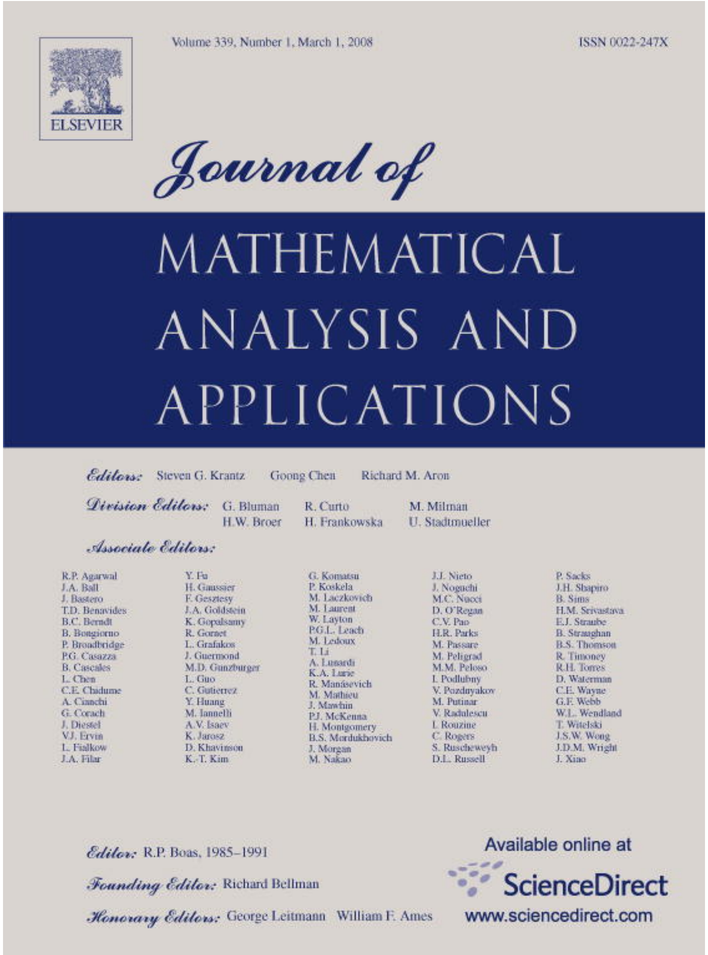
Volume 339, Number 1, March 1, 2008
ISSN 0022-247X
ELSEVIER
Journal of
MATHEMATICAL ANALYSIS AND APPLICATIONS
Editors: Steven G. Krantz Goong Chen Richard M. Aron
Division Editors: G. Bluman R. Curto M. Milman H.W. Broer H. Frankowska U. Stadtmueller
Associate Editors:
Editor: R.P. Boas, 1985–1991
Founding Editor: Richard Bellman
Honorary Editors: George Leitmann William F. Ames
Available online at
ScienceDirect
www.sciencedirect.com

Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 339 (2008) 182–196
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

On coupled transversal and axial motions of two beams with a joint ☆

J.A. Burns [a], E.M. Cliff [a,*], Z. Liu [b], R.D. Spies [c,d]

[a] Interdisciplinary Center for Applied Mathematics, Virginia Polytechnic Institute & State University, Blacksburg, VA 24061-0531, USA
[b] Department of Mathematics and Statistics, University of Minnesota Duluth, Duluth, MN 55812-3000, USA
[c] Instituto de Matemática Aplicada del Litoral, IMAL-CONICET, Güemes 3450, Argentina
[d] Departamento de Matemática, Facultad de Ingeniería Química, Universidad Nacional del Litoral, Santiago del Estero 2829, 3000 Santa Fe, Argentina

Received 29 September 2005
Available online 6 July 2007
Submitted by W.L. Wendland

Abstract

In this paper we develop and analyze a mathematical model for combined axial and transverse motions of two Euler–Bernoulli beams coupled through a joint composed of two rigid bodies. The motivation for this problem comes from the need to accurately model damping and joints for the next generation of inflatable/rigidizable space structures. We assume Kelvin–Voigt damping in the two beams whose motions are coupled through a joint which includes an internal moment. The resulting equations of motion consist of four, second-order in time, partial differential equations, four second-order ordinary differential equations, and certain compatibility boundary conditions. The system is re-cast as an abstract second-order differential equation in an appropriate Hilbert space, consisting of function spaces describing the distributed beam deflections, and a finite-dimensional space that projects important features at the joint boundary. Semigroup theory is used to prove the system is well posed, and that with positive damping parameters the resulting semigroup is analytic and exponentially stable. The spectrum of the infinitesimal generator is characterized.

