Sharp two weight inequalities for commutators of Riemann-Liouville and Weyl fractional integral operators

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Abstract. Let b be a BMO function, $0 < \alpha < 1$ and $I^{+,k}_{\alpha,b}$ the commutator of order k for the Weyl fractional integral. In this paper we prove weighted strong type (p,p) inequalities $(p > 1)$ and weighted endpoint estimates $(p = 1)$ for the operator $I^{+,k}_{\alpha,b}$ and for the pairs of weights of the type $(w, \mathcal{M}w)$, where w is any weight and $\mathcal{M}$ is a suitable one-sided maximal operator. We also prove that, for A^+_∞ weights, the operator $I^{+,k}_{\alpha,b}$ is controlled in the $L^p(w)$ norm by a composition of the one-sided fractional maximal operator and the one-sided Hardy-Littlewood maximal operator iterated k times. These results improve those obtained by an immediate application of the corresponding two-sided results and provide a different way to obtain known results about the operators $I^{+,k}_{\alpha,b}$. The same results can be obtained for the commutator of order k for the Riemann-Liouville fractional integral $I^{-,k}_{\alpha,b}$.

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1. Introduction

We are interested in two weight inequalities of the type

$$\int |Tf|^p w \le C \int |f|^p \mathcal{M}_T w, \quad 1 < p < \infty, \tag{1.1}$$

with no a priori assumption on the weight w and where $\mathcal{M}_T$ is some maximal operator associated with the operator T.

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This type of inequalities have been studied by several authors. For example, if T is a Calderón-Zygmund singular integral operator, C. Pérez ([16]) proved (1.1) with $\mathcal{M}_T = M^{[p]+1}$, where M^k is the Hardy-Littlewood maximal operator M iterated k times and $[p]$ is the integer part of p. Later on, Pérez ([20]) generalized the result in [16] obtaining (1.1) for T equal to the commutator of the Calderón-Zygmund singular integral of order k and in this case $\mathcal{M}_T = M^{[(k+1)p]+1}$.

For the one-sided singular integrals, i.e., singular integrals operators with kernels supported in $(-\infty, 0)$ or $(0, \infty)$, inequalities like (1.1) were proved in [25]. In this case, if the kernel has support in $(-\infty, 0)$, (1.1) can be obtained with $\mathcal{M}_T = (M^-)^{[p]+1}$, where M^- is the one-sided Hardy-Littlewood maximal operator. This result was generalized in [8] for the commutator of the one-sided singular integral of order k with $\mathcal{M}_T = (M^-)^{[(k+1)p]+1}$.

In the case where T is the fractional integral operator, C. Pérez ([19]) proved (1.1) with $\mathcal{M}_T = M_{\alpha p}(M^{[p]})$, where $M_{\alpha p}$ is the fractional maximal operator. This result was generalized in [2] to the commutator of the fractional integral of order k, obtaining in this case that $\mathcal{M}_T = M_{\alpha p}(M^{[(k+1)p]})$. The result in [2] was obtained in the general context of the spaces of homogeneous type.

The first purpose of this paper is to prove an inequality like (1.1) for the commutators of the one-sided fractional integrals (Riemann-Liouville and Weyl fractional integrals). For $0 < \alpha < 1$, $b \in BMO$ and $k = 0, 1, \ldots,$ the k-th order commutators of the one-sided fractional integrals are defined by

$$I^{-,k}_{\alpha,b} f(x) = \int_{-\infty}^{x} (b(x) - b(y))^k \frac{f(y)}{(x-y)^{1-\alpha}} dy$$

and

$$I^{+,k}_{\alpha,b} f(x) = \int_{x}^{\infty} (b(x) - b(y))^k \frac{f(y)}{(y-x)^{1-\alpha}} dy.$$

Let us notice that when $k = 0$ the above operators are the Riemann-Liouville and Weyl fractional integral operators, respectively.

Applying Theorem 1.2 in [2] with $X = \mathbb{R}$, $d(x, y) = |x-y|$ and μ the Lebesgue measure, we obtain (1.1) for $T = I^{-,k}_{\alpha,b}$ and $T = I^{+,k}_{\alpha,b}$ with $\mathcal{M}_T = M_{\alpha p}(M^{[(k+1)p]})$. On the other hand, inequality (1.1) was proved in [24] for T equal to the one-sided fractional integral I^+_α and $\mathcal{M}_T = M^-_{\alpha p}(M^{[p]})$, where $M^-_{\alpha p}$ is the one-sided maximal fractional operator (see Theorem 1.4 in [24]). Our result improves these because we obtain a smaller operator $\mathcal{M}_T$ in the right hand side of (1.1). Precisely, we shall prove the following theorem.

Theorem 1.1. *Let w be any weight, $0 < \alpha < 1$, $1 < p < \infty$ and $k \in \mathbb{N} \cup \{0\}$. If $b \in BMO$ then there exists a positive constant C such that*

$$\int_{\mathbb{R}} |I^{+,k}_{\alpha,b} f(x)|^p w(x) dx \le C \|b\|^{kp}_{BMO} \int_{\mathbb{R}} |f(x)|^p M^-_{\alpha p}((M^-)^{[(k+1)p]} w)(x) dx. \quad (1.2)$$

In this paper, every one-sided result has a corresponding one reversing the orientation of the real line.

Let us observe that from Theorem 1.1 with $k = 0$, we also improve Theorem

1.3 in [24]. In fact, from (1.2) for $k = 0$, by using duality and the trivial boundedness $M_\alpha^- f(x) \le I_\alpha^- f(x)$ we get the following dual inequality for the non linear operator M_α^-

$$\int_{\mathbb{R}} [M_\alpha^- f(x)]^{p'} [M_{\alpha p}^-((M^-)^{[p]} w)(x)]^{1-p'}\, dx \le C \int_{\mathbb{R}} |f(x)|^p w(x)^{1-p'}\, dx.$$

In a similar way as [20], we can show that (1.2) is sharp, in the sense that we cannot replace $[(k+1)p]$ by $[(k+1)p]-1$. In fact, as in [20], we can construct a counterexample by taking $f = w = \chi_{(0,1)}$ and $b(x) = \log|x|$.

One of the main results for proving the above theorem is a pointwise equivalence between two type of maximal operators: the composition of one-sided maximal operators $M_\alpha^-((M^-)^k)$ and a one-sided fractional maximal operator associated with the mean Luxemburg norm in the Orlicz spaces, M^-_{α,ϕ_k}, with $\phi_k(t) = t[\log(e+t)]^k$ (see Section 2 for the corresponding definitions). This equivalence was previously proved in [25] for the case $\alpha = 0$, and in [2] for $0 < \alpha < 1$ but in the two-sided case. The proof for $0 < \alpha < 1$ in the one-sided case is not an obvious generalization of the case $\alpha = 0$ not even in the two-sided case. Section 3 is devoted to show how to adapt the two-sided argument to the one-sided context. From now on, we shall denote $\phi_r(t) = t[\log(e+t)]^r$, for $r > 0$.

As a consequence of the above mentioned equivalence we can obtain the inequality in Theorem 1.1 with $M^-_{\alpha p,\phi_{[(k+1)p]}} w$ instead of $M^-_{\alpha p}((M^-)^{[(k+1)p]} w)$. On the other hand, from the proof of Theorem 1.1 it is easy to see that we can obtain a sharper estimate, namely

$$\int_{\mathbb{R}} |I^{+,k}_{\alpha,b} f(x)|^p w(x) dx \le C \|b\|^{kp}_{BMO} \int_{\mathbb{R}} |f(x)|^p M^-_{\alpha p,\phi_{(k+1)p-1+\delta}} w(x) dx, \tag{1.3}$$

for any $\delta > 0$ and with C depending on δ.

The arguments used to prove Theorem 1.1 are similar to the ones used in [20] and later on in [2]. As in those articles we need to prove previously a one-sided version of the weighted norm inequality between the commutator and a suitable maximal operator. Concretely, we shall prove the following result.

Theorem 1.2. *Let $0 < p < \infty$, $0 < \alpha < 1$ and $k \in \mathbb{N} \cup \{0\}$. If $w \in A^+_\infty$ and $b \in BMO$ then there exists a constant C depending on k and on the A^+_∞ constant of w, such that*

$$\int_{\mathbb{R}} |I^{+,k}_{\alpha,b} f(x)|^p w(x) dx \le C \|b\|^{kp}_{BMO} \int_{\mathbb{R}} [M^+_{\alpha,\phi_k} f(x)]^p w(x) dx,$$

for all f such that the left hand side of the previous inequality is finite.

Just reversing the orientation of $\mathbb{R}$ in the proof of the equivalence between M^-_{α,ϕ_k} and $M^-_\alpha((M^-)^k)$ we get that $M^+_{\alpha,\phi_k} \approx M^+_\alpha((M^+)^k)$. Then, we can write the inequality in the above theorem as

$$\int_{\mathbb{R}} |I^{+,k}_{\alpha,b} f(x)|^p w(x) dx \le C \|b\|^{kp}_{BMO} \int_{\mathbb{R}} [M^+_\alpha((M^+)^k f)(x)]^p w(x) dx. \tag{1.4}$$

The case $k = 0$ of (1.4) was proved by F.J. Martín-Reyes, L. Pick and A. de la Torre in [11]. Let us observe that Theorem 1.2 also improves the result obtained as a consequence of the results in [2] for the one-sided commutators (see also [4] for the case $k = 1$) in two ways: putting in the right hand side a smaller operator and allowing a wider class of weights.

On the other hand, Theorem 1.2 implies another proof of the weighted strong (p, q) inequality for the commutators $I^{+,k}_{\alpha,b}$ obtained in [9] (see Theorem 2.9 in [9]). In fact, we can obtain the following corollary.

Corollary 1.3. *Let $0 < \alpha < 1$, $1 < p < q < \infty$ such that $\frac{1}{p} - \frac{1}{q} = \alpha$, $b \in BMO$, and $k \in \mathbb{N} \cup \{0\}$. Let w be a weight such that $w \in A^+(p, q)$, that is, there exists $C > 0$ such that*

$$\left(\frac{1}{h}\int_{x-h}^{x} w^q\right)^{1/q}\left(\frac{1}{h}\int_{x}^{x+h} w^{-p'}\right)^{1/p'} \le C,$$

for all $h > 0$ and $x \in \mathbb{R}$. Then the operator $I^{+,k}_{\alpha,b}$ satisfies the strong (p, q) inequality

$$\left(\int_{\mathbb{R}} |I^{+,k}_{\alpha,b} f|^q w^q\right)^{1/q} \le C\left(\int_{\mathbb{R}} |f|^p w^p\right)^{1/p}.$$

The proof of the corollary follows easily from (1.4), the strong (p, q) inequality for M^+_α (see [1]) and the weighted L^p boundedness of the maximal operator M^+ for A^+_p weights. In fact, notice that $w \in A^+(p, q)$ is equivalent to $w^q \in A^+_\beta$, $\beta = 1 + q/p'$ (therefore $w^q \in A^+_\infty$) and that $w \in A^+(p, q)$ implies $w^p \in A^+_p$.

