THE STRUCTURE OF INCREASING WEIGHTS ON THE REAL LINE

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ABSTRACT. We examine the structure of a variety of related weight classes on the real line and the positive real axis: doubling measures, A_p weights, the B_p weights of Ariño and Muckenhoupt, and Δ_2 Young functions. We give a number of characterizations of these classes. As applications we compute the Matuszewska-Orlicz indices of a Young function due to Lindberg [27], give a sufficient condition for a function m to be a multiplier of the doubling measures on $\mathbb{R}_+$ and answer a question on quasi-symmetric mappings raised by the first author in [4].

1. INTRODUCTION

The purpose of this paper is to bring together a number of results on the structure of various classes of measures and (increasing) weights on the real line and the positive real axis. These weights are important in a number of different branches of analysis—weighted norm inequalities, Orlicz spaces, quasi-symmetric mappings—and our results incorporate ideas from these fields.

The paper is organized as follows: in Sections 2—5 we describe all of our results in detail. In Section 2 we state the basic definitions and properties of the measures and weights we are interested in. In Section 3 we give our results on the relationship between three classes on the positive real axis: Δ_2 Young functions, A_p weights and the B_p measures of Ariño and Muckenhoupt [1]. As an application we compute the Matuszewska-Orlicz indices (and so the Boyd indices) of the Young function $t^{a+b\sin(\log(|\log(t)|))}$. This function was introduced by Lindberg [27] and considered by a number of other authors. The values of its indices have long been conjectured but, until now, have never been proved. In Section 4 we gather together a number of other

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results about these and other related weight classes. In Section 5 we characterize the so-called engulfing property of sections of convex functions on the real line. This property is important in the study of Monge-Ampère equations (see Gutiérrez and Huang [19]). There is a connection between such convex functions, doubling measures, and quasi-symmetric maps (defined in the introduction to Section 5) and the resulting classification scheme parallels Theorem 3.1 in Section 3. As an application of these ideas we give a sufficient condition on a function to be a multiplier of doubling measures on $\mathbb{R}^+$ and answer in the negative a question on quasi-symmetric functions raised by the first author in [4]. In the remaining sections we give the proofs of our results.

Throughout this paper all notation is standard or will be defined as needed. By weights we will always mean non-negative, locally integrable functions which are positive on a set of positive measure. Given a Lebesgue measurable set E and a weight w, $|E|$ will denote the Lebesgue measure of E and $w(E) = \int_E w\,dx$. Given $1 < p < \infty$, $p' = p/(p-1)$ will denote the conjugate exponent of p. Finally, C will denote a positive constant whose value may change at each appearance.

2. Measures and weights

In this section we give a brief overview of the measures and weights we will be considering.

Doubling measures. Given a non-negative Borel measure μ on $\mathbb{R}$ or $\mathbb{R}^+$, we say that μ is a doubling measure if there exists a constant C_μ such that for every interval I, $\mu(2I) \leq C_\mu \mu(I)$, where $2I$ is the interval with the same center as I and twice the length. It is easy to see that given any $0 < \alpha < 1$, there exists a constant $C = C(C_\mu, \alpha)$ such that $\mu(I) \leq C\mu(\alpha I)$ (where αI has the same center as I and $|\alpha I| = \alpha|I|$). Equivalently, μ is doubling if there exists $M > 1$ such that given adjacent intervals I and J of equal length,

$$M^{-1}\mu(I) \leq \mu(J) \leq M\mu(I). \tag{2.1}$$

If this property only holds for dyadic intervals I and J that have the same dyadic parent, then we say that μ is a dyadic doubling measure.

B_p measures. A non-negative Borel measure μ on $\mathbb{R}^+$ is in the B_p class, $0 < p < \infty$, if there exists a positive constant C such that for every $t > 0$,

$$\int_t^\infty \frac{1}{x^p}\,d\mu \leq \frac{C}{t^p}\int_0^t d\mu. \tag{2.2}$$

The B_p class with μ absolutely continuous and $p > 1$ was introduced by Ariño and Muckenhoupt [1] while studying the Hardy-Littlewood maximal operator on Lorentz spaces. They showed that the B_p condition for absolutely continuous μ is necessary and sufficient for the Hardy-Littlewood maximal operator restricted to decreasing functions to be bounded on $L^p(w, \mathbb{R}^+)$. Sbordone and Wik [36] extended this result to $0 < p \le 1$. A close examination of these proofs show that they hold for any non-negative Borel measure μ satisfying the B_p condition.

Sbordone and Wik further showed that a Borel measure μ is in B_p for some $p > 0$ if and only if it satisfies the B_∞ condition: there exists a constant C such that for every $t > 0$, $\mu([0, 2t]) \le C\mu([0, t])$. This is an "anchored" doubling condition, and so we immediately have that every doubling measure is in B_∞. The reverse inclusion is false: for example, if we define μ by $d\mu(x) = \chi_{[0,1]}(x)dx$, then $\mu \in B_\infty$ but is not doubling.

There is a constructive characterization of B_∞ measures. Let $\mathbb{D}_k$ be the set of all dyadic intervals contained in $I_k = [0, 2^k]$, $k \in \mathbb{Z}$ and let $\mathbb{D} = \bigcup_k \mathbb{D}_k$. Given $I \in \mathbb{D}$, let h_I be the Haar function

$$h_I(x) = \begin{cases} 1 & x \in I^- \\ -1 & x \in I^+, \end{cases}$$

where I^- and I^+ are the dyadic subintervals of I such that $|I^\pm| = |I|/2$.

Theorem 2.1. *A non-negative Borel measure μ satisfies the B_∞ condition if and only if for every $k \in \mathbb{Z}$ the following is true: μ_k, the restriction of μ to I_k, can be written as*

$$\mu_k = \mu(I_k) \prod_{I \in \mathbb{D}_k} (1 + a_I h_I),$$

where the product converges in the weak topology, $|a_I| < 1$, and there exists $\epsilon > 0$ (independent of k) such that if $I = I_j$, $j \le k$, then $a_I < 1 - \epsilon$.*

The proof of Theorem 2.1 is a trivial modification of a characterization of dyadic doubling measures due to Fefferman, Kenig and Pipher [12]. It suffices to note that the B_∞ condition is equivalent to the dyadic B_∞ condition: for all j, $\mu([0, 2^{j+1}]) \le C\mu([0, 2^j])$. Further details are left to the interested reader.

A_p **weights.** A non-negative, locally integrable function w is in the A_p class, $1 < p < \infty$, if there exists a positive constant C such that for every interval $I \subset \mathbb{R}^+$ or $\mathbb{R}$,

$$\left(\frac{1}{|I|}\int_I w(x)\,dx\right)\left(\frac{1}{|I|}\int_I w(x)^{1-p'}\,dx\right)^{p-1} \le C.$$

The union of all the A_p classes is denoted A_∞. The A_p class was introduced by Muckenhoupt [29], who showed that a necessary and sufficient condition for the Hardy-Littlewood maximal operator to be bounded on $L^p(w)$ is $w \in A_p$. The A_p

weights also govern the weighted norm inequalities for the Hilbert transform and a number of other operators. For more information on A_p weights, see Duoandikoetxea [11] or García-Cuerva and Rubio de Francia [18].

Δ_2 Young functions. A non-negative function w on $\mathbb{R}^+$ is a Young function if it is continuous, convex, strictly increasing, $w(0) = 0$, and $w(t)/t \to \infty$ as $t \to \infty$. A Young function satisfies the Δ_2 condition if $w(2t) \leq Cw(t)$ for all $t > 0$. In [4] the first author showed that an increasing weight on $\mathbb{R}^+$ satisfies the Δ_2 condition if and only if it is in A_p for some $p > 1$.

Given a Young function w, let $\bar{w}$ denote its complementary function—that is,

$$\bar{w}(t) = \sup_{s \geq 0}\{st - w(s)\}.$$

The complementary function is also a Young function, and $\bar{w}$ is Δ_2 if and only if w satisfies the $\bigtriangledown_2$ condition: for some $a > 1$, $2aw(t) \leq w(at)$.

The growth of a Young function and its complementary function are measured by its Matuszewska-Orlicz indices. Given a Young function w, define the function H_w by

$$H_w(t) = \sup_{s>0} \frac{w(st)}{w(s)}.$$

Then the upper and lower indices of w are defined by

$$i(w) = \lim_{t\to 0^+} \frac{\log(H_w(t))}{\log(t)}, \qquad I(w) = \lim_{t\to\infty} \frac{\log(H_w(t))}{\log(t)}.$$

The indices of w and $\bar{w}$ have the following properties:

i) $1 \leq i(w) \leq I(w) \leq \infty$;
ii) $I(w)^{-1} + i(\bar{w})^{-1} = i(w)^{-1} + I(\bar{w})^{-1} = 1$;
iii) $I(w) < \infty$ if and only if w is Δ_2 and $i(w) > 1$ if and only if $\bar{w}$ is Δ_2.

For more information on Young functions, see Krasnosel'skiĭ and Rutickiĭ [25] or Rao and Ren [34]. For the indices of Young functions, see Maligranda [28] or Fiorenza and Krbec [14].

3. B_p measures, A_p weights and Δ_2 Young functions

Our main result in this section is to show that there is a very close relationship between B_p measures, A_p weights on $\mathbb{R}^+$ and Δ_2 Young functions.

Theorem 3.1. *Given a non-negative Borel measure μ on $\mathbb{R}^+$, define the functions v and w on $\mathbb{R}^+$ by*

$$v(t) = \int_0^t d\mu \quad \textit{and} \quad w(t) = \int_0^t v(x)\, dx.$$

Then given $p > 0$, the following are equivalent:

- *i)* $\mu \in B_p$*;*
- *ii)* v *is increasing and in* A_{p+1}*;*
- *iii)* w *is a* Δ_2 *Young function and in* A_{p+2}*.*

We prove Theorem 3.1 in Section 6.

Remark 3.2. The implication that (i) implies (ii) in Theorem 3.1 generalizes a result due to Johnson and Neugebauer [22], who showed that the integral of an A_p weight is in A_{p+1}. Their proof was a corollary to a more general result on homomorphisms and A_p weights; the first author gave a direct proof in [4].