Keywords: Mechanics of deformable solids; Abstract differential equations; Semigroups and linear evolution equations

1. Introduction

Inflatable and rigidizable space structures have been the subject of numerous scientific studies for the past fifty years and considerable progress has been made in the development of new materials and technologies for the design and fabrication of these structures. Several proposed space antenna systems will require large ultra-light trusses to provide the "backbone" of the structure. In recent years there has been renewed interest in inflatable/rigidizable space structures [7] because of the efficiency they offer in packaging during boost-to-orbit. It has been recognized that practical precision requirements can only be achieved through the development of new high-fidelity mathematical

☆ Research supported in part by DARPA/STO, NASA Langley Research Center, and the National Institute for Aerospace under Grant NIA 2535, and in part by AFOSR Grants F49620-03-1-0243 and FA9550-07-1-0273.
* Corresponding author.
E-mail addresses: burns@math.vt.edu (J.A. Burns), ecliff@vt.edu (E.M. Cliff), zliu@d.umn.edu (Z. Liu), rspies@imalpde.ceride.gov.ar (R.D. Spies).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.06.047

models and corresponding numerical tools. In particular, there is a need to understand better the dynamical response characteristics, including inherent damping, of truss structures fabricated with these advanced material systems. In addition, several proposed designs [6] make use of joints with special attachment "legs" which lead to the type of model considered in the paper.

In this paper we study an assembly consisting of two beams with Kelvin–Voigt damping, coupled to a simple joint through two legs, and develop an abstract 2nd-order differential equation of the form

$$\ddot{X}(t) + \mathcal{A}\big(S\dot{X}(t) + X(t)\big) = 0,$$

on an appropriate Hilbert space $\mathcal{H}$. The space $\mathcal{H}$ is a product space, consisting of function spaces describing the distributed beam deflections, and a finite-dimensional space that projects important features at the joint boundary. We put this 2nd-order model in a state-space setting and use semigroup theory to prove the model is well posed, and that the semigroup is analytic and exponentially stable.

Very general assemblies of beams have been studied by J. Lagnese et al. [9]. Whereas the beam geometry considered here is a special case of the analysis in [9], the joint model is more complex, as detailed below. Additionally, our beam model includes Kelvin–Voigt damping.

2. Equations of motion

We construct a mathematical model of the coupled joint-legs-beam system shown in Fig. 1. The model describes the transverse and longitudinal motions of two beams coupled to a joint with two legs. Each beam can vibrate in the plane: the transverse (bending) deformation of beam i is given by $w^i(t, s_i)$, while the longitudinal (axial) deformation is given by $u^i(t, s_i)$, where $0 \leqslant s_i \leqslant L_i$, $t \geqslant 0$, $i = 1, 2$. The joint configuration is described by $x(t)$, $y(t)$, the planar Cartesian displacements of the pivot point, and by $\theta_i(t)$, the angle between leg i and the positive x-axis. The joint model is composed of two rigid-bodies (the joint-legs) with an internal moment. As discussed below, the rigid joint model in [9] may be viewed as a limiting case of this model. The physical parameters in the model are given by:

- $L_i, A_i, I_i, E_i, \rho_i$: length, cross-section area, area moment of inertia, Young's modulus and mass density of beam i, $i = 1, 2$ (with $\rho_i, A_i, E_i, I_i > 0$).
- l_i, m_i, I_ℓ^i, d_i: length, mass, mass moment of inertia about the center of mass and distance from pivot to center of mass of joint-leg i, $i = 1, 2$.
- $I_Q^i = I_\ell^i + m_i d_i^2 > 0$: mass moment of inertia of joint-leg i about pivot, $i = 1, 2$.
- μ_i, γ_i, d, k: Kelvin–Voigt damping parameters in the axial motions, in the transverse bending, viscous joint damping, and joint stiffness parameters.
- m_p: mass of the pivot.