Now we turn our attention to the case $p = 1$ of the inequality (1.1). In particular we shall study the following endpoint inequality

$$\int_{\{|Tf|>\lambda\}} w \le C\int \psi\left(\frac{|f|}{\lambda}\right)\mathcal{M}_T w, \quad \lambda > 0, \tag{1.5}$$

where ψ is a Young function.

When T is the commutator of the Calderón-Zygmund singular integral of order k, (1.5) was proved in [21] with $\psi(t) = \phi_k(t)$ and $\mathcal{M}_T = M_{\phi_{k+\varepsilon}}$ for any $\varepsilon > 0$ and with a constant depending on ε. The case $k = 0$ was previously proved in [16].

When T is the fractional integral, two different versions of (1.5) were obtained in [3]. Both with $\psi(t) = t$ but in one of them $\mathcal{M}_T = M_\alpha(M_{\phi_\varepsilon})$ for any $\varepsilon > 0$ and in the other one $\mathcal{M}_T w(x) = M_\alpha w(x) + |x|^\alpha M w(x)$.

The second purpose of this paper is to obtain an inequality like (1.5) for the commutators of the one-sided fractional integral operator. In the following theorem we state our result.

Theorem 1.4. *Let* $0 < \alpha < 1$, $b \in BMO$, *and* $k = 0, 1, \dots$. *Then*

$$\int_{\{x\in\mathbb{R}:|I^{+,k}_{\alpha,b}f(x)|>\lambda\}} w \leq C\int_{\mathbb{R}} \phi_k\left(\frac{||b||^k_{BMO}|f(x)|}{\lambda}\right)(M^-_{\phi_k}w(x) + M^-_{\alpha p,\phi_{k+\varepsilon}}w(x))\,dx,$$

for any weight w, $\lambda > 0$, $\varepsilon > 0$, $1 < p < 1+\frac{\varepsilon}{k+1}$ *and where the constant* C *depends on* ε *and* p.

We shall prove the above theorem following the arguments in [21], that is, by using a Calderón-Zygmund decomposition and an induction argument. It is easy to see that from the above theorem and the corresponding one for the commutator $I^{-,k}_{\alpha,b}$ we obtain a similar result for the two-sided commutators $I^k_{\alpha,b}$ in one dimension. In this case the weight in the right hand side will be $M_{\phi_k}w + M_{\alpha p,\phi_{k+\varepsilon}}w$. From Theorem 1.2 in [2], applying the same type of Calderón-Zygmund decomposition as in [21] and following the same steps that in the proof of Theorem 1.4 we can prove the following result in dimension greater than one (the details are left to the reader).

Theorem 1.5. *Let* $0 < \alpha < 1$, $b \in BMO$, *and* $k = 0, 1, \dots$. *Then*

$$\int_{\{x:|I^{k}_{\alpha,b}f(x)|>\lambda\}} w \leq C\int_{\mathbb{R}^n} \phi_k\left(\frac{||b||^k_{BMO}|f(x)|}{\lambda}\right)(M_{\phi_k}w(x) + M_{\alpha p,\phi_{k+\varepsilon}}w(x))\,dx,$$

for any weight w, $\lambda > 0$, $\varepsilon > 0$, $1 < p < 1+\frac{\varepsilon}{k+1}$ *and where the constant* C *depends on* ε *and* p.

Let us notice that when we formally consider $\alpha = 0$ in the above theorem we recover the corresponding result in [21] for the commutators of the Calderón-Zygmund singular integral. On the other hand, the above theorem in the case $k = 0$ gives (1.5) with $\psi(t) = t$ and $\mathcal{M}_T = M + M_{\alpha p,\phi_\varepsilon}$. This result is an endpoint inequality different than the ones in [3]. As far as we know, our results are not comparable with those in [3].

The article is organized in the following way: in Section 2 we give some definitions and preliminaries. Section 3 is devoted to prove the pointwise equivalence between the one-sided maximal operators previously mentioned in this introduction. In Section 4 we shall prove Theorems 1.1 and 1.2, while Theorem 1.4 will be proved in Section 5.

2. Definitions and preliminaries

The one-sided fractional maximal operators M^+_α and M^-_α, $0 \leq \alpha < 1$ are defined for locally integrable functions f by

$$M^+_\alpha f(x) = \sup_{h>0}\frac{1}{h^{1-\alpha}}\int_x^{x+h}|f(y)|\,dy \quad \text{and} \quad M^-_\alpha f(x) = \sup_{h>0}\frac{1}{h^{1-\alpha}}\int_{x-h}^{x}|f(y)|\,dy.$$

When $\alpha = 0$ in the above operators we get the one-sided Hardy-Littlewood maximal operators and we denote them simply with M^+ and M^-.

The good weights for M^+ and M^- are the one-sided weights A_p^+ and A_p^- introduced by E. Sawyer [26]: $w \in A_p^+$ if there exists a constant $C_p < \infty$ such that for all $a < b < c$

$$\frac{1}{(c-a)^p} \left(\int_a^b w(x)\, dx \right) \left(\int_b^c w(x)^{1-p'}\, dx \right)^{p-1} \leq C_p.$$

In the case $p = 1$, $w \in A_1^+$ if $M^- w(x) \leq C_1\, w(x)$. The class A_∞^+ is defined as the union of all the A_p^+ classes, $A_\infty^+ = \cup_{p \geq 1} A_p^+$. The classes A_p^- are defined in a similar way. (See [26], [10], [12] for more definitions and results.)

We shall use two results of the one-sided weights. It is not difficult to see that $(M_\alpha^- f)^\delta \in A_1^+$ for any $0 \leq \alpha, \delta < 1$ and all locally integrable functions f such that $M_\alpha^- f < \infty$ a.e. (see [15] for the two-sided case). The other result is the following: if $w \in A_1^+$ then $w^{1-r} \in A_r^-$, for all $r > 1$ (see [26] or [10]).

Let us recall some of the needed background on Orlicz spaces. The reader is referred to [23] and [13] for a complete account of this topic. A function $\phi : [0, \infty) \to [0, \infty)$ is a Young function if it is continuous, convex and increasing satisfying $\phi(0) = 0$ and $\phi(t) \to \infty$ as $t \to \infty$.

Given a Young function ϕ, we define the ϕ-mean Luxemburg norm of a function f on I by

$$||f||_{\phi,I} = \inf \left\{ \lambda > 0 : \frac{1}{|I|} \int_I \phi\left(\frac{|f|}{\lambda} \right) \leq 1 \right\}. \tag{2.1}$$

It is well known that if $\phi(t) \leq C\psi(t)$ for all $t \geq t_0$ then $||f||_{\phi,I} \leq C||f||_{\psi,I}$, for all intervals I and functions f. Thus, the behavior of $\phi(t)$ for $t \leq t_0$ is not important. If $\phi \approx \psi$, that is there are constants $t_0, c_1, c_2 > 0$ such that $c_1\phi(t) \leq \psi(t) \leq c_2\phi(t)$ for $t \geq t_0$, the latter estimate implies that $||f||_{\phi,I} \approx ||f||_{\psi,I}$.

Each Young function ϕ has an associated complementary Young function $\tilde{\phi}$ satisfying

$$t \leq \phi^{-1}(t)\tilde{\phi}^{-1}(t) \leq 2t,$$

for all $t > 0$. There is a generalization of Hölder's inequality

$$\frac{1}{|I|} \int_I |fg| \leq ||f||_{\phi,I}||g||_{\tilde{\phi},I}. \tag{2.2}$$

A further generalization of Hölder's inequality (see [13]) that will be useful later is the following: If ϕ, ψ and φ are Young functions and

$$\phi^{-1}(t)\psi^{-1}(t) \leq \varphi^{-1}(t)$$

then

$$\|fg\|_{\varphi,I} \leq 2\|f\|_{\phi,I}\|g\|_{\psi,I}. \tag{2.3}$$

A generalization of Young inequality states that if $\phi^{-1}(t)\psi^{-1}(t) \le \varphi^{-1}(t)$ for $t > 0$, then

$$\varphi(st) \le \phi(s) + \psi(t), \tag{2.4}$$

for all $s, t > 0$.

For each locally integrable function f and $0 \le \alpha < 1$, the one-sided fractional maximal operators associated to the Young function ϕ are defined by

$$M^+_{\alpha,\phi} f(x) = \sup_{x<b} (b-x)^\alpha \|f\|_{\phi,(x,b)} \quad \text{and} \quad M^-_{\alpha,\phi} f(x) = \sup_{a<x} (x-a)^\alpha \|f\|_{\phi,(a,x)}.$$

When $\alpha = 0$ we use the notation M^+_ϕ instead of $M^+_{0,\phi}$ (M^-_ϕ instead of $M^-_{0,\phi}$).

We shall need the following results about these maximal operators. First, notice that if $\phi(t) = t$ then $||f||_{\phi,I} = \frac{1}{|I|}\int_I |f|$ and let us recall that $\phi_r(t) = t[\log(e+t)]^r$. For every $0 \le \alpha < 1$ and every $l \le s$, we get that

$$M^-_\alpha f(x) = M^-_{\alpha,\phi_0} f(x) \le M^-_{\alpha,\phi_l} f(x) \le M^-_{\alpha,\phi_s} f(x). \tag{2.5}$$

Splitting the family of intervals in the definition of $M^-_{\alpha,\phi}$ in two families, those intervals with measure smaller than one and the rest, we can easily prove that, for any Young function ϕ and for every $p > 1$,

$$M^-_{\alpha,\phi} f(x) \le M^-_{\alpha p,\phi} f(x) + M^-_\phi f(x). \tag{2.6}$$

On the other hand, if $I = (a, b)$ and $I^- = (c, a)$ with $|I| = |I^-|$, for any Young function ϕ and nonnegative function f with $M^-_{\alpha,\phi} f(x) < \infty$ a.e., we get that

$$|I|^\alpha M^-_\phi(f\chi_{\mathbb{R}\setminus(I^-\cup I)})(y) \le C M^-_{\alpha,\phi}(f\chi_{\mathbb{R}\setminus(I^-\cup I)})(b), \qquad \text{a.e } y \in I, \tag{2.7}$$

and

$$M^-_{\alpha,\phi}(f\chi_{\mathbb{R}\setminus(I^-\cup I)})(y) \approx \inf_{z\in I} M^-_{\alpha,\phi}(f\chi_{\mathbb{R}\setminus(I^-\cup I)})(z), \qquad \text{a.e } y \in I. \tag{2.8}$$

These results follow easily from the definition of the maximal functions $M^-_{\alpha,\phi}$ and keeping in mind which is the support of $f\chi_{\mathbb{R}\setminus(I^-\cup I)}$.