This paper was written in two stages, with a long gap in between. During this interregnum we learned that the equivalence of (i) and (ii) in Theorem 3.1 (for absolutely continuous μ) was discovered independently by Cerdà and Martín [3], and was implicit in Soria [38]. Their proof is very similar to ours, drawing on results in [4]. We are grateful to the authors for calling our attention to their work.

The indices of a Δ_2 Young function and its complementary function correlate precisely with their A_p classes, as the next two results show.

Theorem 3.3. *Given a* Δ_2 *Young function* w *and* p^*, $1 \le p^* < \infty$, *the following are equivalent:*

- *i)* $I(w) = p^*$*;*
- *ii)* $q > p^*$ *if and only if there exists a constant* C *such that for every* $t > 0$,

$$t^q \int_t^\infty \frac{w(x)}{x^q} \frac{dx}{x} \le C w(t);$$

- *iii)* $q > p^*$ *if and only if for any* $a > 1$,

$$\liminf_{n \to \infty} \left(\frac{w(a^{k-n})}{w(a^k)} \right)^{1/n} > a^{-q},$$

  and the limit infimum is uniform in k*;*
- *iv)* $p^* + 1 = \inf\{q : w \in A_q\}$*;*

In the next result we want $\bar{w}$ to be Δ_2, so we must assume that $i(w) > 1$.

Theorem 3.4. *Given a* Δ_2 *Young function* w *and* p_*, $1 < p_* < \infty$, *the following are equivalent:*

- *i)* $i(w) = p_*$*;*

ii) $q < p_*$ if and only if there exists a constant C such that for every $t > 0$

$$t^q \int_0^t \frac{w(x)}{x^q} \frac{dx}{x} \leq Cw(t);$$

iii) $q < p_*$ if and only if for every $a > 1$,

$$\limsup_{n\to\infty} \left(\frac{w(a^{k-n})}{w(a^k)} \right)^{1/n} < a^{-q},$$

and the limit supremum is uniform in k;

iv) $p_*' + 1 = \inf\{q : \bar{w} \in A_q\}$.

Theorem 3.3 is implicit in the literature. The equivalence of (i) and (ii) is due to Maligranda [28]. The equivalence of (ii), (iii) and (iv) was shown in [4]; there (iii) was only shown for $a = 2$, but the same proof works for any a. We prove Theorem 3.4 in Section 6.

We give two applications of Theorems 3.3 and 3.4. The proof of each is also in Section 6. First, we consider the A_p class of a Δ_2 Young function whose complementary function is also in Δ_2. It follows at once from Theorem 3.1 that in this case both w and $\bar{w}$ can be in A_p only if $p > 2$. In fact, there are sharper lower bounds in terms of the best Δ_2 and ∇_2 constants.

Theorem 3.5. *Let w be a Δ_2 Young function whose complementary function is also in Δ_2, and let D and A be the best constants in the Δ_2 and ∇_2 conditions. Then:*

i) $w \in A_p$ only if $p > 2 + \dfrac{\log 2}{\log A}$;

ii) $\bar{w} \in A_p$ only if $p > 1 + \dfrac{\log D}{\log D - \log 2}$.

As a second application we compute the upper and lower indices of a complicated Young function that has appeared frequently in the literature.

Theorem 3.6. *Given $a, b \in \mathbb{R}^+$, $a > 1 + b\sqrt{2}$, then the Young function*

$$w(t) = t^{a+b\sin(\log(|\log(t)|))}$$

has indices $I(w) = a + b\sqrt{2}$ and $i(w) = a - b\sqrt{2}$.

This example is due to Lindberg [27]. Maligranda [28] showed that $i(w) \geq a - b\sqrt{2}$ and $I(w) \leq a + b\sqrt{2}$. Independently, Fusco and Sbordone [17] considered the special case $a = 4$ and $b = 1$. They showed that $p(w) = 4 - \sqrt{2}$ and $q(w) = 4 + \sqrt{2}$, where p and q are the lower and upper Simonenko indices,

$$p(w) = \inf_{t>0} \frac{tw'(t)}{w(t)} \leq i(w), \quad q(w) = \sup_{t>0} \frac{tw'(t)}{w(t)} \geq I(w).$$

(For more on Simonenko indices, see Maligranda [28].) Fiorenza [13] later extended their result to the general case. Finally, when $a = 4$ and $b = 1$, Fiorenza and Krbec [14] claimed to prove Theorem 3.6; unfortunately, there is an error in their proof and they only establish the inequalities already gotten by Maligranda and Fusco and Sbordone.

4. Additional results for weights on $\mathbb{R}^+$

Another growth condition. If $1 < p < \infty$, then a Young function w is in the class $\mathcal{B}_p$ if there exists $c > 0$ such that

$$\int_c^\infty \frac{w(x)}{x^p}\frac{dx}{x} < \infty.$$

This class was introduced by Pérez [33]. In the literature it is normally denoted by B_p. To avoid confusion with the class B_p of Ariño and Muckenhoupt, we adopt the given notation. The $\mathcal{B}_p$ condition plays an important role in the study of two-weight norm inequalities for classical operators: see [8, 9, 10, 31, 32, 33].

The functions from the class $\mathcal{B}_p$ that often appear in practice are Δ_2 Young functions. As we noted in Section 3, these functions are in A_q for some $q > 1$, so it is reasonable to conjecture that there is some relationship between q and p. By Theorem 3.3, if $w \in A_{p+1}$, then for every $t > 0$,

$$t^p \int_t^\infty \frac{w(x)}{x^p}\frac{dx}{x} \leq Cw(t),$$

so $w \in \mathcal{B}_p$. The converse need not be true; Pérez [33] gave a simple counter-example: $w(t) = t^p \log(e+t)^{-1-\delta}$, $\delta > 0$. It is "almost" in A_{p+1}, however, as the following theorem shows.

Theorem 4.1. *Let w be a Young function such that $w(t) = t^p\psi(t)$, where ψ is a decreasing function. Then $w \in A_q$ for all $q > p+1$.*

Theorem 4.1 includes the principal examples of $\mathcal{B}_p$ weights which appear in practice. However, in general there is no relationship between p and q as the next example shows.

Example 4.2. Given $1 < p < \infty$ and $q > p+1$, there exists a Δ_2 Young function w such that $w \in \mathcal{B}_p$ but w is not in A_q.

We prove Theorem 4.1 and construct Example 4.2 in Section 7.

The classes A_p, B_p and the reverse Hölder inequality. The two classes A_p and B_p govern similar weighted norm inequalities, so it is not surprising that they have similar structural properties as well. One such property is the "$p-\epsilon$" property: if $w \in B_p$ then for some $\epsilon > 0$, $w \in B_{p-\epsilon}$. (This is due to Ariño and Muckenhoupt [1]. Also see Neugebauer [30] and Sbordone and Wik [36].)

For A_p weights, the $p-\epsilon$ property is a consequence of the fact that A_p weights satisfy the reverse Hölder inequality,

$$RH_s: \qquad \left(\frac{1}{|I|}\int_I w(x)^s\,dx\right)^{1/s} \leq \frac{C}{|I|}\int_I w(x)\,dx, \quad \text{for all intervals } I,$$

for some $s > 1$. Therefore, it is surprising that B_p weights do not satisfy a reverse Hölder-type inequality. The natural candidate would be the "anchored" reverse Hölder inequality,

$$RH_s^0: \qquad \left(\frac{1}{t}\int_0^t w(x)^s\,dx\right)^{1/s} \leq \frac{C}{t}\int_0^t w(x)\,dx, \quad \text{for all intervals } I.$$

However, the following is true.

Theorem 4.3. *For any $s > 1$, if $w \in RH_s^0$ then $w \in B_\infty$. However, there exists $w \in B_\infty$ such that w does not satisfy the RH_s^0 condition for any $s > 1$.*

Remark 4.4. Theorem 4.3 is somewhat less surprising if we recall that Ariño and Muckenhoupt [1] showed that if w satisfies the "anchored" A_p condition,

$$A_p^0: \qquad \left(\frac{1}{t}\int_0^t w(x)\,dx\right)\left(\frac{1}{t}\int_0^t w(x)^{1-p'}\,dx\right)^{p-1} \leq C, \quad t > 0,$$

then $w \in B_p$, but that the converse is not true.

Another consequence of the reverse Hölder inequality is that A_p weights can be perturbed by taking powers: if $w \in A_p$ then there exists $s > 1$ such that $w^r \in A_p$ for $0 < r < s$. This property fails dramatically for B_p weights.

Example 4.5. There exists a weight $w \in B_\infty$ such that w^r is not in B_∞ for any $r \neq 1$.

Remark 4.6. If $w \in RH_s^0$, $s > 1$, then by an modifying an argument of Heinonen, Kilpeläinen and Martio [20, pp. 66-8] we can show that $w^r \in RH_{s/r}^0$, $0 < r < s$. Details are left to the reader.

Remark 4.7. Example 4.5 was first discovered while studying the analogous problem for doubling weights. For those results, see [5].

We prove Theorem 4.3 and construct Example 4.5 in Section 8.

B_∞ weights and the minimal operator. The class B_∞ can be characterized in terms of weighted norm inequalities for the minimal operator:

$$mf(x) = \inf_{I \ni x} \frac{1}{|I|} \int_I |f(y)|\, dy,$$

where the infimum is taken over all intervals $I \subset \mathbb{R}^+$ which contain x. The minimal operator was first introduced by Cruz-Uribe and Neugebauer [6] to study the fine structure of A_p weights. They showed that

$$\int_{\mathbb{R}^n} \frac{w(x)}{mf(x)^p}\, dx \leq C \int_{\mathbb{R}^n} \frac{w(x)}{|f(x)|^p}\, dx, \quad 0 < p < \infty,$$

if and only if $w \in A_\infty$. (Here we extend the definition of m to $\mathbb{R}^n$ in the obvious way.) If we restrict f to be an increasing function on $\mathbb{R}^+$, we can prove a result analogous to this and to that of Ariño and Muckenhoupt [1].

