A LOOK AT $BMO_\varphi(\omega)$ THROUGH CARLESON MEASURES

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Abstract. As Fefferman and Stein showed, there is a tight connection between Carleson measures and BMO functions. In this work we extend this type of results to the more general scope of the $BMO_\varphi(\omega)$ spaces. As a by product a weighted version of the Triebel–Lizorkin space $\dot{F}^0_{\infty,2}$ is introduced, which turns to be isomorphic to $BMO(\omega)$ as in the unweighted case.

1. Introduction

Given a growth function φ and a weight ω, we shall consider the $BMO_\varphi(\omega)$ spaces, that is the set of functions whose oscillation, when averaged over balls, is controlled y means of φ and ω, measuring their degree of smoothness. More precisely, we shall say that a locally integrable function f belongs to $BMO_\varphi(\omega)$ if there exists a constant C such that the inequality

$$\frac{1}{\omega(B)}\int_B |f(y)-m_Bf|dy \leq C\varphi(|B|^{\frac{1}{n}}) \tag{1.1}$$

holds for every ball B in $\mathbb{R}^n$, where, as usual, m_Bf denotes the average of f over B respect to the Lebesgue measure. The first appearance of this kind of weighted spaces goes back to [GC] and [MW]. In the last paper, the authors introduced $BMO(\omega)$ ($\varphi \equiv 1$ in our context) as the natural space where weighted L^∞ functions are mapped by $\mathcal{H}$, the Hilbert transform on the line, generalizing the well known BMO space of John and Niremberg. In the more general context $\varphi(t) = t^\beta$, $0 < \beta < 1$, it is shown in [HSV1] that the fractional integral operator I_α

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applies $L^p(w)$ with $p > n/\alpha$ into these spaces, under suitable conditions on the weight. Later on this result was extended to weighted Orlicz spaces [HSV2] giving rise to the spaces under consideration in their full generality. Finally in [M], it is shown that they are preserved by the Hilbert transform on the line.

In their celebrate paper [FS], Fefferman and Stein threw light into the tight connection between BMO-functions and Carleson measures. Let us remind that a measure μ on $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0,\infty)$ is said to be a Carleson measure when a constant C exists such that for any ball $B(x_0, r) \subset \mathbb{R}^n$

$$\mu(B(x_0,r) \times (0,r)) \le Cr^n.$$

With this notation Fefferman-Stein result can be stated as:

$$f \in BMO \Leftrightarrow \int \frac{f(x)}{1+|x|^{n+1}} dx < \infty \text{ and }$$
$$t|\nabla(P_t * f)|^2(x)\, dx\, dt \text{ is a Carleson measure}$$

Later on, W. Smith, in [Sm], proved an extension of this result to the spaces BMO_φ (i.e. $BMO_\varphi(\omega)$ with $\omega = 1$) giving a suitable definition of φ-Carleson measures.

A more recent version of this kind of characterization of functions in BMO appears in Stein's book [S] (see theorem 3, page 159). The precise statement is as follows.

Theorem **1.2.** *Let $\psi \in \mathcal{S}$ with $\int \psi = 0$.*

(a) *If $f \in BMO$ then $d\mu = |f * \psi_t|^2 \frac{dx\,dt}{t}$ is a Carleson measure.*

(b) *Conversely, suppose ψ satisfies also a Tauberian condition, if f is such that $\int \frac{|f(x)|}{1+|x|^{n+1}}\, dx < \infty$ and $d\mu = |f * \psi_t|^2 \frac{dx\,dt}{t}$ is a Carleson measure, then f is in BMO.*

Here, for a Tauberian condition we mean that $\widehat{\psi}$ does not vanish identically in any ray emanating from the origin, and, as usual $\psi_t = t^{-n}\psi(x/t)$.

Also, it is well known that BMO coincides with the Triebel-Lizorkin space $\dot{F}^{0,2}_\infty$ ([FJW]). The above result, even is very close, does not allow to conclude such characterization: one should prove part (b) of the theorem under the more general situation of a distribution in $\mathcal{S}'/\mathcal{P}$ ($\mathcal{P}$ the set of polynomials) instead of the integrability condition on the function f.

In this work we give an extension of the theorem above to the more general spaces $BMO_\varphi(\omega)$ under appropriate assumptions on φ and ω, which, at the same time, allows us to obtain, as a corollary, the identification of $BMO(\omega)$ with a weighted version of $\dot{F}^0_{\infty,2}$.

By the way, it is worth mentioning that Bui and Taibleson defined in [BT] weighted $\dot{F}^\alpha_{\infty,q}$ spaces. However, as we show, for $\alpha = 0$ and $q = 2$, their definition does not give the weighted space $BMO(\omega)$ as expected. In fact we prove that, at least for weights in the Muckenhoupt class A_1, it coincides rather with the unweighted BMO space.

In proving our main theorem we establish a kind of duality inequality envolving generalized Carleson measures and tent spaces. This is achieved by means of an adequate atomic decomposition of the latter spaces.

The structure of the paper is as follows: section 2 contains some basic facts and the statement of our main theorem; sections 3 and 4, respectively, contain some needed results, interesting by themselves, about generalizations of Hardy and tent spaces; the proof of the main theorem is in section 5, and, finally, section 6 is devoted to the above remark on weighted Triebel-Lizorkin spaces.

2. Preliminaries and the main result

We start by reminding some basic notions about growing functions and weights.

For a non-negative and non-decreasing function φ defined in $[0, \infty]$, we shall say that it is of upper type β, if there exists a constant C such that

$$\varphi(\theta t) \leq C\theta^\beta \varphi(t) \tag{2.1}$$

for all $\theta \geq 1$ and $t \geq 0$. If there exists such a number β, we shall denote by $I(\varphi) = \inf\{\beta : \varphi \textit{ is of upper type } \beta\}$. Let us notice that our assumptions on φ guarantees that $I(\varphi) \geq 0$. Similarly, whenever (2.1) holds for $0 \leq \theta \leq 1$, φ is said to be of lower type β.

Next we remind that a weight ω belongs to the Muckenhoupt class A_r, $r > 1$, if there exists a constant C such that for any ball $B \subset \mathbb{R}^n$

$$\frac{1}{|B|}\int_B \omega \; (\frac{1}{|B|}\int_B \omega^{-\frac{1}{r-1}})^{r-1} \leq C$$

with the obvious change for the case $r = 1$. Finally, a weight is in the class A_∞ when it belongs to some A_r, $r > 1$.

As we said in the introduction we shall consider the spaces $BMO_\varphi(\omega)$ for φ a concave function as above, and ω a weight in A_∞. These spaces consist of locally integrable functions on $\mathbb{R}^n$ such that (1.1) holds. Moreover if we set $\|f\|_{BMO_\varphi(\omega)}$ as the infimum of the constants for which (1.1) holds, $BMO_\varphi(\omega)$ turns out to be a Banach space modulo constants.

