Discrete Applied Mathematics 245 (2018) 139–147

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Recent results on containment graphs of paths in a tree

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ARTICLE INFO

Article history:
Available online 25 April 2018

Keywords:
Posets
Comparability graphs
Geometric containment models

ABSTRACT

In this paper, motivated by the questions posed by Spinrad in Spinrad (2003) and Golumbic and Trenk (2004), we investigate those posets that admit a containment model mapping vertices into paths of a tree and their comparability graphs, named *CPT* posets and *CPT* graphs, respectively. We present a necessary condition to be *CPT* and prove it is not sufficient. We provide further examples of *CPT* posets P whose dual P^d is non *CPT*. Thus, we introduce the notion of *dually-CPT* and *strong-CPT* posets. We demonstrate that, unlike what happens with posets admitting a containment model using interval of the line, the dimension and the interval dimension of *CPT* posets is unbounded. On the other hand, we find that the dimension of a *CPT* poset is at most the number of leaves of the tree used in the containment model. We give a characterization of *CPT* (also *dually-CPT* and *strong-CPT*) split posets by a family of forbidden subposets. We prove that every tree is *strong-CPT*.

1. Introduction and previous results

A *partially ordered set* or *poset* is a pair $\mathbf{P} = (X, P)$ where X is a finite non-empty set, whose elements are called *vertices*, and P is a reflexive, antisymmetric and transitive binary relation on X. As usual, we write $x \leq y$ in $\mathbf{P}$ for $(x, y) \in P$; and $x < y$ in $\mathbf{P}$ when $(x, y) \in P$ and $x \neq y$. If $x < y$ or $y < x$, we say that x and y are *comparable* in $\mathbf{P}$ and write $x \perp y$. The sets $\{x \in X : x < z\}$ and $\{x \in X : z < x\}$ are denoted by $D(z)$ and $U(z)$ respectively. We let $D[z] = D(z) \cup \{z\}$ and $U[z] = U(z) \cup \{z\}$. When $D(z) = \emptyset$, we say that z is a *minimal element* of $\mathbf{P}$; and z is *maximal* when $U(z) = \emptyset$. Two vertices z and z' are *twins* (*false twins*), if $D(z) \cup U(z) = D(z') \cup U(z')$ and z and z' are comparable (incomparable).

A *chain* in $\mathbf{P}$ is a subposet whose vertices are pairwise comparable. The *height* of $\mathbf{P}$ is one less than the number of vertices in its maximum chain.

The *restriction* of the relation P to a subset Y of X is denoted by $P(Y)$. We call $\mathbf{P}(Y)$ to the subposet $(Y, P(Y))$ of $\mathbf{P}$.

A *containment model* $M_{\mathbf{P}}$ of a poset $\mathbf{P} = (X, P)$ maps each element x of X into a set M_x in such a way that $x < y$ in $\mathbf{P}$ if and only if M_x is a proper subset of M_y, i.e.

$$x < y \text{ in } \mathbf{P} \Leftrightarrow M_x \subset M_y.$$

We identify the containment model $M_{\mathbf{P}}$ with the set family $(M_x)_{x \in X}$. Notice that a containment model can always be obtained by mapping each vertex x into the set $D[x]$. We say that a model is *injective* when no two vertices are mapped into a same set.

Many classes of posets, grouped together under the generic name of *geometric containment orders*, have been defined by imposing geometric conditions to the sets in which the elements of the poset are mapped: for example, they may be intervals

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https://doi.org/10.1016/j.dam.2018.04.009
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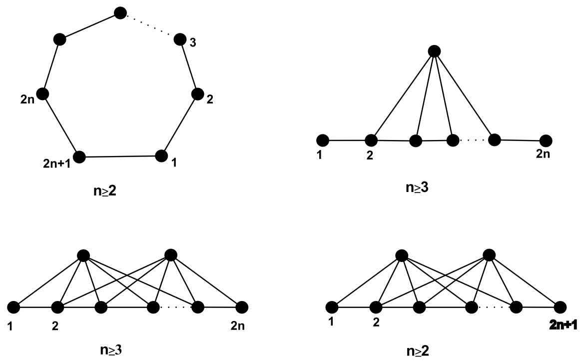

Fig. 1. These graphs together with the complements of the graphs in Fig. 2 constitute the family of minimal forbidden induced subgraphs for comparability graphs.

of the line, angular regions in the plane, d-boxes in the d-Euclidean space, d-spheres in the d-Euclidean space [4,9,16,20]. In [9], it is proved that each poset admits a containment model using subtrees of a star (a tree with a unique vertex with degree greater than one). As reported in [9], Corneil and Golumbic (see [8]) considered those posets that admit a containment model mapping vertices into paths of a tree, and their comparability graphs, called *CPT posets* and *CPT graphs*, respectively. They observed that the 8-wheel W_8 has one of its transitive orientations being a *CPT* poset (where the central vertex is a sink, i.e., its path is contained in each of the other 8 paths), but when reversing the orientations of the edges (the dual, where the central vertex is a source), it is not a *CPT* poset. Their same argument applies to every wheel W_{2k} for $k \geq 3$. Spinrad in [14] and Golumbic and Trenk in [10] called for investigating the properties of *CPT* posets and *CPT* graphs. In this paper, we have initiated such a study, presenting new results on the topic.

