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Arithmetic relations in the set covering polyhedron of circulant clutters

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Abstract

We study the structure of the set covering polyhedron of circulant clutters, $P(\mathscr{C}_n^k)$, especially the properties related to contractions that yield other circulant clutters. Building on work by Cornuéjols and Novick, we show that if $\mathscr{C}_n^k/N$ is isomorphic to $\mathscr{C}_{n'}^{k'}$, then certain algebraic relations must hold and N is the union of particular disjoint simple directed cycles. We also show that this property is actually a characterization. Based on a result by Argiroffo and Bianchi, who characterize the set of null coordinates of vertices of $P(\mathscr{C}_n^k)$ as being one of such N's, we then arrive at other characterizations, one of them being the conditions that hold between the existence of vertices and algebraic relations of certain parameters. With these tools at hand, we show how to obtain by algebraic means some old and new results, without depending on Lehman's work as is traditional in the field.

Keywords: circulant clutter, set covering polyhedron, directed cycle, relative prime numbers.

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1 Background and results

Given a clutter $\mathscr{C}$ with vertices $V(\mathscr{C})$ and edges $E(\mathscr{C})$, we denote by

$$P(\mathscr{C}) = \{x \in \mathbb{R}^n : M(\mathscr{C})x \geq \mathbf{1},\, x \geq \mathbf{0}\},$$

the corresponding set covering polyhedron. Our work is concerned with the circular clutters $\mathscr{C}_n^k$, having vertex set $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$ and edges $\{i, i \oplus 1, \dots, i \oplus (k-1)\}$ for $i \in \mathbb{Z}_n$ ($\oplus$ being addition modulo n), so that it will convenient to regard vectors in $\mathbb{R}^n$ as having coordinates $(x_0, \dots, x_{n-1})$. $G(\mathscr{C}_n^k)$ will denote the directed graph having vertex set $V(\mathscr{C}_n^k) = \mathbb{Z}_n$, and (i, i') is an arc of $G(\mathscr{C}_n^k)$ if and only if $i' \ominus i \in \{k, k+1\}$.

Cornuéjols and Novick [5] described many ideal and minimally non ideal (mni for short) clutters, in particular finding all of the clutters $\mathscr{C}_n^k$ which are ideal or mni. Their results are based on the work by Lehman [6,7], and the following lemma, which is central to our work:

Lemma 1.1 (lemma 4.5 in [5]) *Suppose $2 \leq k \leq n-2$. If a subset N of $V(\mathscr{C}_n^k)$ induces a simple directed cycle, D, in $G(\mathscr{C}_n^k)$, then there exists $n_1, n_2, n_3 \in \mathbb{Z}_+$, $n_1 \geq 1$, such that*

(i) $n n_1 = k n_2 + (k+1)\, n_3$,

(ii) $\gcd(n_1, n_2, n_3) = 1$,

(iii) If $k - n_1 \leq 0$, then $E(\mathscr{C}_n^k / N) = \emptyset$ or $\{\emptyset\}$. If $k - n_1 \geq 1$, then $\mathscr{C}_n^k/N$ is of the form $\mathscr{C}_{n-n_2-n_3}^{k-n_1}$.

In a previous paper [1] we have shown that the condition $\gcd(m, n_1) = 1$ in lemma 1.1 is not only necessary, but also sufficient, for the existence of a simple directed cycle, giving a constructive proof:

Theorem 1.2 *Let n, k and m be given, $1 \leq k \leq n-1$, $1 \leq m \leq n-1$. Then there exists a simple directed cycle D in $G(\mathscr{C}_n^k)$ with $|V(D)| = m$ if and only if the following two conditions are satisfied:*

$$\left\lceil \frac{km}{n} \right\rceil = \left\lfloor \frac{(k+1)\, m}{n} \right\rfloor, \tag{1}$$

and if n_1 is its common value, then $\gcd(m, n_1) = 1$.

Moreover, such a cycle can be constructed in $O(m)$ steps.