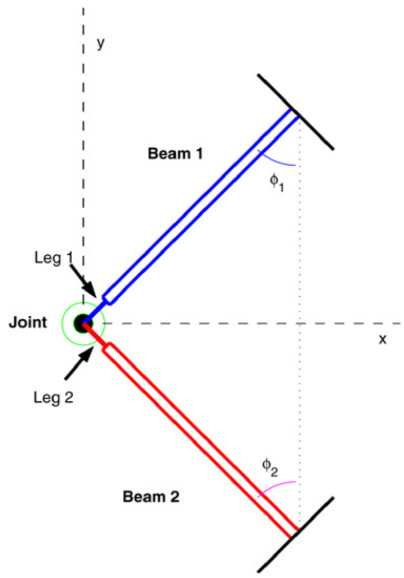

Fig. 1. Joint beam system.

Furthermore, $m = m_1 + m_2 + m_p > 0$ is the total mass of the joint-leg system, and the angles φ_i describe the equilibrium orientation of beam i (see Fig. 1). Finally, $F_i(t)$, $N_i(t)$, $M_i(t)$ represent the axial force, shear force, and bending moment at the end $s_i = L_i$ of beam i, and $M_Q(t)$ is the internal torque exerted on joint-leg 1 by joint-leg 2.

2.1. Differential equations

The bending motions of the beams are described by an Euler–Bernoulli model with Kelvin–Voigt damping, viz.,

$$\rho_i A_i \frac{\partial^2 w^i(t,s_i)}{\partial t^2} + \frac{\partial^2}{\partial s_i^2}\left[E_i I_i \frac{\partial^2 w^i(t,s_i)}{\partial s_i^2} + \gamma_i \frac{\partial^3 w^i(t,s_i)}{\partial s_i^2 \partial t}\right] = 0, \tag{1}$$

$$w^i(t,0) = \frac{\partial w^i(t,0)}{\partial s_i} = 0. \tag{2}$$

The axial motions, again with Kelvin–Voigt damping, are described by

$$\rho_i A_i \frac{\partial^2 u^i(t,s_i)}{\partial t^2} - \frac{\partial}{\partial s_i}\left[E_i A_i \frac{\partial u^i(t,s_i)}{\partial s_i} + \mu_i \frac{\partial^2 u^i(t,s_i)}{\partial s_i \partial t}\right] = 0, \tag{3}$$

$$u^i(t,0) = 0. \tag{4}$$

Since the continuum equations (1)–(4) reflect small deflection theory, we consider the dynamic equations for the joint-leg linearized about $x^0 = y^0 = \dot{x}^0 = \dot{y}^0 = \dot{\theta}_1^0 = \dot{\theta}_2^0 = 0$ and $\theta_1^0 = \frac{\pi}{2} - \varphi_1$, $\theta_2^0 = -\frac{\pi}{2} + \varphi_2$. Principles of Newtonian mechanics applied to the joint elements lead to

$$m\ddot{x}(t) - m_1 d_1 \cos\varphi_1 \ddot{\theta}_1(t) + m_2 d_2 \cos\varphi_2 \ddot{\theta}_2(t) = F_1(t)\sin\varphi_1 - N_1(t)\cos\varphi_1 + F_2(t)\sin\varphi_2 + N_2(t)\cos\varphi_2, \tag{5}$$

$$m\ddot{y}(t) + m_1 d_1 \sin\varphi_1 \ddot{\theta}_1(t) + m_2 d_2 \sin\varphi_2 \ddot{\theta}_2(t) = F_1(t)\cos\varphi_1 + N_1(t)\sin\varphi_1 - F_2(t)\cos\varphi_2 + N_2(t)\sin\varphi_2, \tag{6}$$

$$I_Q^1 \ddot{\theta}_1(t) = M_Q(t) + M_1(t) + l_1 N_1(t) + m_1 d_1\big[\ddot{x}(t)\cos\varphi_1 - \ddot{y}(t)\sin\varphi_1\big], \tag{7}$$

$$I_Q^2 \ddot{\theta}_2(t) = -M_Q(t) + M_2(t) + l_2 N_2(t) - m_2 d_2\big[\ddot{x}(t)\cos\varphi_2 + \ddot{y}(t)\sin\varphi_2\big]. \tag{8}$$