Let us recall that a locally integrable function b belongs to BMO $= BMO(\mathbb{R})$ if

$$\|b\|_{\mathrm{BMO}} = \sup_I \frac{1}{|I|}\int_I |b(x) - b_I|\, dx < \infty,$$

where the supremum runs over all intervals $I \subset \mathbb{R}$ and where b_I stands for the average of b over I. It is easy to prove that a function b is in BMO if for each interval I there exists a constant $c(I)$ such that

$$\sup_I \frac{1}{|I|}\int_I |b(x) - c(I)|\, dx < \infty.$$

Further, this supremum is comparable to $||b||_{BMO}$.

We state some results about BMO functions that we shall use in the article (for the proofs see for example [27]). If $b \in BMO$ then there exists an absolute constant C such that, for any interval $I = (x, x+h)$ and $j \geq 0$,

$$|b_{I_j} - b_I| \leq C(j+1)||b||_{BMO}, \tag{2.9}$$

where $I_j = (x + 2^j h, x + 2^{j+1}h)$. Obviously, the same holds if $I = (x-h, x)$ and $I_j = (x - 2^{j+1}h, x - 2^j h)$.
By applying the John-Nirenberg inequality we get the following facts:

(i) For each p, $1 < p < \infty$, there exists a constant C_p such that

$$\sup_I \left(\frac{1}{|I|} \int_I |b(x) - b_I|^p \, dx \right)^{1/p} \leq C_p ||b||_{BMO}. \tag{2.10}$$

(ii) If $b \in BMO$ then there exists a constant C such that for every interval I,

$$\frac{1}{|I|} \int_I \exp\left(\frac{|b(x) - b_I|}{C||b||_{BMO}} \right) dx < \infty. \tag{2.11}$$

As a consequence of (2.11) we get that for $\tilde{\phi}(t) = \exp(t)$

$$||b - b_I||_{\tilde{\phi}, I} \leq C||b||_{BMO}. \tag{2.12}$$

For a locally integrable funtion f we define the one-sided sharp maximal function as

$$M^{+,\#} f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} \left(f(y) - \frac{1}{h} \int_{x+h}^{x+2h} f \right)^+ dy,$$

where $z^+ = \max(z, 0)$. It is proved in [12] that

$$\begin{aligned} M^{+,\#} f(x) \leq \sup_{h>0} \inf_{a \in \mathbb{R}} \Bigg\{ & \frac{1}{h} \int_x^{x+h} (f(y) - a)^+ \, dy \\ & + \frac{1}{h} \int_{x+h}^{x+2h} (a - f(y))^+ \, dy \Bigg\}. \end{aligned} \tag{2.13}$$

Given an operator T, we use the notation $T_{(\delta)}(f)$, $0 < \delta < 1$, for the operator $[T(|f|^\delta)]^{1/\delta}$.

3. Equivalence between two maximal operators

This section is devoted to prove the following equivalence between two one-sided maximal operators.

Theorem 3.1. *Let $0 \leq \alpha < 1$, $k \in \mathbb{N}$ and $\phi_k(t) = t[\log(e+t)]^k$. Then, there exist constants $C_1, C_2 > 0$ such that*

$$M_\alpha^-((M^-)^k f)(x) \leq C_1 M^-_{\alpha, \phi_k} f(x) \tag{3.1}$$

and

$$M^-_{\alpha,\phi_k}f(x) \le C_2 M^-_\alpha((M^-)^k f)(x), \tag{3.2}$$

for every $x \in \mathbb{R}$.

To adapt the two-sided arguments in the proof of (3.2) we shall need the following two lemmas.

Lemma 3.2. *Let* $0 \le \alpha < 1$ *and* $k \in \mathbb{N}$. *For each interval* $I = (a,x)$ *let us consider the decomposition* $I = I_k^- \cup I_k^+$ *such that* $I_k^- = (a, a + 2^{-k}|I|)$ *and* $I_k^+ = (a + 2^{-k}|I|, x)$, *and let us define the maximal operator*

$$N^-_{\alpha,\phi_k}f(x) = \sup_{I=(a,x)} |I_k^-|^\alpha ||f||_{\phi_k, I_k^-}.$$

Then, there exists a constant C *depending on* k *such that*

$$M^-_{\alpha,\phi_k}f(x) \le C \sum_{j=0}^{k} N^-_{\alpha,\phi_j}f(x),$$

where N^-_{α,ϕ_0} *simply denotes* M^-_{α,ϕ_0}.

Proof. First, notice that for every $\gamma > 1$ there exists a constant $C_{\gamma,k}$ depending on γ and k such that

$$\phi_k(\gamma t) \le C_{\gamma,k} \sum_{j=0}^{k-1} \phi_j(t) + \gamma \phi_k(t). \tag{3.3}$$

In fact,

$$\begin{aligned} \phi_k(\gamma t) &= \gamma t[\log(e + \gamma t)]^k \le \gamma t[\log \gamma + \log(e+t)]^k \\ &\le \gamma t \sum_{j=0}^{k} c_{j,k}[\log\gamma]^{k-j}[\log(e+t)]^j \\ &\le \max_{0\le j\le k-1}\{\gamma c_{j,k}[\log\gamma]^{k-j}\} \sum_{j=0}^{k-1}\phi_j(t) + \gamma\phi_k(t), \end{aligned}$$

where $c_{j,k}$ are constants proceeding from the Newton's formula. Now, by using (3.3) with $\gamma = \gamma_k = 2^k/(2^k-1)$ we shall prove that

$$\gamma_k ||f||_{\phi_k,I} \le C\left[\sum_{j=0}^{k-1} ||f||_{\phi_j,I} + ||f||_{\phi_k,I_k^-}\right] + ||f||_{\phi_k,I_k^+}, \tag{3.4}$$

with $C = \max\{C_{\gamma_k,k}, 1\}$. In fact, if we denote by μ the right hand side of (3.4), using (3.3) with $\gamma = \gamma_k$, the convexity of the functions ϕ_j, $j = 1, \cdots, k$ and (2.1) we have

$$\begin{aligned}
&\frac{1}{|I|}\int_I \phi_k\left(\frac{\gamma_k|f|}{\mu}\right)\\
&\quad\le \frac{C_{\gamma_k,k}}{|I|}\sum_{j=0}^{k-1}\int_I \phi_j\left(\frac{|f|}{\mu}\right)+\frac{\gamma_k}{|I|}\int_{I_k^-}\phi_k\left(\frac{|f|}{\mu}\right)+\frac{\gamma_k}{|I|}\int_{I_k^+}\phi_k\left(\frac{|f|}{\mu}\right)\\
&\quad\le \frac{C_{\gamma_k,k}}{\mu}\sum_{j=0}^{k-1}||f||_{\phi_j,I}\left[\frac{1}{|I|}\int_I\phi_j\left(\frac{|f|}{||f||_{\phi_j,I}}\right)\right]\\
&\qquad+\frac{\gamma_k||f||_{\phi_k,I_k^-}}{2^k\mu}\left[\frac{1}{|I_k^-|}\int_{I_k^-}\phi_k\left(\frac{|f|}{||f||_{\phi_k,I_k^-}}\right)\right]\\
&\qquad+\frac{||f||_{\phi_k,I_k^+}}{\mu}\left[\frac{1}{|I_k^+|}\int_{I_k^+}\phi_k\left(\frac{|f|}{||f||_{\phi_k,I_k^+}}\right)\right]\\
&\quad\le \frac{C_{\gamma_k,k}\sum_{j=0}^{k-1}||f||_{\phi_j,I}+\gamma_k 2^{-k}||f||_{\phi_k,I_k^-}+||f||_{\phi_k,I_k^+}}{\mu}\le 1.
\end{aligned}$$

Then, using again (2.1) we get (3.4). Now, let us observe that by (3.4)

$$\begin{aligned}
|I|^\alpha||f||_{\phi_k,I} &\le \frac{C}{\gamma_k}\sum_{j=0}^{k-1}|I|^\alpha||f||_{\phi_j,I}+\frac{C}{\gamma_k}|I|^\alpha||f||_{\phi_k,I_k^-}+\frac{|I|^\alpha}{\gamma_k}||f||_{\phi_k,I_k^+}\\
&\le \frac{C}{\gamma_k}\sum_{j=0}^{k-1}M^-_{\alpha,\phi_j}f(x)+\frac{C2^{k\alpha}}{\gamma_k}N^-_{\alpha,\phi_k}f(x)+\gamma_k^{\alpha-1}M^-_{\alpha,\phi_k}f(x).
\end{aligned}$$

Taking supremum on $I=(a,x)$ and using that $\gamma_k>1$ we get that

$$M^-_{\alpha,\phi_k}f(x)\le C\left[\sum_{j=0}^{k-1}M^-_{\alpha,\phi_j}f(x)+N^-_{\alpha,\phi_k}f(x)\right].$$

To finish the proof of the lemma notice that, by definition, N^-_{α,ϕ_0} is equal to M^-_{α,ϕ_0} and that M^-_{α,ϕ_0} is pointwise equivalent to the one-sided fractional maximal operator M^-_α. Then, clearly, the lemma holds for $k=1$. For general $k\in\mathbb{N}$ the lemma follows by applying an induction argument over k. □

Lemma 3.3. *Let $I=(a,b)$ be a fix interval and let $I^-=(a,(a+b)/2)$. Then, there exists a constant C such that*

$$\frac{1}{\lambda}\int_{\{x\in I^-:f(x)>\lambda\}} f\le C|\{x\in I: M^-(f\chi_{I^-})(x)>\lambda\}|, \tag{3.5}$$

for any $\lambda\ge\frac{1}{|I^-|}\int_{I^-}f$ and all nonnegative integrable functions f.

