*Publ. Math. Debrecen*
*Manuscript (September, 2008)*

# Separation and contact of sets of different dimensions in a doubling environment

By Hugo Aimar and Liliana Nitti

**Abstract.** In this note we consider two problems related to the geometry of spaces of homogeneous type. First, we give necessary and sufficient geometric conditions on two components of different dimension of a metric measure space, in order to guarantee the doubling property on the whole space. Second, we construct spaces of homogeneous type by pasting measures of different dimensions.

## 1. Introduction and Statement of the Results

Let $(X, d)$ be a metric space, we shall say that a measure $\mu$ defined on the Borel subsets of $X$ is doubling if there exists a positive constant $A$ such that the inequalities

$$0 < \mu(B(x, 2r)) \leq A\mu(B(x, r)) < \infty,$$

---

2000 *Mathematics Subject Classification:* Primary 42B25.

*Key words and phrases:* Spaces of Homogeneous Type, Doubling Property.

The authors were supported by the Consejo Nacional de Investigaciones Científicas y Técnicas de la República Argentina and by the Universidad Nacional del Litoral.

hold for every $x \in X$ and every $r > 0$, where $B(x,r)$ denotes the $d$-ball centered at $x$ with radius $r$: $\{y : d(x,y) < r\}$. If $\mu$ is a given doubling measure on $(X,d)$, we say that $(X,d,\mu)$ is a space of homogeneous type. There is an extensive literature on analysis on these structures, and several examples and applications are given in [CW], where the more general case of quasi-metric spaces is also considered.

We shall also deal with measures that satisfy the property of uniform boundedness of the measure of the unit balls;
$(P)$ : there exist positive and finite constants $a_1$ and $a_2$ such that the inequalities

$$a_1 \leq \mu(B(x,1)) \leq a_2$$

hold for every $x \in X$. Not every space of homogeneous type satisfies property $(P)$. In fact in $(\mathbb{R}, |x-y|, |x|^{\frac{1}{3}}dx)$ we do not have a upper bound for $\mu(B(x,1))$. In $(\mathbb{R}, |x-y|, |x|^{-\frac{1}{3}}dx)$ we do not have a lower bound and both of them are spaces of homogeneous type since $|x|^{\frac{1}{3}}$ is an $A_2(\mathbb{R})$-Muckenhoupt weight. The restriction of the same weights to any compact interval of $\mathbb{R}$ give spaces satisfying property $(P)$. This is always the case if $\mu(B(x,1))$ is continuous in $x \in X$ and $X$ is compact, which is satisfied for some classical fractal sets with the natural Hausdorff measure.

Several examples of spaces of homogeneous type satisfy a condition finer than doubling that in particular implies property $(P)$. We shall say that $(X,d,\mu)$ is a **$\delta$-normal space**, for some positive number $\delta$, if there exist two constants $A_1$ and $A_2$ such that the inequalities

$$A_1 r^\delta \leq \mu(B(x,r)) \leq A_2 r^\delta$$

hold for every $x \in X$ and every $0 < r < diam(X)$. From the point of view of Hausdorff dimension, defined in terms of the distance $d$, the basic difference between doubling and $\delta$-normality is that for the former we only have an upper bound for the dimension of balls, for the later instead, we have that each ball has exactly Hausdorff dimension equal to $\delta$.

A basic result about the integration of powers of the metric on $\delta$-normal spaces, shall be usefull and is contained in our first lemma which shall be proved in Section 3.

**Lemma 1.1.** *Let $(X, d, \mu)$ be a $\delta$-normal space and let $\alpha > -\delta$. Then there exist constants $M_1$ and $M_2$, such that the inequalities*

$$M_1 r^{\delta+\alpha} \leq \int_{B(x,r)} d(x,y)^\alpha \, d\mu(y) \leq M_2 r^{\delta+\alpha}$$

*hold for every $x \in X$ and every $0 < r < diam(X)$.*

In order to introduce the basic notation and terminology and to illustrate the type of problems considered, let us start by the simple case of linear spaces. Precisely, the doubling property for the measure obtained as the sum of Lebesgue measures supported on two linear subspaces of $\mathbb{R}^n$ with different dimensions implies two basic geometric properties : separation and parallelism of the supports of the measures. In fact, for the sake of simplicity, let us think that $n = 3$ and take $H$ a given fixed two dimensional hyperplane in $\mathbb{R}^3$. Let $L$ be the straightline given by $p+tv$, for $t \in \mathbb{R}$, with $p$ and $v$ two fixed vectors $\in \mathbb{R}^3$. Set $\mu = \sigma_H + \lambda_L$, where for each Borel subset $B$ of $\mathbb{R}^3$, $\sigma_H(B)$ is the surface area of $B \cap H$ and $L$ is the length of $B \cap H$ . Then $\mu$ is a doubling measure in $H \cup L$ or in an equivalent way, $H \cup L$, with the Euclidean distance and the measure $\mu$ is a space of homogeneous type, if and only if L is parallel to H and $L$ is not contained $H (p \notin H)$. In order to prove that $\mu$ is a doubling measure provided that $L$ is a straight line parallel to $H$ with $L \nsubseteq H$, we only have to observe that the behavior of the function of $r$,

$\mu(B(x,r))$ is linear when $x \in L$ and $r$ is small, and quadratic in all the other cases, uniformly in $x$. On the other hand, if $L \cap H = \{q\}$, taking $x_n = q + nv$, $n \in \mathbb{N}$ and $r_n = dist(H, x_n)$ we obtain a quadratic growth with respect to $n$ for $\mu(2B_n)$, while $\mu(B_n)$ is linear with respect to $n$. Hence the doubling is impossible. The case $L \subset H$ is even easier. With the above example in mind we shall say that two subsets $X_1$ and $X_2$ in a metric space $(X,d)$ are **separated** if $d(X_1, X_2) > 0$. On the other hand we shall say that $X_1$ is **controlled** by $X_2$ if the function $d(x, X_2)$ is bounded above when $x$ ranges $X_1$. The main result concerning separation and control in a doubling environment is given in the next theorem.

**Theorem 1.1** (Separation and Control)**.** *Let $(X,d)$ be a metric space, with $X = X_1 \cup X_2$, $X_1$ and $X_2$ disjoint, nonempty and $X_1$ closed. For $i = 1,2$, let $\mu_i$ be a doubling Borel measure defined on $X_i$ satisfying property $(P)$. Assume that the function $G(x,y,r) = \frac{\mu_2(B(y,r)\cap X_2)}{\mu_1(B(x,r)\cap X_1)}$ tends to zero as $r \to 0$ uniformly in $x \in X_1$ and $y \in X_2$ and tends to $\infty$ for $r \to \infty$ uniformly in $x \in X_1$ and $y \in X_2$. Then $X_1$and $X_2$ are separated and $X_1$ is controlled by $X_2$, if and only if $(X,d,\mu)$ is a space of homogeneous type with $\mu(E) = \mu_1(E \cap X_1) + \mu_2(E \cap X_2)$ for every Borel subset $E$ of $X$.*

The simplest case for which the measures $\mu_i$, $i = 1,2$, satisfy property $(P)$ and the required conditions on the function $G$ is given by measures in an unbounded $\delta_i-$normal space. In fact, if $(X,d,\mu_i)$ is a $\delta_i-$normal space, $i = 1,2$, with $0 < \delta_1 < \delta_2 < \infty$, we have that $G(x,y,r)$ behaves as $r^{\delta_2 - \delta_1}$ uniformly in $x \in X_1$ and $y \in X_2$. Hence $G(x,y,r)$ tends uniformly to zero as $r \to 0$ and uniformly to infinity for $r \to \infty$. Aside from the linear situation described before stating Theorem 1.1, examples of this situation arise also naturally in non linear or even in fractal settings. In fact the results in [M], show that the fractal sets produced by the Hutchinson iteration scheme (see [H]), under the open set condition, are spaces of homogeneous type with the right Hausdorff measure which are $\delta$-normal for some positive $\delta$. That is the case of middle thirds Cantor sets and Sierpinsky gaskets. Periodic extensions of such sets imbedded in $\mathbb{R}^2$ equiped with the usual distance, provide a whole family of examples of such spaces with two components of different non-integer dimension.