Note that in these equations $\theta_i(t)$ denotes the perturbation in the angle between leg i and the positive x-axis. The present model includes elastic and viscous terms for the internal moment $M_Q(t)$ given by

$$M_Q(t) = k\big(\theta_2(t) - \theta_1(t)\big) + d\big(\dot{\theta}_2(t) - \dot{\theta}_1(t)\big), \tag{9}$$

but it is clearly possible to extend this to more general models. Note that increasing k renders the joint stiffer and as $k \to \infty$ this joint model approximates the *Rigid Joint* in [9, §8.1.1]. Similarly with $d = k = 0$ the internal moment vanishes, and in the singular limit $I_Q^i \to 0$ the model approximates the *Pinned Joint* in [9, §8.1.2]. However for finite admissible values of the joint parameters, the joint model here differs from those in [9].

2.2. Compatibility conditions

Geometric compatibility between the joint-leg and the $s_i = L_i$ end of the beam requires that:
for beam 1–leg 1:

$$\begin{cases} x(t) - l_1\theta_1(t)\cos\varphi_1 + w^1(t,L_1)\cos\varphi_1 + u^1(t,L_1)\sin\varphi_1 = 0, \\ y(t) + l_1\theta_1(t)\sin\varphi_1 - w^1(t,L_1)\sin\varphi_1 + u^1(t,L_1)\cos\varphi_1 = 0, \\ \theta_1(t) + w_s^1(t,L_1) = 0, \end{cases} \tag{10}$$

whereas, for beam 2–leg 2:

$$\begin{cases} x(t) + l_2\theta_2(t)\cos\varphi_2 - w^2(t,L_2)\cos\varphi_2 + u^2(t,L_2)\sin\varphi_2 = 0, \\ y(t) + l_2\theta_2(t)\sin\varphi_2 - w^2(t,L_2)\sin\varphi_2 - u^2(t,L_2)\cos\varphi_2 = 0, \\ \theta_2(t) + w_s^2(t,L_2) = 0. \end{cases} \tag{11}$$

These conditions require that the Cartesian position of the beam-tip and the joint-leg-tip remain the same, and that the end-slope of the beam remain aligned with the joint-leg.

Furthermore, the Kelvin–Voigt constitutive model for the material requires: bending moment at the interfaces

$$E_\iota I_\iota w^\iota_{ss}(t, L_\iota) + \gamma_\iota \dot{w}^\iota_{ss}(t, L_\iota) = M_\iota(t), \quad \iota = 1, 2, \tag{12}$$

shear forces at the interfaces

$$\big(E_\iota I_\iota w^\iota_{ss} + \gamma_\iota \dot{w}^\iota_{ss}\big)_s(t, L_\iota) = N_\iota(t), \quad \iota = 1, 2, \tag{13}$$

and axial forces at the interfaces

$$\big(A_\iota E_\iota u^\iota + \mu_\iota \dot{u}^\iota\big)_s(t, L_\iota) = F_\iota(t), \quad \iota = 1, 2. \tag{14}$$

The seemingly cumbersome notation for the spatial derivatives in (13), (14) is necessary because, while each sum in parentheses is smooth, the summands need not be [1,10].

Note that the geometric compatibility equations (10), (11) can also be written in the form

$$\begin{cases} u^1(t, L_1) = -x(t)\sin\varphi_1 - y(t)\cos\varphi_1, \\ w^1(t, L_1) = -x(t)\cos\varphi_1 + y(t)\sin\varphi_1 + l_1\theta_1(t), \\ w^1_s(t, L_1) = -\theta_1(t), \end{cases} \tag{15}$$

$$\begin{cases} u^2(t, L_2) = -x(t)\sin\varphi_2 + y(t)\cos\varphi_2, \\ w^2(t, L_2) = x(t)\cos\varphi_2 + y(t)\sin\varphi_2 + l_2\theta_2(t), \\ w^2_s(t, L_2) = -\theta_2(t). \end{cases} \tag{16}$$

3. A second-order abstract differential equation

We denote by $H^n(0, L)$ the usual Sobolev space of functions in $L^2(0, L)$ with derivatives up to order n in $L^2(0, L)$. The space $H^n_\ell(0, L)$ denote the space of functions in $H^n(0, L)$ that vanish, together with all derivatives up to the order $n - 1$, at the left end ($s = 0$).