Proof. It is well known (see for example [6], pp. 423) that

$$\frac{1}{\lambda}\int_{\{x:f(x)>\lambda\}} f \leq \frac{1}{\lambda}\int_{\{x:M^-f(x)>\lambda\}} f = |\{x : M^-f(x) > \lambda\}|.$$

Applying the above result to $f\chi_{I^-}$ we get that

$$\frac{1}{\lambda}\int_{\{x\in I^-:f(x)>\lambda\}} f \leq C|\{x : M^-(f\chi_{I^-})(x) > \lambda\}|.$$

Now, the lemma follows since if $x < a$ then $M^-(f\chi_{I^-})(x) = 0$ and if $x > b$ then $M^-(f\chi_{I^-})(x) \leq \frac{1}{|I^-|}\int_{I^-} f \leq \lambda$.

□

Proof of Theorem 3.1. Without loss of generality we may assume that $f \geq 0$. To prove (3.1) let us consider an interval $I = (a, x)$. Notice that by inequality (4.4) in [2]

$$\frac{1}{|I|}\int_I (M^-)^k f(y)\,dy \leq \frac{1}{|I|}\int_I M^k f(y)\,dy \leqslant C||f||_{\phi_k,I}, \tag{3.6}$$

for any f such that $\operatorname{supp}(f) \subset I$. Now, writting $f = f_1 + f_2$ where $f_1 = f\chi_{2I}$ with $2I = (2a - x, x)$ we get that

$$|I|^{\alpha-1}\int_I (M^-)^k f \leq |I|^{\alpha-1}\int_I (M^-)^k f_1 + |I|^{\alpha-1}\int_I (M^-)^k f_2 = A + B.$$

By (3.6)

$$A \leq C|2I|^{\alpha-1}\int_{2I} (M^-)^k f_1(y)\,dy \leq C|2I|^{\alpha}||f||_{\phi_k,2I} \leq CM^-_{\alpha,\phi_k}f(x).$$

Using the equivalence for $\alpha = 0$ (see Proposition 1 in [25]) and (2.7) we get that

$$B \leq |I|^{\alpha-1}\int_I M^-_{\phi_{k-1}} f_2(y)\,dy \leq CM^-_{\alpha,\phi_{k-1}}f_2(x) \leq CM^-_{\alpha,\phi_k}f(x).$$

Then (3.1) follows taking supremum on $a < x$.

To prove (3.2) we proceed as in [22] (see also [2]). By Lemma 3.2 it suffices to show that

$$N^-_{\alpha,\phi_k}f(x) \leq CM^-_\alpha((M^-)^k f)(x),$$

for any $k \in \mathbb{Z}$. Let $I = (a, x)$ be any interval. Notice that it is enough to show that there is a constant C_k such that

$$\|f\|_{\phi_k,I_k^-} \leq \frac{C_k}{|I|}\int_I (M^-)^k f(y)\,dy. \tag{3.7}$$

Let $\lambda_k = \lambda_k(f) = \frac{1}{|I|}\int_I (M^-)^k f$. To prove (3.7) we shall show that there is a constant $C_k > 1$ such that

$$\frac{1}{|I_k^-|}\int_{I_k^-} \phi_k\Big(\frac{f}{C_k\lambda_k}\Big) \leq 1. \tag{3.8}$$

To prove (3.8) we shall use induction on k and the formula

$$\int_I \Phi(|f|)\,d\nu = \int_a^\infty \Phi'(\lambda)\nu(\{x\in I : |f(x)|>\lambda\})\,d\lambda,$$

which holds for any increasing continuously differentiable function Φ and where a is such that $\Phi(a)=0$. In fact, for $k=1$ and $g=\frac{f}{C_1\lambda_1}$ we have

$$\begin{aligned}
\frac{1}{|I_1^-|}\int_{I_1^-}\phi_1\left(\frac{f}{C_1\lambda_1}\right) &= \frac{1}{|I_1^-|}\int_{I_1^-} g[\log(e+g)] \\
&= \frac{1}{|I_1^-|}\int_{1-e}^\infty \frac{1}{e+\lambda} g(\{x\in I_1^- : g(x)>\lambda\})\,d\lambda \\
&= \frac{1}{|I_1^-|}\left(\int_{1-e}^1 + \int_1^\infty\right)\frac{1}{e+\lambda}\int_{\{x\in I_1^- : g(x)>\lambda\}} g(x)\,dx\,d\lambda \\
&= I+II.
\end{aligned}$$

Since $f(y)\le M^-f(y)$ a.e.,

$$\begin{aligned}
I &\le \frac{\log(1+e)g(I_1^-)}{|I_1^-|} = \frac{\log(1+e)}{|I_1^-|C_1\lambda_1}\int_{I_1^-} f(y)\,dy \\
&= \frac{2\log(1+e)\int_{I_1^-} f(y)\,dy}{C_1\int_I M^-f(y)\,dy} \le \frac{2\log(1+e)}{C_1} < 1,
\end{aligned}$$

if we choose $C_1 > 2\log(1+e)$. On the other hand, by Lemma 3.3, since $\lambda > 1 > \frac{1}{|I_1^-|}\int_{I_1^-} g$ there exists a constant C such that

$$\begin{aligned}
II &= \frac{1}{|I_1^-|}\int_1^\infty \frac{1}{e+\lambda}\int_{\{x\in I_1^- : g(x)>\lambda\}} g(x)\,dx\,d\lambda \\
&\le \frac{C}{|I_1^-|}\int_1^\infty \frac{\lambda}{e+\lambda}|\{x\in I : M^-g(x)>\lambda\}|\,d\lambda \\
&\le \frac{C}{|I_1^-|}\int_0^\infty |\{x\in I : M^-g(x)>\lambda\}|\,d\lambda \\
&\le \frac{2C}{|I|}\int_I M^-g(x)\,dx = \frac{2C}{\lambda_1 C_1|I|}\int_I M^-f(x)\,dx = \frac{2C}{C_1} < 1,
\end{aligned}$$

provided that $C_1 > 2C$. Thus we have proved the case $k=1$. Suppose that $k>1$ and the result holds for $k-1$. If $g=\frac{f}{C_k\lambda_k}$ then

$$\begin{aligned}
\frac{1}{|I_k^-|}\int_{I_k^-}\phi_k\left(\frac{f}{C_k\lambda_k}\right) &= \frac{1}{|I_k^-|}\int_{I_k^-} g[\log(e+g)]^k \\
&= \frac{k}{|I_k^-|}\int_{1-e}^\infty \frac{[\log(e+\lambda)]^{k-1}}{e+\lambda} g(\{x\in I_k^- : g(x)>\lambda\})\,d\lambda \\
&= \frac{k}{|I_k^-|}\left(\int_{1-e}^1 + \int_1^\infty\right) = I+II.
\end{aligned}$$

Notice that,

$$I \quad \leq \quad \frac{[\log(1+e)]^k}{|I_k^-|} g(I_k^-) = \frac{2^k[\log(1+e)]^k}{C_k \int_I (M^-)^k f} \int_{I_k^-} f \leq \frac{2^k[\log(1+e)]^k}{C_k} < 1,$$

if we choose $C_k > 2^k[\log(1+e)]^k$. Notice that by this election of C_k, $\frac{1}{|I_k^-|}\int_{I_k^-} g \leq 1$. Then, applying Lemma 3.3 we get

$$\begin{aligned}
II &= \frac{k}{|I_k^-|}\int_1^\infty \frac{[\log(e+\lambda)]^{k-1}}{e+\lambda}\int_{\{x\in I_k^- : g(x)>\lambda\}} g(x)\,dx\,d\lambda \\
&\leq \frac{kC}{|I_k^-|}\int_1^\infty \frac{\lambda[\log(e+\lambda)]^{k-1}}{e+\lambda}|\{x\in I_{k-1}^- : M^-g(x)>\lambda\}|d\lambda \\
&\leq \frac{kC}{|I_k^-|}\int_1^\infty [\log(e+\lambda)]^{k-1}|\{x\in I_{k-1}^- : M^-g(x)>\lambda\}|d\lambda \\
&\leq \frac{kC}{|I_k^-|}\int_0^\infty \phi_{k-1}'(\lambda)|\{x\in I_{k-1}^- : M^-g(x)>\lambda\}|d\lambda \\
&\leq \frac{kC}{|I_k^-|}\int_{I_{k-1}^-} \phi_{k-1}(M^-g) \\
&\leq \frac{2kC}{|I_{k-1}^-|}\int_{I_{k-1}^-} \phi_{k-1}\left(\frac{M^-f}{C_k\lambda_k(f)}\right).
\end{aligned}$$

Let us observe that $\lambda_k(f) = \lambda_{k-1}(M^-f)$. Then, choosing $C_k > 2kCC_{k-1}$, using that the function ϕ_{k-1} is convex and the induction hypothesis we obtain that

$$\begin{aligned}
II &\leq 2kC\frac{C_{k-1}}{C_k}\frac{1}{|I_{k-1}^-|}\int_{I_{k-1}^-}\phi_{k-1}\left(\frac{M^-f}{C_{k-1}\lambda_{k-1}(M^-f)}\right) \\
&\leq 2kC\frac{C_{k-1}}{C_k} < 1.
\end{aligned}$$

In this way, inequality (3.8) is proved and so is inequality (3.2).

□

4. Proof of Theorems 1.1 and 1.2

We begin by proving the following pointwise estimate.

Lemma 4.1. *Let $0<\alpha<1$, $b\in BMO$, $k\in\mathbb{N}$ and $0<\delta<\epsilon<1$. Then there exists a constant C such that*

$$M_{(\delta)}^{+,\#}(I_{\alpha,b}^{+,k}f)(x) \leq C\left(\sum_{m=0}^{k-1}\|b\|_{BMO}^{k-m}M_{(\epsilon)}^+(I_{\alpha,b}^{+,m}f)(x) + \|b\|_{BMO}^k M_{\alpha,\phi_k}^+ f(x)\right),$$

for $x\in\mathbb{R}$ and $f\geq 0$.