The second problem considered in this note is the following. Under the assumption of contact of sets $X_1$ and $X_2$ of different dimensions, how to modify, introducing weights, the normal measures in order to get a doubling measure for the whole space $X_1 \cup X_2$? Let us start by showing an elementary example for the case of two linear manifolds in the n-dimensional Euclidean space.

Let $X_i, i = 1, 2$, be two linear manifolds in $\mathbb{R}^n$ with dimensions $n_i$ and $0 < n_1 < n_2 < n$. Assume that $X_1$ and $X_2$ intersect at the only point $p \in \mathbb{R}^n$. Let $\mu_i$ be the $n_i$-dimensional Lebesgue measure supported on $X_i$.

From Theorem 1.1 we know that the measure

$$\mu^0(E) = \mu_1(E \cap X_1) + \mu_2(E \cap X_2)$$

can not be doubling. If instead of $\mu^0$ we take a measure of the form

$$\mu^\alpha(E) = \int_{E\cap X_1} |x - p|^\alpha \, d\mu_1 + \mu_2(E \cap X_2), \tag{1.1}$$

some elementary situations suggest that $\mu^\alpha$ could be doubling for some specific value of $\alpha$.

In fact assume that $n = 3$ and that $X_1$ is a straight line, $X_2$ is a hyperplane and that $p$ is the only point in the intersection of $X_1$ and $X_2$. It is easy to show that $\mu^1$ is doubling in this particular situation, moreover the only $\alpha$ for with $\mu^\alpha$ is doubling in $X_1 \cup X_2$ is $\alpha = 1$. Actually as a corollary of our theorem we shall prove that (1.1) is doubling as a measure defined on the space $X_1 \cup X_2$ with the restriction of the usual distance , if and only if $\alpha = n_2 - n_1$.

Before stating our result, let us start by introducing an abstract metric notion of local contact for two given sets.

Let $(X, d)$ be a metric space. Assume that $X = X_1 \cup X_2$ with $X_i$, $i = 1, 2$, non-empty, disjoint open subsets of $X$ such that $d(X_1, X_2) = 0$. Since $\inf\{d(x_1, x_2) : x_1 \in X_1, x_2 \in X_2\} = 0$, we can take two sequences $\mathbf{x}^1 = \{x_k^1 : k \in \mathbb{N}\} \subset X_1$ and $\mathbf{x}^2 = \{x_k^2 : k \in \mathbb{N}\} \subset X_2$ such that $d(x_k^1, x_k^2) \to 0$ as $k \to \infty$. In this situation we shall say that the pair of sequences $(\mathbf{x}^1, \mathbf{x}^2)$ is **admissible**. The system $(X_1, X_2, d)$ is said to satisfy property $\mathcal{C}$, (contact at only one "point"), if every admissible pair of sequences $(\mathbf{x}^1, \mathbf{x}^2)$ is also a pair of Cauchy sequences in $X_1$ and $X_2$ respectively.

Notice that if $(X_1, X_2, d)$ satisfies **property** $\mathcal{C}$ and $(\mathbf{x}^1, \mathbf{x}^2)$ and $(\mathbf{y}^1, \mathbf{y}^2)$ are two admissible pairs, we have that both $d(x_k^1, y_k^1)$ and $d(x_k^2, y_k^2)$ tend to zero as $k \to \infty$. In fact by taking two new sequences $\mathbf{z}^1$ and $\mathbf{z}^2$ by alternating the terms of the sequences $\mathbf{x}^1$ and $\mathbf{y}^1$, and the sequences $\mathbf{x}^2$ and $\mathbf{y}^2$ respectively, we get that $(\mathbf{z}^1, \mathbf{z}^2)$ is again an admissible pair. From property $\mathcal{C}$, $\mathbf{z}^1$ and $\mathbf{z}^2$ must be two Cauchy sequences. In other words for every $\epsilon > 0$, there exists $K(\epsilon)$ such that $d(z_k^i, z_m^i) < \epsilon$, for every $k$, $m \geq K(\epsilon)$ and $i =$ 1,2. From the construction of $z^i$, we have that for $k$ large enough $d(x_k^i, y_k^i) < \epsilon$.

Since we are not assuming completness of the space this definition is a substitute of the heuristic idea of contact at only one point of the two components $X_1$ and $X_2$.

The next result which follows from property $\mathcal{C}$ allows us to define a "distance to the contact".

**Lemma 1.2.** *Assume that the system* $(X_1, X_2, d)$ *satisfies property* $\mathcal{C}$. *Let* $(\mathbf{x}^1, \mathbf{x}^2)$ *be an admissible pair of sequences. Then for each* $x$ *in* $X$ *the sequences of real numbers* $\{d(x_k^1, x) : k \in \mathbb{N}\}$ *and* $\{d(x_k^2, x) : k \in \mathbb{N}\}$ *have the same finite positive limit. Moreover this limit is independent of the particular admissible pair of sequences chosen.*

The above lemma, allows us to define the function $d : X \to \mathbb{R}^+$, by $d(x) = lim_{k\to\infty} d(x, x_k^1) = lim_{k\to\infty} d(x, x_k^2)$ for an admissible pair $(\mathbf{x}^1, \mathbf{x}^2)$. Set $d_i$ denote $d_i(x) = d(x)$ for $x \in X_i$ and $d_i(x) = 0$ for $x \notin X_i$, in other words $d_i = \chi_{X_i} d$ where $\chi_E$ denotes the indicator function of the set $E$.

Notice that $d_1(x) \geq d(x, X_2)$ and $d_2(x) \geq d(x, X_1)$. Generally reversed inequalities like $d_1(x) \leq C_2 d(x, X_1)$ for every $x \in X$ do not hold. When these type of reversed inequalities hold we say that $X_1$ and $X_2$ have contact of order zero. Let us formalize the generalized idea of contact of order zero for the two components of $X$. We say that the system $(X_1, X_2, d)$ with property $\mathcal{C}$ satisfies $\mathcal{C}_o$ (**contact of order zero**) if if there exists a constant $C$ such that the inequality

$$d_1(x) \leq C d(x, X_2),$$

holds for every $x \in X$. Notice that since $d_1(x) \geq d(x, X_2)$, we have that $C \geq 1$. The next lemmas show that the roles of $X_1$ and $X_2$ in $\mathcal{C}_0$ can be interchanged.

**Lemma 1.3.** *If the system* $(X_1, X_2, d)$ *satisfy* $\mathcal{C}$, *then property* $\mathcal{C}_0$ *is equivalent to the existence of a constant* $c > 0$ *such that for every* $x \in X_1$, $B(x, c d_1(x)) \cap X_2 = \emptyset$.

**Lemma 1.4.** *Assume that the system* $(X_1, X_2, d)$ *satisfies* $\mathcal{C}$. *Then* $(X_1, X_2, d)$ *satisfies* $\mathcal{C}_0$ *if and only if* $(X_2, X_1, d)$ *satisfies* $\mathcal{C}_0$.