The comparability graph $G_{\mathbf{P}}$ of a poset $\mathbf{P} = (X, P)$ is the simple graph with vertex set $V(G_{\mathbf{P}}) = X$ and edge set $E(G_{\mathbf{P}}) = \{xy : x \perp y\}$. We say that two posets are *associated* if their comparability graphs are isomorphic. A graph G is a *comparability graph* if there exists some poset $\mathbf{P}$ such that $G = G_{\mathbf{P}}$.

A *transitive orientation* $\overrightarrow{E}$ of a graph $G = (V, E)$ is an assignment of one of the two possible directions, $\overrightarrow{xy}$ or $\overrightarrow{yx}$, to each edge $xy \in E$ in such a way that if $\overrightarrow{xy} \in \overrightarrow{E}$ and $\overrightarrow{yz} \in \overrightarrow{E}$ then $\overrightarrow{xz} \in \overrightarrow{E}$. The graphs whose edges can be transitively oriented are exactly the comparability graphs [6]. Furthermore, given a transitive orientation $\overrightarrow{E}$ of a graph $G = (V, E)$, we let $\mathbf{P}_{\overrightarrow{E}}$ denote the poset $(V, P_{\overrightarrow{E}})$ where $u < v$ in $\mathbf{P}_{\overrightarrow{E}}$ if and only if $\overrightarrow{uv} \in \overrightarrow{E}$. The comparability graph of $\mathbf{P}_{\overrightarrow{E}}$ is G. Thereby, the transitive orientations of G are put in one-to-one correspondence with the posets whose comparability graphs are G.

Gallai provides the following characterization of comparability graphs by a family of minimal forbidden induced subgraphs.

Theorem 1 (*[5]*)**.** *A graph is a comparability graph if and only if none of its induced subgraphs is isomorphic to a graph in Fig. 1 or to the complement of a graph in Fig. 2.*

For further information on comparability graphs see [2,7,16].

Dushnik and Miller defined the *dimension of a poset* $\mathbf{P}$, denoted by $dim(\mathbf{P})$, as the minimum number of linear orders whose intersection is $\mathbf{P}$ [3]. Trotter et al. proved that if $\mathbf{P}$ and $\mathbf{P}'$ are associated posets then $dim(\mathbf{P}) = dim(\mathbf{P}')$, leading to the definition of *dimension of a comparability graph* [19,16].

The *dual* of a poset $\mathbf{P} = (X, P)$ is the poset $\mathbf{P}^d = (X, P^d)$ where $x < y$ in $\mathbf{P}^d$ if and only if $y < x$ in $\mathbf{P}$. Notice that $\mathbf{P}$ and $\mathbf{P}^d$ are associated and, obviously, $dim(\mathbf{P}) = dim(\mathbf{P}^d)$.

In [3], it was proved that $dim(\mathbf{P}) \leq 2$ if and only if $\mathbf{P}$ admits a containment model mapping vertices into intervals of the line. Therefore, posets with dimension at most 2 also appear in the literature as *containment orders of intervals*, we will write *CI* posets for short. Comparability graphs of interval containment orders, or *CI* graphs, have been widely studied and characterized in different ways.

Theorem 2 (*[3,13]*)**.** *The following statements are equivalent.*

1. *G is a CI graph.*
2. *G is a comparability graph with $dim(G) \leq 2$.*
3. *G and its complement $\overline{G}$ are comparability graphs.*
4. *G is a permutation graph [2].*

The previous theorem together with Gallai's characterization of comparability graphs provides a characterization of *CI* graphs by induced forbidden subgraphs. In addition, observe the following simple result:

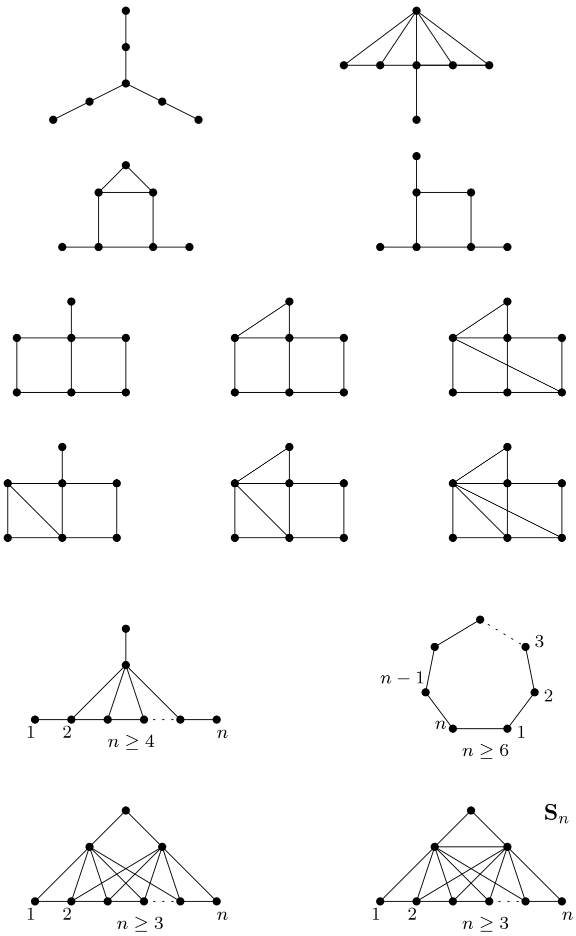

Fig. 2. The complements of these graphs together with the graphs in Fig. 1 constitute the family of minimal forbidden induced subgraphs for comparability graphs.

