^1\|, \beta^{\frac{\ell}{2}} \|D^2 y^1\|, \beta^{\frac{\ell}{2}} |\tau_1 - \tau_2| \rightarrow 0. \tag{3.4}$$

The components of (3.2) related to the beam equations are

$$\beta^\ell \left[i\beta u^i - v^i \right] \rightarrow 0 \quad \text{in } H_\ell^1(0, L_i), \tag{3.5}$$

$$\beta^\ell \left[i\beta v^i - \delta_1^i D^2(u^i + \delta_2^i v^i) \right] \rightarrow 0 \quad \text{in } L^2(0, L_i), \tag{3.6}$$

$$\beta^\ell \left[i\beta w^i - y^i \right] \rightarrow 0 \quad \text{in } H_\ell^2(0, L_i), \tag{3.7}$$

$$\beta^\ell \left[i\beta y^i + \delta_3^i D^4(w^i + \delta_4^i y^i) \right] \rightarrow 0 \quad \text{in } L^2(0, L_i), \tag{3.8}$$

for $i = 1, 2$. In these equations, $\delta_j^i, i = 1, 2, j = 1, \dots, 4$, are nonnegative constants depending on the physical parameters of our system. More precisely, $\delta_1^i = \frac{E_i}{\rho_i}, \delta_2^i = \frac{\mu_i}{E_i A_i}, \delta_3^i = \frac{E_i I_i}{\rho_i A_i}$ and $\delta_4^i = \frac{\gamma_i}{E_i I_i}$. Note that with our assumptions on the damping parameters we have $\delta_2^2 = \delta_4^2 = 0$. Since we are assuming Kelvin–Voigt damping on beam-1, in the absence of external excitation, its energy will decay to zero, for all initial conditions in the state space. In fact, from (3.4), (3.5) and (3.7) it follows immediately that

$$\|Du^1\|, \|v^1\|, \|D^2w^1\|, \|y^1\| \rightarrow 0. \tag{3.9}$$

However, there is no “direct” damping in beam-2. All we have is damping in beam-1 and in the joint, indirectly passed to beam-2 through the joint dynamics and compatibility conditions. We want to analyze the effect of these damping sources on beam-2. In order to do that, we will start by applying a multiplier method to the equations for beam-2. Using (3.5) and (3.7) to replace $i\beta v_2$ and $i\beta y_2$ in (3.6) and (3.8) by $-\beta^2 u^2$ and $-\beta^2 w^2$, respectively, then taking the L^2 -inner product with $s_2 Du^2$ and $s_2 Dw^2$ (here s_2 is the spatial variable in beam-2), respectively, recalling that $\delta_2^2 = \delta_4^2 = 0$ and integrating by parts, we get

$$-L_2|\beta u^2(L_2)|^2 - L_2\delta_1^2|Du^2(L_2)|^2 + \|\beta u^2\|^2 + \delta_1^2\|Du^2\|^2 \rightarrow 0, \tag{3.10}$$

$$\begin{aligned} & -L_2|\beta w^2(L_2)|^2 + 2\text{Re}(\delta_3^2[L_2D^3w^2(L_2) - D^2w^2(L_2)]D\bar{w}^2(L_2)) \\ & - L_2\delta_3^2|D^2w^2(L_2)|^2 + \|\beta w^2\|^2 + 3\delta_3^2\|D^2w^2\|^2 \rightarrow 0. \end{aligned} \tag{3.11}$$

Remark. The above two equations indicate that whether the energy of beam-2 decays to zero is equivalent to whether the boundary terms decay to zero. In particular, it is necessary that both the extensional force and the bending moment of beam-2 at the $s_2 = L_2$ end (i.e. F_2 and M_2) converge to zero.

It follows from (3.4) and the Trace theorem that

$$\beta^{\frac{\ell}{2}}v^1(L_1), \beta^{\frac{\ell}{2}}y^1(L_1), \beta^{\frac{\ell}{2}}Dy^1(L_1) \rightarrow 0. \tag{3.12}$$

Hence, by the compatibility conditions (2.9), we obtain

$$\beta^{\frac{\ell}{2}}p, \beta^{\frac{\ell}{2}}q, \beta^{\frac{\ell}{2}}\tau_1 \rightarrow 0. \tag{3.13}$$