The next statement contains our main result concerning doubling and contact. For $i = 1, 2$ let $(X_i, d, \mu_i)$ be a $n_i$- normal space with constants $A_1^i$ and $A_2^i$ and $0 < n_1 < n_2 < \infty$, set $A_1 = \min\{A_1^1, A_1^2\}$ and $A_2 = \max\{A_2^1, A_2^2\}$. For $\alpha_1 > -n_1$ and $\alpha_2 > -n_2$, let us define

$$\mu^{\alpha_1,\alpha_2}(E) = \int_{E\cap X_1} d_1(x)^{\alpha_1}\, d\mu_1(x) + \int_{E\cap X_2} d_2(x)^{\alpha_2}\, d\mu_2(x) \tag{1.2}$$

for every Borel subset $E$ of $X = X_1 \cup X_2$.

Notice that if $(X_1, X_2, d)$ satisfies $\mathcal{C}_0$, then

$$\mu^{\alpha_1,\alpha_2}(E) \simeq \int_{E\cap X_1} d(x, X_2)^{\alpha_1}\, d\mu_1(x) + \int_{E\cap X_2} d(x, X_1)^{\alpha_2}\, d\mu_2(x).$$

**Theorem 1.2** (Doubling and contact). *Assume that $(X_1, X_2, d)$ satisfies $\mathcal{C}_0$. For $i = 1, 2$ let $(X_i, d, \mu_i)$ be a $n_i$- normal space with $0 < n_1 < n_2 < \infty$, $\alpha_1 > -n_1$, $\alpha_2 > -n_2$, let $\mu^{\alpha_1,\alpha_2}$ be the measure defined by (1.2). Then $(X_1 \cup X_2, d, \mu^{\alpha_1,\alpha_2})$ is a space of homogenous type if and only if $\alpha_1 - \alpha_2 = n_2 - n_1$.*

Let us point out that from the structure results contained in [MS], all the above theorems can immediately be extended to quasi-metric spaces.
In Section 2 we give the proof of Theorem 1.1. In Section 3 we prove Theorem 1.2 and Lemmas 1.1 1.2, 1.3 and 1.4 among same other technical results.

## 2. Proof of Theorem 1.1

We shall first prove in two independent results that in a mixed dimension environment, doubling implies separation and that doubling implies the control of the set of larger dimension over the set of smaller dimension. We point out that none of them needs property $(P)$ which shall only be used in the proof of the converse.

Let us start by proving that doubling implies separation.

**Theorem 2.1** (Separation)**.** *Let $(X, d)$ be a metric space. Assume that $X = X_1 \cup X_2$ with $X_1$ and $X_2$ nonempty, $X_1$ closed and $X_1 \cap X_2 = \emptyset$. Let us assume also that for $i = 1, 2$, $\mu_i$ is a Borel measure defined on $X_i$ such that the function $G(x, y, r) = \frac{\mu_2(B(y,r)\cap X_2)}{\mu_1(B(x,r)\cap X_1)}$ tends to zero as $r \to 0$ uniformly in $x \in X_1$ and $y \in X_2$. Let $\mu(E) = \mu_1(E \cap X_1) + \mu_2(E \cap X_2)$ for every Borel set $E$ of $X$. Then if $(X, d, \mu)$ is a space of homogeneous type, we have that $X_1$and $X_2$ are separated.*

Proof. Assume that $d(X_1, X_2) = 0$. Pick a sequence $\{y_n\} \subset X_2$ such that each $r_n = d(y_n, X_1) > 0$ for all $n$ and $r_n \to 0$ as $n \to \infty$. Now, for each $n \in N$ we have a point $x_n \in X_1 \cap B(y_n, 2r_n)$. Hence $B(x_n, r_n) \subset B(y_n, 4r_n)$. So that

$$\mu(B(y_n, 4r_n)) \geq \mu_1(B(y_n, 4r_n)) \cap X_1) \geq \mu_1(B(x_n, r_n) \cap X_1)$$

for every $n \in \mathbb{N}$. On the other hand $\mu(B(y_n, r_n)) = \mu_2(B(y_n, r_n))$, from the above inequalities we have

$$\frac{\mu(B(y_n, 4r_n))}{\mu(B(y_n, r_n))} \geq \frac{\mu_1(B(x_n, r_n) \cap X_1)}{\mu_2(B(y_n, r_n))} = (G(x_n, y_n, r_n))^{-1}$$

Since the right hand side in the last inequality tends to infinity as $n \to \infty$, the doubling property is impossible for $\mu$. $\square$

**Theorem 2.2** (Control)**.** *Let $(X, d)$ be a metric space. Assume that $X = X_1 \cup X_2$ with $X_1$ and $X_2$ nonempty, $X_1$ closed and $X_1 \cap X_2 = \emptyset$. Let us assume also that for $i = 1, 2$ $\mu_i$ is a Borel measure defined on $X_i$ such the function $G(x, y, r)$ tends to $\infty$ as $r \to \infty$ uniformly in $x \in X_1$ and $y \in X_2$ . Let $\mu(E) = \mu_1(E \cap X_1) + \mu_2(E \cap X_2)$ for every Borel set $E$ of $X$. Then if $(X, d, \mu)$ is a space of homogeneous type, we have that $X_1$ is controlled by $X_2$.*

Proof. Assume that the function $d(x, X_2)$ for $x \in X_1$ is unbounded. Let us pick a sequence $\{x_n\}$ of points in $X_1$ with $d(x_n, X_2) \to \infty$ as $n \to \infty$. Taking $r_n = d(x_n, X_2)$ we have that $B(x_n, r_n) \subset X_1$ and $B(x_n, 4r_n) \supset B(y_n, r_n)$ with $y_n \in X_2 \cap B(x_n, 2r_n)$, hence

$$\mu(B(x_n, 4r_n) \geq \mu_2(B(y_n, r_n) \cap X_2)$$

On the other hand

$$\mu(B(x_n, r_n)) = \mu_1(B(x_n, r_n)).$$

So that, since $\mu$ is doubling on $(X, d)$ we have

$$\begin{aligned} G(x_n, y_n, r_n) = \frac{\mu_2(B(y_n, r_n) \cap X_2)}{\mu_1(B(x_n, r_n) \cap X_1)} \quad &\leq \quad \frac{\mu_2(B(x_n, 4r_n) \cap X_2)}{\mu_1(B(x_n, r_n))} \\ \leq \frac{\mu(B(x_n, 4r_n))}{\mu(B(x_n, r_n))}, \end{aligned}$$

which is bounded even when $r_n \to \infty$. □

As a corollary of the above two theorems we get the first half of Theorem 1.1

**Corollary 2.1.** *Let $(X, d)$ a metric space. Assume that $X = X_1 \cup X_2$ with $X_1$ and $X_2$ disjoint, nonempty and $X_1$ closed. Let us assume also that for $i = 1, 2$, $\mu_i$ is a Borel measure defined on $X_i$ such that the function $G(x, y, r)$ tends to zero as $r \to 0$ and tends to $\infty$ for $r \to \infty$ uniformly in $x \in X_1$ and $y \in X_2$. Let $\mu(E) = \mu_1(E \cap X_1) + \mu_2(E \cap X_2)$ for every $E$ of $X$. If $(X, d, \mu)$ is a space of homogeneous type, then $X_1$ and $X_2$ are separated and $X_1$ is controlled by $X_2$.*

The next theorem shows us that under the assumption of property $(P)$, the converse of Corollary 2.1 holds.