Remark 1. Let $\mathbf{P}$ and $\mathbf{P}'$ be associated posets. Then, $\mathbf{P}$ is a *CI* poset if and only if $\mathbf{P}'$ is a *CI* poset. In particular, $\mathbf{P}$ is a *CI* poset if and only if $\mathbf{P}^d$ is a *CI* poset.

We identify paths of a graph with their vertex sets. Consequently, we say that a path W is contained in a path W' and write $W \subseteq W'$ when every vertex of W is a vertex of W'. Also, $W \subset W'$ means $W \neq W'$ and $W \subseteq W'$.

Definition 3. A poset $\mathbf{P} = (X, P)$ is a containment order of paths in a tree, or *CPT* poset for brevity, if it admits a containment model $M_{\mathbf{P}} = (W_x)_{x \in X}$ where every W_x is a path of a tree T, which is called the host tree of the model.

Clearly the class of *CPT* posets contains the class of *CI* posets.

As we said before, motivated by the questions posed by Spinrad in [14] and Golumbic in [10], we further the study of *CPT* posets and their comparability graphs. In Section 2, we work with posets and, in Section 3, with graphs. We present a necessary condition to be *CPT* and show it is not sufficient. We show additional examples of a *CPT* posets P whose dual P^d is not *CPT*. Thus, we introduce the notion of *dually-CPT* and *strong-CPT* posets. We demonstrate that, unlike what happens with *CI* posets, the dimension and the interval dimension of *CPT* posets is unbounded. On the other hand, we find that the number of leaves of any host tree is an upper bound of the dimension. We give a characterization of *CPT* (also dually-*CPT* and strong-*CPT*) split posets by a family of forbidden subposets.

2. *CPT*, *dually-CPT* and *strong-CPT* posets

We begin by giving a necessary condition for being a *CPT* poset.

Lemma 4. *If z is a vertex of a CPT poset $\mathbf{P}$ then $\mathbf{P}(D[z])$ is a CI poset.*

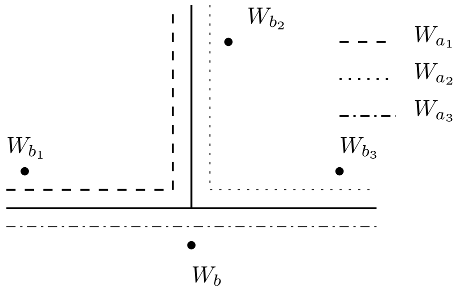

Fig. 3. A *CPT* model of $\mathbf{N}^d$. Vertices b_1, b_2, b_3 and b are mapped into one-vertex paths.

Proof. Let $(W_x)_{x\in X}$ be a *CPT* model of $\mathbf{P} = (X, P)$. Since $x \in D(z)$ implies $W_x \subset W_z$, we have that every W_x with $x \in D[z]$ is an interval of W_z. Thus $(W_x)_{x\in D[z]}$ is an interval containment model of $\mathbf{P}(D[z])$. □

Minimal posets with dimension 3 (any proper subposet has dimension at most 2) are called 3-*irreducible*, all such posets were independently determined by Kelly [12] and by Trotter and Moore [17]. A consequence of the previous lemma is that every 3-irreducible poset plus a least upper bound is a non *CPT* poset.

In what follows we offer examples of posets that are non *CPT*, although they satisfy the necessary condition given by Lemma 4. We also analyze the effect of adding twins or false twins.

We depicted posets using *Hasse diagrams*. Notice that the Hasse diagram and the comparability graph of a poset with height one are isomorphic.

Lemma 5. *The poset* $\mathbf{N}^d$ *depicted in Fig. 4 is CPT. In every CPT model of* $\mathbf{N}^d$, *the vertex labeled as b is mapped into a one-vertex path.*

Proof. In Fig. 3, there is a *CPT* model of $\mathbf{N}^d$; thus it is a *CPT* poset.

Let T be the host tree of any *CPT* model of $\mathbf{N}^d$. Notice no two vertices can be mapped into a same path of T. Let $W_{a_1} : q_1, q_2, \ldots, q_k$ be the path representing the vertex a_1 of $\mathbf{N}^d$, where q_i are vertices of T. Since $b_1 < a_1$, $b_2 < a_1$ and $b < a_1$ in $\mathbf{N}^d$, it follows that W_{b_1}, W_{b_2} and W_b are subpaths of W_{a_1}. For $1 \le i \le j \le k$, we will write $[q_i, q_j]$ to denote the subpath of W_{a_1} with vertices $q_i, q_{i+1}, \ldots, q_{j-1}, q_j$. Since W_{a_3} is a path of T containing W_{b_1} and W_b, but not containing W_{b_2}; and W_{a_2} is a path of T containing W_{b_2} and W_b, but not containing W_{b_1}, we can assume that $W_{b_1} : [q_{i_{b_1}}, q_{j_{b_1}}]$, $W_b : [q_{i_b}, q_{j_b}]$, $W_{b_2} : [q_{i_{b_2}}, q_{j_{b_2}}]$, with $i_{b_1} \le j_{b_1} < j_b < j_{b_2}$ and $i_{b_1} < i_b < i_{b_2} \le j_{b_2}$.

Since W_{b_3} is not contained in W_{a_1} there exists a vertex h of T that belongs to $(W_{a_3} \cap W_{a_2}) - W_{a_1}$. Let q_{i_h} be the vertex of W_{a_1} closest to h in T.