Next we introduce a generalization of the notion of Carleson measures. For φ and ω as above, we shall say that a measure $d\mu$ on R^{n+1}_+ is a (φ,ω)-Carleson measure when a constant exists such that

$$\int_{\hat{B}} |d\mu| \leq C\omega(B)\varphi^2(|B|^{1/n}) \tag{2.2}$$

for any ball $B \subset \mathbb{R}^n$. Here $\hat{B}$ denotes the tent corresponding to $B = B(x_0, r)$, that is $\hat{B} = \{(x,t) \in \mathbb{R}^{n+1}_+ : |x - x_0| + t < r\}$. As usual we denote by $[d\mu]_{\varphi,\omega}$ the infimum of the constants appearing in (2.2). This definition is a weighted extension of the notion given in [Sm]. Now we are in position to state our main result.

Theorem **2.3.** *: Let φ be a non-negative, non-decreasing concave function defined on $[0,\infty)$ with $I(\varphi) < 1$. Let $q = (1 + I(\varphi))/n$ and ω a weight on A_q. Further, let ψ be a function in $\mathcal{S}(\mathbb{R}^n)$ with null integral. Then we have*
(a) If $f \in BMO_\varphi(\omega), d\mu = |\psi_t * f|^2(x)\frac{t^n}{\omega(B(x,t))}dx\frac{dt}{t}$ is a (φ,ω)-Carleson measure with

$$[d\mu]_{\varphi,\omega} \leq C\|f\|^2_{BMO_\varphi(\omega)}.$$

(b) Assume further ψ satisfies a Tauberian condition. Then any distribution $f \in \mathcal{S}'/\mathcal{P}$ such that $d\mu =| \psi_t * f|^2(x)\frac{t^n}{\omega(B(x,t))}dx\frac{dt}{t}$ is a (ψ,ω) -Carleson measure can be seen as a $BMO_\varphi(\omega)$ function and

$$\|f\|^2_{BMO_\varphi}(\omega) \leq C[d\mu]_{\varphi,(\omega)}.$$

We notice that part (a) is a generalization of (a) in theorem (1.2) while part (b) looks slightly different. However we may obtain as a corollary of our theorem such an extension.

Corollary **2.4.** *Let φ and ω as above and ψ a function in $\mathcal{S}(\mathbb{R}^n)$ tauberian and with null integral. Then if f is such that $\int_{\mathbb{R}^n} \frac{|f(x)|}{1+|x|^{n+1}}dx <$*

∞ and $d\mu = |f * \psi_t|^2(x)\frac{t^n}{\omega(B(x,t))}dx\frac{dt}{t}$ is a (φ,ω)-Carleson measure, f is also in $BMO_\varphi(\omega)$ with $\|f\|^2_{BMO_\varphi(\omega)} \leq C[d\mu]_{\varphi,\omega}$.

This corollary follows from the theorem just by noting that a function f satisfying $\int_{\mathbb{R}^n} \frac{|f(x)|}{1+|x|^{n+1}}dx < \infty$ defines an element of $\mathcal{S}'/\mathcal{P}$.

3. Some basic facts about $H^q_\eta(\omega)$

In this section we present some results concerning weighted Hardy-Orlicz atomic spaces that will be useful to our purposes. Mostly, they are spread in literature, perhaps not with the degree of generality we need here. Anyway, we state them and outline their proofs for the sake of completeness.

In the sequel we shall work with a non-negative, increasing and concave function η with $\eta(0) = 0$ and lower type $\ell > \frac{n}{n+1}$. Given such η and a weight $\omega \in A_q$, we shall say that the function a is an (η, q, ω)-atom if a is supported in a ball B, has zero average and

$$\|a\|_{L^q(\omega)} \leq \frac{|B|}{(\omega(B))^{1/q'}}\ \eta^{-1}(\frac{1}{|B|}). \tag{3.1}$$

With this notion we define the atomic space $H^q_\eta(\omega)$ as the set of distributions $f \in \mathcal{S}'$ that can be written as $f = \Sigma_{i=1}^\infty b_i$ (in the sense of distributions), where $\{b_i\}$ is a sequence of multiples of (η, q, ω)-atoms such that

$$\sum_{i=1}^\infty |B_i|\eta(\frac{\omega(B_i)^{1/q'}}{|B_i|}\|b_i\|_{L^q(\omega)}) < \infty,$$

where B_i is a ball containing the support of b_i. For any such decomposition we introduce the quantity

$$\Lambda_q(\{b_i\}) = \inf\{\lambda : \sum |B_i|\eta(\frac{\omega(B_i)^{1/q'}}{\lambda^{1/\ell}|B_i|}\|b_i\|_{L^q(\omega)}) \leq 1\}$$

and we denote by $[f]_{H^q_\eta(\omega)}$ the infimum of $\Lambda_q(\{b_i\})$ taken over all decompositions of f. It is easy to check that $[.]_{H^q_\eta(\omega)}$ defines a quasi-metric invariant under traslations and positive homogeneous when raised to the $(1/\ell)^{th}$-power.

Let us observe that any function g in $L^q(\omega)$, supported in a ball and with zero average, belongs to $H^q_\eta(\omega)$ and moveover if it satisfies (3.1) then $[g]_{H^q_\eta(\omega)} \leq 1$.

The first result we need is quite standard.

PROPOSITION 3.2. *: Let L be a functional in the dual of $H^q_\eta(\omega)$, then there exists $h \in BMO_\varphi(\omega)$ with $\varphi(t^{1/n}) = 1/t\eta^{-1}(1/t)$ such that*

$$L(g) = \int h(x)g(x)dx$$

for any $g \in L^q(\omega)$ with compact support and zero average.
Moreover

$$\|h\|_{BMO_\varphi(\omega)} \leq [L] = inf\{C : |L(f)| \leq C[f]^{1/\ell}_{H^q_\eta(\omega)}\}. \tag{3.3}$$

Proof. : As usual, it is easy to see that for any ball B, L defines a bounded linear functional on $L^q_0(B,\omega)$, the subspace of functions in $L^q(\omega)$ supported in B with zero average, since for such f we have

$$|L(f)| \leq C[f]^{1/\ell}_{H^q_\eta(\omega)} \leq C\frac{\omega(B)^{1/q'}}{|B|\eta^{-1}(\frac{1}{|B|})}\|f\|_{L^q(\omega)}.$$

Extending L by the Hahn-Banach Theorem we know that there exists a function $h_B \in L^{q'}(\omega^{1-q'})$ supported in B, such that

$$L(f) = \int_B h_B f = \int_B (h_B - m_B h_B)f, \quad f \in L^q_0(B,\omega)$$

and moreover we have

$$\left(\frac{1}{\omega(B)}\int_B |h_B - m_B h_B|^{q'}\omega^{1-q'}\right)^{1/q'} \leq C\varphi(|B|^{1/n}). \tag{3.4}$$

Taking now an increasing sequence of balls, by a standard argument, a function h may be defined, modulo constants, satisfying (3.4) for any ball. Since $\omega \in A_q$ it is known that such inequality implies $h \in BMO_\varphi(\omega)$, producing an equivalent norm, (see [M]). Therefore (3.3) also follows. □

The next result shows that functions in $\mathcal{S}$ with zero moments of any order are dense in our spaces. Related results appear in [ST], however their spaces are not quite the same as ours.