Theorem 4.8. *Given a non-negative Borel measure μ, the following are equivalent:*

i) $\mu \in B_\infty$;

ii) there exists a constant C such that for any p, $0 < p < \infty$, and any non-negative, increasing function f,

$$\mu(\{x \in \mathbb{R}^+ : mf(x) < 1/t\}) \leq \frac{C}{t^p} \int_{\mathbb{R}^+} \frac{d\mu}{f(x)^p};$$

iii) there exists a constant C such that for any p, $0 < p < \infty$, and any non-negative, increasing function f,

$$\int_{\mathbb{R}^+} \frac{d\mu}{mf(x)^p} \leq C \int_{\mathbb{R}^+} \frac{d\mu}{f(x)^p}.$$

Cruz-Uribe, Neugebauer and Olesen [7] proved two-weight norm inequalities for the minimal operator, and based on their results we originally conjectured that a necessary and sufficient condition for the two-weight version of Theorem 4.8 to hold is

$$\frac{1}{t} \int_0^t u(x)\, dx \leq C \left(\frac{1}{t} \int_0^t v(x)^{1/(p+1)}\, dx \right)^{p+1}. \tag{4.1}$$

But in the one-weight case, by Theorem 4.3 this implies that $w^{1/(p+1)} \in B_\infty$, and Example 4.5 shows that this is not necessary for w to be in B_∞. In Proposition 9.1 we show that inequality (4.1) is sufficient for the weak-type inequality; the problem of finding necessary and sufficient conditions for the weak and strong-type inequalities remains open.

All of these results are proved in Section 9.

5. Doubling measures, quasi-symmetric maps, and the engulfing property

In this section we are primarily interested in weights on $\mathbb{R}$ instead of $\mathbb{R}^+$. For brevity, if μ is a doubling measure, we will write $\mu \in \mathbf{D}$.

A homeomorphism $W : \mathbb{R} \to \mathbb{R}$ is called *quasi-symmetric* if there exists a constant $M = M(W) \geq 1$ such that

$$\frac{1}{M} \leq \frac{W(x+t) - W(x)}{W(x) - W(x-t)} \leq M, \qquad x \in \mathbb{R},\ t > 0. \tag{5.1}$$

In this case we write $W \in \mathbf{QS}$. Given a measure μ, define $W_\mu : \mathbb{R} \to \mathbb{R}$ by $W_\mu(t) = \int_0^t d\mu$. By (2.1) and (5.1), if $\mu \in \mathbf{D}$, then W_μ is a quasi-symmetric mapping. Conversely, given an increasing quasi-symmetric mapping W, the measure μ_W defined by $\mu_W(I) = W(y) - W(x)$, where $I = (x, y) \subset \mathbb{R}$, is a doubling measure (again by (2.1) and (5.1)). This gives us a correspondence between doubling measures and increasing quasi-symmetric mappings (modulo additive constants). In this section we add another element into the **D**–**QS** scheme by considering the class of convex functions gotten by integrating quasi-symmetric maps. Our first task will be to characterize these convex functions in terms of their sections.

In order to introduce our approach let us first consider strictly convex functions in higher dimensions. (A function is strictly convex if its graph does not contain any line-segments.) Every strictly convex differentiable function $w : \mathbb{R}^n \to \mathbb{R}$ has an associated *Monge-Ampère measure* μ_w defined on any Borel set $E \subset \mathbb{R}^n$ by

$$\mu_w(E) = |\nabla w(E)|. \tag{5.2}$$

Following L. Caffarelli [2], we associate with w a family of open convex sets called *sections* defined by

$$S_w(x, t) = \{y \in \mathbb{R}^n : w(y) < w(x) + \langle \nabla w(x), y - x\rangle + t\}, \tag{5.3}$$

where $x \in \mathbb{R}^n$ and $t > 0$. The measure μ_w has the *(DC)-doubling property* if there exist constants $\alpha \in (0, 1)$ and $C > 1$ such that

$$\mu_w(S_w(x, t)) \leq C\mu_w(\alpha S_w(x, t)), \tag{5.4}$$

for every section $S_w(x, t)$. Here $\alpha S_w(x, t)$ is the open convex set obtained by α-contraction of $S_w(x, t)$ with respect to its center of mass. The sections of w satisfy the *engulfing property* if there exists a constant $K > 1$ such that for every section $S_w(x, t)$,

$$y \in S_w(x, t) \Rightarrow S_w(x, t) \subset S_w(y, Kt).$$

We will write $w \in \mathbf{Eng}$ if w is a strictly convex differentiable function w whose sections have the engulfing property. In [19], C. Gutiérrez and Q. Huang proved that the (DC)-doubling property for μ_w implies the engulfing property for the sections of w, where the engulfing constant K depends only on α and C in (5.4), and the dimension n. In [15], two of the authors proved the converse: the engulfing property with constant K implies the (DC)-doubling property with some $\alpha = \alpha(K, n)$ and $C = C(K, n)$.

If we restrict ourselves to dimension 1, then the strict convexity of w implies that every open bounded interval $I \subset \mathbb{R}$ is a section $S_w(x, t)$ of w for some $x \in I$ and $t > 0$ (x is gotten, for instance, from the mean value theorem). Further, for μ_w the (DC)-doubling condition (5.4) is equivalent to $\mu_w \in \mathbf{D}$. Therefore, μ_w is doubling if and only if $w \in \mathbf{Eng}$. Thus, every function $w \in \mathbf{Eng}$ generates a doubling measure $\mu_w \in \mathbf{D}$. Conversely, given $\mu \in \mathbf{D}$, define

$$w_\mu(x) = \int_0^x W_\mu(t)\, dt. \tag{5.5}$$

Then $w_\mu : \mathbb{R} \to \mathbb{R}$ is a strictly convex differentiable function; and, by (5.2), its Monge-Ampère measure coincides with μ. Since $\mu \in \mathbf{D}$, μ satisfies the (DC)-doubling condition. Therefore, $w_\mu \in \mathbf{Eng}$, and we get the **D**–**QS**–**Eng** scheme. (Compare this with the equivalences in Theorem 3.1.) An n-dimensional version of the **D**–**QS**–**Eng** scheme has been introduced by Kovalev, Maldonado, and Wu in [24].

In Section 10 we prove the following characterization of the engulfing property. Given $w : \mathbb{R} \to \mathbb{R}$ strictly convex and differentiable, set

$$w_x(y) := w(y) - w(x) - w'(x)(y - x), \quad x, y \in \mathbb{R}.$$

Theorem 5.1. *Let $w : \mathbb{R} \to \mathbb{R}$ be a strictly convex differentiable function. The following are equivalent:*

i) (Engulfing property of the sections of w.) There exists a constant $K > 1$ such that

$$x \in S_w(y, t) \Rightarrow S_w(y, t) \subset S_w(x, Kt), \quad x, y \in \mathbb{R}, t > 0.$$

ii) There exists a constant $K' > 1$ such that

$$x \in S_w(y, t) \Rightarrow y \in S_w(x, K't), \quad x, y \in \mathbb{R}, t > 0.$$

iii) There exists a constant $K'' > 1$ such that for every $x, y \in \mathbb{R}$

$$\frac{K'' + 1}{K''} w_x(y) \leq \left(w'(x) - w'(y)\right)(x - y) \leq (K'' + 1) w_x(y).$$

Remark 5.2. Theorem 5.1 has been generalized to n dimensions in [15]. The proof of this generalization uses Theorem 5.1, and an earlier version of this result appeared in the unpublished manuscript [16]. We stress that this is the first time that Theorem 5.1 is published and that the generalization in [15] makes heavy use of Theorem 5.1. That is, the n-dimensional version follows from the the one-dimensional version (Theorem 5.1) which has not been published anywhere else.

As an immediate consequence of Theorem 5.1 we get two corollaries. The first shows that we may assume without loss of generality that w is a smooth function in computations.

Corollary 5.3. *Let $w \in \mathbf{Eng}$ with constant K. If $\rho \in C_0^\infty(\mathbb{R})$ and $\rho \geq 0$ (but not identically zero), then $\rho * w \in \mathbf{Eng}$ with constant $\tilde{K}$, where $\tilde{K}$ depends only on K (and not on ρ.)*

The second gives another characterization of quasi-symmetric mappings that we will need to construct Example 11.1.

Corollary 5.4. *An increasing homeomorphism $W : \mathbb{R} \to \mathbb{R}$ is quasi-symmetric if and only if there exists $\lambda \in (0, 1/2)$ such that for every $x, y \in \mathbb{R}$, $x < y$*

$$\lambda W(y) + (1-\lambda)W(x) \leq \frac{1}{y-x}\int_x^y W(s)\,ds \leq \lambda W(x) + (1-\lambda)W(y).$$

Growth conditions. Let $w \in \mathbf{Eng}$ with constant K, and $w(0) = w'(0) = 0$. We do not lose generality with this assumption since it can always be gotten by subtracting a suitable affine function from w; doing this does not change its associated Monge-Ampère measure. By part *(iii)* in Theorem 5.1, for $x > 0$ we have

$$\frac{K+1}{K} w(x) \leq w'(x)x \leq (K+1)w(x); \tag{5.6}$$

if we divide this expression by $xw(x)$ and recognize the derivatives of the corresponding logarithms, we get that the functions $w(x)/x^{1+1/K}$ and $w(x)/x^{K+1}$ are increasing and decreasing respectively on $\mathbb{R}^+$. As an immediate consequence of this we get estimates of the Matuszewska-Orlicz indices of w as functions on $\mathbb{R}^+$: $1 + K^{-1} \leq i(w)$ and $K + 1 \geq I(w)$. (See Maligranda [28] or Fiorenza and Krbec [14] for further details.)