The following result—with little changes—is part of the proof given by Cornuéjols and Novick of lemma 1.1, and reveals an important part of the structure of cycles in $G(\mathscr{C}_n^k)$:

Proposition 1.3 *Suppose* $1 \le k \le n-1$*,* D *is a simple directed cycle in* $G(\mathscr{C}_n^k)$*, having* n_2 *arcs of length* k *and* n_3 *arcs of length* $k+1$*,* n_1 *is defined as in lemma 1.1(i),* $N = V(D)$ *is written in the canonical form*

$$N = \{i_0, i_1, \dots, i_{m-1}\} \quad \textit{so that} \quad i_0 < i_1 < \dots < i_{m-1}, \tag{2}$$

and $s : N \to N$ *is defined by* $s(i) = i'$ *if* (i, i') *is an arc of* D*.*

Then, if $\underset{m}{\oplus}$ *indicates addition modulo* m*,*

$$s(i_j) = i_{j \underset{m}{\oplus} n_1} \quad \textit{for } j = 0, \dots, m-1, \tag{3}$$

or, equivalently, $|\{i, i \oplus 1, \dots, s(i) \ominus 1\} \cap N| = n_1$*.*

The following is a generalization to the case of several cycles:

Theorem 1.4 *Suppose* $1 \le k \le n-1$*,* $1 \le m \le n-1$*.*

If $D_1, \dots, D_d$ *are disjoint simple directed cycles in* $G(\mathscr{C}_n^k)$*, all having length* m/d*,* $N = \cup_u V(D_u)$ *is written in the canonical form (2), and* $m = |N|$*, then:*

(a) $\lceil km/n \rceil = \lfloor (k+1)\, m/n \rfloor$*,*

(b) if $n_1 = \lceil km/n \rceil$*, then* $\gcd(m, n_1) = d$*,*

(c) if $s : N \to N$ *is defined by* $s(i) = i'$ *if* (i, i') *is an arc of* D_u *for some* $u = 1, \dots, d$*, then equation (3) holds, i.e.,* $s(i_j) = i_{j \underset{m}{\oplus} n_1}$ *for* $j = 0, \dots, m-1$*.*

Conversely,

Theorem 1.5 *Suppose the conditions (a) and (b) above are satisfied,* $N \subset \mathbb{Z}_n$ *is written in the canonical form (2), and* $s : N \to N$ *defined by the condition (3) satisfies*

$$(i, s(i)) \quad \textit{is an arc of } G(\mathscr{C}_n^k) \textit{ for all } i \in N. \tag{4}$$

Then there exist (uniquely determined) d *disjoint simple directed cycles in* $G(\mathscr{C}_n^k)$*,* $D_1, \dots, D_d$*, all having the same length* m/d*, such that* $N = \cup_u V(D_u)$*.*

If the previous conditions hold, then we have the following interlacing property of the cycles:

Lemma 1.6 *Suppose the assumptions of theorem 1.4 hold. Then in each cyclic interval of* N *(expressed as in (2)) of length* d*, there is exactly one point of each cycle.*

We now generalize theorem 1.2 to the case of several disjoint cycles.

Theorem 1.7 *Let* n, k, m *be given, with* $1 \le k \le n-1$ *and* $0 \le m \le n-1$*.*

There exist d disjoint cycles in $G(\mathscr{C}_n^k)$ each of length m/d if and only if equation (1) holds and $\gcd(m, n_1) = d$*, where n_1 is the common value.*

Lemma 1.1 relates the existence of a simple cycle D to the condition $\gcd(n_1, m) = 1$ and contractions $\mathscr{C}_n^k/V(D)$, and we have seen that existence of several disjoint cycles is related to some algebraic conditions. The following result shows that contractions are also related.

Theorem 1.8 *Suppose $n, k, m, n', k' \in \mathbb{N}$ and $N \subset \mathbb{Z}_n$ are given, such that $2 \leq k \leq n-2$, $m = |N|$, $1 \leq m \leq n-2$, and $1 \leq k' < n'$. Then the following are equivalent:*

(a) $\mathscr{C}_n^k/N \sim \mathscr{C}_{n'}^{k'}$.