PROPOSITION **3.5.** *$\mathcal{S}_\infty = \{f \in \mathcal{S} : supp\ \hat{f} \subset \{x : \epsilon < |x| < \frac{1}{\epsilon}$, for some $\epsilon > 0\}$ is a dense subspace of $H^q_\eta(\omega)$ as long as $q < 2 + \frac{1}{n} - \frac{1}{\ell}$.*

Proof. As in [Se] given $B_0 = B(x_0, r)$ a ball, a function $g \in L^q(\omega)$ with zero integral can be split, pointwisely and in the sense of $\mathcal{S}'$, as

$$g = \sum_{k\geq 0}(g - m_k)\chi_{E_k} + \sum_{k\geq 0}\beta_k R_k, \tag{3.6}$$

where $E_0 = B_0$, $E_k = B(x_0, r2^k) - B(x_0, r2^{k-1}) = B_k - B_{k-1}, m_k = \frac{1}{|E_k|}\int_{E_k} g$, $\beta_k = \sum_{i\geq k+1} m_i|E_i| = \int_{B_k^c} g$ and $R_k = |E_{k+1}|^{-1}\chi_{E_k+1} - |E_k|^{-1}\chi_{E_k}$.

Clearly each term in the sums is multiple of an atom. Moreover if $g \in \mathcal{S}$ it is easy to check that this decomposition implies that $g \in H^q_\eta(\omega)$. Thus $\mathcal{S}_\infty$ is a subspace of $H^q_\eta(\omega)$.

To obtain the density we observe that it is enough to approximate functions in $L^q(\omega)$ with compact support and zero average. Let b be one such function and σ a radial function in $\mathcal{S}$ such that $\hat{\sigma}(\xi) = 1$ for $|\xi| \leq 1$ and $\hat{\sigma}(\xi) = 0$ for $|\xi| \geq 2$. For any $t, 0 < t \leq 1$, the function $\sigma_t * b - \sigma_{1/t} * b$ belongs to $\mathcal{S}_\infty$ and moreover we will show that

$$\|\sigma_t * b - b\|_{H^q_\eta}(\omega) \longrightarrow 0 \tag{3.7}$$

and

$$\|\sigma_{1/t} * b\|_{H^q_\eta(\omega)} \longrightarrow 0 \tag{3.8}$$

when t goes to zero.

To this end we use the above decomposition for $g = \sigma_t * b - b$, $B_0 = 2B^*$, with B^* a ball containing the support of b. We denote m_k^t and β_k^t the corresponding coefficients.

For $x \in E_k, k \geq 1$, using the decay of σ we get the estimate

$$|\sigma_t * b|(x) \leq C(N, \sigma, B_0, \omega)\frac{t^{N-n}}{2^{kN}}$$

for any positive integer N. Then

$$\|(\sigma_t * b - b - m_k^t)\mathcal{X}_{E_k}\|_{L^q(\omega)} \leq Ct^{N-n}2^{k(n-N)}, \quad k \geq 1.$$

Besides, for $k = 0$, using that σ_t is an approximation to the identity and that $\omega \in A_q$, we get

$$\|\mathcal{X}_{E_0}(\sigma_t * b - b)\|_{L^q(\omega)} \to 0 \quad for \quad t \to 0.$$

Therefore, setting $h_k^t = (\sigma_t * b - b - m_k^t)\chi_{E_k}$, choosing N large enough and using again that $\omega \in A_q$, we easily obtain that $\Lambda_q(\{h_k^t\}) \longrightarrow 0$, as desired.

Also, for any $k \geq 0$, due to the decay of σ we get

$$|\beta_k^t| = |\int_{B_k^c} \sigma_t * b| \leq C t^{N-n} 2^{k(n-N)} \|b\|_1$$

and hence

$$\|\beta_k^t R_k\|_{L^q(\omega)} \leq C t^{N-n} 2^{-kN} (\omega(B_k))^{1/q}.$$

Arguing as above we also get $\Lambda_q(\{\beta^{t_k} R_k\} \longrightarrow 0$. Then (3.7) is proved.

To prove (3.8) we use (3.6) again, now for $g = \sigma_{1/t} * b$, and we denote by $\tilde{m}_k^t$ and $\tilde{\beta}_k^t$ the corresponding coefficients.

First for $k \geq 1$, using the smoothness and decay of σ, we have for $x \in E_k$

$$|\sigma_{1/t} * b|(x) \leq C \frac{t^{n+1-M}}{2^{kM}} \|b\|_{L^q(\omega)}$$

for M as large as we want. Choosing $M = n + 1 - \delta$, with $0 < \delta < 1$, we get

$$\|(\sigma_{1/t} * b)\mathcal{X}_{E_k}\|_{L^q(\omega)} \leq C t^{\delta} 2^{-k(1-\delta)} \|b\|_{L^q(\omega)}$$

As for $k = 0$ we clearly have

$$\|(\sigma_{1/t} * b)\mathcal{X}_{E_0}\|_{L^q(\omega)} \leq C t^n \|b\|_{L^q(\omega)}$$

Therefore, settting $\tilde{h}_k^t = (\sigma_{1/t} * b - \tilde{m}_k^t)\chi_{E_k}$ and using that η is of lower type ℓ we obtain

$$\sum_{k\geq 0} |B_k| \eta(\frac{\omega(B_k)^{1/q'}}{|B_k|} \|\tilde{h}_k^t\|_{L^q(\omega)}) \leq C\eta(t^{\delta}) \sum_{k\geq 0} 2^{k(n+\ell(n(q-2)+\delta-1))}$$

Since $q < 2 + \frac{1}{n} - \frac{1}{\ell}$, we may choose δ small enough to make the last series convergent. This shows that $\Lambda_q(\{\tilde{h}_k^t\}) \longrightarrow 0$.