Proof. As in [5] (see also [19]), writing $b(x)-b(y)=(b(x)-\lambda)-(b(y)-\lambda)$, where λ is an arbitrary constant, we can obtain the following decomposition

$$I^{+,k}_{\alpha,b}f(x)=I^+_\alpha((b-\lambda)^k f)(x)+\sum_{m=0}^{k-1}C_{m,k}(b(x)-\lambda)^{k-m}I^{+,m}_{\alpha,b}f(x). \tag{4.1}$$

Observe that $0<\delta<1$ implies $||a|^\delta-|c|^\delta|\le|a-c|^\delta$ for $a,c\in\mathbb{R}$. Then, given $x\in\mathbb{R}$ and $h>0$ and taking into account (2.13), it is enough to show that for some constant a depending on x and h, there exists $C>0$ such that

$$\begin{aligned}&\left(\frac{1}{h}\int_x^{x+2h}|I^{+,k}_{\alpha,b}f(y)-a|^\delta dy\right)^{1/\delta}\\ &\quad\le\; C\left(\sum_{m=0}^{k-1}\|b\|^{k-m}_{BMO}M^+_{(\epsilon)}(I^{+,m}_{\alpha,b}f)(x)+\|b\|^k_{BMO}M^+_{\alpha,\phi_k}f(x)\right). \end{aligned}\tag{4.2}$$

Now, let us fix x and $h>0$ and let $J=[x,x+4h]$. Then we write $f=f_1+f_2$, where $f_1=f\chi_J$. Taking $\lambda=b_J=\frac{1}{4h}\int_x^{x+4h}b$, $a=I^+_\alpha((b-b_J)^kf_2)(x+2h)$ and using (4.1) we have that

$$\left(\frac{1}{h}\int_x^{x+2h}|I^{+,k}_{\alpha,b}f(y)-a|^\delta dy\right)^{1/\delta}\le I_1+I_2+I_3, \tag{4.3}$$

where

$$I_1=C\sum_{m=0}^{k-1}\left(\frac{1}{h}\int_x^{x+2h}|b(y)-b_J|^{(k-m)\delta}|I^{+,m}_{\alpha,b}f(y)|^\delta dy\right)^{1/\delta},$$

$$I_2=C\left(\frac{1}{h}\int_x^{x+2h}|I^+_\alpha((b-b_J)^kf_1)(y)|^\delta dy\right)^{1/\delta}$$

and

$$I_3=\left(\frac{1}{h}\int_x^{x+2h}|I^+_\alpha((b-b_J)^kf_2)(y)-I^+_\alpha((b-b_J)^kf_2)(x+2h)|^\delta dy\right)^{1/\delta}.$$

Choosing $1<r<\epsilon/\delta$, using Hölder's inequality and (2.10), it follows that

$$\begin{aligned}I_1&\le C\sum_{m=0}^{k-1}\left(\frac{1}{h}\int_x^{x+2h}|b-b_J|^{(k-m)\delta r'}\right)^{1/\delta r'}\left(\frac{1}{h}\int_x^{x+2h}|I^{+,m}_{\alpha,b}f|^{\delta r}\right)^{1/\delta r}\\ &\le C\sum_{m=0}^{k-1}\|b\|^{k-m}_{BMO}M^+_{(\delta r)}(I^{+,m}_{\alpha,b}f)(x)\\ &\le C\sum_{m=0}^{k-1}\|b\|^{k-m}_{BMO}M^+_{(\epsilon)}(I^{+,m}_{\alpha,b}f)(x).\end{aligned}$$

Now, we estimate I_2. Since I^+_α is of weak type $(1,(1-\alpha)^{-1})$, Kolmogorov's inequality and (2.2) with $\phi_k(t)=t[\log(e+t)]^k$ and $\tilde{\phi}_k(t)=\exp(t^{1/k})$ yield

$$\begin{aligned} I_2 &\leq \frac{C}{h^{1-\alpha}}\int_x^{x+4h}|b-b_J|^k f(y)dy \\ &\leq Ch^\alpha \||b-b_J|^k\|_{\tilde{\phi}_k,J}\|f\|_{\phi_k,J}. \end{aligned}$$

Then, by (2.12) we get that

$$I_2 \quad\leq\quad C\|b-b_J\|^k_{\tilde{\phi},J}M^+_{\alpha,\phi_k}f(x)\leq C\|b\|^k_{BMO}M^+_{\alpha,\phi_k}f(x).$$

Notice that, by Jensen's inequality

$$\begin{aligned} I_3 &\leq \frac{1}{h}\int_x^{x+2h}\left|\int_{x+4h}^\infty (b(t)-b_J)^k f(t)\left((t-y)^{\alpha-1}-(t-(x+2h))^{\alpha-1}\right)\,dt\right|dy \\ &\leq \frac{1}{h}\int_x^{x+2h}\sum_{j=2}^\infty\int_{x+2^jh}^{x+2^{j+1}h}|b(t)-b_J|^k|f(t)|\left|(t-y)^{\alpha-1}-(t-(x+2h))^{\alpha-1}\right|dtdy. \end{aligned}$$

Now, by using the mean value theorem we have that

$$I_3\leq Ch^{\alpha-1}\sum_{j=2}^\infty 2^{j(\alpha-2)}\int_{x+2^jh}^{x+2^{j+1}h}|b(t)-b_J|^k|f(t)|\,dt.$$

Let $J_j=[x+2^jh,x+2^{j+1}h]$. Then

$$\begin{aligned} I_3 &\leq Ch^{\alpha-1}\sum_{j=2}^\infty 2^{j(\alpha-2)}\int_{x+2^jh}^{x+2^{j+1}h}|b(t)-b_{J_j}|^k|f(t)|\,dt \\ &\quad+Ch^{\alpha-1}\sum_{j=2}^\infty 2^{j(\alpha-2)}|b_{J_j}-b_J|^k\int_{x+2^jh}^{x+2^{j+1}h}|f(t)|\,dt=I+II. \end{aligned}$$

Observe that from the generalized Hölder's inequality and (2.12) we obtain

$$\begin{aligned} I &\leq Ch^\alpha\sum_{j=2}^\infty 2^{j(\alpha-1)}||b-b_{J_j}||^k_{\tilde{\phi},J_j}||f||_{\phi_k,J_j} \\ &\leq C||b||^k_{BMO}\sum_{j=2}^\infty 2^{-j}(2^jh)^\alpha||f||_{\phi_k,J_j}\leq C||b||^k_{BMO}M^+_{\alpha,\phi_k}f(x). \end{aligned}$$

On the other hand, using (2.9) it is easy to see that

$$\begin{aligned} II &\leq Ch^{\alpha-1}\sum_{j=2}^\infty 2^{j(\alpha-2)}\left(\int_{x+2^jh}^{x+2^{j+1}h}|f(t)|\,dt\right)(j+1)^k||b||^k_{BMO} \\ &\leq C||b||^k_{BMO}M^+_\alpha f(x). \end{aligned}$$

Putting together the above estimates we are done. □

Proof of Theorem 1.2. We shall prove the theorem proceeding by induction on k. As we mentioned in the introduction, the case $k=0$ was proved in [11]. So let us assume that the theorem is true for all $j \le k-1$ and let us prove the case $j=k$. Applying Theorem 4 in [12] and Lemma 4.1 we have that, for every δ small enough and any ϵ with $\delta < \epsilon < 1$,

$$\begin{aligned} ||I^{+,k}_{\alpha,b} f||_{p,w} &\le ||M^+_{(\delta)}(I^{+,k}_{\alpha,b} f)||_{p,w} \le C||M^{+,\#}_{(\delta)}(I^{+,k}_{\alpha,b} f)||_{p,w} \\ &\le C\sum_{j=0}^{k-1} ||b||^{k-j}_{BMO} ||M^+_{(\epsilon)}(I^{+,j}_{\alpha,b} f)||_{p,w} + C||b||^k_{BMO}||M^+_{\alpha,\phi_k} f||_{p,w}. \end{aligned}$$

Notice that the condition $M^+_{(\delta)}(I^{+,k}_{\alpha,b} f) \in L^p(w)$ in Theorem 4 ([12]) is satisfied. In fact, observe that we work with functions f such that $I^{+,k}_{\alpha,b} f \in L^p(w)$. On the other hand, since $w \in A^+_\infty$, there exists $r > 1$ such that $w \in A^+_r$. Then, for all $\delta > 0$ small enough we have that $r < \frac{p}{\delta}$ and thus, $w \in A^+_{\frac{p}{\delta}}$. Therefore $M^+_{(\delta)}(I^{+,k}_{\alpha,b} f) \in L^p(w)$. Now, we choose $\epsilon > \delta$ such that $r < \frac{p}{\epsilon}$. So that $w \in A^+_{\frac{p}{\epsilon}}$ and we get

$$||M^+_{(\epsilon)}(I^{+,j}_{\alpha,b} f)||^p_{p,w} = \int_{\mathbb{R}} [M^+(|I^{+,j}_{\alpha,b} f|^\epsilon]^{\frac{p}{\epsilon}} w \le C\int_{\mathbb{R}} [|I^{+,j}_{\alpha,b} f|^\epsilon]^{\frac{p}{\epsilon}} w = C||I^{+,j}_{\alpha,b} f||^p_{p,w}.$$

Then, by recurrence and taking into account (2.5),

$$\begin{aligned} ||I^{+,k}_{\alpha,b} f||_{p,w} &\le C\sum_{j=0}^{k-1} ||b||^{k-j}_{BMO}||I^{+,j}_{\alpha,b} f||_{p,w} + C||b||^k_{BMO}||M^+_{\alpha,\phi_k} f||_{p,w} \\ &\le C\sum_{j=0}^{k-1} ||b||^{k-j}_{BMO}||b||^j_{BMO}||M^+_{\alpha,\phi_j} f||_{p,w} + C||b||^k_{BMO}||M^+_{\alpha,\phi_k} f||_{p,w} \\ &\le C||b||^k_{BMO}||M^+_{\alpha,\phi_k} f||_{p,w}. \end{aligned}$$

$\square$

Proof of Theorem 1.1. The proof of this theorem is similar to the corresponding one in [2] and follows the lines of Pérez's articles (see for example [20]), but we include it for the sake of completeness. First, let us observe that we only need to consider the case $\alpha p < 1$ (see, for example, the beginning of the proof of Theorem 1.2 in [2]). By a duality argument, it is enough to show that

$$\int_{\mathbb{R}} |I^{-,k}_{\alpha,b} f|^{p'} (M^-_{\alpha p}(M^-)^{[(k+1)p]} w)^{1-p'} \le C\int_{\mathbb{R}} |f|^{p'} w^{1-p'}. \tag{4.4}$$

As mentioned in Section 2, for $0 < \alpha < 1$ and $0 \le \delta < 1$ the function $(M^-_\alpha g)^\delta$ belongs to A^+_1. Thus, choosing $r > p'$ and $\delta = (p'-1)/(r-1)$,

$$[M^-_{\alpha p}((M^-)^{[(k+1)p]} w)(x)]^{1-p'} = \left\{[M^-_{\alpha p}((M^-)^{[(k+1)p]} w)(x)]^{\frac{p'-1}{r-1}}\right\}^{1-r} \in A^-_r \subset A^-_\infty.$$