Since $\beta^{\frac{\ell}{2}}|\tau_1 - \tau_2|$ converges to zero (see (3.4)), so does $\beta^{\frac{\ell}{2}}\tau_2$. In summary, we have proved that

$$\left\| \beta^{\frac{\ell}{2}}\zeta \right\|_{\mathbb{R}^4} = \left\| \beta^{\frac{\ell}{2}} \begin{pmatrix} p \\ q \\ \tau_1 \\ \tau_2 \end{pmatrix} \right\|_{\mathbb{R}^4} \rightarrow 0. \tag{3.14}$$

Now using the compatibility condition (2.10), we also get estimates of beam-2 at the $s_2 = L_2$ boundary, namely that

$$\beta^{\frac{\ell}{2}}v^2(L_2), \beta^{\frac{\ell}{2}}y^2(L_2), \beta^{\frac{\ell}{2}}Dy^2(L_2) \rightarrow 0. \tag{3.15}$$

The component of (3.2) related to the joint dynamics yields

$$\beta^\ell C^T [i\beta\zeta + M^{-1}B\zeta - M^{-1}CF] \rightarrow 0 \quad \text{in } \mathbb{R}^6$$

and therefore

$$\beta^\ell [i\beta\zeta + M^{-1}B\zeta - M^{-1}CF] \rightarrow 0 \quad \text{in } \mathbb{R}^4, \tag{3.16}$$

since C is a full-rank matrix. Applying (3.14) to (3.16) leads to

$$\beta^{\frac{\ell}{2}-1}CF \rightarrow 0 \quad \text{in } \mathbb{R}^4, \tag{3.17}$$

i.e.,

$$\beta^{\frac{\ell}{2}-1} \begin{pmatrix} F_1 \sin \varphi_1 - N_1 \cos \varphi_1 + F_2 \sin \varphi_2 + N_2 \cos \varphi_2 \\ F_1 \cos \varphi_1 + N_1 \sin \varphi_1 - F_2 \cos \varphi_2 + N_2 \sin \varphi_2 \\ M_1 + \ell_1 N_1 \\ M_2 + \ell_2 N_2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.18}$$

Let us recall the remark following Eq. (3.11). To get $(F_2, M_2, N_2 \rightarrow 0)$ from (3.18), we must have $F_1, M_1 \rightarrow 0$ (or $F_1, N_1 \rightarrow 0$) and $\ell \geq 2$.

By the Gagliardo–Nirenberg inequality, we obtain the estimate

$$|\beta^{\frac{\ell-2}{4}} F_1| \leq c \|\beta^{\frac{\ell}{2}} D(u^1 + v^1)\|^{\frac{1}{2}} \left(\frac{\|D(u^1 + v^1)\|_{H^1}}{\beta} \right)^{\frac{1}{2}} \rightarrow 0, \tag{3.19}$$

which follows from the fact that the first term on the right hand side converges to zero, and the second term is bounded which can be seen from (3.6). Similarly, we obtain

$$\begin{aligned} |\beta^{\frac{3\ell-2}{8}} M_1| &\leq c |\beta^{\frac{3\ell-2}{8}}| \|D^2(w^1 + y^1)\|^{\frac{1}{2}} \|D^2(w^1 + y^1)\|_{H^1}^{\frac{1}{2}} \\ &\leq c \|\beta^{\frac{\ell}{2}} D^2(w^1 + y^1)\|^{\frac{3}{4}} \left(\frac{\|D^2(w^1 + y^1)\|_{H^2}}{\beta} \right)^{\frac{1}{4}} \rightarrow 0, \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} |\beta^{\frac{\ell-6}{8}} N_1| &\leq c |\beta^{\frac{\ell-6}{8}}| \|D^3(w^1 + y^1)\|^{\frac{1}{2}} \|D^3(w^1 + y^1)\|_{H^1}^{\frac{1}{2}} \\ &\leq c \|\beta^{\frac{\ell}{2}} D^2(w^1 + y^1)\|^{\frac{3}{4}} \left(\frac{\|D^3(w^1 + y^1)\|_{H^1}}{\beta} \right)^{\frac{3}{4}} \rightarrow 0, \end{aligned} \tag{3.21}$$

where the convergence to zero in both (3.20) and (3.21) follows from the fact that the first factor tends to zero and the second is bounded by virtue of (3.8).