Define the Hilbert space $\mathcal{H}_z \doteq L^2(0, L_1) \times L^2(0, L_2) \times L^2(0, L_1) \times L^2(0, L_2)$ with the inner product

$$\big\langle \big(w^1, w^2, u^1, u^2\big)^T, \big(\xi^1, \xi^2, v^1, v^2\big)^T \big\rangle_{\mathcal{H}_z} \doteq \sum_{i=1}^{2} \rho_i A_i \big[\langle w^i, \xi^i\rangle + \langle u^i, v^i\rangle\big],$$

where $\langle\cdot,\cdot\rangle$ denotes the usual L^2-inner product. We define on $\mathcal{H}_z$ the operator $\mathcal{A}_1$ by

$$\mathrm{dom}(\mathcal{A}_1) \doteq H^2_\ell \cap H^4(0, L_1) \times H^2_\ell \cap H^4(0, L_2) \times H^1_\ell \cap H^2(0, L_1) \times H^1_\ell \cap H^2(0, L_2),$$

$$\mathcal{A}_1 \doteq \begin{pmatrix} \frac{E_1 I_1}{\rho_1 A_1} D^4 & 0 & 0 & 0 \\ 0 & \frac{E_2 I_2}{\rho_2 A_2} D^4 & 0 & 0 \\ 0 & 0 & -\frac{E_1}{\rho_1} D^2 & 0 \\ 0 & 0 & 0 & -\frac{E_2}{\rho_2} D^2 \end{pmatrix},$$

where $D \doteq \frac{\mathrm{d}}{\mathrm{d}s}$. From standard arguments it follows that $\mathrm{dom}(\mathcal{A}_1)$ is dense in $\mathcal{H}_z$.

With this notation Eqs. (1)–(4) can be written as the abstract second-order ordinary differential equation in $\mathcal{H}_z$,

$$\ddot{z}(t) + \mathcal{A}_1\big(S_1\dot{z}(t) + z(t)\big) = 0, \tag{17}$$

where

$$z(t) \doteq \begin{pmatrix} w^1(t,\cdot) \\ w^2(t,\cdot) \\ u^1(t,\cdot) \\ u^2(t,\cdot) \end{pmatrix}, \qquad S_1 \doteq \begin{pmatrix} \frac{\gamma_1}{E_1 I_1} & 0 & 0 & 0 \\ 0 & \frac{\gamma_2}{E_2 I_2} & 0 & 0 \\ 0 & 0 & \frac{\mu_1}{E_1 A_1} & 0 \\ 0 & 0 & 0 & \frac{\mu_2}{E_2 A_2} \end{pmatrix}. \tag{18}$$

For the joint dynamics we begin by defining two boundary projection operators P_1^B and P_2^B from $\mathcal{H}_z$ to $\mathbb{R}^6$ by

$$\begin{aligned}
&\operatorname{dom}(P_1^B) \doteq H^2(0,L_1)\times H^2(0,L_2)\times H^1(0,L_1)\times H^1(0,L_2),\\
&\operatorname{dom}(P_2^B) \doteq H^4(0,L_1)\times H^4(0,L_2)\times H^2(0,L_1)\times H^2(0,L_2),\\
&P_1^B(w^1,w^2,u^1,u^2)^T \doteq \left(-w_s^1(L_1), w^1(L_1), -w_s^2(L_2), w^2(L_2), -u^1(L_1), -u^2(L_2)\right)^T,\\
&P_2^B(w^1,w^2,u^1,u^2)^T \doteq \left(w_{ss}^1(L_1), w_{sss}^1(L_1), w_{ss}^2(L_2), w_{sss}^2(L_2), u_s^1(L_1), u_s^2(L_2)\right)^T,
\end{aligned}$$

where the subscript "s" denotes derivative with respect to the spatial variable s. Note that $\operatorname{dom}(\mathcal{A}_1)$ is contained in both $\operatorname{dom}(P_1^B)$ and $\operatorname{dom}(P_2^B)$, so that both projections are well defined on $\operatorname{dom}(\mathcal{A}_1)$.