Furthermore, it is immediate that the sections of the function $x \mapsto w(-x)$ also have the engulfing property with the same constant. Consequently, if we set $m_w := \min\{w(1), w(-1)\}$ and $M_w := \max\{w(1), w(-1)\}$, we have

$$m_w|x|^{1+K} \leq w(x) \leq M_w|x|^{1+1/K}, \qquad \text{if } |x| \leq 1,$$

and

$$m_w|x|^{1+1/K} \leq w(x) \leq M_w|x|^{1+K}, \qquad \text{if } |x| \geq 1.$$

If we combine this with (5.6) and use that $w'(x)x = |w'(x)x|$ we get a growth condition on w':

$$\frac{K+1}{K} m_w|x|^K \leq |w'(x)| \leq (K+1)M_w|x|^{1/K}, \qquad \text{if } |x| \leq 1, \tag{5.7}$$

and

$$\frac{K+1}{K} m_w |x|^{1/K} \le |w'(x)| \le (K+1) M_w |x|^K, \qquad \text{if } |x| \ge 1. \tag{5.8}$$

It is easy to prove that if $w \in \mathbf{Eng}$ then, for fixed $y \in \mathbb{R}$, the function $x \mapsto w(x+y)$ also is in **Eng**, with the same constant. Hence, (5.7) yields that w' is $1/K$-Hölder continuous. Furthermore, we have the following result.

Theorem 5.5. *Let $w \in \mathbf{Eng}$ with constant K and $w(0) = w'(0) = 0$, then for every $x > 0$,*

$$\frac{1}{x^2} \int_0^x w''(t) t^2 \, dt \simeq \int_0^x \frac{w'(t)}{t} \, dt \simeq \frac{w(x)}{x} \simeq w'(x),$$

where the constants depend only on K.

The uniform Δ_2 condition. Condition (5.6) implies that w is a Δ_2 function; to see this notice that $\int_x^{2x} w'(t)/w(t) \, dt \le (K+1) \int_x^{2x} 1/t \, dt$. In fact, (5.6) also implies that w' is a Δ_2 function.

However, (5.6) is stronger than the Δ_2 condition. For instance, the strictly convex function $\phi(x) = \sqrt{x^2+1} - 1$ is Δ_2, but it is easy to see that it does not verify (5.6) for any $K > 1$. The next result shows that in order to have $w \in \mathbf{Eng}$, w must satisfy the Δ_2 condition about every point (not only about 0) and must also be essentially symmetric about every point.

Theorem 5.6. *Let $w : \mathbb{R} \to \mathbb{R}$ be a strictly convex differentiable function. Then $w \in \mathbf{Eng}$ if and only if it satisfies*

$$w_y(y-a) \simeq w_y(y+a), \quad y \in \mathbb{R}, a > 0. \tag{5.9}$$

and

$$w_y(y+a) \simeq w_y(y+2a), \quad y \in \mathbb{R}, a > 0. \tag{5.10}$$

Remark 5.7. Condition (5.9) says that w is essentially symmetric about every point, and condition (5.10) says that it satisfies the Δ_2 condition at each point in $\mathbb{R}$.

Applications to doubling measures. As an application of the techniques developed to characterize the engulfing property, we give a sufficient conditions for multipliers of doubling measures. For simplicity we restrict ourselves to $\mathbb{R}^+$, though we note in passing that these results can be extended to the real line using the techniques given by Schröder [37]. The multipliers of A_p weights were characterized by Johnson and Neugebauer in [21], and the multipliers for the class of monotone doubling weights were classified by the first author in [4]. However, very little is known for multipliers of doubling measures.

To state our result we need one definition. Given a non-negative measurable function m on $\mathbb{R}^+$, we say that $m \in \mathbf{M}$ if there exists a constant $C_m > 1$ such that

$$m(2t) \le C_m m(t), \quad t > 0, \tag{5.11}$$

and

$$m(s) \le C_m m(t), \quad 0 < s < t. \tag{5.12}$$

Note that if m is increasing, then (5.12) is automatic and (5.11) is the Δ_2 condition.

Theorem 5.8. *Let $m \in \mathbf{M}$ with constant C_m, and $\mu \in \mathbf{D}$ with doubling constant C_μ. Then $m\mu \in \mathbf{D}$ with doubling constant depending only on C_μ and C_m.*

As a second application we consider the following problem. In [4, Theorem 6.5], the first author proved that if μ is a doubling measure on $\mathbb{R}^+$, then $W(t) = \mu([0,t))$ is an A_p weight for all $p > p_0$, where p_0 depended only on the doubling constant of μ. The function W has several additional properties: it is strictly increasing and (locally) Hölder continuous. Motivated by this, he posed the following question: If W is a strictly increasing function on $\mathbb{R}^+$ that is in A_p for some $p > 1$, $W(0) = 0$, and is (locally) Hölder continuous, then is the measure μ defined by $\mu([0,t]) = W(t)$ a doubling measure? By applying our techniques we have shown that the answer to this question is no, and we give a counterexample in Example 11.1.

We prove Theorems 5.1, 5.5 and 5.6, and Corollaries 5.3 and 5.4 in Section 10. We prove Theorem 5.8 and construct Example 11.1 in Section 11.

6. Properties of Δ_2 Young Functions

In this section we prove Theorems 3.1, 3.4, 3.5, and 3.6. The proof of Theorem 3.1 requires two lemmas from [4]: we alluded to the first in the Introduction; the second is part of Theorem 3.3 and we restate it here for convenience.

Lemma 6.1. *Given an increasing function w on $\mathbb{R}^+$, then $w \in A_\infty$ if and only if w satisfies the Δ_2 condition.*

Lemma 6.2. *Given p, $1 < p < \infty$, an increasing function w on $\mathbb{R}^+$ is in A_p if and only if for all $t > 0$,*

$$t^{p-1} \int_t^\infty \frac{w(x)}{x^p}\, dx \le C w(t). \tag{6.1}$$

Proof of Theorem 3.1. We first show that (i) and (ii) are equivalent. Suppose first that $\mu \in B_p$. Then v is an increasing function, so we can apply Lemma 6.2. By

Fubini's theorem and inequality (2.2),

$$\begin{aligned}\int_t^\infty \frac{v(x)}{x^{p+1}}\,dx &= \int_t^\infty \int_0^x d\mu(y) x^{-(p+1)}\,dx \\ &= \int_0^t \int_t^\infty x^{-(p+1)}\,dx\,d\mu(y) + \int_t^\infty \int_y^\infty x^{-(p+1)}\,dx d\mu(y) \\ &= \frac{1}{pt^p}\int_0^t d\mu(y) + \frac{1}{p}\int_t^\infty y^{-p}\,d\mu(y) \le Ct^{-p}v(t).\end{aligned}$$

Conversely, if $v \in A_{p+1}$ then essentially the same calculation shows that

$$Ct^{-p}\int_0^t d\mu(y) = Ct^{-p}v(t) \ge \int_t^\infty \frac{v(x)}{x^{p+1}}\,dx \ge \frac{1}{p}\int_t^\infty y^{-p}\,d\mu(y).$$

so $\mu \in B_p$.

We now show that (ii) and (iii) are equivalent. It suffices to note that if $v \in A_{p+1}$, then $v \in B_{p+1}$. (This is due to Ariño and Muckenhoupt [1]; a simple proof is implicit in the proof of Lemma 6.2 in [4].) But then by the above argument $w \in A_{p+2}$.

To prove the converse, suppose that $w \in A_{p+2}$. Then w satisfies the Δ_2 condition, and so, since v is increasing,

$$Cw(t) \ge w(2t) \ge \int_t^{2t} v\,dx \ge tv(t).$$

Similarly, $w(t) \le tv(t)$. Therefore

$$t^p \int_t^\infty \frac{v(x)}{x^{p+1}}\,dx \le Ct^p \int_t^\infty \frac{w(x)}{x^{p+2}}\,dx \le Ct^{-1}w(t) \le Cv(t).$$

Hence by Lemma 6.2, $v \in A_{p+1}$. □

Proof of Theorem 3.4. The equivalence of (i) and (ii) is due to Maligranda [28]; we note in passing that for this to be true we do not have to assume that $i(w) > 1$.

The equivalence of (i) and (iv) follows from Theorem 3.3: $i(w) = p_*$ is equivalent to $I(\bar{w}) = p'_*$, which in turn is equivalent to (iv).

To complete the proof we will show that (ii) and (iii) are equivalent. Suppose that (iii) is true. Fix $q < p_*$, $t > 0$ and fix $k \in \mathbb{Z}$ such that $2^{k-1} < t \le 2^k$. Then since w

is increasing,

$$t^q \int_0^t \frac{w(x)}{x^q} \frac{dx}{x} \le 2^{kq} \int_0^{2^k} \frac{w(x)}{x^q} \frac{dx}{x} = 2^{kq} \sum_{j=-\infty}^{k} \int_{2^{j-1}}^{2^j} \frac{w(x)}{x^q} \frac{dx}{x}$$

$$\le 2^{kq} \sum_{j=-\infty}^{k} \log(2) \frac{w(2^j)}{2^{(j-1)q}} = \log(2) w(2^k) \sum_{n=0}^{\infty} 2^{(n+1)q} \frac{w(2^{k-n})}{w(2^k)}.$$

By hypothesis, there exists $L < 2^{-q}$ and $N > 0$, independent of k, such that if $n \ge N$,

$$\frac{w(2^{k-n})}{w(2^k)} \le L^n.$$

Therefore, since w is Δ_2,

$$t^q \int_0^t \frac{w(x)}{x^q} \frac{dx}{x} \le \log(2) C w(2^{k-1}) \sum_{n=0}^{\infty} (L 2^q)^n \le C w(t).$$

To prove that (ii) implies (iii), fix $a > 1$ and $k \in \mathbb{Z}$, and fix t such that $a^k \le t < a^{k+1}$. Fix q and $r > 1$ such that $q < qr < p_*$. Then arguing almost exactly as we did in the first part of the proof, we have that

$$\int_0^t \frac{w(x)}{x^{qr}} \frac{dx}{x} \le C \log(a) \sum_{n=0}^{\infty} \frac{w(a^{k-n})}{a^{qr(k-n)}}.$$

Therefore, by (ii) we have that

$$C \ge w(t)^{-1} t^q \int_0^t \frac{w(x)}{x^{qr}} \frac{dx}{x} \ge \sum_{n=0}^{\infty} \frac{w(a^{k-n})}{w(a^k)} a^{qrn}.$$

Since the series converges, by the root test we must have that

$$\limsup_{n\to\infty} \left(\frac{w(a^{k-n})}{w(a^k)} \right)^{1/n} \le a^{-qr} < a^{-q}.$$

Further, since this sum is uniformly bounded for all k, the limit supremum must hold uniformly in k.