A similar argument proves the convergence to zero of $\Lambda_q(\{\tilde{\beta}_k^t R_k\})$, finishing the proof of the proposition. $\square$

4. Some basic facts on tent spaces $T_\eta(\omega)$

In what follows for a measurable function G defined on $\mathbb{R}^{n+1}_+$ we set

$$\mathcal{V}(G)(x) = (\int_{\Gamma(x)} |G(y,t)|^2 \frac{dydt}{t^{n+1}})^{1/2} \tag{4.1}$$

where $\Gamma(x)$ denotes the cone $\{(y,t) : |x-y| < t\}$.

For a non-negative increasing and concave function with $\eta(0) = 0$ and lower type $\ell > n/(n+1)$ and a weight ω in $L^1_{loc}(\mathbb{R}^n)$, we introduce the tent space $T_\eta(\omega)$ as those functions G such that

$$[\mathcal{V}(G)\omega]_{L^\eta} \equiv [G]_{T_\eta(\omega)} < \infty,$$

where by $[g]_{L^\eta}$ we mean $inf\{\lambda : \int \eta(g/\lambda^{1/\ell}) \le 1\}$.

The main goal of this section is to get an atomic decomposition of $T_\eta(\omega)$, extending the result contained in [CMS] for $\eta(t) = t$ and $\omega \equiv 1$. To this end we first introduce the notion of atoms.

Given a ball $B = B(x,r) \subset \mathbb{R}^n$ we denote by $\hat{B}$ the tent over B, i.e. $\hat{B} = \{(y,t) : |x-y| + t < r\}$. Now, a function $a(y,t)$ is said to be an atom whenever is supported in some $\hat{B}$ and

$$\left(\int_0^\infty |a(y,t)|^2 \frac{\omega(B(y,t))}{t^n} \frac{dydt}{t}\right)^{1/2} \le \frac{|B|}{(\omega(B))^{1/2}} \eta^{-1}(\frac{1}{|B|}). \tag{4.2}$$

Observe that if we set $W(y,t) = \omega(B(y,t))/t^{n+1}$ the left hand side of (4.2) is just $\|a\|_{L^2(W)}$. Also, due to the concavity of η, it is easy to check that atoms do belong to $T_\eta(\omega)$ and moreover $[a]_{T_\eta(\omega)} \le 1$. With this notation we obtain the following result.

Theorem **4.3** (Atomic decomposition of $T_\eta(\omega)$)**.** *Let η be a function as above and ω a weight in $A_{2-(1/\ell-1/n)}$. Given $F \in T_\eta(\omega)$, there exists a sequence of multiple of atoms, $\{b_j\}$, such that*

$$F = \sum b_j \qquad a.e.$$

Moreover, if we denote by B_j the ball associate to b_j such that supp $b_j \subset \hat{B}_j$ and define

$$\Lambda(\{b_j\}) = inf\{\lambda > 0 : \sum |B_j|\eta(\frac{(\omega(B_j))^{1/2}}{\lambda^{1/\ell}|B_j|}\, \|b_j\|_{L^2(W)}) \leq 1\}, \tag{4.4}$$

we have $\Lambda(\{b_j\}) < \infty$ *and*

$$inf \Lambda(\{c_j\}) \leq C[F]_{T_\eta(\omega)} \tag{4.5}$$

where the infimum is taken over all possible decompositions of F*.*

Before proving the theorem we need some technical results.

LEMMA **4.6.** *Let* ρ *be a non-negative increasing function of finite upper type and* ω *a weight in* A_∞*. Then*

a) *There exists a constant* C_0 *such that*

$$\rho(\frac{1}{|Q|}\int_Q \omega) \leq C_0 \frac{1}{|Q|}\int_Q \rho(\omega)$$

for any cube $Q \subset \mathbb{R}^n$

b) *If in addition* ρ *is concave, then for any* $C > 0, \rho(C\omega)$ *belongs to* A_∞ *with an uniform constant.*

The proofs are straightforward using the following characterization of A_∞(see for example [CF]).

There exist $0 < \alpha, \beta < 1$ such that for any cube $Q \subset \mathbb{R}^n$

$$|\{x \in Q : \omega(x) > \beta m_Q \omega\}| \geq \alpha |Q|.$$

The next proposition gives a weighted version of a clue estimate given in [CMS]. Before stating it we introduce the definition over the tent of a general measurable set $\Omega \subset \mathbb{R}^n$ as the union of the tents $\hat{B}$ for all the balls $B \subset \Omega$.

PROPOSITION **4.7.** *Let* ω *be a weight in* A_∞ *and* B_0 *a ball in* $\mathbb{R}^n$*. Then there exists a constant* C *such that for every measurable function* F *defined on* $\mathbb{R}^n \times (0,\infty)$ *and every measurable set* $E \subset B_0$*, we have*

$$\int_{\hat{B}_0 - \hat{\Omega}} |F(x,t)|^2 \frac{\omega(B(x,t))}{t^n}\frac{dxdt}{t} \leq C \int_{B_0 - E} \mathcal{V}^2(F)(x)\omega(x)dx,$$

where $\Omega = \{x \in B_0 : M(\mathcal{X}_E)(x) > \frac{1}{2}\}$*, with* M *the Hardy-Littlewood maximal operator.*

Proof. Set $Z = \{(x,y,t) \in (B_0 - E) \times \widehat{B}_0 - \widehat{\Omega} : |x-y| < t\}$ and $Z_{(y,t)} = (B_0 - E) \cap B(y,t)$. We claim that there exists $\alpha > 0$, such that for any $(y,t) \in \widehat{B}_0 - \widehat{\Omega}$.

$$\omega(Z_{(y,t)}) \geq \alpha\omega(B(y,t)). \tag{4.8}$$

In fact, if $(y,t) \in \widehat{B}_0 - \widehat{\Omega}$, there exists $x_0 \notin \Omega$ with $x_0 \in B(y,t) \subset B_0$. So $M(\mathcal{X}_E)(x_0) \leq \frac{1}{2}$, and, in particular

$$|E \cap B(y,t)| \leq \frac{1}{2}|B(y,t)|.$$

Therefore $|(B_0 - E) \cap B(y,t)| \geq \frac{1}{2}|B(y,t)|$ and (4.8) follows from the A_∞ condition.

Then, we have

$$\begin{aligned}
\int_{\widehat{B}_0 - \widehat{\Omega}} |F(y,t)|^2 \frac{\omega(B(y,t))}{t^n}\,\frac{dxdt}{t} &\leq \frac{1}{\alpha}\int_{\widehat{B}_0 - \widehat{\Omega}} |F(y,t)|^2 \int_{Z_{(y,t)}} \omega(x)dx \frac{dydt}{t^{n+1}} \\
&= \frac{1}{\alpha}\int_Z |F(y,t)|^2 \omega(x)dx\,\frac{dydt}{t^{n+1}} \\
&\leq \frac{1}{\alpha}\int_{B_0 - E} \omega(x)(\int_{\Gamma(x)} |F(y,t)|^2\,\frac{dydt}{t^{n+1}})\,dx \\
&= \frac{1}{\alpha}\int_{B_0 - E} \mathcal{V}^2(F)(x)\omega(x)\,dx
\end{aligned}$$

□

Now, we are in position to proceed with the decomposition into atoms.