Next consider the (4×4) matrix M and the (4×6) matrix C given by

$$M \doteq \begin{pmatrix} mI_2 & P \\ P^T & \operatorname{diag}(I_Q^1, I_Q^2) \end{pmatrix}, \quad P \doteq \begin{pmatrix} -m_1 d_1 \cos\varphi_1 & m_2 d_2 \cos\varphi_2 \\ m_1 d_1 \sin\varphi_1 & m_2 d_2 \sin\varphi_2 \end{pmatrix} \tag{19}$$

and

$$C \doteq \begin{pmatrix} 0 & -\cos\varphi_1 & 0 & \cos\varphi_2 & \sin\varphi_1 & \sin\varphi_2 \\ 0 & \sin\varphi_1 & 0 & \sin\varphi_2 & \cos\varphi_1 & -\cos\varphi_2 \\ 1 & l_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & l_2 & 0 & 0 \end{pmatrix}. \tag{20}$$

Furthermore, define $\mathcal{H}_b \doteq [\ker(C)]^\perp = \operatorname{range}(C^T)$ with the inner product

$$\langle b,a\rangle_{\mathcal{H}_b} \doteq b^T\left(C^T M^{-1} C\right)^\dagger a = \left\langle b, \left(C^T M^{-1} C\right)^\dagger a\right\rangle_{\mathbb{R}^6},$$

where $(C^T M^{-1} C)^\dagger$ denotes the Moore–Penrose generalized inverse of the matrix $C^T M^{-1} C$. It can be easily checked that C has full rank, $\ker(C)$ has dimension 2 and therefore $[\ker(C)]^\perp$ is a four-dimensional subspace of $\mathbb{R}^6$. Since $\ker(C^T M^{-1} C) = \ker(C)$, the matrix $C^T M^{-1} C$ is a one-to-one operator from $[\ker(C)]^\perp$ onto itself. The Moore–Penrose generalized inverse of $C^T M^{-1} C$ is precisely the inverse of this operator. An immediate calculation shows that the matrix $C^T M^{-1} C$ restricted to $[\ker(C)]^\perp$ is strictly positive definite, and therefore so is $(C^T M^{-1} C)^\dagger$. In the sequel, we make free and repeated use of the four "Moore–Penrose identities" (see, for instance, [4, §2.1])

$$TT^\dagger T = T, \qquad T^\dagger T T^\dagger = T^\dagger, \qquad T^\dagger T = \widetilde{Q}_1, \qquad TT^\dagger = \widetilde{Q}_2, \tag{21}$$

where T is a densely-defined, bounded linear operator from a Hilbert space $\mathcal{X}$ into a Hilbert space $\mathcal{Y}$ and $\widetilde{Q}_1$ and $\widetilde{Q}_2$ are the orthogonal projections of $\mathcal{X}$ onto $[\ker(T)]^\perp$ and of $\mathcal{Y}$ onto $\operatorname{range}(T)$, respectively. In the infinite-dimensional case, $\widetilde{Q}_2$ is the orthogonal projection onto the closure of $\operatorname{range}(T)$ and in the right-hand side of the last identity above, $\widetilde{Q}_2$ must be restricted to $\operatorname{dom}(T^\dagger) \doteq \operatorname{range}(T) + [\operatorname{range}(T)]^\perp$. In our case $T = C^T M^{-1} C$, $\mathcal{X} = \mathcal{Y} = \mathbb{R}^6$, $\ker(T) = \ker(C)$, $\operatorname{range}(T) = \operatorname{range}(C^T) = [\ker(C)]^\perp$. Thus $\widetilde{Q}_1 = \widetilde{Q}_2$ is the orthogonal projection of $\mathbb{R}^6$ onto $[\ker(C)]^\perp = \operatorname{range}(C^T)$.