This completes the proof. □

Proof of Theorem 3.5. By the ∇_2 condition, for all $t > 0$, $w(t/A) \le w(t)/(2A)$, so by induction, $w(A^{-n}t) \le w(t)/(2A)^n$, $n \ge 1$. Let $t = A^k$; then

$$\left(\frac{w(A^{k-n})}{w(A^k)} \right)^{1/n} < \frac{1}{2A}.$$

On the other hand, by Lemma 6.1 and Theorem 3.3, we must have that for some $p > 1$,

$$\liminf_{n\to\infty} \left(\frac{w(A^{k-n})}{w(A^k)}\right)^{1/n} > A^{1-p}.$$

If we combine these two inequalities and solve for p, we see that

$$p > 2 + \frac{\log 2}{\log A}.$$

If we repeat this argument, *mutatis mutandis*, for $\bar{w}$, using the Δ_2 condition and (iii) in Theorem 3.4, we get the desired bound for $\bar{w}$. □

Proof of Theorem 3.6. As we noted above, it has already been shown that $I(w) \leq a + b\sqrt{2}$ and $i(w) \geq a - b\sqrt{2}$. Therefore, we only need to prove the reverse inequalities. Let $k > n$ be large positive integers; their exact values will be fixed below. Then

$$\left(\frac{w(e^{k-n})}{w(e^k)}\right)^{1/n} = \exp\left(-a + b\left(\frac{(k-n)\sin\log(k-n) - k\sin\log(k)}{n}\right)\right).$$

If we rewrite $\log(k-n) = \log(k) + \log\left(1 - \frac{n}{k}\right)$, then by a simple calculation we have that

$$\begin{aligned} &\frac{(k-n)\sin\log(k-n) - k\sin\log(k)}{n} \\ &= \frac{(k-n)\sin\log(k)\cos\log\left(1-\frac{n}{k}\right) + (k-n)\cos\log(k)\sin\log\left(1-\frac{n}{k}\right) - k\sin\log(k)}{n}. \end{aligned}$$

Since $k > n$, we have that

$$\sin\log\left(1 - \frac{n}{k}\right) = -\frac{n}{k} + O\left(\frac{n^2}{k^2}\right),$$

$$\cos\log\left(1 - \frac{n}{k}\right) = 1 + O\left(\frac{n^2}{k^2}\right).$$

If we substitute this into the above expression and simplify we get

$$\frac{(k-n)\sin\log(k-n) - k\sin\log(k)}{n} = -\sin\log(k) - \cos\log(k) + O\left(\frac{n}{k}\right).$$

Hence,

$$\left(\frac{w(e^{k-n})}{w(e^k)}\right)^{1/n} = \exp\left(-a - b(\sin\log(k) + \cos\log(k)) + O\left(\frac{n}{k}\right)\right).$$

The function $\sin\log(x) + \cos\log(x)$ has maximum value $\sqrt{2}$ and minimum value $-\sqrt{2}$, and since sine is periodic, it takes on these values for arbitrarily large values of x.

Therefore, if we take any $\epsilon > 0$, then for any large $n > 0$, there exist k_1, $k_2 > n$ such that

$$-a - b(\sin\log(k_1) + \cos\log(k_1)) + O\left(\frac{n}{k_1}\right) < -a - b\sqrt{2} + \epsilon \tag{6.2}$$

$$-a - b(\sin\log(k_2) + \cos\log(k_2)) + O\left(\frac{n}{k_2}\right) > a - b\sqrt{2} - \epsilon. \tag{6.3}$$

If we combine inequality (6.2) with (iii) of Theorem 3.3, we see that for every $q > I(w)$, we have that

$$\exp(-a - b\sqrt{2} + \epsilon) > e^{-q},$$

so $q > a + b\sqrt{2} - \epsilon$. Since this is true for all such q and all $\epsilon > 0$, we conclude that $I(w) \geq a + b\sqrt{2}$.

We argue in exactly the same way using (6.3) and (iii) of Theorem 3.4 to conclude that $i(w) \leq a - b\sqrt{2}$. This completes the proof. □

7. The Class $\mathcal{B}_p$

In this section we prove Theorem 4.1 and construct Example 4.2.

Proof of Theorem 4.1. Since w is a Young function, it is differentiable except possibly on a countable set, and $v(t) = w'(t)$ is an increasing function. (See Wheeden and Zygmund [39, p. 120].) By Theorem 3.1 it will suffice to show that $v \in A_r$ for any $r > p$. But since ψ is decreasing, $\psi' \leq 0$, so for $t > 0$,

$$v(t) = w'(t) = pt^{p-1}\psi(t) + t^p\psi'(t) \leq pt^{p-1}\psi(t).$$

Therefore, for $r > p$ and any $t > 0$, since ψ is decreasing,

$$\begin{aligned} t^{r-1}\int_t^\infty \frac{v(x)}{x^r}\,dx &\leq pt^{r-1}\int_t^\infty \frac{\psi(x)}{x^{r-p+1}}\,dx \leq pt^{r-1}\psi(t)\int_t^\infty x^{p-r-1}\,dx \\ &\leq Ct^{p-1}\psi(t) \leq Ct^{-1}w(t) \leq Cv(t). \end{aligned}$$

The last inequality holds since v is increasing. □

Construction of Example 4.2. Given an increasing weight v, define w as in Theorem 3.1. If $v \in A_\infty$ then arguing as in the proof of Theorem 3.1, $w(t) \approx tv(t)$. Therefore, by Theorem 3.1, to get the desired counter-example it will suffice to fix r, $0 < r < p-1$, and to construct an increasing function v such that $v(t) \leq Ct^r$, $t \geq 1$, and $v \in A_\infty$, but such that v is not in A_{q-1}. Without loss of generality we may assume that $q - 2 = jr$ for some integer $j \geq 2$. (The reason for this assumption will become clear below.)

Define v as follows: let $I_0 = [0,1]$ and for $n \geq 1$ let $I_n = (2^{n-1}, 2^n]$. Now set

$$v(t) = \sum_{n=0}^{\infty} a_n \chi_{I_n}(t);$$

we will choose the a_n's so that v has the desired properties.

We first define a strictly increasing sequence $\{N_i\}$ by letting $N_0 = 0$, and for all i odd, let $N_i = (j-1)i + N_{i-1}$ and $N_{i+1} = ji + N_{i-1}$.

Now define the a_n's as follows: let $a_0 = 1$. If i is odd and $N_{i-1} \leq n \leq N_i$, let $a_n = 2^{r(N_{i-1})}$; if $N_i < n \leq N_{i+1}$ let $a_n = a_{N_i} 2^{(q-2)(n-N_i)}$. Then

$$a_{N_{i+1}} = 2^{rN_{i-1}} 2^{(q-2)(N_{i+1}-N_i)} = 2^{rN_{i-1}+jri} = 2^{rN_{i+1}}.$$

Since $q-2 > r$, it follows immediately from our choice of the a_n's that $v(t) \leq 2^{q-2}t^r$, and that $v(2t) \leq 2^{q-2}v(t)$, so by Lemma 6.1, $v \in A_\infty$.

To show that v is not in A_{q-1} we apply Lemma 6.2 with $t = 2^k$. Then the left-hand side of inequality (6.1) becomes

$$\begin{aligned} t^{q-2} \int_t^\infty \frac{v(x)}{x^{q-1}}\,dx =& 2^{k(q-2)} \sum_{n=k+1}^{\infty} \int_{2^{n-1}}^{2^n} x^{1-q}\,dx \\ =& \frac{2^{q-2}-1}{q-2} 2^{k(q-2)} \sum_{n=k+1}^{\infty} a_n 2^{n(2-q)}. \end{aligned}$$

Since the righthand side of inequality (6.1) is equal to a_k, it will suffice to show that for any K greater than 0 there exists k such that

$$\sum_{n=k+1}^{\infty} a_n 2^{n(2-q)} > K a_k 2^{k(2-q)}.$$

Let $k = N_i$, i odd. Then

$$\sum_{n=k+1}^{\infty} a_n 2^{n(2-q)} > \sum_{n=N_i+1}^{N_{i+1}} a_{N_i} 2^{(q-2)(n-N_i)} 2^{n(2-q)} = a_k 2^{k(2-q)}(N_{i+1} - N_i).$$

By our choice of the N_i's, we can make the difference $N_{i+1} - N_i$ as large as desired. This completes our proof. □

8. Properties of B_p weights

In this section we prove Theorem 4.3 and construct Example 4.5. We are grateful to C.J. Neugebauer for showing us the original proof of Theorem 4.3.

Proof of Theorem 4.3. Let C be the constant in the RH_s^0 condition for w, and fix $\alpha > 0$ such that $(1-\alpha)^{1/s'} = (2C)^{-1}$. Then by Hölder's inequality and the RH_s^0 condition, for any $t > 0$, if we let $E = [\alpha t, t]$ then

$$\int_{\alpha t}^t w(x)\,dx = \int_0^t w(x)\chi_E\,dx \leq \left(\int_0^t w(x)^s\,dx\right)^{1/s} |E|^{1/s'}$$
$$\leq C(1-\alpha)^{1/s'} \int_0^t w(x)\,dx = \frac{1}{2}\int_0^t w(x)\,dx.$$

Hence

$$\int_0^t w(x)\,dx \leq 2\int_0^{\alpha t} w(x)\,dx,$$

and so by induction,

$$\int_0^t w(x)\,dx \leq 2^n \int_0^{\alpha^n t} w(x)\,dx.$$

For some $n \geq 1$, $\alpha^n \leq 1/2$; therefore $w \in B_\infty$.

To prove that B_∞ weights need not satisfy the RH_s^0 condition, let $w(t) = \chi_{[0,1]}(t)$. The B_∞ condition is immediate. On the other hand, if we take $t > 1$ and $s > 1$, then the RH_s^0 condition would imply that $t^{-1/s} \leq Ct^{-1}$, which is false for t sufficiently large. □

Construction of Example 4.2. Let $I_0 = [0,1]$ and for $n \geq 1$ let $I_n = (2^{n-1}, 2^n]$. Let J_n be the interval of width 2^{-n} with the same center as I_n. Define the increasing sequence $\{N_i\}$ by $N_0 = 0$ and $N_i = N_{i-1} + i$.