(b) $|E(\mathscr{C}_n^k/N)| = |V(\mathscr{C}_n^k/N)|$.

(c) There exist d disjoint simple directed cycles of $G(\mathscr{C}_n^k)$, $D_1, D_2, \ldots, D_d$, having the same length, such that $N = \cup_u V(D_u)$.

From a geometric point of view, given a clutter $\mathscr{C}$ and a subset $N \subset V(\mathscr{C})$, we may interpret $P(\mathscr{C}/N)$ as the intersection of $P(\mathscr{C})$ with the subspace $\{x \in \mathbb{R}^n : x_i = 0\ \forall i \in N\}$. Since the conditions $x_i \geq 0$ for all $i \in N$, are some of the inequalities defining $P(\mathscr{C}/N)$, vertices in $P(\mathscr{C}/N)$ (considered as subset of $\mathbb{R}^n$) are already vertices of $P(\mathscr{C})$. Thus, if for $x \in \mathbb{R}^n$ we let $N(x) = \{i \in \mathbb{Z}_n : x_i = 0\}$ and $m(x) = |N(x)|$, we see that a vertex x of $P(\mathscr{C}_n^k)$ with $N(x) \neq \emptyset$ will have a corresponding vertex in $P(\mathscr{C}_n^k/N(x))$ (regarded now as a subset of $\mathbb{R}^{n-m(x)}$), all of whose coordinates are positive. The remarkable fact, as shown by Argiroffo and Bianchi [3], is that for all vertices $x \in P(\mathscr{C}_n^k)$, $N(x)$ is such that $\mathscr{C}_n^k/N(x)$ is a circulant clutter. The following is a variant of their result:

Theorem 1.9 *The point x is a vertex of $P(\mathscr{C}_n^k)$ if and only if there exist n' and k', such that $1 \leq k' < n'$, $\mathscr{C}_n^k/N(x) \sim \mathscr{C}_{n'}^{k'}$, $\gcd(n', k') = 1$, and $x_i = 1/k'$ for all $i \notin N(x)$. (Here we allow $N(x) = \emptyset$.)*

Using our previous results, we obtain alternative characterizations of the vertices of $P(\mathscr{C}_n^k)$:

Theorem 1.10 *Suppose n and k are given, with $1 \leq k < n$. For $x \in \mathbb{R}^n$, let $N(x)$ be written in the canonical form (2) (if $N(x) \neq \emptyset$), and let $m = |N(x)|$.*

Then, x is a vertex of $P(\mathscr{C}_n^k)$ if and only if the following conditions hold:

(i) $m \leq n-2$, and the equality (1) is satisfied,

(ii) if n_1 is the common value in the equality (1), then $n_1 < k$ and $\gcd(n-m, k-n_1) = 1$,

(iii) $x_i = 1/(k - n_1)$ *for all* $i \notin N(x)$*, and*

(iv) *if* $m > 0$ *and* $d = \gcd(m, n_1)$*, then there exist* d *disjoint simple directed cycles in* $G(\mathscr{C}_n^k)$*,* $D_1, \dots, D_d$*, all of length* m/d*, and such that* $N(x) = \cup_u V(D_u)$*.*

Alternatively, we could change the condition (iv) to:

(iv') if $m > 0$*, and* s *is defined by equation* (3)*, then equation* (4) *holds.*

Corollary 1.11 *Let* n*,* k *and* m *be given non negative integers, such that* $1 \le k < n$ *and* $1 \le m < n - 1$*. Then,* $P(\mathscr{C}_n^k)$ *has a vertex with exactly* m *zero coordinates and the remaining coordinates taking the value* $1/k'$ *if and only if (i) the equation* (1) *holds, (ii) if* n_1 *is its common value, then* $k' = k - n_1$ *is positive, and (iii)* $\gcd(n - m, k') = 1$*.*