Since h is a vertex of W_{a_3}, and W_{b_1} and W_b are contained in W_{a_3}, then $i_h \le i_{b_1}$ or $i_h \ge j_b$. If $i_h \le i_{b_1}$, since h is a vertex of W_{a_2} and W_{b_2} is contained in W_{a_2}, we have that W_{b_1} is contained in W_{a_2}, which contradicts b_1 and a_2 are incomparable in $\mathbf{N}^d$. Therefore, $j_b \le i_h$.

Analogously, since h is a vertex of W_{a_2}, and W_{b_2} and W_b are contained in W_{a_2}, then $i_h \le i_b$ or $i_h \ge j_{b_2}$. If $i_h \ge j_{b_2}$, since h is a vertex of W_{a_3} and W_{b_1} is contained in W_{a_3}, we have that W_{b_2} is contained in W_{a_3}, which contradicts a_3 and b_2 are incomparable in $\mathbf{N}^d$. Therefore, $i_h \le i_b$.

We conclude that $i_b = j_b = i_h$, so W_b is a one-vertex path. □

Lemma 6. *Let* $\mathbf{N}$ *and* $\mathbf{M}$ *be the posets depicted in Fig. 4. The following statements hold.*

1. $\mathbf{N}(D[z])$ *is CI for every vertex z of* $\mathbf{N}$ *and* $\mathbf{M}$.
2. $\mathbf{M}$ *is obtained by adding the vertex b′ (twin of b) to* $\mathbf{N}^d$.
3. $\mathbf{N}$ *is non CPT*.
4. $\mathbf{M}$ *is non CPT*.

Proof. It is straightforward to prove 1 and 2.

Item 4 follows by Lemma 5 and the fact that the path $W_{b'}$ has to be a proper subpath of W_b. To prove 3 we will use the following fact.

Fact 1. *Given three intervals of a line, there exist two of them such that if I is any interval containing both then I also contains the remaining third.*

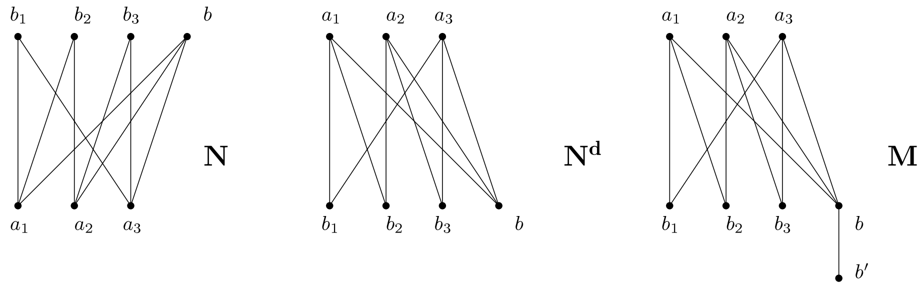

Fig. 4. Poset **N** is non *CPT*. Its dual $\mathbf{N}^d$ is *CPT*. The poset **M** obtained from $\mathbf{N}^d$ by adding b' is non *CPT*. Notice that b and b' are twins.

Assume, in order to derive a contradiction, that the poset **N** depicted in Fig. 4 is *CPT*. Thus there exist a tree T and a path W_x of T for each vertex x of **N** satisfying $x < y$ in **N** if and only if $W_x \subset W_y$.

Since $a_1 < b$, $a_2 < b$ and $a_3 < b$ in **N**, we have that W_{a_1}, W_{a_2} and W_{a_3} are intervals of W_b. Thus, by Fact 1, two of them, w.l.o.g. say W_{a_1} and W_{a_2}, satisfy that every subpath of W_b containing W_{a_1} and W_{a_2} also contains W_{a_3}. Since $W_{b_2} \cap W_b$ is a subpath of W_b containing W_{a_1} and W_{a_2}, we have that $W_{a_3} \subseteq W_{b_2} \cap W_b$, therefore $W_{a_3} \subseteq W_{b_2}$. On the other hand, it is clear that $W_{a_3} \neq W_{b_2}$, thus $W_{a_3} \subset W_{b_2}$, which contradicts the fact that a_3 and b_2 are incomparable. □

Corollary 7. *The property of being a CPT poset is invariant under the operation of adding false twins; but it can be lost adding twin vertices.*

Since any poset and its dual are associated, Lemma 6 also shows that the strong property of *CI* posets emphasized in Remark 1 does not hold for *CPT* posets, as first observed by Corneil and Golumbic [8].

Corollary 8 (*In Contrast to Remark 1*)**.** *The fact that a poset* $\mathbf{P}'$ *is associated with a CPT poset* **P** *does not imply that* $\mathbf{P}'$ *is CPT. Moreover,* **P** *is CPT does not imply that* $\mathbf{P}^d$ *is CPT.*

It motivates the following definitions.

Definition 9. A poset **P** is *dually-CPT* if it is *CPT* and $\mathbf{P}^d$ is also *CPT*.

Definition 10. A poset **P** is *strong-CPT* if it is *CPT* and every other poset associated with **P** is also *CPT*.

It clear that

$$CI \subseteq \text{strong-}CPT \subseteq \text{dually-}CPT \subseteq CPT. \quad (1)$$

Notice that the poset N^d in Lemma 6 is *CPT* but it is non *dually-CPT*; therefore the last inclusion in (1) is strict.

Let **B** be the poset with seven vertices a, b_i and c_i for $1 \leq i \leq 3$, such that $b_i < c_i$ and $a < c_i$ for every i. The comparability graph of **B** is a tree and $dim(\mathbf{B}) = 3$ [16]; therefore the class of posets whose comparability graph is a tree is not contained in the class of *CI* posets. Next theorem shows that such posets are *strong-CPT* and, consequently, that the first inclusion in (1) is strict.