Now define w by

$$w(t) = \sum_{n=0}^\infty a_n(t),$$

where for i odd,

$$a_n(t) = \begin{cases} 4^n \chi_{J_n}(t) & N_{i-1} \leq n < N_i \\ 2\chi_{I_n}(t) & N_i \leq n < N_{i+1}. \end{cases}$$

Since for each $n \geq 0$, $w(I_n) = 2^n$, it follows that $w \in B_\infty$.

To see that w^r is not in B_∞, suppose first that $r < 1$. If we let $t = 2^{N_i - 1}$, i odd, then $w^r([t, 2t]) = w^r(I_{N_i}) = 2^r 2^{N_i - 1}$, but

$$w^r([0,t]) = \sum_{n=0}^{N_i - 1} w^r(I_n) \leq \sum_{n=0}^{N_{i-1}} 2^r 2^{n-1} + \sum_{n=0}^{N_i - 1} 4^{rn} 2^{-n}$$
$$\leq 2^r 2^{N_{i-1}} + C2^{(2r-1)N_i}.$$

Therefore,

$$\frac{w^r([0,t])}{w^r([t,2t])} \leq 2^{1+N_{i-1}-N_i} + C2^{(2r-2)N_i+1}.$$

Since $r < 1$, and since $N_i - N_{i-1} = i$, we can make the righthand side as small as desired. However, this contradicts the B_∞ condition, so w^r is not in B_∞.

If $r > 1$ let $t = 2^{N_i-1}$, i even. Then a similar computation shows that $w^r([t,2t]) = 2^{(2r-1)N_i+1}$ and $w^r([0,t]) \leq C2^{(2r-1)N_{i-1}} + C2^r 2^{N_i}$. Then the ratio $w^r([0,t])/w^r([t,2t])$ can again be made as small as desired and so w^r is not in B_∞. □

9. The minimal operator

In this section we prove weighted norm inequalities for the minimal operator.

Proof of Theorem 4.8. To prove that (1) implies (3), we use the result of Ariño and Muckenhoupt [1] and Sbordone and Wik [36] mentioned in the Introduction: given $q > 0$, if $\mu \in B_q$ and g is non-negative and decreasing then

$$\int_{\mathbb{R}^+} Mg(x)^q \, d\mu \leq C \int_{\mathbb{R}^+} g(x)^q \, d\mu, \tag{9.1}$$

where M is the Hardy-Littlewood maximal operator. But by Hölder's inequality, if f is non-negative and decreasing, then for any $r > 0$,

$$\mathcal{m}f(x)^{-p} \leq M(f^{-r})(x)^{p/r}, \quad x \in \mathbb{R}^+. \tag{9.2}$$

If $\mu \in B_\infty$ then $\mu \in B_q$ for some $q > 0$, so fix $r > 0$ such that $p/r = q$. Then (3) follows from inequalities (9.1) and (9.2).

That (3) implies (2) is immediate.

Finally, we prove that (2) implies (1). Fix $t > 0$ and let

$$f(x) = \begin{cases} 1 & x \in [0,t], \\ \beta & x \in (t,2t] \\ \infty & x > 2t, \end{cases}$$

where $\beta > 1$ will be fixed below. Then $\mathcal{m}f(x) \geq 1$, so (2) implies that

$$\mu([0,2t]) \leq C \int_{\mathbb{R}^+} \frac{d\mu}{f^p} \leq C\mu([0,t]) + \frac{C}{\beta^p}\mu([t,2t]).$$

Fix β so that $C/\beta^p \leq 1/2$. Then re-arranging terms we get $\mu([0,2t]) \leq C\mu([0,t])$. Since C is independent of t, $\mu \in B_\infty$. □

We conclude this section by giving a sufficient condition for a two-weight, weak-type inequality. The proof is adapted from the corresponding result in Cruz-Uribe, Neugebauer and Olesen [7]. We are grateful to C.J. Neugebauer for showing it to us.

Proposition 9.1. *Fix p, $0 < p < \infty$. If (u, v) is a pair of weights such that*

$$\frac{1}{t}\int_0^t u(x)\,dx \leq C\left(\frac{1}{t}\int_0^t v(x)^{1/(p+1)}\,dx\right)^{p+1},$$

then for any non-negative, increasing function f,

$$u(\{x \in \mathbb{R}^+ : \mathcal{m}f(x) < 1/t\}) \leq \frac{C}{t^p}\int_{\mathbb{R}^+}\frac{v(x)}{f(x)^p}\,dx.$$

Proof. Fix $t > 0$. Since f is increasing and $\mathcal{m}f$ is upper semi-continuous, there exists $r_t > 0$ such that $E_t = \{x \in \mathbb{R}^+ : \mathcal{m}f(x) < 1/t\} = [0, r_t)$. Again since f is increasing,

$$\frac{1}{r_t}\int_0^{r_t} f(x)\,dx \leq 1/t.$$

Therefore, by our hypothesis and by Hölder's inequality,

$$\begin{aligned}
u(E_t) = \int_0^{r_t} u(x)\,dx &\leq \frac{r_t^p}{t^p}\left(\int_0^{r_t} u(x)\,dx\right)\left(\int_0^{r_t} f(x)\,dx\right)^{-p} \\
&\leq \frac{C}{t^p}\left(\int_0^{r_t} v(x)^{1/(p+1)}\,dx\right)^{p+1}\left(\int_0^{r_t} f(x)\,dx\right)^{-p} \\
&\leq \frac{C}{t^p}\int_0^{r_t}\frac{v(x)}{f(x)^p}\,dx \leq \frac{C}{t^p}\int_{\mathbb{R}^+}\frac{v(x)}{f(x)^p}\,dx.
\end{aligned}$$

□

10. Characterizations of the engulfing property

In this section we prove Theorems 5.1, 5.5 and 5.6, and Corollaries 5.3 and 5.4.

Proof of Theorem 5.1. The proof of $(i) \Rightarrow (ii)$ is obvious since $y \in S_w(y, t)$ for every $y \in \mathbb{R}$ and $t > 0$. Thus (ii) holds with $K' = K$.

Proof of $(ii) \Rightarrow (iii)$. Given $x, y \in \mathbb{R}$ and $\varepsilon > 0$, we have

$$w(x) < w(x) + \varepsilon = w(y) + w'(y)(x - y) + w(x) - w(y) - w'(y)(x - y) + \varepsilon,$$

(note that the convexity of w implies $w(x) - w(y) - w'(y)(x - y) \geq 0$), this means that $x \in S_w(y, w(x) - w(y) - w'(y)(x - y) + \varepsilon)$. By property (ii), we must have $y \in S_w(x, K'(w(x) - w(y) - w'(y)(x - y) + \varepsilon))$, which means

$$w(y) \leq w(x) + w'(x)(y - x) + K'w(x) - K'w(y) - K'w'(y)(x - y) + K'\varepsilon.$$

Letting ε go to 0 we get

$$(K'+1)w(y) \le (K'+1)w(x) + (w'(x) + K'w'(y))(y-x). \tag{10.1}$$

Now interchanging the roles of x and y, we obtain

$$(K'+1)w(x) \le (K'+1)w(y) + (w'(y) + K'w'(x))(x-y). \tag{10.2}$$

From (10.1) and (10.2),

$$\frac{w'(x)+K'w'(y)}{K'+1} \le \frac{w(x)-w(y)}{x-y} \le \frac{w'(y)+K'w'(x)}{K'+1}. \tag{10.3}$$

By using the first inequality we get

$$\frac{1}{K'+1}\left(w'(x)-w'(y)\right)(x-y) \le w(x)-w(y)-w'(y)(x-y). \tag{10.4}$$

The second inequality yields

$$w(x)-w(y)-w'(x)(x-y) \le \frac{1}{K'+1}\left(w'(y)-w'(x)\right)(x-y), \tag{10.5}$$

which implies

$$w(x)-w(y)-w'(y)(x-y) \le \frac{K'}{K'+1}\left(w'(x)-w'(y)\right)(x-y). \tag{10.6}$$

Now (iii) follows from (10.6) and (10.4) (with x and y exchanged) and $K''=K'$.

Proof of $(ii) \Rightarrow (i)$. Let us take $x \in S_w(y,t)$ (consequently $y \in S_w(x,K't)$). We need to produce K such that $S_w(y,t) \subset S_w(x,Kt)$. Take $z \in S_w(y,t)$ (consequently $y \in S_w(z,K't)$). Suppose first that $(z-y)(x-y) \ge 0$, then

$$\begin{aligned}
w(x)-&w(z)-w'(z)(x-z)\\
&= w(y)-w(z)-w'(z)(y-z) + (w(x)-w(y)-w'(z)(x-y))\\
&\le K't + w(x)-w(y)-w'(z)(x-y)\\
&= K't + w(x)-w(y)-w'(y)(x-y) + w'(y)(x-y) - w'(z)(x-y)\\
&\le K't + t + \left(w'(y)-w'(z)\right)(x-y) =: A(x,y,z)\\
&\le (K'+1)t.
\end{aligned}$$

Thus, $x \in S_w(z,(K'+1)t)$ and then property (ii) implies $z \in S_w(x, K'(K'+1)t)$. The second case to consider is $(z-y)(x-y) < 0$ and $|y-z| < |x-y|$. Then

$$\begin{aligned} w(z) - w(x) - w'(x)(z-x) \\ &= w(z) - w(y) - w'(y)(z-y) \\ &+ (w(y) - w(x) + w'(y)(z-y) - w'(x)(z-x)) \\ &\le t + w(y) - w(x) - w'(x)(y-x) \\ &+ w'(x)(y-x) + w'(y)(z-y) - w'(x)(z-x) \\ &\le t + K't + (w'(x) - w'(y))(y-z) \\ &\le (K'+1)t + (w'(x) - w'(y))(x-y) \qquad (10.7) \\ &\le (K'+1)t + (K'+1)(w(x) - w(y) - w'(y)(x-y)) \\ &\le (K'+1)t + (K'+1)t = 2(K'+1)t, \end{aligned}$$

where we used part (iii) in (10.7). Now, consider the case $(z-y)(x-y) < 0$ and $|y-z| > |x-y|$ we then have

$$\begin{aligned} A(x,y,z) &\le K't + t + (w'(y) - w'(z))(y-z) \\ &\le (K'+1)t + (K'+1)(w(z) - w(y) - w'(y)(z-y)) \le 2(K'+1)t. \end{aligned}$$

Hence, $x \in S_w(z, 2(K'+1)t)$ and by (ii) we get $z \in S_w(x, 2K'(K'+1))$. Therefore, the engulfing property follows with constant $K = 2K'(K'+1)$.