Picking $\ell = 6$, from (3.19) and (3.21) we get that $F_1, N_1 \rightarrow 0$, but picking $\ell = 2$, from (3.19) and (3.20) we get that $F_1, M_1 \rightarrow 0$. Hence we choose $\ell = 2$ for a faster rate $\frac{1}{2}$. Going back to (3.18), we then deduce that

$$F_2, M_2, N_2 \rightarrow 0,$$

and therefore

$$\|v^2\|, \|y^2\|, \|Du^2\|, \|D^2w^2\| \rightarrow 0. \tag{3.22}$$

Here we have replaced $\|\beta u^2\|$ and $\|\beta w^2\|$ by $\|v^2\|$ and $\|y^2\|$, respectively by virtue of (3.5) and (3.7). Combining (3.4), (3.14) and (3.22), we reach the contradiction $\|X\|_{\mathcal{H}} \rightarrow 0$. Hence, condition (1.2) holds.

Now, if condition (1.1) is false, then there exist $\beta \in \mathbb{R}$ and a sequence $\{X_n\}_{n=1}^\infty \subset D(\mathcal{A})$ with $\|X_n\|_{\mathcal{H}} = 1 \forall n$ such that

$$\lim_{n \rightarrow \infty} \|(i\beta - \mathcal{A})X_n\|_{\mathcal{H}} = 0. \tag{3.23}$$

We can repeat the above arguments in exactly the same way (notice that we have intentionally avoided using the fact that $\beta \rightarrow \infty$) to once again obtain the contradiction $\|X\|_{\mathcal{H}} \rightarrow 0$. Hence, condition (1.1) also holds. The proof is then completed. ■

4. Polynomial stability for $b = 0$

In this section, we will consider the case of $b = 0$, i.e., there is no rotational damping in the joint. It turns out that the polynomial stability still holds, but the solution decays at a slower rate.

Theorem 4.1. *The semigroup $S(t)$ for the case of $b = 0$ is polynomially stable with order $\frac{1}{14}$.*

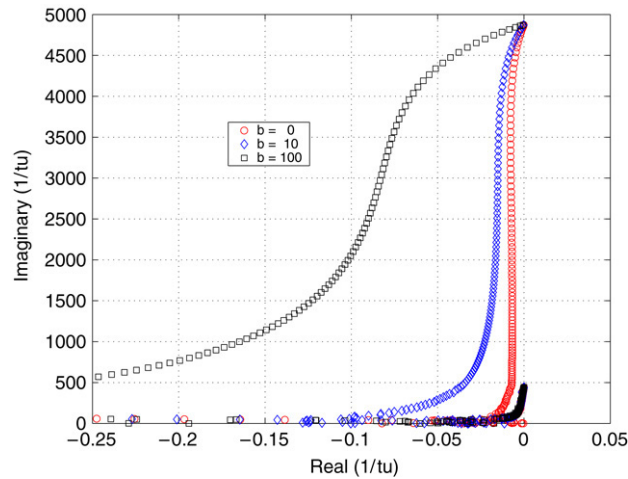


Fig. 5.1. Comparison of eigenvalues for $b = 0, 10, 100$.

Proof. We will modify the proof in the last section. The argument up to (3.4) is still valid. But the argument for $\beta^{\frac{\ell}{2}}\tau_2 \rightarrow 0$ has to be changed. Dividing (3.16) by $\beta^{\frac{\ell}{2}-1}$, $\ell \geq 2$ and then multiplying by the matrix M , we obtain

$$i\beta^{\frac{\ell}{2}}M\zeta + i\beta^{\frac{\ell}{2}-1}B\zeta - i\beta^{\frac{\ell}{2}-1}CF \rightarrow 0 \quad \text{in } \mathbb{R}^4, \tag{4.1}$$

Since we already have, see (3.13),

$$\beta^{\frac{\ell}{2}}p, \beta^{\frac{\ell}{2}}q, \beta^{\frac{\ell}{2}}\tau_1 \rightarrow 0,$$

(4.1) is further simplified to

$$i\beta^{\frac{\ell}{2}} \begin{pmatrix} m_2d_2\tau_2 \cos \varphi_2 \\ m_2d_2\tau_2 \sin \varphi_2 \\ 0 \\ I_Q^2\tau_2 \end{pmatrix} + \beta^{\frac{\ell}{2}-1} \begin{pmatrix} 0 \\ 0 \\ -\tau_2 \\ \tau_2 \end{pmatrix} - \beta^{\frac{\ell}{2}-1} \begin{pmatrix} F_1 \sin \varphi_1 - N_1 \cos \varphi_1 + F_2 \sin \varphi_2 + N_2 \cos \varphi_2 \\ F_1 \cos \varphi_1 + N_1 \sin \varphi_1 - F_2 \cos \varphi_2 + N_2 \sin \varphi_2 \\ M_1 + \ell_1 N_1 \\ M_2 + \ell_2 N_2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.2}$$