**Theorem 2.3.** *Let $(X,d)$ a metric space. Assume that $X = X_1 \cup X_2$ with $X_1$ and $X_2$ disjoint, nonempty and $X_1$ closed. Let us assume that for $i = 1,2$ $\mu_i$ is a doubling Borel measure defined on $X_i$ and that each $\mu_i$ verifies property $(P)$. If the function $G(x,y,r)$ tends to zero as $r \to 0$ uniformly in $x \in X_1$ and $y \in X_2$ and tends to $\infty$ for $r \to \infty$ uniformly in $x \in X_1$ and $y \in X_2$, if $X_1$ and $X_2$ are separated and if $X_1$ is controlled by $X_2$, then $(X,d,\mu)$ is a space of homogeneous type with $\mu(E) = \mu_1(E \cap X_1) + \mu_2(E \cap X_2)$*

Proof. To prove that $\mu$ is doubling, we shall study the family of functions of $r > 0$, $\psi_x(r) = \frac{\mu(B(x,2r))}{\mu(B(x,r))}$ for $x \in X$. We will see that this family is uniformly bounded above. Suppose first that $x \in X_1$. Let $c_1$ and $c_2$ be positive constants such that $c_1 \leq d(x, X_2) \leq c_2$, for every $x \in X_1$. Notice that if $0 < r < \frac{c_1}{2}$, $B(x,2r) \cap X_2 = \emptyset$, for $x \in X_1$ then $\mu(B(x,2r))$ and $\mu(B(x,r))$ are defined only by the term corresponding to $\mu_1$. Thus, $\psi_x(r) = \frac{\mu_1(B(x,2r))}{\mu_1(B(x,r))}$ is bounded uniformly because $\mu_1$ is doubling. Now we suppose that $r > 2c_2$. Therefore there exists $y \in X_2$ such that $d(x,y) < \frac{r}{2}$. Then $B(y, \frac{r}{2}) \subset B(x,r)$, hence we obtain that

$$\psi_x(r) = \frac{\mu(B(x,2r))}{\mu(B(x,r))} = \frac{\mu_1(B(x,2r) \cap X_1) + \mu_2(B(x,2r) \cap X_2)}{\mu_1(B(x,r) \cap X_1) + \mu_2(B(x,r) \cap X_2)}$$

then

$$\psi_x(r) \leq A_1 + \frac{\mu_2(B(x,2r) \cap X_2)}{\mu_2(B(y, \frac{r}{2}) \cap X_2)}$$

where $A_1$ is the doubling constant for $\mu_1$. Notice that $B(x,2r) \subset B(y,4r)$, and from the above inequalities we have that

$$\psi_x(r) \quad \leq \quad A_1 + \frac{\mu_2(B(y,4r))}{\mu_2(B(y, \frac{r}{2}))} \tag{2.1}$$

Since $\mu_2$ is also doubling we get again that $\psi_x$ is bounded above. Suppose now that $\frac{c_1}{2} \leq r \leq 2c_2$

$$\begin{aligned}
\psi_x(r) &\leq \frac{\mu_1(B(x,2r)\cap X_1)+\mu_2(B(x,2r)\cap X_2)}{\mu_1(B(x,r)\cap X_1)} \\
&\leq A_1 + \frac{\mu_2(B(x,2r)\cap X_2)}{\mu_1(B(x,r)\cap X_1)} \\
&\leq A_1 + \frac{\mu_2(B(x,4c_2)\cap X_2)}{\mu_1(B(x,\frac{c_1}{2})\cap X_1)}.
\end{aligned}$$

Observe that since $d(x, X_2) < c_2$, there exists $y \in X_2$ such that $d(y, x) < 2c_2$, therefore the ball $B(x, 4c_2) \subset B(y, 8c_2)$. Applying this to the last inequality, we obtain that

$$\psi_x(r) \leq A_1 + \frac{\mu_2(B(y,8c_2)\cap X_2)}{\mu_1(B(x,\frac{c_1}{2})\cap X_1)}.$$

Since each $\mu_i$ satisfies property $(P)$ and is doubling we get a uniform bound for $\psi_x(r)$.

Let us now assume that $x \in X_2$. Notice that for $0 < 2r < d(x, X_1)$ the uniform boundedness of $\psi_x(r)$ follows from the doubling property of $\mu_2$, since $B(x, 2r) \cap X_1 = \emptyset$. Assume now that $d(x, X_1) < 2r$, hence

$$\psi_x(r) \leq \frac{\mu_1(B(x,2r)\cap X_1)}{\mu_2(B(x,r)\cap X_2)} + \frac{\mu_2(B(x,2r)\cap X_2)}{\mu_2(B(x,r)\cap X_2)}.$$

The second term on the right hand side of this inequality is bounded because $\mu_2$ is doubling. On the other hand, since $B(x, 2r) \cap X_1 \neq \emptyset$, then there exists $y \in B(x, 2r) \cap X_1$, such that the ball $B(x, 2r) \subset B(y, 4r)$, then

$$\psi_x(r) \leq A_2 + \frac{\mu_1(B(y,4r)\cap X_1)}{\mu_2(B(x,r)\cap X_2)}.$$

Since $G \to \infty$ for $r \to \infty$ uniformly in $x$ and $y$, there exists $M \in \mathbb{N}$ so that if $r > M$,we have

$$\frac{\mu_1(B(y,4r) \cap X_1)}{\mu_2(B(x,r) \cap X_2)} \leq 1.$$

On the other hand, if $r \leq M$, since $X_1$ and $X_2$ are separated, $2r > d(x, X_1) > c_1$, we have that $\frac{c_1}{2} \leq r < M$ and we can iterate property $(P)$ to obtain the desired boundedness of $\psi_x(r)$. □

## 3. Proof of Theorem 1.2

We start with the proof of Lemmas 1.1 to 1.4.

**Proof of Lemma 1.1**

We observe that since the space has no atoms, given the ball $B(x,r)$, with $x \in X$ and $r < diam(X)$, then

$$\begin{aligned}\mu^\alpha(B(x,r)) :&= \int_{B(x,r)} d(x,y)^\alpha \, d\mu(y)\\ &= \sum_{j=0}^{\infty} \int_{rM^{-j-1} \leq d(x,y) < rM^{-j}} d(x,y)^\alpha d\mu(y),\end{aligned}$$

where $M$ is a constant greater than one that we will fix later. We shall obtain upper and lower estimates for the general term of the last series. Notice that in the integral defining the $j-th$ term of the series we have that $d(x,y) \simeq rM^{-j}$, then

$$\begin{aligned}&\int_{rM^{-j-1} \leq d(x,y) < rM^{-j}} d(x,y)^\alpha \, d\mu(y)\\ &\qquad \simeq r^\alpha M^{-\alpha j}[\mu(B(y, rM^{-j})) - \mu(B(x, rM^{-j-1}))].\end{aligned}$$

Since $(X, d, \mu)$ is a $\delta$-normal space and $r < diam(X)$ we have that

$$a_1(rM^{-i})^\delta \leq \mu(B(x, rM^{-i})) \leq a_2(rM^{-i})^\delta$$

for every $i = 0, 1, 2, ...$By choosing $M = (\frac{2a_2}{a_1})^{\frac{1}{\delta}}$ and applying the above inequalities with $i = j$ and $i = j+1$, we have

$$\begin{aligned}\frac{a_1}{2}(rM^{-j})^\delta \leq (a_1 - \frac{a_2}{M^\delta})(rM^{-j})^\delta &\leq \mu(B(x, rM^{-j})) - \mu(B(x, rM^{-j-1})) \\ \leq (a_2 - \frac{a_1}{M^\delta})(rM^{-j})^\delta &\leq a_2(rM^{-j})^\delta.\end{aligned}$$