Proof of Theorem 4.3. For $k \in \mathbb{Z}$ let $E_k = \{x : \mathcal{V}(F)(x) > 2^k\}$ and $\Omega_k = \{x : M(\mathcal{X}_{E_k})(x) > \frac{1}{2}\}$. It is not hard to check that, except for a zero measure set, $supp\,(F) \subset \cup\widehat{\Omega}_k$. In fact, for any Lebesgue point (x,t) not belonging to any $\widehat{\Omega}_k$, there exists a sequence $\{y_k\} \subset B(x,t)$ with $M(\mathcal{X}_{E_k})(y_k) \leq \frac{1}{2}$. Therefore for any $k, |B(x,t) \cap \{z : \mathcal{V}(F)(z) \leq 2^k\}| \geq \frac{1}{2}|B(x,t)|$, and taking the limit for k tending to - ∞, we get

$$|B(x,t) \cap \{z : \mathcal{V}(F)(z) = 0\}| \geq \frac{1}{2}|B(x,t)|$$

.

From here we easily conclude that for some $y \in B(x,t), F = 0$ a.e. in $\Gamma(y)$ and hence $F(x,t) = 0$.

Now, for each k we make a Whitney decomposition of Ω_k into cubes Q_k^j. Next we choose a family of corresponding concentric balls B_k^j

containing Q_k^j with radious C-times the diameter of Q_k^j, in such a way that for the sets

$$A_k^j = \widehat{B_k^j} \cap (Q_k^j \times (0,\infty)) \cap (\widehat{\Omega}_k - \widehat{\Omega}_{k+1})$$

it holds

$$\widehat{\Omega}_k - \widehat{\Omega}_{k+1} \subset \cup_j A_k^j.$$

In fact, it is not difficult to see that it is enough to take C greater than $C_0 + 1$, where C_0 is the constant of the Whitney covering.

Now we define $b_k^j = F\chi_{A_k^j}$. It is clear that they are multiples of atoms and that $F = \sum b_k^j$. It remains to show that $\Lambda(\{b_k^j\}) \le C[F]_{T_\eta(\omega)}$. First observe that, by proposition (4.7),

$$\begin{aligned}
\|b_k^j\|^2_{L^2(W)} &\le \int_{\widehat{B_k^j} - \widehat{\Omega}_{k+1}} |F(y,t)|^2 \frac{\omega(B(y,t))}{t^n} \frac{dydt}{t} \\
&\le C \int_{B_k^j - E_{k+1}} |\mathcal{V}(F)(x)|^2 \omega(x) dx \\
&\le C 2^{2(k+1)} \omega(B_k^j).
\end{aligned}$$

If we set $\gamma = [F]^{1/\ell}_{T_\eta(\omega)}$, by lemma (4.6), we get

$$\begin{aligned}
\sum_{k,j} |B_k^j| \eta\Big(\frac{\omega(B_k^j)^{1/2}}{|B_k^j|\gamma} \|b_k^j\|_{L^2(W)}\Big) &\le C \sum_{k,j} |Q_k^j| \eta\Big(2^{k+1} \frac{\omega(Q_k^j)}{\gamma |Q_k^j|}\Big) \\
&\le C \sum_{k,j} \int_{Q_k^j} \eta\Big(\omega(z) \frac{2^{k+1}}{\gamma}\Big) dz \\
&\le C \sum_k \int_{\Omega_k} \eta\Big(\omega(z) \frac{2^{k+1}}{\gamma}\Big) dz.
\end{aligned}$$

But, by part b) of the same Lemma, there exists $p > 1$ such that $\eta(C\omega) \in A_p$ with an uniform constant. Therefore the Hardy-Littlewood maximal operator is of weak type (p,p) with respect to $\eta(C\omega)$ with an uniform constant. Thus, we have

$$\int_{\{M(\chi_{E_k}) > \frac{1}{2}\}} \eta\Big(\frac{2^{k+1}}{\gamma}\omega(z)\Big) dz \le C \int_{E_k} \eta\Big(\frac{2^{k+1}}{\gamma}\omega(z)\Big) dz.$$

With this estimate, the above sum over k is bounded by

$$
\begin{aligned}
C\sum_k \int_{E_k} \eta\left(\frac{2^{k+1}}{\gamma}\omega(z)\right) dz &\leq C\int_{\mathbb{R}^n} \sum_{k<log_2\mathcal{V}(F)(z)} \eta(\frac{2^{k+1}}{\gamma}\omega(z))dz \\
&\leq C\int_{\mathbb{R}^n} (\sum_{k<log_2\mathcal{V}(F)(z)} \int_{2^{k+1}}^{2^{k+2}} \eta(\frac{s}{\gamma}\omega(z))\frac{ds}{s})dz \\
&\leq C\int_{\mathbb{R}^n} (\int^{4\omega(z)\mathcal{V}(F)(z)\mathcal{V}(F)(z)/\gamma} \eta(s)\frac{ds}{s})dz \\
&\leq C\int_{\mathbb{R}^n} \eta(\frac{\mathcal{V}(F)(z)}{\gamma}\omega(z))dz,
\end{aligned}
$$

where we have used that the positive lower type of η implies $\int_0^t \eta(s)\frac{ds}{s} \leq C\eta(t)$. This shows our assertion. $\square$

5. Proof of the main result

Proof of Theorem 2.3. Part (a). Let $B = B(x_0, r)$ be a ball in $\mathbb{R}^n$. We split f as

$$f = (f - m_B f)\chi_{\tilde{B}} + (f - m_B f)\chi_{\tilde{B}^c} + m_B f = f_1 + f_2 + f_3,$$

where $\tilde{B} = B(x_0, 2r)$. Since ψ_t has zero average, $\psi_t * f_3 \equiv 0$. For f_1, we have

$$
\begin{aligned}
I &= \int_{\hat{B}} |\psi_t * f_1|^2(y)\frac{t^n}{\omega(B(y,t))}\,\frac{dydt}{t} \\
&\leq C\int_{\hat{B}} |\psi_t * f_1|^2(y)\frac{\omega^{-1}(B(y,t))}{t^n}\,\frac{dydt}{t} \\
&= C\int_{\hat{B}} |\psi_t * f_1|^2(y)\left(\int_{B(y,t)} \omega^{-1}(z)dz)\frac{dydt}{t^{n+1}}\right) \\
&= C\int_{B(x_0,r)} \omega^{-1}(z)\left(\int_{\Gamma(z)} |\psi_t * f_1|^2(y)\frac{dydt}{t^{n+1}}\right).
\end{aligned}
$$