Theorem 11. *Every poset whose comparability graph is a tree is strong-CPT.*

Proof. Let **P** be a poset such that $G_{\mathbf{P}}$ is a tree. Notice that **P** has height one and so every vertex of **P** is a minimal or a maximal element.

We will built inductively an injective *CPT* model of **P** in which every minimal vertex is mapped into a one-vertex path. Let v_0 be a leaf of **P** and $(W_v)_{v \in V - \{v_0\}}$ be such a *CPT* model of $G - v_0$ on a host tree T. Let v_1 be the only vertex of **P** comparable with v_0. There are two cases to be considered.

If v_0 is minimal in **P** then v_1 is maximal, therefore W_{v_1} is contained in no path of the model. Let q be an end vertex of W_{v_1}. We proceed adding to T a new vertex q' adjacent to q, replacing W_{v_1} by $W_{v_1} \cup \{q'\}$, doing $W_{v_0} = \{q'\}$ and letting the remaining paths unchanged.

If v_0 is maximal in **P** then v_1 is minimal, thus W_{v_1} is a one-vertex path in T, say $W_{v_1} = \{q\}$. In this case, we proceed adding to T a new vertex q' adjacent to q, doing $W_{v_0} = \{q, q'\}$ and letting the remaining paths unchanged. □

We do not know if the middle inclusion in (1) is strict, then we let the following problem.

Open problem 12. *Is there a dually-CPT poset which is non strong-CPT?*

Notice also that the property of being *strong-CPT* is not necessarily hereditary by subposet, it may happen that a subposet $\mathbf{P}'$ of $\mathbf{P}$ admits more associated posets than $\mathbf{P}$ itself. However we do not know if this is the case for *strong-CPT* posets.

Open problem 13. *Is the property of being strong-CPT hereditary by subposets?*

Clearly, a negative answer to Open Problem 12 implies a positive answer to Open Problem 13.
In Section 2.2 we solve both problems in the class of split posets.

2.1. Dimension and interval dimension

In [14], it is asked whether *CPT* posets have bounded dimension. A negative answer is given by Theorem 14. Following [16], we let $\mathbf{P}(1, k; n)$ denote the poset formed by the 1-element and the k-element subsets of $\{1, 2, \ldots, n\}$ ordered by inclusion. There it is proved that $dim(\mathbf{P}(1, n-1; n)) = n$. Posets $\mathbf{P}(1, n-1; n)$ (also known as *Hiraguchi's posets*) are standard examples of posets with dimension as large as desired, but, by Lemma 6, they are not *CPT* posets for $n \geq 4$. Notice that, for $n \geq 4$, the poset $\mathbf{N}$ depicted in Fig. 4 is a subposet of $\mathbf{P}(1, n-1; n)$. The proof of Theorem 14 makes use of the posets $\mathbf{P}(1, 2; n)$ which are *CPT*.

Theorem 14. *For every positive integer d, there exists a CPT poset* $\mathbf{P}$ *with* $dim(\mathbf{P}) > d$.

Proof. We will show that, for each n, the poset $\mathbf{P}(1, 2; n)$ is *CPT*, thus the proof follows from the fact that $dim(\mathbf{P}(1, 2; n)) > lglg(n)$ [16].

Let T be the star with center q and leaves $q_1, q_2, \ldots, q_n$. Map every 1-element subset $\{i\}$ into the path $W_i = \{q_i\}$ of T, and map every 2-element subset $\{i, j\}$ into the path $W_{ij} = \{q_i, q, q_j\}$. □

The *interval dimension* of a poset is defined analogously to dimension but considering interval orders instead of linear orders [16]. Since linear orders are interval orders, it follows that $Idim(\mathbf{P}) \leq dim(\mathbf{P})$ for every poset $\mathbf{P}$. The difference between both dimensions may be large; however, if $\mathbf{P}$ is a poset of height one then it is at most one [15]. Thus, since the posets $\mathbf{P}(1, 2; n)$ used in the proof of Theorem 14 have height one, we obtain the following corollary.

Corollary 15. *For every positive integer d, there exists a CPT poset* $\mathbf{P}$ *with* $Idim(\mathbf{P}) > d$.

On the other hand, the dimension of a *CPT* poset is bounded by the number of *leaves* (vertices with degree one) of the host tree.

Theorem 16. *If a poset* $\mathbf{P}$ *admits a CPT model in a host tree T with k leaves then* $dim(\mathbf{P}) \leq k$.

Proof. In [18] it is proved that the dimension of the poset formed by the connected subgraphs of a graph G ordered by vertex inclusion equals the number of non-cut vertices of G. The proof follows taking $G = T$. □

Next lemma shows that posets used in the proof of Theorem 14 are non *dually-CPT*, thus the following problem remains unsolved.