Our results can be used to study many families of circulant clutters. For example, the following is a characterization of ideal and mni circulant clutters in purely arithmetical terms:

Proposition 1.12 *If* $n \ge 3$ *and* $1 \le k \le n-1$*, then* $\mathscr{C}_n^k$ *is ideal or mni if and only if for every* m*,* $1 \le m \le n-2$*, for which (i)* $\lceil km/n \rceil = \lfloor (k+1)\, m/n \rfloor$*, and (ii) if* $n_1 = \lceil km/n \rceil$ *and* $\gcd(n - m, k - n_1) = 1$*, then necessarily* $n_1 = k - 1$*.*

If these conditions are satisfied, then $\mathscr{C}_n^k$ *is mni if* $\gcd(n, k) = 1$*, and otherwise is ideal.*

As already mentioned, Cornuéjols and Novick [5] gave a complete description of all the ideal and mni circulant clutters. Using properties of the Farey series, we may obtain the same results, without using Lehman's theorems.

There have been many efforts to introduce and study more general classes of clutters encompassing ideal and mni clutters. Of interest to us are near-ideal clutters, introduced by Argiroffo in her Ph.D. thesis [2] (see also [4]). Near-ideal circulant clutters may be defined as those for which $P(\mathscr{C}_n^k) \cap \{x \in \mathbb{R}^n : \mathbf{1} \cdot x \ge \lceil n/k \rceil\}$ is the convex hull of the 0-1 vertices of $P(\mathscr{C}_n^k)$.

We have:

Proposition 1.13 *Suppose* n *and* k *are given,* $n > k \ge 3$*. Then,*

(i) $\mathscr{C}_n^k$ *is not near-ideal if and only if there exist* n' *and* k' *such that*

$$k > k' > 1, \quad \gcd(k', n') = 1, \quad \frac{n'}{k'} > \left\lceil \frac{n}{k} \right\rceil, \quad \frac{n}{k+1} \ge \frac{n'}{k'+1}.$$

(ii) *For any* $\nu \ge 2$*,* $\mathscr{C}_{\nu k}^k$ *is not near-ideal except for* $\mathscr{C}_6^3$*,* $\mathscr{C}_9^3$ *and* $\mathscr{C}_8^4$*, which are ideal.*

(iii) *If* $k \geq \frac{2}{3}n - 1$*, then* $\mathscr{C}_n^k$ *is near-ideal.*

(iv) *If* $k \geq 3$ *and* $n \geq 13k$*, then* $\mathscr{C}_n^k$ *is not near-ideal.*

(ii) was observed in [3]. A result very similar to (iii), with the bound $k \geq \lfloor 2n/3 \rfloor$, was obtained by Argiroffo [2], using techniques involving blockers.

It is rather simple to construct a table to show the values of n and k for which $\mathscr{C}_n^k$ is near-ideal. Examining such a table reveals that we cannot hope for an exhaustive classification of near-ideal circulant clutters, similar to that given by Cornuéjols and Novick for ideal and mni circulant clutters.

As a final application, we show that the 0-1 vertices of $P(\mathscr{C}_n^k)$ always have fewer non zero coordinates than fractional vertices, and the number of non zero coordinates of 0-1 vertices are consecutive:

Proposition 1.14 *Suppose* x *is a* 0-1 *vertex of* $P(\mathscr{C}_n^k)$*, with* $|N(x)| = m$*, and* x' *is another vertex, not necessarily* 0-1*, with* $|N(x')| = m'$*. Then,*

(i) If x' *is a fractional vertex, then* $m' < m$*.*

(ii) If x' *is* 0-1 *and* $m < m'$*, then, for any* $m'' \in \mathbb{N}$ *with* $m < m'' < m'$*, there exists a* 0-1 *vertex,* x''*, of* $P(\mathscr{C}_n^k)$ *with* $|N(x'')| = m''$*.*

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