Since $\omega \in A_2$ from the theory of vector valued singular integrals we have that the operator $S_\psi f(z) = (\int_{\Gamma(z)} |\psi_t * f_1|^2(y)\frac{dydt}{t^{n+1}})^{1/2}$ is bounded from $L^2(\omega^{-1})$ in $L^2(\omega^{-1})$, then

$$
\begin{aligned}
I \leq C\int_{\mathbb{R}^n} |f_1(z)|^2\omega^{-1}(z)dz &= C\int_{\tilde{B}} |f(z) - m_B f|^2\omega^{-1}(z)dz \\
&\leq C\omega(B)\varphi^2(|B|^{1/n})\|f\|^2_{BMO_\varphi(\omega)}.
\end{aligned}
$$

The last inequality is due to the equivalence of norms in $BMO_\varphi(\omega)$ (see[M]). Now, for f_2, denoting by $B_k = B(x_0, 2^k r)$, we have

(5.1)
$$\begin{aligned}|\psi_t * f_2|(y) &\le \sum_{k=2}^{\infty}\int_{B_k - B_{k-1}} |f(x) - m_B f|\,|\psi_t(y-x)|dx\\ &\le \sum_{k=2}^{\infty}\int_{B_k - B_{k-1}} |f(x) - m_{B_k} f||\psi_t(y-x)|dx\end{aligned}$$
$$+\sum_{k=2}^{\infty}\sum_{j=1}^{\infty}\frac{1}{|B_j|}\left(\int_{B_j}|f(z)-m_{B_j}f|dz\right)\left(\int_{B_k-B_{k-1}}|\psi_t(y-x)|dx\right) = D_1 + D_2.$$

Using that $\psi \in \mathcal{S}$ and the fact that φ is of upper type β, we get for $(y,t) \in \widehat{B}$

$$\begin{aligned}D_1 &\le C\sum_{k=2}^{\infty}\int_{B_k-B_{k-1}}|f(x)-m_{B_k}f|\frac{t^\alpha}{(t+|y-x|)^{n+\alpha}}\,dx\\ &\le C\left(\frac{t}{r}\right)^\alpha \sum_{k=2}^{\infty}\frac{1}{2^{k\alpha}}\frac{1}{|B_k|}\int_{B_k}|f(x)-m_{B_k}f|\,dx\\ &\le C\left(\frac{t}{r}\right)^\alpha \|f\|_{BMO_\varphi(\omega)}\sum_{k=2}^{\infty}\frac{1}{2^{k\alpha}}\frac{\omega(B_k)}{|B_k|}\varphi(2^k r)\\ &\le C\left(\frac{t}{r}\right)^\alpha \|f\|_{BMO_\varphi(\omega)} r^\alpha\int_{B^c}\frac{\omega(z)\varphi(|z-x_0|)}{|z-x_0|^{n+\alpha}}\,dz\\ &\le Ct^\alpha\|f\|_{BMO_\varphi(\omega)}r^{-\beta}\varphi(r)\int_{B^c}\frac{\omega(z)}{|z-x_0|^{n+\alpha-\beta}}dz.\end{aligned}$$

Taking $\alpha = n+\beta$ and recalling a well known property of A_2 weights, we obtain

$$D_1 \le Ct^{n+\beta}\|f\|_{BMO_\varphi(\omega)}r^{-\beta}\varphi(r)\frac{\omega(B)}{|B|^2} = C\|f\|_{BMO_\varphi(\omega)}(\frac{t}{r})^{n+\beta}\varphi(r)\frac{\omega(B)}{|B|}.$$

To estimate D_2 we observe that

$$\int_{B_k-B_{k-1}}|\psi_t(y-x)|dx \le C\frac{t^\alpha}{(t+2^k r)^{n+\alpha}}|B_k|.$$

Therefore, using $\omega \in A_2$ implies $\omega \in A_{2-\epsilon}$ and taking $\alpha = n+\beta$ again,

$$\begin{aligned}
D_2 &\le C\|f\|_{BMO_\varphi(\omega)} \sum_{k=2}^{\infty} \frac{t^\alpha}{(2^k r)^\alpha} \sum_{j=1}^{k} \frac{\omega(B_j)}{|B_j|}\varphi(2^j r) \\
&\le C\|f\|_{BMO_\varphi(\omega)} \left(\frac{t}{r}\right)^\alpha \frac{\omega(B)}{|B|}\varphi(r) \sum_{k=2}^{\infty} \frac{1}{2^{k\alpha}} \sum_{j=1}^{k} 2^{jn(2-\epsilon)} 2^{j(\beta-n)} \\
&\le C\|f\|_{BMO_\varphi(\omega)} \left(\frac{t}{r}\right)^{n+\beta} \varphi(r)\frac{\omega(B)}{|B|}.
\end{aligned}$$

So we obtain the same estimate for D_1 and D_2. Then, integrating over $\hat{B}$, we get

$$\int_{\hat{B}} |\psi_t * f_2|^2(y) \frac{\omega^{-1}(B(y,t))}{t^n} \frac{dydt}{t} \le C\|f\|^2_{BMO_\varphi(\omega)} \frac{\varphi^2(r)}{r^{2(n+\beta)}} \omega(B)|B| \int_0^r t^{n+2\beta} \frac{dt}{t}.$$

Finally, from this estimate and that obtained for I we finish the proof of (a).

Now we turn into the proof of b). Under our assumptions on ψ there exists $\bar{\psi} \in \mathcal{S}$ with $\int \bar{\psi} = 0$ such that for any $g \in \mathcal{S}$

$$g_\epsilon = \int_\epsilon^{1/\epsilon} \bar{\psi}_t * \psi_t * g \frac{dt}{t} \longrightarrow g$$

pointwisely (see, for example, [S], page 159). Furthermore for $g \in \mathcal{S}_\infty$ (see proposition (3.5) for the definition), we may follow the same steps as in [FJW], page 122, to conclude that the above convergence occurs also in the topology of $\mathcal{S}$. Therefore for f as in the hypothesis, $g \in \mathcal{S}_\infty$, and denoting $\tilde{g}(x) = g(-x)$, we have

$$\begin{aligned}
(f, \tilde{g}) &= \lim_{\epsilon \longrightarrow 0} (f, \tilde{g}_\epsilon) \\
&= \lim_{\epsilon \longrightarrow 0} \int_\epsilon^{1/\epsilon} (f, \bar{\psi}_t * \psi_t * \tilde{g}) \frac{dt}{t} \\
&= \lim_{\epsilon \longrightarrow 0} \int_\epsilon^{1/\epsilon} (\bar{\psi}_t * f, \psi_t * g) \frac{dt}{t} \qquad (5.2) \\
&= \lim_{\epsilon \longrightarrow 0} \int_\epsilon^{1/\epsilon} \left(\int (\psi_t * f)(x)(\bar{\psi}_t * g)(x)dx\right) \frac{dt}{t},
\end{aligned}$$