Open problem 17. *Determine whether the dimension of dually-CPT posets is bounded above by a constant. The same question for strong-CPT posets.*

Lemma 18. *For every* $n \geq 4$, *the dual of the poset* $\mathbf{P}(1, 2; n)$ *is non CPT.*

Proof. Notice that $\mathbf{P}(1, 2; 4)$ is a subposet of $\mathbf{P}(1, 2; n)$ for $n \geq 4$; therefore, it is enough to prove that $\mathbf{H} = \mathbf{P}(1, 2; 4)^d$ is non *CPT*. Suppose the contrary and take a *CPT* model of $\mathbf{H}$. For every vertex $\{i\}$ or $\{i, j\}$ of $\mathbf{H}$, let W_i, respectively $W_{i,j}$, be the corresponding subpath of a tree T. The path W_1 contains the paths $W_{1,2}$, $W_{1,3}$ and $W_{1,4}$. Notice that no two of these three paths have the same left end-point or the same right end-point. By the symmetry of $\mathbf{H}$, we can assume, without lost of generality, that the left end-point of $W_{1,2}$ is the closest to the left end-point of W_1; and that the right end-point of $W_{1,3}$ is the closest to the right end-point of W_1.

Notice also that $W_{1,2} \subset W_2$, $W_{1,3} \not\subseteq W_2$; $W_{1,3} \subseteq W_3$, $W_{1,2} \not\subseteq W_3$; and, because of the vertex $\{2, 3\}$, $W_2 \cap W_3 \not\subseteq W_1$. Therefore, there exist a vertex t of the host tree T between the right end-point $r_{1,2}$ of $W_{1,2}$ and the left end-point $l_{1,3}$ of $W_{1,3}$; and a vertex s of T adjacent to t such that the path W_2 contains $l_{1,2}$, t and s; while the path W_3 contains $r_{1,3}$, t and s. See Fig. 5.

Since W_4 must intersect the three paths W_1, W_2 and W_3 we have that the vertex t of the host tree also belongs to W_4. On the other hand, because

$W_{1,4} \subset W_1 \cap W_4$, $W_{1,4} \not\subseteq W_2$, $W_{1,4} \not\subseteq W_3$; and
$W_{1,2} \not\subseteq W_4$ and $W_{1,3} \not\subseteq W_4$,

we have that the left end-point of $W_1 \cap W_4$ must be between $l_{1,2}$ and t, while the right end-point of $W_1 \cap W_4$ must be between t and $r_{1,3}$. It implies that both $W_4 \cap W_2$ and $W_4 \cap W_3$ are contained in W_1 in contradiction with the existence of the vertices $\{2, 4\}$ and $\{3, 4\}$, respectively. □

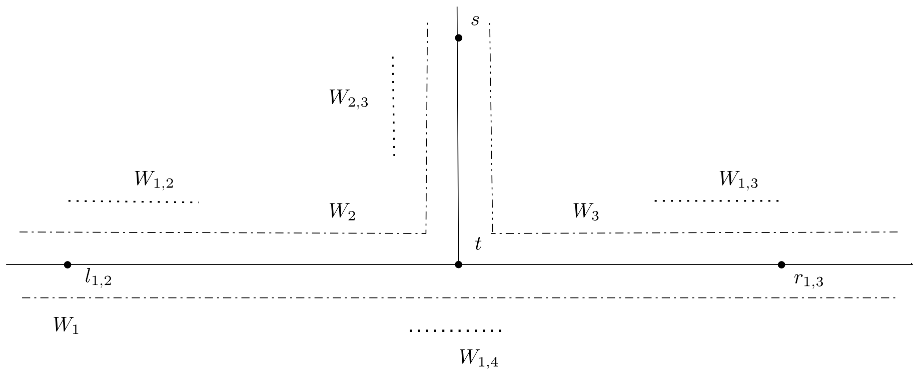

Fig. 5. $\mathbf{P}(1,2;4)$ is non *CPT*.

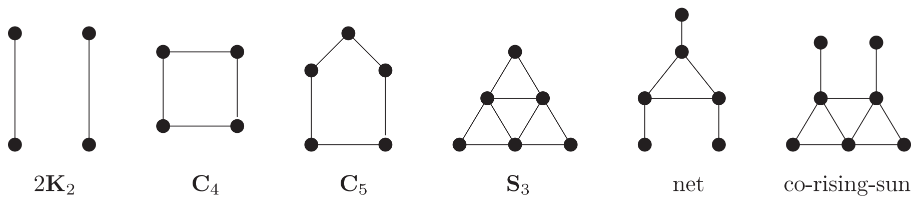

Fig. 6. Forbidden induced subgraphs for split comparability graphs.

2.2. Split posets

A graph whose vertex set can be partitioned into a clique and a stable set is called *split*. The class of split graphs is exactly the class of $\{2K_2, C_4, C_5\}$-free graphs; therefore a graph is split and comparability if and only if none of its induced subgraphs is isomorphic to a graph in Fig. 6 [2]. A poset whose comparability graph is split is called a *split poset* [11].

A poset $\mathbf{P}$ is split if and only if its Hasse diagram consists of a maximal chain $x_1 < x_2 < \cdots < x_k < y_1 < y_2 < \cdots < y_m$, termed *main chain* of $\mathbf{P}$, and any other vertex z of $\mathbf{P}$ satisfies one of the following conditions:

- z has a unique neighbor, this neighbor is some of the vertices y_j in the main chain of $\mathbf{P}$, and $z < y_j$.
- z has a unique neighbor, this neighbor is some of the vertices x_i in the main chain of $\mathbf{P}$, and $x_i < z$.
- z has exactly two neighbors, they are vertices x_i and y_j in the main chain of $\mathbf{P}$, and $x_i < z < y_j$.

The vertex y_m of the main chain is called the *top vertex* of the poset.

In the present section, split posets which are *CPT*, dually-*CPT* or strong-*CPT* are characterized by forbidden subposets.