Hence

$$\mu^\alpha(Bx, r) \simeq r^{\alpha+\delta} \sum_{j=0}^{\infty} M^{-(\alpha+\delta)j} \simeq r^{\alpha+\delta}$$

for $\alpha > -\delta$. $\square$

**Proof of Lemma 1.2**

Assume that $x \in X_1$. Notice that for each $x \in X_1$ and for each sequence $\mathbf{x}^1 = \{x_k^1 : k \in \mathbb{N}\} \subset X_1$ , which is the first component of an admissible pair, since $|d(x_k^1, x) - d(x_m^1, x)| \leq d(x_k^1, x_m^1)$, we have that the sequence of real numbers $\{d(x_k^1, x) : k \in \mathbb{N}\}$ converges to a real number $d(x)$. On the other hand if $\mathbf{x}^2 = \{x_k^2 : k \in \mathbb{N}\}$, is the second component of an admissible pair we have that $|d(x_k^2, x) - d(x)| \leq |d(x_k^2, x) - d(x_k^1, x)| + |d(x_k^1, x) - d(x)|$, hence we also have that $d(x) = \lim_{n\to\infty} d(x_k^2, x)$. From the remark following the definition of property $\mathcal{C}$, we have that $d(x)$ is independent of the particular admissible pair $(\mathbf{x}^1, \mathbf{x}^2)$. Notice also that since $X_1$ is open, $d(x) > 0$ on $X_1$. For $x \in X_2$ the argument is the same. $\square$

**Proof of Lemma 1.3**

Let us first prove that $\mathcal{C}_0$ for the system $(X_1, X_2, d)$ implies that there exists a constant $c > 0$ such that for every $x \in X_1$ the ball $B(x, cd(x))$, has no point in $X_2$. We know from property $\mathcal{C}_0$, that for some constant $C > 0$ and every $x \in X_1$, $d(x) \leq Cd(x, X_2)$. Take $c = C^{-1}$. If $B(x, cd(x)) \cap X_2 \neq \emptyset$, for some $x \in X_1$, then there would exist $y \in X_2$ such that $d(y, x) < cd(x)$ for that particular $x \in X_1$. Hence $d(x, y) < cd(x) \leq cCd(x, X_2) = d(x, X_2)$, which is impossible since $y \in X_2$.

To show the sufficiency of that condition, take $C = c^{-1}$ and notice that for every $x \in X_1$ the ball $B(x, cd(x))$ is contained in $X_1$. Which in turn implies that $d(x) \leq Cd(x, X_2)$ as desired. □

**Proof of Lemma 1.4**

We assume that $(X_1, X_2, d)$ satisfies $\mathcal{C}_0$. From Lemma 1.3, we only need to prove the existence of a constant $c' > 0$ such that for every $y \in X_2$ we have that $B(y, c'd(y)) \cap X_1 = \emptyset$. Take $0 < c' < 1$ for which $\frac{c'}{1-c'} < c$, where $c$ is the constant for which $B(x, cd(x)) \cap X_2 = \emptyset$ for every $x \in X_1$. Assume that for some $y \in X_2$, $B(y, c'd(y)) \cap X_1 \neq \emptyset$. Hence, there exists $x \in X_1$ such that $x \in B(y, c'd(y))$. Notice that

$$d(y) \leq \frac{1}{1-c'} d(x)$$

In fact if $(\mathbf{x}^1, \mathbf{x}^2)$ is an admissible pair of sequences, we have $d(y, x_n^2) \leq d(y, x) + d(x, x_n^2) \leq c'd(y) + d(x, x_n^1) + d(x_n^1, x_n^2)$. Let $n \to \infty$, to obtain the desired inequality. Hence $d(x, y) < \frac{c'}{1-c'} d(x) < cd(x)$ or $B(x, cd(x)) \cap X_2 \neq \emptyset$, which is a contradiction. □

Recall that in the introduction we defined the functions $d_i = d\chi_{X_i}$, $i = 1, 2$. Let us show that the functions $d_i^\alpha$ with $i = 1, 2$, and $\alpha > -n_i$, are locally integrable, and moreover that the measure $\mu_i^\alpha(E) = \int_{X_i \cap E} d_i(x)^\alpha d\mu_i(x)$ is doubling on $(X_i, d)$. These results are contained in the next theorem.

**Theorem 3.1.** *Let $(X_1, X_2, d)$ be as in Theorem 1.2. Then for every $\alpha >$*

*$-n_i$ for $i = 1, 2$, the measure $\mu_i^\alpha$ on the Borel subsets of $X_i$ satisfies the following estimate for $y \in X_i$ and $0 < \gamma < (\frac{A_1}{A_2})^{\frac{1}{n_i}}$*

*(1) $\mu_i^\alpha(B(y,r)) \geq a_1 r^{\alpha+n_i}$, if $d_i(y) \leq \frac{\gamma r}{5}$ and $0 < r < diam(X_i)$,*

*(2) $\mu_i^\alpha(B(y,r)) \geq a_1 d_i(y)^\alpha r^{n_i}$, if $d_i(y) > \frac{\gamma r}{5}$ and $0 < r < diam(X_i)$,*

*(3) $\mu_i^\alpha(B(y,r)) \leq a_2 r^{\alpha+n_i}$, if $d_i(y) \leq 5r$,*

*(4) $\mu_i^\alpha(B(y,r)) \leq a_2 d_i^\alpha(y) r^{n_i}$, if $d_i(y) > 5r$.*

*The constants $a_1$ and $a_2$ depend only on the geometric constants. As a consequence, each $(X_i, d, \mu_i^\alpha)$ is a space of homogeneous type.*

Proof. To prove the theorem, let us work out the case $i = 1$. Since $(X_1, d, \mu_1)$ is a $n_1$ -normal space, with constants $A_1$ and $A_2$, the inequalities $A_1 r^{n_1} \leq \mu_1(B(x,r)) \leq A_2 r^{n_1}$ hold for every $x \in X_1$ and $0 < r < diam(X_1)$. We shall obtain upper and lower estimates for

$$\int_{B(y,r)} d_1(x)^\alpha d\mu_1(x) \tag{3.1}$$

in terms of $r > 0$ and $y \in X_1$. Let us start with the lower estimates in the case in which $d_1(y) \leq \frac{\gamma}{5} r$, where $0 < \gamma < (\frac{A_1}{A_2})^{\frac{1}{n_1}}$. Set $C(y,r) = B(y,r) \setminus B(y, \gamma r)$; $r > 0$, $y \in X_1$. Since $d_1(y) \leq \frac{\gamma}{5} r$, then, for $n$ large enough we have that $d(y, x_n^1) \leq \frac{\gamma}{4} r$, where $(\mathbf{x}^1, \mathbf{x}^2)$ is an admissible pair. Hence for those values of $n$ and $x \in C(y,r)$, we must have the inequalities $2r \geq d(x, x_n^1) \geq \frac{3\gamma}{4} r$. So that $2r \geq d_1(x) \geq \frac{3\gamma}{4} r$. Hence $\int_{B(y,r)} d_1^\alpha d\mu_1 \geq \int_{C(y,r)} d_1^\alpha d\mu_1 \geq c r^\alpha \mu_1(C(y,r)) \geq c r^\alpha [A_1 r^{n_1} - A_2 \gamma^{n_1} r^{n_1}] = c' r^{\alpha+n_1}$, which follows from our choice of $\gamma$.