Applying Theorem 1.2 with the orientation reversed we get

$$\begin{aligned}
&\int |I^{-,k}_{\alpha,b} f(x)|^{p'} [M^-_{\alpha p}((M^-)^{[(k+1)p]} w)(x)]^{1-p'} dx \\
&\leq \; C \int [M^-_{\alpha,\phi_k} f(x)]^{p'} [M^-_{\alpha p}((M^-)^{[(k+1)p]} w)(x)]^{1-p'} dx, \qquad (4.5)
\end{aligned}$$

and then, by Theorem 3.1, it is enough to show that

$$\int [M^-_{\alpha,\phi_k} f(x)]^{p'} [M^-_{\alpha p,\phi_{[(k+1)p]}} w(x)]^{1-p'} dx \quad \leq \quad C \int |f(x)|^{p'} w(x)^{1-p'} dx.$$

Defining $g = f w^{-1/p}$, the above inequality may be stated as

$$\int [M^-_{\alpha,\phi_k} (g w^{1/p})(x)]^{p'} [M^-_{\alpha p,\phi_{[(k+1)p]}} w(x)]^{1-p'} dx \quad \leq \quad C \int |g(x)|^{p'} dx.$$

Now, we shall use that, for large t,

$$\begin{aligned}
\phi_k^{-1}(t) \approx \frac{t}{[\log(e+t)]^k} \quad &= \quad \frac{t^{1/p}}{[\log(e+t)]^{k+(p-1+\epsilon)/p}} \times t^{1/p'} [\log(e+t)]^{(p-1+\epsilon)/p} \\
&= \quad \psi_k^{-1}(t) \times \varphi^{-1}(t),
\end{aligned}$$

where $\psi_k(t) \approx t^p [\log(e+t)]^{(k+1)p-1+\epsilon}$ and $\varphi(t) \approx t^{p'} [\log(e+t)]^{-(1+(p'-1)\epsilon)}$ (see [14]). Thus, by (2.3),

$$(x-a)^\alpha \| g w^{1/p} \|_{\phi_k,(a,x)} \leq C (x-a)^\alpha \|g\|_{\varphi,(a,x)} \|w^{1/p}\|_{\psi_k,(a,x)}.$$

Choosing $\epsilon > 0$ so that $(k+1)p - 1 + \epsilon = [(k+1)p]$ we have that

$$(x-a)^\alpha \| g w^{1/p} \|_{\phi_k,(a,x)} \leq C \|g\|_{\varphi,(a,x)} \left((x-a)^{\alpha p} \|w\|_{\phi_{[(k+1)p]},(a,x)} \right)^{1/p}.$$

Therefore

$$M^-_{\alpha,\phi_k} (g w^{1/p})(x) \leq C M^-_\varphi g(x) [M^-_{\alpha p,\phi_{[(k+1)p]}} w(x)]^{1/p}.$$

Moreover, since φ satisfies condition $B_{p'}$ (that is, there is a positive constant c such that $\int_c^\infty \frac{\varphi(t)}{t^{p'+1}}\, dt < \infty$), applying Theorem 1.7 in [17] we get that

$$\begin{aligned}
\int [M^-_{\alpha,\phi_k} (g w^{1/p})(x)]^{p'} [M^-_{\alpha p,\phi_{[(k+1)p]}} w(x)]^{1-p'} dx \quad &\leq \quad C \int |M^-_\varphi g(x)|^{p'} dx \\
&\leq \quad C \int |M_\varphi g(x)|^{p'} dx \\
&\leq \quad C \int |g(x)|^{p'} dx,
\end{aligned}$$

where $M_\varphi g(x) = \sup_{x \in I} ||g||_{\varphi,I}$. This concludes the proof of the theorem. □

Remark 4.2. Observe that from the proof of Theorem 1.1 we can obtain the following sharper inequality

$$\int_{\mathbb{R}} |I^{+,k}_{\alpha,b} f(x)|^p w(x) dx \leq C \|b\|^{kp}_{BMO} \int_{\mathbb{R}} |f(x)|^p M^-_{\alpha p,\phi_\eta} w(x) dx,$$

with $\eta = (k+1)p - 1 + \varepsilon$, $\varepsilon > 0$, and where the constant C depends on ε. In fact, to see this we only need to show that $(M^-_{\alpha p,\phi_\eta} w)^{1-p'} \in A^-_\infty$. Let us sketch the proof.

First, notice that $M^-_{\alpha p,\phi_\eta} w \approx [M^-_{\alpha,\psi_\eta}(w^{1/p})]^p$, where $\psi_\eta(t) = t^p[\log(e+t)]^\eta \approx \phi_\eta(t^p)$. Then, if we prove that $[M^-_{\alpha,\psi_\eta}(w^{1/p})]^\delta \in A^+_1$ for any $\delta \in (0,1)$, we will get that $[M^-_{\alpha p,\phi_\eta} w]^{1-p'} \approx [M^-_{\alpha,\psi_\eta}(w^{1/p})]^{-p'} \in A^-_\infty$ since

$$[M^-_{\alpha,\psi_\eta}(w^{1/p})]^{-p'} = \{[M^-_{\alpha,\psi_\eta}(w^{1/p})]^\delta\}^{1-(1+p'/\delta)} \in A^-_{1+p'/\delta} \subset A^-_\infty.$$

Observe that, the fact $[M^-_{\alpha,\psi_\eta}(w^{1/p})]^\delta \in A^+_1$ follows trivially from the inequalities

$$C_1 M^-_{\alpha,\psi_\eta} w(x) \le M^-_\alpha(M^-_{\psi_\eta} w)(x) \le C_2 M^-_{\alpha,\psi_\eta} w(x). \tag{4.6}$$

To prove the first inequality of (4.6) we define the maximal operator

$$N^-_{\alpha,\psi_\eta} w(x) = \sup_{I=(a,x)} |I^-|^\alpha ||w||_{\psi_\eta,I^-},$$

where $I^- = (a, x - |I|/2^p)$. As in Lemma 3.2 we can prove that there exists a constant C such that

$$M^-_{\alpha,\psi_\eta} w(x) \le C N^-_{\alpha,\psi_\eta} w(x). \tag{4.7}$$

In fact, notice that $\psi_0^p(2t) = 2^p \psi_0^p(t)$, $\psi_\eta(2t) \le 2^p(\log 2)^\eta \psi_0(t) + 2^p\psi_\eta(t)$ for $0 < \eta \le 1$, and, in general, if $k < \eta \le k+1$, with $k \in \mathbb{N}$,

$$\psi_\eta(2t) \le C\left\{\sum_{j=0}^{k} \psi_j(t) + \sum_{j=0}^{k-1} \psi_{\eta+j-k}(t)\right\} + 2^p\psi_\eta(t).$$

Then, following the arguments in the proof of Lemma 3.2 with $\gamma_k = 2$ and using that $\alpha p \in (0,1)$ we get (4.7). Now, let $I^+ = I \setminus I^-$. Then

$$\begin{aligned} |I^-|^\alpha ||w||_{\psi_\eta,I^-} &\le |I^-|^\alpha |I^+|^{-1} \int_{I^+} ||w||_{\psi_\eta,(a,y)}\, dy \\ &\le 2^p |I|^{\alpha-1} \int_I M^-_{\psi_\eta} w(y)\, dy \le 2^p M^-_\alpha(M^-_{\psi_\eta} w)(x). \end{aligned}$$

Putting together the above inequalities and (4.7) we get the desired inequality. The second inequality in (4.6) follows as in the proof of (3.1). In fact, taking into account (2.7), we only need to show that

$$\frac{1}{|I|}\int_I M^-_{\psi_\eta} w \le \frac{1}{|I|}\int_I M_{\psi_\eta} w \le C||w||_{\psi_\eta,I},$$

for any function w with support in I. The last inequality follows, with standard arguments, by using a weak type inequality of M_{ψ_η} (see for example [2]).

5. Proof of Theorem 1.4

Proof. Without loss of generality we may assume that $f \geq 0$, $f \in L^1(\mathbb{R})$ and $||b||_{BMO} = 1$. Let $\lambda > 0$ and let $I_j = (a_j, b_j)$ be the connected components of $\Omega = \{x \in \mathbb{R} : M^+ f(x) > \lambda\}$. Then

$$\Omega = \cup_j I_j \quad \text{and} \quad \frac{1}{|I_j|}\int_{I_j} f = \lambda \tag{5.1}$$

(see for example [6] p. 423). If $x \notin \Omega$, then $\frac{1}{h}\int_x^{x+h} f \leq \lambda$ for all $h > 0$, and therefore $f(x) \leq \lambda$ a.e $x \in \mathbb{R} \setminus \Omega$. Let us write $f = g + h$, with g defined by

$$g(x) = \begin{cases} f(x), & x \in \mathbb{R} \setminus \Omega \\ f_{I_j}, & x \in I_j \end{cases}$$

where $f_I = \frac{1}{|I|}\int_I f$ and $h(x) = \sum_j h_j(x)$, with $h_j(x) = (f(x) - f_{I_j})\chi_{I_j}(x)$. Observe that $g(x) \leq \lambda$, a.e.. Let us define

$$\tilde{\Omega} = \cup_j (I_j^- \cup I_j) = \cup_j \tilde{I}_j,$$

where $I_j^- = (c_j, a_j)$ and $|I_j^-| = |I_j|$. We will use the notation

$$w^*(x) = w(x)\chi_{\mathbb{R}\setminus\tilde{\Omega}} \quad \text{and} \quad w_j(x) = w(x)\chi_{\mathbb{R}\setminus\tilde{I}_j}.$$

Now, we prove the theorem proceeding by induction on k. We start by proving the case $k = 0$. Notice that

$$\begin{aligned} w(\{x \in \mathbb{R} : |I_\alpha^+ f(x)| > \lambda\}) &\leq w(\{x \in \mathbb{R}\setminus\tilde{\Omega} : |I_\alpha^+ g(x)| > \lambda/2\}) + w(\tilde{\Omega}) \\ &\quad + w(\{x \in \mathbb{R}\setminus\tilde{\Omega} : |I_\alpha^+ h(x)| > \lambda/2\}) \\ &= I + II + III. \end{aligned}$$

Given $\varepsilon > 0$ we choose p such that $1 < p < 1 + \varepsilon$. We apply (1.3) with $\delta = \varepsilon + 1 - p > 0$. Then we have that

$$\begin{aligned} I &\leq \frac{C}{\lambda^p}\int_{\mathbb{R}} |I_\alpha^+ g(x)|^p w^*(x)\,dx \\ &\leq \frac{C}{\lambda^p}\int_{\mathbb{R}} [g(x)]^p M^-_{\alpha p,\phi_\varepsilon} w^*(x)\,dx \leq \frac{C}{\lambda}\int_{\mathbb{R}} g(x) M^-_{\alpha p,\phi_\varepsilon} w^*(x)\,dx \\ &\leq \frac{C}{\lambda}\int_{\mathbb{R}\setminus\Omega} f(x) M^-_{\alpha p,\phi_\varepsilon} w^*(x)\,dx + \frac{C}{\lambda}\sum_j \int_{I_j} f_{I_j} M^-_{\alpha p,\phi_\varepsilon} w_j(x)\,dx. \end{aligned}$$