where for the last equality we use that $\psi_t * f$ and $\bar{\psi}_t * g$ are C^∞ functions and that, as we will see bellow, the integral is absolutely convergent. We claim that for any pair of measurable functions on $\mathbb{R}^{n+1}_+$, say F

and G, we have

$$\int_{\mathbb{R}^{n+1}_+} |F(x,t)||G(x,t)|\frac{dxdt}{t} \le C[dF]_{\varphi,\omega}[G]^{1/\ell}_{T_\eta(\omega)}, \tag{5.3}$$

where $dF = |F(x,t)|^2 \frac{t^n}{\omega(B(x,t))}\frac{dxdt}{t}$, $\eta^{-1}(\frac{1}{t}) = \frac{1}{t\varphi(t^{1/n})}$ and ℓ is the lower type of η, that is $\ell = 1/(1-\beta/n)$. In fact if $G \in T_\eta(\omega)$, in view of the atomic decomposition (see Theorem 4.3), G can be written a.e. as

$$G(x,t) = \sum_j b_j(x,t)$$

in such a way that

$$\Lambda(\{b_j\}) \le C[G]_{T_\eta(\omega)}.$$

Therefore, if B_j is the ball associate to b_j such that $supp(bj) \subset \hat{B}_j$, we get

$$\begin{aligned} |\int_{\mathbb{R}^{n+1}_+} F(x,t)G(x,t)\frac{dxdt}{t}| &\le \sum_j \left(\int_{\hat{B}_j} |F(x,t)|^2 \frac{t^n}{\omega(B(x,t))}\frac{dxdt}{t}\right)^{1/2} \\ &\quad \left(\int_{\hat{B}_j} |b_j(x,t)|^2 \frac{\omega(B(x,t))}{t^n}\frac{dx}{t}\right)^{1/2} \\ &\le [dF]_{\varphi,\omega} \sum_j \omega(B_j)^{1/2}\varphi(|B_j|^{1/n})\|b_j\|_{L^2(W)}. \end{aligned}$$

Now, is σ denotes the last sum, it is easy to check that

$$\sum_j |B_j|\eta\left(\frac{\omega(B_j)^{1/2}}{|B_j|\sigma}\|b_j\|_{L^2(W)}\right) \ge 1.$$

In fact, replacing $\varphi(|B_j|^{1/n})$ by $1/(|B_j|\eta^{-1}(1/|B_j|))$, the above inequality follows using the fact that η is of upper type less than or equal to one. Therefore, in view of the definition of Λ, we get

$$\sigma^\ell \le \Lambda(\{b_j\})$$

finishing the proof of (5.3). Now, applying this inequality in (5.2), we have

$$\begin{aligned} |(f,\tilde{g})| &\le C \quad [d\mu]^{1/2}_{\varphi,\omega}\left[\left(\int_{\Gamma(.)} |\bar{\psi}_t * g|^2(y)\frac{dy\,dt}{t^{n+1}}\right)^{1/2}\omega(.)\right]^{1/\ell}_{L^\eta(\mathbb{R}^n)} \\ &= \quad C[d\mu]^{1/2}_{\varphi,\omega}\left[(S_{\bar{\psi}}g)\omega\right]^{1/\ell}_{L^\eta(\mathbb{R}^n)}, \end{aligned}$$

where $d\mu$ denotes the measure associate to $\psi_t * f$. Then, it is clear that part b) of our theorem follows from the above inequality by using Proposition (3.2), provided we can prove that

$$[(S_{\bar\psi} g)]_{L^\eta(\mathbb{R}^n)} \leq C[g]_{H^q_\eta(\omega)} \tag{5.4}$$

In order to check that (5.4) holds, we recall, as in the proof of part a), that it is well known that S_ψ can be studied as a Calderón-Zygmund operator taken values on $L^2(\mathbb{R}^n \times \mathbb{R}^+, \frac{dydt}{t^{n+1}})$. Then, the theory of vector valued singular integrals allows us to assert that it is bounded on $L^q(\omega)$ for any $\omega \in A_q$. Now, for a function b in $L^q(\omega)$ with compact support on a ball $B_0 = B(x_0, r_0)$ and zero average, Jensens's inequality leads us to the following estimate

$$\begin{aligned} \int_{\tilde B_0} \eta(S_{\bar\psi} b(x)\omega(x))dx &\leq C|B_0|\eta\left(\frac{\omega(B_0)^{1/q'}}{|B_0|}\|S_{\bar\psi} b\|_{L^q(\omega)}\right) \\ &\leq C|B_0|\eta\left(\frac{\omega(B_0)^{1/q'}}{|B_0|}\|b\|_{L^q(\omega)}\right) \end{aligned} \tag{5.5}$$

where $\tilde B_0 = B(x_0, 2r_0)$. On the other hand, it also known that

$$S_{\bar\psi} b(x) \leq C\left(\frac{r_0}{|x-x_0|}\right)^{n+1} \frac{\|b\|_{L^q(\omega)}}{\omega(B_0)^{1/q}},$$

for $x \notin \tilde B_0$. Then, a standard reasoning using this estimate, Jensen's inequality and the fact that $\omega \in A_q$ allows us to get

$$\begin{aligned} \int_{\tilde B_0^c} \eta(S_{\bar\psi} b(x)\omega(x))dx &\leq C|B_0|\sum_{j=2}^\infty 2^{jn} \\ &\quad \times\left(\frac{\omega(B_0)^{1/q'}}{|B_0|}\frac{\|b\|_{L^q(\omega)}}{2^{(2n-nq+1)j}}\right) \\ &\leq C|B_0|\eta\left(\frac{\omega(B_0)^{1/q'}}{|B_0|}\|b\|_{L^q(\omega)}\right) \end{aligned} \tag{5.6}$$

because of our assumptions on q and ℓ. Therefore if b is an atom in $H^q_\eta(\omega)$, (5.5) and (5.6) imply (5.4) for $g = b$. Consequently (5.4) holds for every g in $H^q_\eta(\omega)$, finishing our proof. □

6. A weighted Triebel–Lizorkin space

As we said in section 1, Bui and Taibleson introduced in [BT] a weighted version of the Triebel–Lizorkin spaces $\dot{F}^{\alpha}_{\infty,q}$. Specifically, they take $\psi \in \mathcal{S}$ with *supp* $\hat{\psi} \subset \{\xi : \frac{1}{2} \leq |\xi| \leq 2\}$ and $\sum_{j=-\infty}^{\infty} |\hat{\psi}(2^{-j}\xi)|^2 = 1$ for $|\xi| \neq 0$. Then, for $\omega \in A_\infty$, the space $\dot{F}^{\alpha,\omega}_{\infty,q}, \alpha \in \mathbb{R}, 0 < q < \infty$ is defined as the set of f in $\mathcal{S}'/\mathcal{P}$ such that

$$\sup_Q \left(\frac{1}{\omega(Q)} \int_Q \sum_{j=-log_2(\ell(Q))}^{\infty} (2^{j\alpha} |\psi_{2^{-j}} * f|(x))^q \omega(x) dx \right)^{1/q} < \infty, \tag{6.1}$$

where Q denotes a dyadic cube in $\mathbb{R}^n$ with lenght side $\ell(Q)$. For the case $\alpha = 0$ and $q = 2$ we have the following result.