Proof of $(iii) \Rightarrow (ii)$**.** Suppose $x \in S_w(y,t)$, that is, $w(x) - w(y) - w'(y)(x-y) < t$. Now, by using the first inequality in (iii),

$$w(y) - w(x) - w'(x)(y-x) \le \frac{K''}{K''+1}(w'(x) - w'(y))(x-y),$$

and by the second inequality in (iii),

$$\begin{aligned} (w'(x) - w'(y))(x-y) &= (w'(y) - w'(x))(y-x) \\ &\le (K''+1)(w(x) - w(y) - w'(y)(x-y)) < (K''+1)t, \end{aligned}$$

consequently, $w(y) - w(x) - w'(x)(y-x) < K''t$, which means $y \in S_w(x, K''t)$; and (ii) follows with $K' = K''$. □

Proof of Corollary 5.3. Set $\psi(x) = \int w(x-u)\rho(u)\,du$. By Theorem 5.1,

$$\begin{aligned}
&\frac{K''+1}{K''}\left(\psi(y)-\psi(x)-\psi'(x)(y-x)\right)\\
&=\frac{K''+1}{K'}\int\left(w(y-u)-w(x-u)-w'(x-u)(y-u-(x-u))\right)\rho(u)\,du\\
&\leq\int(w'(y-u)-w'(x-y))(y-x)\rho(u)\,du\\
&\leq(K''+1)\int\left(w(y-u)-w(x-u)-w'(x-u)(y-u-(x-u))\right)\rho(u)\,du\\
&\leq(K''+1)\left(\psi(y)-\psi(x)-\psi'(x)(y-x)\right).
\end{aligned}$$

Again by Theorem 5.1, $\psi \in \mathbf{Eng}$ with some $\tilde{K}$ depending only on K. □

Proof of Corollary 5.4. By definition, W is quasi-symmetric if and only if $\mu := W'$ (weak-derivative, that is, $W(t) - W(0) = \int_0^t d\mu$) is a doubling measure. Equivalently, the sections of w_μ (as defined in (5.5)) verify the engulfing property, with some constant K. We only need to apply Theorem 5.1 to w_μ, and express (10.3) in terms of w_μ with $\lambda = 1/(K'+1)$ and $w'_w = W$. Which yields the desired result. □

To prove Theorem 5.5 we need the following lemma that is a classical result on quasi-symmetric mappings. We give a new proof that is short, quantitative, and more geometric.

Lemma 10.1. *Let $w \in \mathbf{Eng}$ with constant K, and suppose that $w(0) = w'(0) = 0$. Then, for $0 < \sigma < \tau$,*

$$\frac{1}{K}\left(\frac{\tau}{\sigma}\right)^{1/K} \leq \frac{w'(\tau)}{w'(\sigma)} \leq K\left(\frac{\tau}{\sigma}\right)^{K}. \tag{10.8}$$

Proof. We begin with the first inequality. By (5.6) we have

$$\frac{w'(\tau)}{w'(\sigma)} = \frac{\sigma}{\tau}\frac{\tau w'(\tau)}{\sigma w'(\sigma)} \geq \frac{\sigma}{\tau}\frac{(1+1/K)}{K+1}\frac{w(\tau)}{w(\sigma)}.$$

Now, since the engulfing property implies that the function $f(t)/t^{1+1/K}$ is increasing on $(0,\infty)$, we get

$$\frac{\sigma}{\tau}\frac{(1+1/K)}{(K+1)}\frac{w(\tau)}{w(\sigma)} \geq \frac{\sigma}{\tau}\frac{(1+1/K)}{(K+1)}\left(\frac{\tau}{\sigma}\right)^{1+1/K} = \frac{1}{K}\left(\frac{\tau}{\sigma}\right)^{1/K}.$$

For the second inequality, we use (5.6) again

$$\frac{w'(\tau)}{w'(\sigma)} = \frac{\sigma}{\tau}\frac{\tau w'(\tau)}{\sigma w'(\sigma)} \leq \frac{\sigma}{\tau}\frac{(K+1)}{(1+1/K)}\frac{w(\tau)}{w(\sigma)}.$$

Now we use that the function $w(t)/t^{1+K}$ is decreasing to get

$$\frac{(K+1)}{(1+1/K)}\frac{\sigma}{\tau}\frac{w(\tau)}{w(\sigma)} \leq \frac{(K+1)}{(1+1/K)}\left(\frac{\tau}{\sigma}\right)^K = K\left(\frac{\tau}{\sigma}\right)^K.$$

□

Proof of Theorem 5.5. Let us prove first the equivalence $\frac{1}{x^2}\int_0^x w''(t)t^2\,dt \simeq \frac{w(x)}{x}$. Let $f \in \mathbf{Eng}$ with constant K and $f(0) = f'(0) = 0$, then, given $0 \leq \sigma < \tau$, we have

$$\int_\sigma^\tau t^2 f''(t)dt = t^2 f'(t)|_\sigma^\tau - 2\int_\sigma^\tau tf'(t)dt \leq \tau^2 f'(\tau) \leq (K+1)\tau f(\tau),$$

where we used (5.6) for the last inequality. On the other hand, for $0 < \sigma < \tau$, and using (5.6) again

$$\begin{aligned}\int_\sigma^\tau t^2 f''(t)dt &\geq (1+1/K)\int_\sigma^\tau tf(t)f''(t)/f'(t)\,dt \\ &= (1+1/K)\int_\sigma^\tau tf(t)(\ln f')'(t)\,dt \geq (1+1/K)\sigma f(\sigma)\left(\ln f'(t)|_\sigma^\tau\right) \\ &= (1+1/K)\sigma f(\sigma)\ln(f'(\tau)/f'(\sigma)).\end{aligned}$$

But, by Lemma 10.1, $f'(\tau)/f'(\sigma) \geq \frac{1}{K}\left(\frac{\tau}{\sigma}\right)^{1/K}$. Next, set $\sigma = \tau/(2K^K)$ and use that the function $f(t)/t^{1+K}$ is decreasing, (so that $f(\sigma) \geq (\tau/\sigma)^{1+K} f(\tau)$) to get

$$\begin{aligned}\int_{\tau/(2K^K)}^\tau t^2 f''(t)dt &\geq (1+1/K)\frac{\tau}{2K^K}\left(\frac{1}{2K^K}\right)^{1+K} f(\tau)\frac{1}{K}\ln 2 \\ &= (\ln 2)\frac{K+1}{K^2}\left(\frac{1}{2K^K}\right)^{K+2}\tau f(\tau).\end{aligned}$$

Finally, take $\tau = 1$ and define $C_K = (K+1)$, $c_K = (\ln 2)\frac{K+1}{K^2}\left(\frac{1}{2K^K}\right)^{K+2}$ to get

$$c_K f(1) \leq \int_0^1 t^2 f''(t)dt \leq C_K f(1). \tag{10.9}$$

Now, given $x \in \mathbb{R}$ and w as in the statement of the theorem, define $f(t) := w(tx)$. By relating the sections of f and w it is immediate to verify that $f \in \mathbf{Eng}$ with constant K (independent of x). If we now apply (10.9) to this f we get

$$c_K w(x) \leq \int_0^1 (tx)^2 w''(tx)dt \leq C_K w(x). \tag{10.10}$$

After the change of variables $u = tx$, we get the first equivalence.

We now prove the equivalence $\int_0^x \frac{w'(t)}{t}\,dt \simeq \frac{w(x)}{x}$. Clearly, $\int_0^x w'(t)/t\,dt \geq 1/x \int_0^x w'(t)\,dt = w(x)/x$. On the other hand, by (5.6) we have

$$\left(\frac{w(t)}{t}\right)' \geq \frac{1}{K}\frac{w(t)}{t^2},$$

and (5.6) gives $w(t)/t \geq w'(t)/(K+1)$. Thus

$$\left(\frac{w(t)}{t}\right)' \geq \frac{1}{K(K+1)}\frac{w'(t)}{t},$$

integrating from 0 to x, we get $w(x)/x \geq 1/(K^2+K)\int_0^x w'(t)/t\,dt$.

The equivalence $w'(x)x \simeq w(x)$ is just (5.6). □

Proof of Theorem 5.6. Assume first that $w \in \mathbf{Eng}$. Then, in the notation of Theorem 5.1 for all $y \in \mathbb{R}$ and $a > 0$ we have (with constants depending only on K)

$$\frac{w_y(y-a)}{w_y(y+a)} \simeq \frac{(w'(y-a)-w'(y))(-a)}{(w'(y+a)-w'(y))a} = \frac{\mu_w(y-a,y)}{\mu_w(y,y+a)} \simeq 1.$$

Hence,

$$w_y(y-a) \simeq w_y(y+a), \quad y \in \mathbb{R}, a > 0,$$

which is (5.9).

Similarly, by Theorem 5.1 we get

$$\frac{w_y(y+a)}{w_y(y+2a)} \simeq \frac{(w'(y+a)-w'(y))a}{(w'(y+2a)-w'(y))2a} \simeq \frac{\mu_w(y,y+a)}{\mu_w(y,y+2a)} \simeq 1.$$

Hence,

$$w_y(y+a) \simeq w_y(y+2a), \quad y \in \mathbb{R}, a > 0, \tag{10.11}$$

which is (5.10).