Let us now obtain a lower bound for 3.1 when $d_1(y) > \frac{\gamma}{5} r$. Notice that, in this case, $d_1(x) \backsim d_1(y)$ for every $x \in B(y, \frac{\gamma r}{10})$ where the equivalence depends only on the geometric constants of the space. So that $\int_{B(y,r)} d_1^\alpha d\mu_1 \geq \int_{B(y, \frac{\gamma r}{10})} d_1^\alpha d\mu_1 \geq$

$cd_1(y)^\alpha r^{n_1}$. Hence our lower estimates for 3.1 with $\alpha > -n_1$, $y \in X_1$ and $0 < r < diam(X_1)$ are given by a constant times

(1) $r^{\alpha+n_1}$ when $d_1(y) \leq \frac{\gamma r}{5}$,

(2) $d_1^\alpha r^{n_1}$ when $d_1(y) > \frac{\gamma r}{5}$.

Let us now prove (3). It is clear that for $n$ large enough, we have the inequality $d(y, x_n^1) \leq 6r$. Hence for these values of $n$ the ball $B(y,r) \subset B(x_n^1, 8r)$ so that $\chi_{B(y,r)}(x) d_1^\alpha(x) \leq \liminf_{n\to\infty}(\chi_{B(x_n^1,8r)}(x) d^\alpha(x, x_n^1))$. Hence applying Fatou's Lemma and Lemma 1.1 we have

$$\int_{B(y,r)} d_1(x)^\alpha d\mu_1(x) \leq \liminf \int_{B(x_n^1,8r)} d(x,x_n^1)^\alpha d\mu_1(x) \simeq r^{\alpha+n_1}.$$

To prove (4), we notice that if $d_1(y) > 5r$, there exists $M \in \mathbb{N}$ such that for every $n \geq M$, $d(y, x_n^1) > 2r$, and for these values of $n$ and for every $x \in B(y,r)$ we have $d_1(x) \simeq d_1(y)$. Hence $\int_{B(y,r)} d_1(x)^\alpha d\mu_1(x) \simeq d_1(y)^\alpha r^{n_1}$. It only remains to prove that $\mu_1^\alpha$ is doubling. We shall consider the family of functions $\varphi_y(r) = \mu_1^\alpha(B(y,r))$, $y \in X_1$ and $r > 0$. Notice that:

(a) $0 < r \leq \frac{d_1(y)}{5}$, from (4) and (2), we obtain $a_1 d_1(y)^\alpha r^{n_1} \leq \varphi_y(r) \leq a_2 d_1^\alpha(y) r^{n_1}$.

(b) $\frac{d_1(y)}{5} < r < diam(X_1)$, is valid that $a_1 r^{n_1+\alpha} \leq \varphi_y(r) \leq a_2 r^{n_1+\alpha}$,

In fact, we note that from (3), we obtain that $\varphi_y(r) \leq a_2 r^{n_1+\alpha}$. To get a lower estimate, we consider the cases, $\frac{d_1(y)}{5} < r \leq 5\frac{d_1(y)}{\gamma}$ and $5\frac{d_1(y)}{\gamma} \leq r < diam(X_1)$. If $X_1$ is unbounded, for the first case, from (2) and the fact that $r \approx d_1(y)$ we obtain that $a_1 r^{n_1+\alpha} \leq \varphi_y(r)$ and for the second, from (1), we obtain the same estimate. If $X_1$ is bounded, the situation is as easy as before.

(c) $X_1$ is bounded and $r \geq diam(X_1)$, we have that $\varphi_y(r) = \mu^{\alpha_1}(X_1)$

Using (a),(b) and (c) the uniform boundedness of

$$\frac{\varphi_y(2r)}{\varphi_y(r)} \tag{3.2}$$

as function of $r$ for $y \in X_1$, follows immediately. $\square$

Before starting with the proof of Theorem 1.2, let us write as a theorem, with some detail, the behavior of $\mu^{\alpha_1,\alpha_2}$ on balls of $(X_1 \cup X_2, d)$ as a function of the center and the radius of the ball, both in the bounded and unbounded cases. In the next statement we shall use the following notation. Set $n(x)$ and $\alpha(x)$ to denote the simple functions given by

$$n(x) = \begin{cases} n_1; & \text{if } x \in X_1 \\ n_2; & \text{if } x \in X_2 \end{cases} \tag{3.3}$$

and

$$\alpha(x) = \begin{cases} \alpha_1; & \text{if } x \in X_1 \\ \alpha_2; & \text{if } x \in X_2. \end{cases} \tag{3.4}$$

**Theorem 3.2.** *Assume that $(X_1, X_2, d)$ satisfies $\mathcal{C}_0$. For $i = 1, 2$ let $\mu_i$ be a Borel measure on $(X_i, d)$ such that $(X_i, d, \mu_i)$ is a $n_i$-normal space, with $0 < n_1 < n_2 < \infty$. For $\alpha_1 > -n_1$, and $\alpha_2 > -n_2$, let $\mu^{\alpha_1,\alpha_2}$ be the measure define by (1.2). Assume that $\alpha_1 - \alpha_2 = n_2 - n_1$ and set $\beta = \alpha_1 + n_1 = \alpha_2 + n_2 = \alpha(x) + n(x)$. Then, there exist constants $0 < a < A < \infty$ and $1 > c > 0$ depending only on $\alpha_i$, such that given $x \in X = X_1 \cup X_2$ and $r > 0$ we have*

*3.2.1* $ad(x)^{\alpha(x)}r^{n(x)} \leq \mu^{\alpha_1,\alpha_2}(B(x,r)) \leq Ad(x)^{\alpha(x)}r^{n(x)}$

*for $x \in X$ and $r < cd(x)$;*

*3.2.2* $ar^{\beta} \leq \mu^{\alpha_1,\alpha_2}(B(x,r)) \leq Ar^{\beta}$

*for $cd(x) \leq r \leq S := diam(X_1) + diam(X_2)$;*

*3.2.3* $\mu^{\alpha_1,\alpha_2}(B(x,r)) = \mu^{\alpha_1,\alpha_2}(X)$

*for* $r > S$*.*

PROOF. Take $c = \frac{\gamma}{5C}$, with $\gamma$ chosen in such a way that the result of Theorem 3.1 holds for $i = 1$ and $i = 2$ and $C$ is the constant for property $\mathcal{C}_0$ for the system $(X_1, X_2, d)$, notice that $c < 1$. Let us start by proving 3.2.1. Take $x \in X_i$ and $0 < r < cd(x) = \frac{\gamma d(x)}{5C} < diam(X_i)$. From $\mathcal{C}_0$, we obtain also that $r < \frac{\gamma d(x,X_j)}{5}$, $j \neq i$. Hence $X_j \cap B(x,r) = \emptyset$ and $\mu^{\alpha_1,\alpha_2}(B(x,r)) = \mu^{\alpha_i}((B(x,r))$. So that from (a) in the proof of Theorem 3.1 we have the desired inequalities;

$$a_1 d^{\alpha(x)}(x) r^{n(x)} \leq \mu^{\alpha_1,\alpha_2}(B(x,r)) \leq a_2 d^{\alpha(x)}(x) r^{n(x)}.$$

Proof of 3.2.3. Notice that from property $\mathcal{C}$ of the system $(X_1, X_2, d)$ we have that $diam(X_1 \cup X_2) \leq diam(X_1) + diam(X_2)$. In fact given an admissible pair $(\mathbf{x}^1, \mathbf{x}^2)$ and two points, $x \in X_1$ and $y \in X_2$, we have that $d(x,y) \leq d(x, x_n^1) + d(x_n^1, x_n^2) + d(x_n^2, y)$, hence $diam(X_1 \cup X_2) \leq diam(X_1) + diam(X_2) + d(x_n^1, x_n^2)$ for every $n \in \mathbb{N}$. So that, if $r > diam(X_1) + diam(X_2)$ then $B(x,r) = X_1 \cup X_2$.