It is clear that we only have to estimate the second term. By (2.8) we get that

$$\begin{aligned}\int_{I_j} f_{I_j} M^-_{\alpha p,\phi_\varepsilon} w_j(x)\,dx &\le \left(\int_{I_j} f(x)dx\right) \frac{1}{|I_j|}\int_{I_j} M^-_{\alpha p,\phi_\varepsilon} w_j(x)\,dx \\ &\le \left(\int_{I_j} f(x)\,dx\right) \inf_{z\in I_j} M^-_{\alpha p,\phi_\varepsilon} w_j(z) \\ &\le \int_{I_j} f(x) M^-_{\alpha p,\phi_\varepsilon} w_j(x)\,dx. \qquad (5.2)\end{aligned}$$

Hence, $I \le \frac{C}{\lambda}\int_{\mathbb{R}} f(x) M^-_{\alpha p,\phi_\varepsilon} w(x)\,dx$. Now, we shall estimate II. Notice that

$$II = w(\tilde{\Omega}) = w(\cup_j \tilde{I}_j) \le \sum_j [w(I_j^-) + w(I_j)].$$

For each j we have

$$w(I_j^-) = \frac{w(I_j^-)}{|I_j^-|}\frac{1}{\lambda}\int_{I_j} f(x)\,dx \le \frac{C}{\lambda}\int_{I_j} f(x) M^- w(x)\,dx.$$

If we now use that M^+ is of weak-type (1,1) with respect to the pair of weights (w, M^-w) (see [26] or [10]) we get that

$$\sum_j w(I_j) = w(\Omega) \le \frac{C}{\lambda}\int_{\mathbb{R}} f(x) M^- w(x)\,dx.$$

Then $II \le \frac{C}{\lambda}\int_{\mathbb{R}} f(x) M^- w(x)\,dx$. To estimate III we use the fact that $\int_{I_j} h_j = 0$ and we obtain

$$\begin{aligned} III &\le \frac{C}{\lambda}\sum_j \int_{\mathbb{R}\setminus\tilde{\Omega}} |I^+_\alpha h_j(x)| w(x)\,dx \le \frac{C}{\lambda}\sum_j \int_{\mathbb{R}\setminus\tilde{I}_j} |I^+_\alpha h_j(x)| w_j(x)\,dx \\ &= \frac{C}{\lambda}\sum_j \int_{-\infty}^{c_j} \left|\int_{I_j}\left(\frac{h_j(y)}{(y-x)^{1-\alpha}} - \frac{h_j(y)}{(a_j-x)^{1-\alpha}}\right) dy\right| w_j(x)\,dx \\ &\le \frac{C}{\lambda}\sum_j \int_{I_j} |h_j(y)| \sum_{m=0}^{\infty}\int_{I_{m,j}} \left|\frac{1}{(y-x)^{1-\alpha}} - \frac{1}{(a_j-x)^{1-\alpha}}\right| w_j(x)\,dx\,dy,\end{aligned}$$

where $I_{m,j} = (a_j - 2^{m+1}r_j, a_j - 2^m r_j)$ with $r_j = |I_j|$. Now, using the mean value theorem for each $y \in I_j$ we get that

$$\begin{aligned}\int_{I_{m,j}} \left|\frac{1}{(y-x)^{1-\alpha}} - \frac{1}{(a_j-x)^{1-\alpha}}\right| w_j(x)\,dx &\le C\frac{r_j}{(2^m r_j)^{2-\alpha}}\int_{I_{m,j}} w_j(x)\,dx \\ &\le C\frac{r_j}{(2^m r_j)^{2-\alpha}}\int_{a_j-2^{m+1}r_j}^{y} w_j(x)\,dx \\ &\le C 2^{-m} M^-_\alpha w_j(y).\end{aligned}$$

Then

$$
\begin{aligned}
III &\leq \frac{C}{\lambda}\sum_j \int_{I_j} |h_j(y)| M_\alpha^- \, w_j(y)\, dy \\
&\leq \frac{C}{\lambda}\sum_j \int_{I_j} f(y) M_\alpha^- \, w_j(y)\, dy + \frac{C}{\lambda}\sum_j \int_{I_j} f_{I_j} M_\alpha^- \, w_j(y)\, dy.
\end{aligned}
$$

Using (2.8) as in (5.2) in the second term of the last inequality,

$$
III \leq \frac{C}{\lambda}\sum_j \int_{I_j} f(y) M_\alpha^- \, w_j(y)\, dy \leq \frac{C}{\lambda}\int_{\mathbb{R}} f(y) M_\alpha^- \, w(y)\, dy.
$$

Therefore, collecting the estimates for I, II and III and using (2.6) we get the theorem in the case $k = 0$.

Let now $k \in \mathbb{N}$ and suppose that the theorem is true for $j < k$. Then with the same notation as in the proof of the case $k = 0$,

$$
\begin{aligned}
w(\{x \in \mathbb{R} : |I_{\alpha,b}^{+,k} f(x)| > \lambda\}) &\leq w(\{x \in \mathbb{R} \setminus \tilde{\Omega} : |I_{\alpha,b}^{+,k} g(x)| > \lambda/2\}) + w(\tilde{\Omega}) \\
&\quad + w(\{x \in \mathbb{R} \setminus \tilde{\Omega} : |I_{\alpha,b}^{+,k} h(x)| > \lambda/2\}) \\
&= I + II + III.
\end{aligned}
$$

Given $\varepsilon > 0$ we choose p such that $1 < p < 1 + \frac{\varepsilon}{k+1}$ and we apply (1.3) with $\delta = \varepsilon - (k+1)(p-1) > 0$. Then, as in the case $k = 0$, we obtain that

$$
I \leq \frac{C}{\lambda}\int_{\mathbb{R}} f(x) M_{\alpha p, \phi_{k+\epsilon}}^- w(x)\, dx
$$

and

$$
II \leq \frac{C}{\lambda}\int_{\mathbb{R}} f(x) M^- w(x)\, dx.
$$

To estimate III we write

$$
\begin{aligned}
\sum_j I_{\alpha,b}^{+,k} h_j(x) &= \sum_j (b(x) - b_{I_j})^k I_\alpha^+ h_j(x) + \sum_j I_\alpha^+\big((b - b_{I_j})^k h_j\big)(x) \\
&\quad + \sum_{l=1}^{k-1} C_{k,l} I_{\alpha,b}^{+,l}\Big(\sum_j (b - b_{I_j})^{k-l} h_j\Big)(x).
\end{aligned}
$$

The above decomposition follows from (4.1) as in [21]. Then

$$
\begin{aligned}
III &\le w(\{x \in \mathbb{R}\setminus\tilde{\Omega} : |\sum_j (b(x)-b_{I_j})^k I_\alpha^+ h_j(x)| > \frac{\lambda}{6}\}) \\
&\quad + w(\{x \in \mathbb{R}\setminus\tilde{\Omega} : |\sum_j I_\alpha^+\big((b-b_{I_j})^k h_j\big)(x)| > \frac{\lambda}{6}\}) \\
&\quad + w(\{x \in \mathbb{R}\setminus\tilde{\Omega} : |\sum_{l=1}^{k-1} C_{k,l} I_{\alpha,b}^{+,l}\big(\sum_j (b-b_{I_j})^{k-l} h_j\big)(x)| > \frac{\lambda}{6}\}) \\
&= (III)^a + (III)^b + (III)^c.
\end{aligned}
$$

In a similar way as in the estimate of III for $k=0$, we get that

$$
\begin{aligned}
(III)^a &\le \frac{C}{\lambda}\sum_j \int_{\mathbb{R}\setminus\tilde{\Omega}} |b(x)-b_{I_j}|^k |I_\alpha^+ h_j(x)| w(x)\,dx \\
&\le \frac{C}{\lambda}\sum_j \int_{\mathbb{R}\setminus\tilde{\Omega}} |b(x)-b_{I_j}|^k \left| \int_{I_j} \frac{h_j(y)}{(y-x)^{1-\alpha}}\,dy - \int_{I_j} \frac{h_j(y)}{(a_j-x)^{1-\alpha}}\,dy \right| w(x)\,dx \\
&\le \frac{C}{\lambda}\sum_j \int_{I_j} |h_j(y)| \int_{\mathbb{R}\setminus\tilde{I}_j} \left|(y-x)^{\alpha-1} - (a_j-x)^{\alpha-1}\right| |b(x)-b_{I_j}|^k w_j(x)\,dx\,dy \\
&\le \frac{C}{\lambda}\sum_j \int_{I_j} |h_j(y)| \sum_{m=0}^{\infty} \int_{I_{m,j}} \left|(y-x)^{\alpha-1} - (a_j-x)^{\alpha-1}\right| |b(x)-b_{I_j}|^k w_j(x)\,dx\,dy,
\end{aligned}
$$

where $I_{m,j} = (a_j - 2^{m+1}r_j, a_j - 2^m r_j)$. Using again the mean value theorem, we get

$$
\begin{aligned}
\int_{I_{m,j}} &\left| \frac{1}{(y-x)^{1-\alpha}} - \frac{1}{(a_j-x)^{1-\alpha}} \right| |b(x)-b_{I_j}|^k w_j(x)\,dx \\
&\le C\frac{r_j}{(2^m r_j)^{2-\alpha}} \int_{I_{m,j}} |b(x)-b_{I_{m,j}}|^k w_j(x)\,dx \\
&\quad + C\frac{r_j}{(2^m r_j)^{2-\alpha}} |b_{I_j} - b_{I_{m,j}}|^k \int_{I_{m,j}} w_j(x)\,dx \\
&= (III)_1^a + (III)_2^a.
\end{aligned}
$$