Although the following lemma can be easily proved using *modular decomposition* [5], we will give a demonstration based on transitive orientations for the convenience of those readers not familiar with modules. The posets $\mathbf{S}_4$ and $\mathbf{S}$ are the ones depicted in Fig. 7. Notice that the comparability graph of $\mathbf{S}_4$ is the graph S_4 in Fig. 2.

Lemma 19. *(i) If S_4 is an induced subgraph of a comparability graph $G_{\mathbf{P}}$ then the poset $\mathbf{S}_4$ is a subposet of $\mathbf{P}$. (ii) If $\mathbf{S}$ is a subposet of a poset $\mathbf{P}$ and $G_{\mathbf{P}} = G_{\mathbf{P}'}$ then either $\mathbf{S}$ or $\mathbf{S}^d$ is a subposet of $\mathbf{P}'$.*

Proof. To prove (i) notice that S_4 admits only two transitive orientations: one corresponding to the poset $\mathbf{S}_4$ and its reverse corresponding to the dual of $\mathbf{S}_4$. Since $\mathbf{S}_4$ and $\mathbf{S}_4^d$ are isomorphic, the proof follows.

In order to prove (ii), call y to the top vertex of $\mathbf{S}$ and observe that the subposet of $\mathbf{S}$ obtained by removing y is the poset $\mathbf{S}_4$, thus S_4 is an induced subgraph of $G_{\mathbf{P}}$, and so of $G_{\mathbf{P}'}$. Therefore, by (i), $\mathbf{S}_4$ is a subposet of $\mathbf{P}'$. To complete the proof, notice that the vertex y must be comparable with every vertex of $\mathbf{S}_4$. This implies y is in the main chain of the split poset $\mathbf{P}'$, and either y is greater than every vertex of $\mathbf{S}_4$ or y is less than every vertex of $\mathbf{S}_4$. In the first case $\mathbf{S}$ is a subposet of $\mathbf{P}'$ and in the latter $\mathbf{S}^d$ is a subposet of $\mathbf{P}'$. □

Theorem 20. *Let $\mathbf{P}$ be a split poset. The following conditions are equivalent.*

(i) $\mathbf{P}$ *is CPT*.
(ii) $\mathbf{P}(D[y])$ *is CI, where y is the top vertex of* $\mathbf{P}$.
(iii) $\mathbf{P}$ *does not contain the poset* $\mathbf{S}$ *depicted in Fig. 7 as a subposet.*

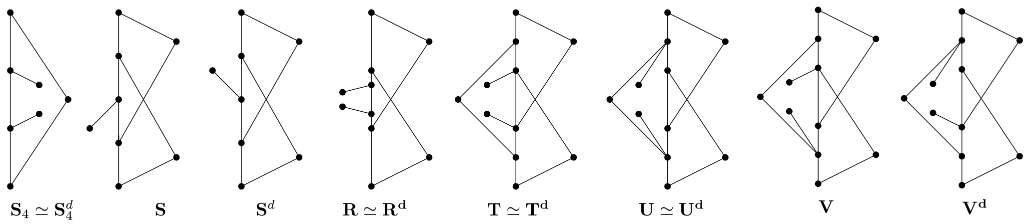

Fig. 7. Examples of split posets. $\mathbf{S}_4$ is non *CI* [16], but we have proved that $\mathbf{S}_4$ is *CPT*. **S** is non *CPT*, but $\mathbf{S}^d$ is *CPT*; so the comparability graph of **S** is a *CPT* graph. The posets **R**, **T**, **U** and **V** are non *CPT*; even more, any poset associated with **R**, **T**, **U** or **V** is non *CPT*. Therefore, the comparability graphs of these posets are non *CPT*. The symbol $\simeq$ means *isomorphic*.

Proof. By Lemma 4, (*i*) implies (*ii*).

Let **P** be a split poset satisfying (*ii*). Denote by Z the set of vertices of **P** that are not in $D[y]$. Since **P** is split, if $z \in Z$ then there exists a unique vertex x_z in $D[y]$ adjacent to z; moreover, $x_z < z$ in **P**.

Let T be the host tree of a *CI* model M of $\mathbf{P}(D[y])$, w.l.o.g. we can assume M is injective. We obtain a *CPT* model M' of **P** by adding to T a pendant vertex q_z adjacent to one end vertex of the path W_{x_z} of M, and mapping z into the path $W_z = W_{x_z} \cup \{q_z\}$; for every $z \in Z$. We have proved that (*ii*) implies (*i*).

Now, in order to derive a contradiction, assume that **S** is a subposet of **P**. Clearly, the top vertex of **S** must be a vertex of the main chain of **P**, therefore **S** is a subposet of $\mathbf{P}(D[y])$. Thus the non *CI* poset $\mathbf{S}_4$ depicted in Fig. 7 is a subposet of $\mathbf{P}(D[y])$; it contradicts the fact that $\mathbf{P}(D[y])$ is *CI*. This proves (*ii*) implies (*iii*).