Let us next prove 3.2.2. Since, even when the estimates are not difficult, there are several different situations for $x$ and $r$ that deserve to be considered in separate form, we shall introduce some notation. Set $m = \min\{diam(X_1), diam(X_2)\}$ and $M = \max\{diam(X_1), diam(X_2)\}$. In the general situation that we are considering any one of the two components of the space $X$ could be bounded or unbounded. So that regarding the effective variation of $r > 0$, we have three basic intervals (or half-lines) given by: $J_1 = (0, m)$, $J_2 = [m, M]$, $J_3 = (M, S]$, where $S = diam(X_1) + diam(X_2)$.

Notice that if $m = +\infty$, $J_1$ is the half line $\mathbb{R}^+$ and $J_2$ and $J_3$ are empty. Also, if $m < +\infty$ and $M = +\infty$, then $J_2$ is the closed half line starting at $m$ and $J_3$

is empty. The only case in which $J_3$ is non-empty is when both components and hence the whole space $X$ are bounded.

Now, since we are considering case 3.2.2 in the statement of Theorem 3.2, the given point $x \in X$ and the radius $r > 0$ are related by $cd(x) \leq r \leq S$, we shall still divide this interval in two subintervals to provide the desired estimates. Set $I_1 = [cd(x), \frac{5}{\gamma}d(x)]$ and $I_2 = [\frac{5}{\gamma}d(x), S]$. Of course when $S = +\infty$, $I_2$ becomes the open half line starting at $\frac{5}{\gamma}d(x)$. With these two partitions in mind we consider the intersection sets $K_{p,q} = I_p \cap J_q$ for $p = 1, 2$ and $q = 1, 2, 3$.

Estimates for $\mu^{\alpha_1,\alpha_2}$ in $K_{1,1}$:

Since $r \in I_1$, we have that $r^{n(x)+\alpha(x)} = r^{\beta} \cong d(x)^{\alpha(x)} r^{n(x)}$. Let $i = 1, 2$ be such that $x \in X_i$. From (a) and (b) in the proof of Theorem 3.1 we obtain that

$$\begin{aligned} a_1 r^{n(x)+\alpha(x)} &\leq \mu^{\alpha_1,\alpha_2}(B(x,r)) = \\ \int_{B(x,r)\cap X_1} d(y)^{\alpha_1}\, d\mu_1(y) + \int_{B(x,r)\cap X_2} d(y)^{\alpha_2}\, d\mu_2(y) &\leq \\ a_2 r^{n(x)+\alpha(x)} + \int_{B(x,r)\cap X_j} d(y)^{\alpha_j}\, d\mu_j(y) & \end{aligned} \tag{3.5}$$

for $j \neq i$. Then we only need to get an upper estimate for

$$\int_{B(x,r)\cap X_j} d(y)^{\alpha_j} d\mu_j(y)$$

when $B(x,r) \cap X_j \neq \emptyset$, for $j \neq i$. We consider an admissible pair $(\mathbf{x}^1, \mathbf{x}^2)$. Since $d(x) \leq \frac{r}{c}$, there exists $N_0 \in \mathbb{N}$, such that, for every $n \geq N_0$, we have that $d(x, x_n^i) \leq \frac{2r}{c}$, furthermore, there exists $N_1 \in \mathbb{N}$, such that $d(x_n^i, x_n^j) \leq \frac{r}{c}$ for $n \geq N_1$. Then for $z \in B(x_n^j, r)$ we have both: $B(x,r) \subset B(z, rk)$, with $k = 2 + \frac{3}{c}$ and $d(z) \leq r$,. From (b) in the proof of the Theorem 3.1 applied to the ball $B(z, rk)$, we get

$$\int_{B(x,r)\cap X_j} d(y)^{\alpha_j} d\mu_j(y) \leq \mu^{\alpha_j}(B(x,r)) \leq \mu^{\alpha_j}(B(z,rk)) \leq A r^{\beta}$$

Estimates for $\mu^{\alpha_1,\alpha_2}$ in $K_{1,2}$:

We are assuming $K_{1,2} \neq \emptyset$, hence $J_2 \neq \emptyset$. So that $m < \infty$. Let $i \in \{1,2\}$ be such that $diam(X_i) = m$. We shall use $j$ to denote the only element of $\{1,2\}$ with $j \neq i$. So $diam(X_j) \geq diam(X_i)$. Let us first assume that $x \in X_i$. Since $r \in I_1$ we have that $cd(x) \leq r \leq \frac{5}{\gamma} d(x)$. Since $r \in J_2$ we also have that $diam(X_i) = m \leq r < M = diam(X_j)$. But, since we are assuming $x \in X_i$, we have that $d(x) \leq m$. Hence $r \simeq m$. Hence

$$\mu^{\alpha_1,\alpha_2}(B(x,r)) \simeq \mu^{\alpha_1,\alpha_2}(B(x,m))$$

which for $x \in X_i$ is bounded above and below by constants depending only of $\alpha_1, \alpha_2$ and the geometric constants of the space. Since $m^\beta \simeq r^\beta$ we also have the desired result 3.2.2. for $r \in K_{1,2}$ and $x \in X_i$.

Let us keep considering $r \in K_{1,2}$; and $m = diam(X_i)$. Assume now that $x \in X_j$. Since $cd(x) \leq r \leq \frac{5}{\gamma} d(x)$ and $x \in X_j$, this means that $cd_j(x) \leq r \leq \frac{5}{\gamma} diam(X_j)$. So that we can apply the argument used in (b) of the proof of Theorem 3.1 to see that

$$\int_{B(x,r)\cap X_j} d_j(y)^{\alpha_j}\, d\mu_j(y) \simeq r^\beta.$$

Again, since $r \in J_2$ we have that

$$\int_{B(x,r)\cap X_i} d_i(y)^{\alpha_i}\, d\mu_i(y) \leq \int_{X_i} d_i(y)^{\alpha_i}\, d\mu_i(y) < \infty.$$

So that $\mu^{\alpha_1,\alpha_2}(B(x,r)) \simeq r^\beta$ for $r \in K_{1,2}$.

Estimates for $\mu^{\alpha_1,\alpha_2}$ in $K_{1,3}$:

Since we are assuming $r > M$ inequalities (3.2.2) are immediate.