By the generalized Hölder's inequality with $\phi_k(t) = t[\log(e+t)]^k$ and $\tilde{\phi}_k(t) \approx e^{t^{1/k}}$, and using (2.12) we get

$$
(III)_1^a \le C\frac{r_j}{(2^m r_j)^{1-\alpha}} ||b||_{BMO}^k ||w_j||_{\phi_k, I_{m,j}} \le C\frac{1}{2^m} M_{\alpha,\phi_k}^- w_j(y),
$$

for all $y \in I_j$. Now, applying (2.9) we get that

$$
(III)_2^a \le C\frac{(m+1)^k}{2^m} M_\alpha^- w_j(y),
$$

for all $y \in I_j$. Then

$$(III)^a \leq \frac{C}{\lambda} \sum_j \int_{I_j} |h_j(y)| \left(\sum_{m=0}^{\infty} \frac{1}{2^m} M^-_{\alpha,\phi_k} w_j(y) + \sum_{m=1}^{\infty} \frac{(m+1)^k}{2^m} M^-_{\alpha} w_j(y) \right) dy$$
$$\leq \frac{C}{\lambda} \sum_j \int_{I_j} |h_j(y)| M^-_{\alpha,\phi_k} w_j(y)\, dy \leq \frac{C}{\lambda} \int_{\mathbb{R}} f(y) M^-_{\alpha,\phi_k} w(y)\, dy,$$

where the last inequality follows as in the estimation of III in the case $k=0$. To estimate $(III)^b$ we shall use the case $k=0$ and we obtain

$$\begin{aligned}(III)^b \leq& \frac{C}{\lambda} \sum_j \int_{I_j} |b(x) - b_{I_j}|^k |f(x) - f_{I_j}| [M^-_{\alpha p,\phi_\varepsilon} w_j(x) + M^- w_j(x)]\, dx \\ \leq& \frac{C}{\lambda} \sum_j [\inf_{I_j} M^-_{\alpha p,\phi_\varepsilon} w_j + \inf_{I_j} M^- w_j] \int_{I_j} |b(x) - b_{I_j}|^k f(x) dx \\ &+ \frac{C}{\lambda} \sum_j [\inf_{I_j} M^-_{\alpha p,\phi_\varepsilon} w_j + \inf_{I_j} M^- w_j] \int_{I_j} |b(x) - b_{I_j}|^k f_{I_j} dx \\ =& (III)^b_1 + (III)^b_2.\end{aligned}$$

To estimate $(III)^b_2$ we will use (2.8) and (2.10). Then we have

$$\begin{aligned}(III)^b_2 &\leq \frac{C}{\lambda} \sum_j \frac{1}{|I_j|} [\inf_{I_j} M^-_{\alpha p,\phi_\varepsilon} w_j + \inf_{I_j} M^- w_j] \int_{I_j} |b(x) - b_{I_j}|^k \int_{I_j} f(y) dy dx \\ &\leq \frac{C}{\lambda} \sum_j \frac{1}{|I_j|} \int_{I_j} |b(x) - b_{I_j}|^k dx \int_{I_j} f(y) [M^-_{\alpha p,\phi_\varepsilon} w_j(y) + M^- w_j(y)] dy \\ &\leq \frac{C}{\lambda} \sum_j ||b||^k_{BMO} \int_{I_j} f(y) [M^-_{\alpha p,\phi_\varepsilon} w_j(y) + M^- w_j(y)] dy \\ &\leq \frac{C}{\lambda} \int_{\mathbb{R}} f(y) [M^-_{\alpha p,\phi_\varepsilon} w(y) + M^- w(y)] dy.\end{aligned}$$

To estimate $(III)^b_1$ we use again the generalized Hölder's inequality for ϕ_k and $\tilde{\phi}_k$

$$\begin{aligned}(III)^b_1 &\leq \frac{C}{\lambda} \sum_j [\inf_{I_j} M^-_{\alpha p,\phi_\varepsilon} w_j + \inf_{I_j} M^- w_j] \int_{I_j} |b(x) - b_{I_j}|^k f(x) dx \\ &\leq \frac{C}{\lambda} \sum_j [\inf_{I_j} M^-_{\alpha p,\phi_\varepsilon} w_j + \inf_{I_j} M^- w_j] |I_j| ||(b - b_{I_j})^k||_{\tilde{\phi}_k, I_j} ||f||_{\phi_k, I_j} \\ &\leq \frac{C}{\lambda} ||b||^k_{BMO} \sum_j [\inf_{I_j} M^-_{\alpha p,\phi_\varepsilon} w_j + \inf_{I_j} M^- w_j] |I_j| ||f||_{\phi_k, I_j}.\end{aligned}$$

Now, the inequality

$$||f||_{\phi_k, I} \leq \inf_{\mu > 0} \left\{ \mu + \frac{\mu}{|I|} \int_I \phi_k \left(\frac{|f|}{\mu} \right) \right\},$$

(see [7] and also [23]) and (5.1) gives us that

$$\begin{aligned}\frac{1}{\lambda}|I_j|||f||_{\phi_k,I_j} &\le \frac{1}{\lambda}|I_j|\inf_{\mu>0}\left\{\mu+\frac{\mu}{|I_j|}\int_{I_j}\phi_k\left(\frac{|f|}{\mu}\right)\right\}\\ &\le |I_j|+\int_{I_j}\phi_k\left(\frac{f}{\lambda}\right)=\frac{1}{\lambda}\int_{I_j}f+\int_{I_j}\phi_k\left(\frac{f}{\lambda}\right)\le 2\int_{I_j}\phi_k\left(\frac{f}{\lambda}\right).\end{aligned}$$

Then

$$\begin{aligned}(III)_1^b &\le C\sum_j\int_{I_j}\phi_k\left(\frac{f(x)}{\lambda}\right)[M^-_{\alpha p,\phi_\varepsilon}w_j(x)+M^-w_j(x)]dx\\ &\le C\int\phi_k\left(\frac{f(x)}{\lambda}\right)[M^-_{\alpha p,\phi_\varepsilon}w(x)+M^-w(x)]dx.\end{aligned}$$

To conclude the proof we only have to estimate $(III)^c$ where we will use the induction argument:

$$\begin{aligned}(III)^c &\le w(\{x\in\mathbb{R}\setminus\tilde{\Omega}:|\sum_{l=1}^{k-1}C_{k,l}I^{+,l}_{\alpha,b}\big(\sum_j(b-b_{I_j})^{k-l}f\chi_{I_j}\big)(x)|>\frac{\lambda}{12}\})\\ &\quad+w(\{x\in\mathbb{R}\setminus\tilde{\Omega}:|\sum_{l=1}^{k-1}C_{k,l}I^{+,l}_{\alpha,b}\big(\sum_j(b-b_{I_j})^{k-l}f_{I_j}\chi_{I_j}\big)(x)|>\frac{\lambda}{12}\})\\ &=(III)_1^c+(III)_2^c.\end{aligned}$$

By induction,

$$\begin{aligned}&(III)_1^c\\ &\le C\sum_{l=1}^{k-1}\int_{\mathbb{R}}\phi_l\left(\frac{f(x)}{\lambda}\sum_j(b(x)-b_{I_j})^{k-l}\chi_{I_j}(x)\right)[M^-_{\phi_l}w^*(x)+M^-_{\alpha p,\phi_{l+\varepsilon}}w^*(x)]dx\\ &\le C\sum_{l=1}^{k-1}\sum_j\int_{I_j}\phi_l\left(\frac{f(x)}{\lambda}(b(x)-b_{I_j})^{k-l}\right)[M^-_{\phi_l}w_j(x)+M^-_{\alpha p,\phi_{l+\varepsilon}}w_j(x)]dx\\ &\le C\sum_{l=1}^{k-1}\sum_j[\inf_{I_j}M^-_{\phi_l}w_j+\inf_{I_j}M^-_{\alpha p,\phi_{l+\varepsilon}}w_j]\int_{I_j}\phi_l\left(\frac{f(x)}{\lambda}(b(x)-b_{I_j})^{k-l}\right)dx.\end{aligned}$$

Now observe that $\phi_k^{-1}(t)\approx\frac{t}{[\log(e+t)]^k}$ and $\tilde{\phi}_k^{-1}(t)\approx[\log(e+t)]^k$. Then

$$\phi_k^{-1}(t)\tilde{\phi}_{k-l}^{-1}(t)\le C\,\phi_l^{-1}(t).$$

Using (2.4), (2.11) and (5.1) we get that

$$\begin{aligned}
&\int_{I_j} \phi_l \left(\frac{f(x)}{\lambda} (b(x) - b_{I_j})^{k-l} \right) dx \\
&\qquad \le C \int_{I_j} \phi_k \left(\frac{f(x)}{\lambda} \right) dx + C \int_{I_j} \tilde{\phi}_{k-l} \left((b(x) - b_{I_j})^{k-l} \right) dx \\
&\qquad \le C \int_{I_j} \phi_k \left(\frac{f(x)}{\lambda} \right) dx + C|I_j| \\
&\qquad \le C \int_{I_j} \phi_k \left(\frac{f(x)}{\lambda} \right) dx + C \int_{I_j} \frac{f(x)}{\lambda} \, dx \le C \int_{I_j} \phi_k \left(\frac{f(x)}{\lambda} \right) dx.
\end{aligned}$$

Then

$$\begin{aligned}
(III)_1^c &\le C \sum_{l=1}^{k-1} \sum_j [\inf_{I_j} M^-_{\phi_l} w_j + \inf_{I_j} M^-_{\alpha p, \phi_{l+\varepsilon}} w_j] \int_{I_j} \phi_k \left(\frac{f(x)}{\lambda} \right) dx \\
&\le C \sum_{l=1}^{k-1} \sum_j \int_{I_j} \phi_k \left(\frac{f(x)}{\lambda} \right) [M^-_{\phi_l} w(x) + M^-_{\alpha p, \phi_{l+\varepsilon}} w(x)] dx.
\end{aligned}$$

The term $(III)_2^c$ is controlled in the same way, just observe that by (2.11) and Jensen's inequality

$$\begin{aligned}
\int_{I_j} \phi_l \left(\frac{f_{I_j}}{\lambda} (b(x) - b_{I_j})^{k-l} \right) dx &\le C|I_j| \phi_k \left(\frac{f_{I_j}}{\lambda} \right) + C|I_j| \\
&\le C \int_{I_j} \phi_k \left(\frac{f(x)}{\lambda} \right) dx.
\end{aligned}$$

Then, using (2.5),

$$(III)^c \le C \int \phi_k \left(\frac{f(x)}{\lambda} \right) [M^-_{\phi_{k-1}} w(x) + M^-_{\alpha p, \phi_{k-1+\varepsilon}} w(x)] \, dx.$$

By (2.5) again and (2.6) we have that

$$III \le C \int \phi_k \left(\frac{f(x)}{\lambda} \right) [M^-_{\alpha p, \phi_{k+\varepsilon}} w(x) + M^-_{\phi_k} w(x)] \, dx.$$

Now, putting together the estimates of I, II and III, we are done. $\square$

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