Proposition **6.2.** *The space BMO is contained in $\dot{F}^{0,\omega}_{\infty,2}$ for any ω in A_2. Moreover, if in addition, we assume $\omega \in A_1$, then both spaces coincide.*

In view of the above proposition and Theorem (2.3), we consider that $\dot{F}^{\alpha,\omega}_{\infty,q}$ should be rather defined as the set of f in $\mathcal{S}'/\mathcal{P}$ such that

$$\sup_B \left(\frac{1}{\omega(B)} \int_{\hat{B}} (t^{-\alpha} |\psi_t * f|(x))^q \frac{t^n}{\omega(B(x,t))} \frac{dxdt}{t} \right)^{1/q} < \infty.$$

When $\alpha = 0$ and $q = 2$, note that for $\omega \in A_{1+1/n}$ and ψ in $\mathcal{S}$ satisfying a tauberian condition, Theorem 2.3 allows us to obtain two facts: first, a weighted version of the well known result $BMO \simeq \dot{F}^0_{\infty,2}$, and, second, as a consequence, that the definition of the space does not depend on the choice of ψ

Proof of Proposition 6.2 Let f be in BMO. Given a cube Q, we split f as follows

$$f = (f - m_{Q,\omega} f)\mathcal{X}_{\tilde{Q}} + (f - m_{Q,\omega} f)\chi_{\tilde{Q}^c} + m_{Q,\omega} f = f_1 + f_2 + f_3,$$

where $m_{Q,\omega} f = \frac{1}{\omega(Q)} \int_Q f\omega$ and $\tilde{Q}$ denotes the concentric cube with Q and length side $2\ell(Q)$. In order to prove that f satisfies (6.1) for $\alpha = 0$ and $q = 2$, we first estimate

$$\begin{aligned} I = \int_Q \sum_{j=-log_2 \ell(Q)}^{\infty} |\psi_{2^{-j}} * f_1|^2(x)\omega(x)dx &\leq \int_{\mathbb{R}^n} \sum_{j=-\infty}^{\infty} |\psi_{2^{-j}} * f_1|^2(x)\omega(x)dx \\ &= \int_{\mathbb{R}^n} (G(f_1)(x))^2 \omega(x) dx. \end{aligned}$$

But, J. García–Cuerva and J. M. Martell prove (see [GM]) that G can be viewed as a vector valued Calderón-Zygmund operator, which, since $\omega \in A_2$, it is bounded from $L^2(\omega)$ into $L^2_{\ell^2}(\omega)$. So, we have

$$I \leq C \int_{\mathbb{R}^n} |f_1(x)|^2 \omega(x)dx = C \int_{\tilde{Q}} |f - m_{Q,\omega}f|^2 \omega(x)dx \\ \leq C\|f\|^2_{BMO}\, \omega(Q),$$

where the last inequality follows from the fact that (see[MW])

$$sup_Q\Big(\frac{1}{\omega(Q)} \int_Q |f - m_{Q,\omega}f|^p \omega\Big)^{1/p} \simeq \|f\|_{BMO}$$

On the other hand, denoting by Q_k the concentric cube with Q and length side $2^k\ell(Q)$, a similar reasoning to that applied for getting estimate (5.1) allows us to get

$$\begin{aligned} |\psi_{2^{-j}} * f_2|(y) \leq & C\left(\frac{2^{-j}}{\ell(Q)}\right)^\alpha \left(\sum_{k=2}^\infty \frac{1}{2^{k\alpha}} \frac{1}{|Q_k|} \int_{Q_k - Q_{k-1}} |f(x) - m_{Q_k,\omega}f|dx \right. \\ & + \sum_{k=2}^\infty \sum_{i=2}^k \frac{1}{\omega(Q_i)} \int_{Q_i} |f(x) - m_{Q_i,\omega}f|\omega(x)dx \\ & \left. \times \int_{Q_k - Q_{k-1}} \psi_{2^{-j}}(y-x)dx \right), \end{aligned}$$

for every $j \in Z$, for $\alpha > 0$ fixed. Then, since $\omega \in A_2$, Hölder's inequality and the fact that the norms are equivalent, yield

$$|\psi_{2^{-j}} * f_2|(y) \leq C\left(\frac{2^{-j}}{\ell(Q)}\right)^\alpha \|f\|_{BMO}.$$

Now, from this inequality, we get

$$\begin{aligned} \int_Q \sum_{j=-log_2\ell(Q)}^\infty |\psi_{2^{-j}} * f_2|^2(x)\omega(x)dx \ & \leq \ C\frac{\omega(Q)}{(\ell(Q))^\alpha}\|f\|_{BMO} \sum_{j=-log_2\ell(Q)}^\infty 2^{-j\alpha} \\ & \leq \ C\omega(Q)\|f\|_{BMO}. \end{aligned}$$

So, the above estimate, the obtained for I and the fact that $\psi_2 - j * f_3 = 0$ prove $f \in \dot{F}^{0,\omega}_{\infty,2}$ that is $BMO \subset F^{0,\omega}_{\infty,q}$.

Now, assuming $\omega \in A_1$ and using the well known unweighted result $BMO \simeq \dot{F}^0_{\infty,2}$ (see [FJ], for instance), we can write

$$\begin{aligned}
\|f\|^2_{BMO} &\leq \sup_Q \frac{1}{|Q|} \int_Q \sum_{j=-log_2 \ell(Q)}^{\infty} |\psi_{2^{-j}} * f|^2(x) dx \\
&\leq C \sup_Q \frac{1}{\omega(Q)} \int_Q \sum_{j=-log_2 \ell(Q)}^{\infty} |\psi_{2^{-j}} * f|^2(x) \omega(x) dx.
\end{aligned}$$

Clearly, this implies $\dot{F}^{0,\omega}_{\infty,2} \subset BMO$. $\square$

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