Estimates for $\mu^{\alpha_1,\alpha_2}$ in $K_{2,1}$ :

Let us consider $x \in X_i$, notice that from (4) and (2) of Theorem 3.1, we have that

$$\begin{aligned} a_1 r^{n(x)+\alpha(x} + \int_{B(x,r)\cap X_j} d_j^{\alpha_j}(y)\, d\mu_j(y) \leq \mu^{\alpha_1,\alpha_2}(B(x,r)) \leq \\ a_2 r^{n(x)+\alpha(x)} + \int_{B(x,r)\cap X_j} d_j^{\alpha_j}(y)\, d\mu_j(y) \end{aligned} \tag{3.6}$$

In order to estimate $\int_{B(x,r)\cap X_j} d_j^{\alpha_j}(y)\, d\mu_j(y)$, we consider a sequence $x_n^j$ in $X_j$ such that $\mathbf{x}^i$ and $\mathbf{x}^j$ are the components of an admissible pair. Since $d(x) < \frac{r\gamma}{5}$ then exists $N_0' \in \mathbb{N}$ such that for every $n \geq N_0'$ is valid that $d(x, x_n^i) \leq \frac{r\gamma}{4}$ and exists $N_1' \in \mathbb{N}$ such for every $n \geq N_1'$, $d(x_n^i, x_n^j) \leq \frac{r\gamma}{4}$. Taking $n \geq \max\{N_0', N_1'\}$, we obtain that $d(x, x_n^j) \leq \frac{r\gamma}{2}$, hence $B(x,r) \cap X_j \subset B(z, (1 + r\frac{\gamma}{2})) \cap X_j$ with $z \in B(x_n^j, \frac{\gamma r}{5}) \cap X_j$, and $B(z, \frac{r\gamma}{5}) \cap X_j \subset B(x,r) \cap X_j$, hence

$$\begin{aligned} \int_{B(z,\frac{r\gamma}{5})\cap X_j} d_j^{\alpha_j}(y)\, d\mu_j(y) &\leq \int_{B(x,r)\cap X_j} d_j^{\alpha_j}(y)\, d\mu_j(x) \\ &\leq \int_{B(z,r(1+\gamma\frac{1}{2}))\cap X_j} d_j^{\alpha_j}(y)\, d\mu_j(y) \end{aligned}$$

Since the measure $\mu^{\alpha_j}$ is doubling, there exist constants $A_1$ and $A_2$ , such that from the above inequalities

$$\begin{aligned} A_1 \int_{B(z,r)\cap X_j} d_j^{\alpha_j}(y)\, d\mu_j(y) &\leq \int_{B(x,r)\cap X_j} d_j^{\alpha_j}(y)\, d\mu_j(y) \\ &\leq A_2 \int_{B(z,r)\cap X_j} d_j^{\alpha_j}(y)\, d\mu_j(y). \end{aligned}$$

Since $d_j(z) \leq \frac{r\gamma}{5}$, using (1) and (3) of Theorem 3.1, it follows from 3.6,

$$\mu^{\alpha_1,\alpha_2}(B(x,r)) \approx r^{n_1+\alpha_1} \approx r^{\beta}$$

Estimates for $\mu^{\alpha_1,\alpha_2}$ in $K_{2,2}$ :

For this case , the result is obtained directly from 3.6 and Theorem 3.1.

The estimates for $\mu^{\alpha_1,\alpha_2}$ in $K_{2,3}$, are similar to those in $K_{1,3}$. □

Proof of Theorem 1.2

Let us start by showing that the equation $\alpha_1 - \alpha_2 = n_2 - n_1$ is necessary for the doubling of $\mu^{\alpha_1,\alpha_2}$. Assume that $\mu^{\alpha_1,\alpha_2}$ is doubling. Let us consider the admissible pair $(\mathbf{x}^1, \mathbf{x}^2)$. For each $j$, we take the balls $B(x_j^1, r_j)$ where $r_j = d_1(x_j^1)$. Notice that for every $j$, the balls $B(x_j^1, 2r_j) \cap X_2 \neq \emptyset$. On the other hand $B(x_j^1, \lambda r_j) \cap X_2 \supset B(\xi, \frac{r_j}{C}) \cap X_2$ where $C$ is the constant of property $\mathcal{C}_0$, $\lambda = \frac{1+2C}{C}$ and $\xi \in B(x_j^1, 2r_j) \cap X_2$. Now

$$\frac{\mu^{\alpha_1,\alpha_2}(B(x_j^1, 2\lambda r_j))}{\mu^{\alpha_1,\alpha_2}(B(x_j^1, r_j))} \geq \frac{\int_{B(\xi,\frac{r_j}{C})\cap X_2} d_2^{\alpha_2}(x) d\mu_2(x)}{\int_{B(x_j^1, r_j)} d_1^{\alpha_1}(x) d\mu_1(x)} \tag{3.7}$$

and applying the results obtained for the lowers and the upper estimates in Theorem 3.1, we obtain that

$$\frac{\mu^{\alpha_1,\alpha_2}(B(x_j^1, 2\lambda r_j))}{\mu^{\alpha_1,\alpha_2}(B(x_j^1, r_j))} \geq \frac{c_1 r_j^{\alpha_2+n_2}}{c_2 r_j^{\alpha_1+n_1}}.$$

Since $\mu^{\alpha_1,\alpha_2}$ is doubling we have $r_j^{\alpha_2+n_2} \leq C' r_j^{\alpha_1+n_1}$, which, for $j \to \infty$, implies $\alpha_2 + n_2 \geq \alpha_1 + n_1$. By a symmetric argument starting with $(x_j^2)$ we prove that the doubling of $\mu^{\alpha_1,\alpha_2}$ implies $\alpha_2 + n_2 \leq \alpha_1 + n_1$.

The proof of the sufficiency of $n_1 + n_2 = \alpha_1 + \alpha_2$ is based on Theorem 3.2. In fact, given $x \in X$ and $r > 0$, we may consider the following cases which can be handled according to the estimates in Theorem 3.2.

(i) $2r < cd(x)$, follows directly from 3.2.1;

(ii) $r < cd(x) \leq 2r \leq S$, follows from 3.2.1, 3.2.2, and the fact that $r \simeq d(x)$;

(iii) $r < cd(x)$ and $2r \geq S$, using 3.2.3 we have

$$\mu^{\alpha_1,\alpha_2}(B(x, 2r)) = \mu^{\alpha_1,\alpha_2}(X_1) + \mu^{\alpha_1,\alpha_2}(X_2)$$

and

$$
\begin{aligned}
\mu^{\alpha_1,\alpha_2}(B(x,r)) \geq ad(x)^{\alpha(x)} r^{n(x)} &> \frac{ar^{\alpha(x)+n(x)}}{c^{\alpha(x)}} \\
&= \frac{a}{c^{\alpha(x)}} r^{\beta} \\
&\geq \frac{a}{2^{\beta} c^{\alpha(x)}} S^{\beta} \\
&\geq \bar{c}(\mu^{\alpha_1}(X_1) + \mu^{\alpha_2}(X_2))
\end{aligned}
$$

for some positive constant which does not depend on $r$ neither on $x$.

(iv) $cd(x) \leq r < 2r < S$, follows directly from 3.2.2

(v) $cd(x) \leq r < S \leq 2r$ is similar to $(iii)$,

(vi) if $r \geq S$ there is nothing to prove since $B(x, 2r) = B(x, r)$. $\square$

## References

[CW] Coifman, R. and Weiss, Extensions of Hardy spaces and their use in analysis, *Bulletin of the American Mathematical Society* **83** (1977), 569–645.

[H] Huthinson J. E., Fractal and self-similarity, *Indiana University Mathematics* **30** (1981), 713–47.

[M] Mosco, U., Variational Fractals, *Ann. Scola Norm. Sup. Pisa CL. Sci(4)* **XXV** (1997), 683–712.

[MS] Macías, R.A and Segovia, C., Lipschitz Function on Spaces of Homogenous Type, *Advances in Mathematics* **33** (1979), 271–309.

INSTITUTO DE MATEMÁTICA APLICADA DEL LITORAL, GÜEMES 3450, 3000 SANTA FE, ARGENTINA

*E-mail:* haimar@santafe-conicet.gov.ar

FACULTAD DE HUMANIDADES Y CIENCIAS - DEPTO. DE MATEMÁTICA, UNL, PJE. EL POZO, 3000 SANTA FE, ARGENTINA

*E-mail:* rnitti@fhuc.unl.edu.ar