From the third component of the above equation, we see that $\beta\tau_2 \rightarrow 0$ is equivalent to $\beta M_1, \beta N_1 \rightarrow 0$. This requires that $\ell = 14$ as estimated in (3.20) and (3.21) which are still valid when $b = 0$. With this choice of ℓ , $F_1 \rightarrow 0$ due to (3.19), and (4.2) becomes

$$\begin{pmatrix} F_2 \sin \varphi_2 + N_2 \cos \varphi_2 \\ -F_2 \cos \varphi_2 + N_2 \sin \varphi_2 \\ M_2 + \ell_2 N_2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.3}$$

This proves that

$$F_2, M_2, N_2 \rightarrow 0.$$

We then finish the proof by repeat the argument from (3.22) to the end of proof of Theorem 3.1. ■

5. Numerical results

A finite dimensional approximation scheme of the joint-leg-beam system (2.16) was given in [5]. We compute the eigenvalues of the approximating system for $\mu_2 = \gamma_2 = 0$. Fig. 5.1 compares the cases of $b = 0, 10, 100$. It can be seen that with the rotational damping in the joint, all eigenvalues move to the left more and more as b increases. But the high frequency ones only move to the left slightly. Fig. 5.2 gives a zoom-in view of these high frequency eigenvalues. Therefore, the rotational damping in the joint is more effective to vibration of lower frequency modes than the high

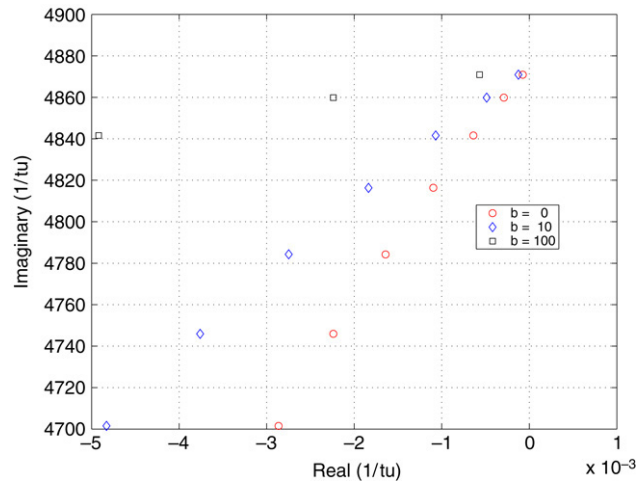


Fig. 5.2. Zoom-in view of comparison of high frequency eigenvalues.

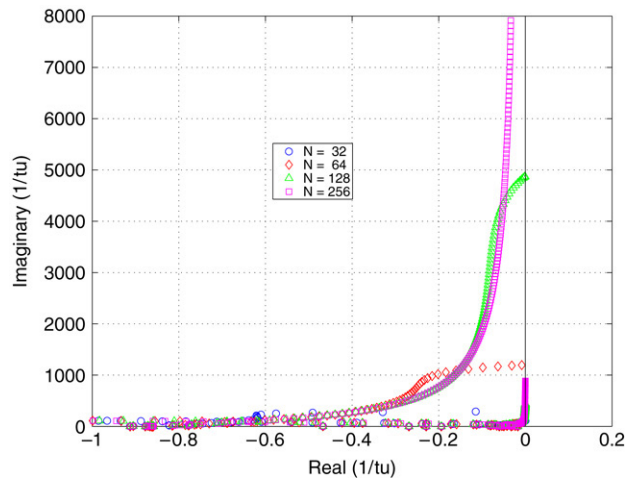


Fig. 5.3. The trend of eigenvalues for $b = 100$.

ones. Fig. 5.3 plots the eigenvalues for the case $b = 100$ with several values of the dimension of approximating system. We observe a trend that there is a branch whose imaginary part goes to infinity as the real part approaches zero. Therefore, system (2.16) cannot be exponentially stable when there is no damping in beam-2.

Acknowledgements

This work was supported by DARPA/SPO, NASA LaRC and the National Institute of Aerospace under grant VT-03-1, 2535.

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