Finally, let **P** be a split poset satisfying (*iii*) and assume that $\mathbf{P}(D[y])$ is non *CI*. Thus, by Theorems 1 and 2, the comparability graph of $\mathbf{P}(D[y])$ contains an induced subgraph H that is either the complement of a graph in Fig. 1 or a graph in Fig. 2. On the other hand, since the comparability graph of $\mathbf{P}(D[y])$ is split, we have that H is split; therefore H does not contain as induced subgraph any of the graphs in Fig. 6. A simple direct comparison of such graphs shows that H must be the graph S_4 in Fig. 2 (S_4 is the only graph between the complements of graphs in Fig. 1 and the graphs in Fig. 2 which does not contain as induced subgraph a graph in Fig. 6). By (*i*) of Lemma 19, $\mathbf{P}(D[y])$ contains the poset $\mathbf{S}_4$ in Fig. 7 as subposet. Notice that y cannot be one of the vertices of $\mathbf{S}_4$, thus y is greater than every vertex of $\mathbf{S}_4$ and so **S** is a subposet of $\mathbf{P}(D[y])$, which contradicts (*iii*). □

Theorem 21. *Let* **P** *be a split poset. The following conditions are equivalent.*

(i) **P** *is strong-CPT*.
(ii) **P** *is dually-CPT*.
(iii) **P** *contains neither the poset* **S** *in* Fig. 7 *nor* $\mathbf{S}^d$ *as subposet.*

Proof. The implication (*i*) then (*ii*) holds even for non split posets, and (*ii*) implies (*iii*) follows trivially from Theorem 20.

Let **P** be a split poset satisfying (*iii*). By Theorem 20, **P** is *CPT*. Assume, in order to derive a contradiction, that **P** is non strong-*CPT*; thus there exists a split poset $\mathbf{P}'$ such that $G_{\mathbf{P}'} = G_{\mathbf{P}}$ and $\mathbf{P}'$ is non *CPT*. By Theorem 20, $\mathbf{P}'$ contains the poset **S** as subposet, then, by (*ii*) of Lemma 19, **P** contains either **S** or $\mathbf{S}^d$ as subposet, contrary to assumption. □

3. *CPT* graphs and subclasses

Comparability graphs of *CPT* posets are called *CPT graphs* or *containment graphs of paths in a tree*. Clearly, a graph $G = (V, E)$ is *CPT* if and only if there exist a transitive orientation $\overrightarrow{E}$ of G, a tree T and a family $(W_v)_{v \in V}$ of paths of T satisfying that

$$\overrightarrow{uv} \in \overrightarrow{E} \Leftrightarrow W_u \subset W_v.$$

Strong- CPT graphs and *dually- CPT graphs* are the comparability graphs of the strong-*CPT* posets and dually-*CPT* posets, respectively.

As a corollary of Theorem 11 we have the following result.

Theorem 22. *Every tree is a strong-CPT graph.*

By just comparing the family of forbidden induced subgraphs for split and *CI* graphs, it can be established that a split graph is *CI* if and only if it does not contain $G_{\mathbf{S}_4}$ in Fig. 8 as induced subgraph. We obtain a characterization by forbidden induced subgraphs of split dually-*CPT* and split strong-*CPT* graphs.

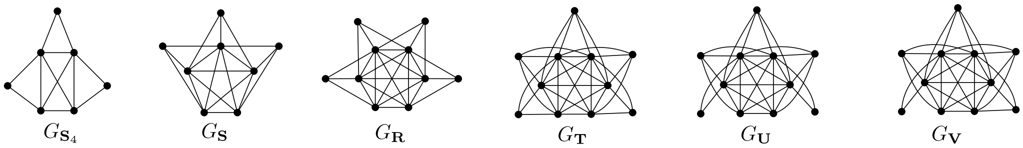

Fig. 8. The comparability graphs of posets $\mathbf{S}_4$, $\mathbf{S}$, $\mathbf{R}$ and $\mathbf{T}$, $\mathbf{U}$ and $\mathbf{V}$ in Fig. 7. Notice that the graph $G_{\mathbf{S}_4}$ is the same named S_4 in Fig. 2.

Theorem 23. *Let G be a comparability split graph. The following statements are equivalent.*

(i) *G is strong-CPT.*
(ii) *G is dually-CPT.*
(iii) *G does not contain $G_{\mathbf{S}}$ in Fig. 8 as induced subgraph.*

Proof. It is straightforward consequence of Theorem 21 and the fact that a comparability graph $G_{\mathbf{P}}$ has an induced subgraph isomorphic to $G_{\mathbf{S}}$ if and only if $\mathbf{P}$ has a subposet isomorphic to $\mathbf{S}$ or to $\mathbf{S}^d$. □

Notice that although Theorem 20 provides a characterization of split *CPT* posets by forbidden subposets, we leave open the problem of characterizing split *CPT* graphs by induced subgraphs. A trivial corollary of Theorem 20 is that the posets **R**, **T**, **U** and **V** in Fig. 7 are non *CPT*; even more, it can be proved that any poset **P** associated with **R**, **T**, **U** or **V** has a subposet isomorphic to **S**, thus **P** is non *CPT*. This implies that the comparability graphs of these posets depicted in Fig. 8 are forbidden induced subgraphs for the class of split *CPT* graphs. It is also possible to prove that they are minimal (a *CPT* graph is obtained by removing any one of their vertices). We conjecture there is no others.

Conjecture 24. *A split graph G is CPT if and only if no induced subgraph of G is isomorphic to the graphs $G_{\mathbf{R}}$, $G_{\mathbf{T}}$, $G_{\mathbf{U}}$ and $G_{\mathbf{V}}$ in Fig. 8.*

Observation: Some results in the present paper were presented at the VIII Latin-American Algorithms, Graphs and Optimization Symposium, and a summary appeared in the proceedings [1].

Acknowledgments

We thank the anonymous referees for their comments and suggestions which helped to improve the manuscript.

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