Conversely, suppose that w satisfies conditions (5.9) and (5.10). We will show that $w_y(x) \simeq w_y'(x)(x-y)$ for every $x, y \in \mathbb{R}$; the desired result then follows from Theorem 5.1. If $y < x$, set $b = x - y > 0$. Then, since $w_y(y) = w_y'(y) = 0$, by convexity of w_y we have

$$w_y'(x) = w_y'(y+b) \leq \frac{w_y(y+2b)}{b} \leq C\frac{w_y(y+b)}{b},$$

where we used (5.10). On the other hand, by the convexity of w_y,

$$w_y'(y+b) \geq \frac{w_y(y+b)}{b}.$$

Thus, $w_y(x) \simeq w_y'(x)(x-y)$ when $y < x$. For the case $y > x$ note that (5.9) and (5.10) imply

$$w_y(y-2a) \leq Cw_y(y-a), \quad y \in \mathbb{R}, a > 0.$$

Set $a = y - x$ and now use that convexity implies $-w_y(y-a) \le w_y(y-2a)/a$. The rest follows as in the first case. □

11. Doubling measures

In this section we prove Theorem 5.8 and construct Example 11.1.

Proof of Theorem 5.8. Given positive real numbers σ and τ, set $\tilde{\tau} = (3\tau+\sigma)/4$, $\tilde{\sigma} = (3\sigma+\tau)/4$. That is, if $I = (\sigma,\tau)$, then $(1/2)I = (\tilde{\sigma},\tilde{\tau})$.

As usual, set $w(x) = \int_0^x \int_0^t d\mu(s)dt$. Then $w \in \textbf{Eng}$ with a constant K depending only on the doubling constant for μ. We denote by $C(C_m, K)$ a constant depending only on C_m and K, which could be different at each occurrence. We first consider intervals (σ,τ) such that $\tau/\sigma \le M_K := (2K)^K + 1$. In this case we have

$$\int_\sigma^\tau m(t)w''(t)\,dt \le C_m m(\tau)\int_\sigma^\tau w''(t)\,dt \le C_\mu C_m m(\tau)\int_{\tilde{\sigma}}^{\tilde{\tau}} w''(t)\,dt,$$

where we used (5.12) and w'' doubling. Now, since $0 < \sigma < \tilde{\sigma} < \tilde{\tau} < \tau < M_K\sigma$, by iteration of (5.11), there exists a constant $C = C(C_m, K)$, depending only on C_m and K, such that $m(\tau) \le Cm(\sigma)$. Then,

$$C_\mu C_m m(\tau)\int_{\tilde{\sigma}}^{\tilde{\tau}} w''(t)\,dt \le C(C_m,K)m(\sigma)\int_{\tilde{\sigma}}^{\tilde{\tau}} w''(t)\,dt;$$

using (5.12) again we see that $m(\sigma) \le C_m m(t)$ for $t \in (\tilde{\sigma},\tilde{\tau})$. Thus, we get the doubling condition

$$\int_\sigma^\tau m(t)w''(t)\,dt \le C(C_m,K)\int_{\tilde{\sigma}}^{\tilde{\tau}} m(t)w''(t)\,dt. \tag{11.1}$$

We now consider intervals (σ,τ) with $\tau/\sigma \ge M_K$. For $\beta \in (0,1)$, set

$$\sigma_\beta = \frac{\tau+\sigma}{2} - \beta\frac{\tau-\sigma}{2} \quad \text{and} \quad \tau_\beta = \frac{\tau+\sigma}{2} + \beta\frac{\tau-\sigma}{2},$$

so that if $I = (\sigma,\tau)$, then $\beta I = (\sigma_\beta, \tau_\beta)$. Notice that $\tau/\sigma \ge M_K$ implies

$$\frac{\tau_\beta}{\sigma_\beta} = \frac{(1+\beta)\frac{\tau}{\sigma} + (1-\beta)}{(1-\beta)\frac{\tau}{\sigma} + (1+\beta)} \ge \frac{(1+\beta)M_K + (1-\beta)}{(1-\beta)M_K + (1+\beta)}.$$

Fix $\alpha = \alpha(K) \in (0,1)$ close to 1 and depending only on K such that

$$\frac{(1+\alpha)M_K + (1-\alpha)}{(1-\alpha)M_K + (1+\alpha)} \ge (2K)^K;$$

then $\tau_\alpha/\sigma_\alpha \geq (2K)^K$. We will show that for intervals (σ,τ) with $\tau/\sigma \geq M_K$ we have

$$\int_\sigma^\tau m(t)w''(t)\,dt \leq C(K,C_m)\int_{\sigma_\alpha}^{\tau_\alpha} m(t)w''(t)\,dt. \tag{11.2}$$

Set $\varsigma_\alpha := \tau_\alpha/(2K)^K \geq \sigma_\alpha$.

$$\begin{aligned}\int_{\sigma_\alpha}^{\tau_\alpha} m(t)w''(t)\,dt &\geq \int_{\varsigma_\alpha}^{\tau_\alpha} m(t)w''(t)\,dt = \int_{\varsigma_\alpha}^{\tau_\alpha} m(t)w'(t)\frac{w''(t)}{w'(t)}\,dt\\ &\geq \frac{1}{C_m}m(\varsigma_\alpha)w'(\varsigma_\alpha)\ln(w'(\tau_\alpha)/w'(\varsigma_\alpha)).\end{aligned}$$

By Lemma (10.1), we have

$$\frac{w'(\tau_\alpha)}{w'(\varsigma_\alpha)} \geq \frac{1}{K}\left(\frac{\tau_\alpha}{\varsigma_\alpha}\right)^{1/K} = 2.$$

Thus,

$$\int_{\sigma_\alpha}^{\tau_\alpha} m(t)w''(t)\,dt \geq \frac{\ln 2}{C_m}m(\varsigma_\alpha)w'(\varsigma_\alpha).$$

Then by (5.11) and since by (5.6) w' is Δ_2, there exists a constant C_0, depending only on K and C_m, such that $m(\tau_\alpha) \leq C_0 m(\varsigma_\alpha)$ and $w'(\tau_\alpha) \leq C_0 w'(\varsigma_\alpha)$. Hence,

$$\int_{\sigma_\alpha}^{\tau_\alpha} m(t)w''(t)\,dt \geq C(C_m,K)m(\tau_\alpha)w'(\tau_\alpha).$$

Since $\tau \leq 2\tau_\alpha$, for some constant C_1 depending only on C_m and K, we get $m(\tau) \leq C_1 m(\tau_\alpha)$ and $w'(\tau) \leq C_1 w'(\tau_\alpha)$. Hence,

$$\int_{\sigma_\alpha}^{\tau_\alpha} m(t)w''(t)\,dt \geq C(C_m,K)m(\tau)w'(\tau).$$

On the other hand,

$$\int_\sigma^\tau m(t)w''(t)\,dt \leq C_m m(\tau)(w'(\tau)-w'(\sigma)) \leq C_m m(\tau)w'(\tau),$$

and we get (11.2). Notice that, since α is close to 1, we can take $\alpha \geq 1/2$. In this case we have $(\tilde\sigma,\tilde\tau) \subset (\sigma_\alpha,\tau_\alpha)$ and we can replace $(\tilde\sigma,\tilde\tau)$ by $(\sigma_\alpha,\tau_\alpha)$ in the right hand side of (11.1). Therefore, $\mu(\sigma,\tau) \leq C\mu(\sigma_\alpha,\tau_\alpha)$, for all $\sigma,\tau > 0$, and $m\mu$ is a doubling measure on $\mathbb{R}^+$. □

Example 11.1. There exists a function W on $\mathbb{R}^+$ such that $W(0)=0$, W is increasing and locally Hölder continuous, but the measure μ defined by $\mu([0,t]) = W(t)$ is not a doubling measure.

Proof. For $x \in \mathbb{R}^+$ define

$$W(x) = n + (x - n)^{n+1}$$

if $x \in [n, n+1)$, $n = 0, 1, 2, \ldots$ It is easily seen that $x/2 \le W(x) \le x$, for every $x \in \mathbb{R}^+$. Since the weight $w_0(x) = x$ is doubling in $\mathbb{R}^+$, and W is comparable to w_0 on $\mathbb{R}^+$, we get that W is a doubling weight on $\mathbb{R}^+$ as well. Since W is a strictly increasing doubling weight, by Corollary 4.4 in [4], W is an A_∞ weight on $\mathbb{R}^+$. Clearly, W is locally Lipschitz. Thus, W verifies all the hypotheses required in the question. Now, suppose that μ is a doubling measure. Let $\overline{W}$ be the odd extension of W to the whole real line, since $\mu \in \mathbf{D}$ we have $\overline{W} \in \mathbf{QS}$ (see Lemma 7.1 in [26]). In particular, $\overline{W}$ must verify the inequalities (5.6) in Corollary 5.4 for some $\lambda \in (0, 1/2)$. But, taking $x = n$ and $y = n + 1$ ($n \in \mathbb{N}$ arbitrary), we get

$$\frac{1}{y - x} \int_x^y W(t)\, dt = \int_n^{n+1} W(t)\, dt = n + \int_n^{n+1} (t - n)^{n+1}\, dt = n + \frac{1}{n + 2}.$$

On the other hand,

$$\lambda W(y) + (1 - \lambda) W(x) = \lambda(n + 1) + (1 - \lambda)n = \lambda + n.$$

Thus, by the first inequality in (5.6) we get $\lambda \le 1/(n + 2)$, which cannot hold for every n. Thus, we reach the contradiction. □

Remark 11.2. The class of weights W on $\mathbb{R}^+$ which gives a positive answer to the original question is determined by inequality (5.6) in Corollary 5.4.

Remark 11.3. This construction of Example 11.1 is interesting for its application of the techniques developed above. However, we also want to note that by using Theorem 3.1 we can easily construct another counterexample. Let μ be absolutely continuous with Radon-Nikodym derivative $m(t) = \chi_{[0,1]}(t) + t^{-1}\chi_{(1,\infty)}(t)$. Then simple computations show that μ is not doubling; $\mu \in B_p$ for any $p > 0$ so $v \in A_{p+1}$; and v is strictly increasing and Lipschitz.

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