With this notation the linearized equations for the joint-legs system (5)–(8) can be written in the form

$$M\ddot{\eta}(t) + B\dot{\eta}(t) + A\eta(t) = CF(t), \tag{22}$$

where $\eta \doteq (x(t), y(t), \theta_1(t), \theta_2(t))^T$, M and C are the matrices defined in (19), (20), and

$$A \doteq \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&k&-k\\0&0&-k&k \end{pmatrix}, \qquad B \doteq \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&d&-d\\0&0&-d&d \end{pmatrix}, \qquad F(t) \doteq \begin{pmatrix} M_1(t)\\N_1(t)\\M_2(t)\\N_2(t)\\F_1(t)\\F_2(t) \end{pmatrix}.$$

Also, with the notation above the geometric compatibility conditions (15), (16) are simply written in the form

$$P_1^B z(t) = C^T\eta(t), \tag{23}$$

while the dynamic boundary compatibility conditions (12)–(14) take the form

$$EP_2^B\big(z(t)+S_1\dot z(t)\big)=F(t), \tag{24}$$

where $E \doteq \mathrm{diag}(E_1I_1, E_1I_1, E_2I_2, E_2I_2, E_1A_1, E_2A_2)$ and the operator $S_1 : \mathcal{H}_z \mapsto \mathcal{H}_z$ is given in display (18). For later use we define the (matrix) operator $G : \mathcal{H}_b \mapsto \mathcal{H}_b$ by $G \doteq \mathrm{diag}(\gamma_1, \gamma_1, \gamma_2, \gamma_2, \mu_1, \mu_2)$.

For ease of exposition we restrict attention to the case of an ideal pinned joint in which no damping and no stiffness are presented in the joint, i.e. $d = k = 0$ and therefore $B = A = 0$. Multiply Eq. (22) by $C^T M^{-1}$ to obtain

$$C^T\ddot\eta(t)=C^TM^{-1}CF(t).$$

By (23) the left-hand side of the above equation equals $P_1^B\ddot z(t)$ while by (24), the right-hand side equals to $C^TM^{-1}CEP_2^B(z(t)+S_1\dot z(t))$. We arrive at the following ordinary differential equation for these two boundary projection vectors:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\big(P_1^Bz(t)\big)=C^TM^{-1}CEP_2^B\big(z(t)+S_1\dot z(t)\big). \tag{25}$$

To assemble the combined system we define the Hilbert space $\mathcal{H} \doteq \mathcal{H}_z \times \mathcal{H}_b$ with the usual inner product inherited from those in $\mathcal{H}_z$ and $\mathcal{H}_b$. In this Hilbert space we define the elastic operator $\mathcal{A}$ by

$$\mathrm{dom}(\mathcal{A})\doteq\left\{\begin{pmatrix} z\\ b\end{pmatrix}\in\mathrm{dom}(\mathcal{A}_1)\times\mathcal{H}_b \;\middle|\; P_1^Bz=b\right\}\quad\text{and}\quad \mathcal{A}\doteq\begin{pmatrix}\mathcal{A}_1 & 0\\ -C^TM^{-1}CEP_2^B & 0\end{pmatrix}. \tag{26}$$

Clearly $\mathrm{dom}(\mathcal{A})$ is dense in $\mathcal{H}$ because $\mathrm{dom}(\mathcal{A}_1)$ is dense in $\mathcal{H}_z$.

Thus, Eqs. (17) and (25) can be written as the second-order, homogeneous, linear elastic system

$$\ddot X(t)+\mathcal{A}\big(S\dot X(t)+X(t)\big)=0\quad\text{on }\mathcal{H}, \tag{27}$$

where $X(t) \doteq \binom{z(t)}{b(t)}$ and $S \doteq \begin{pmatrix} S_1 & 0\\ 0 & E^{-1}G\end{pmatrix}$. Note S_1 is a nonnegative, self-adjoint, bounded, linear operator on $\mathcal{H}_z$. Also if (and only if) γ_1, γ_2, μ_1 and μ_2 are all positive, then S_1 is strictly positive and in the undamped case ($\gamma_1 = \gamma_2 = \mu_1 = \mu_2 = 0$) one has $S_1 = 0$, $G = 0$ and therefore $S = 0$. In any case, $S^{\frac{1}{2}}$ is a well-defined, nonnegative, self-adjoint, bounded linear operator on $\mathcal{H}$. Also, it is clear that S_1 commutes with $\mathcal{A}_1$.

Remark 1. It is interesting to note that in this framework, the joint-leg dynamics (Eqs. (5)–(8)) and the dynamic compatibility conditions (Eqs. (12)–(14)) are reflected in an ODE coupling both boundary projection vectors $b_1(t) \doteq P_1^Bz(t)$ and $b_2(t) \doteq P_2^Bz(t)$ (Eq. (25)). On the other hand, the geometric compatibility conditions (Eqs. (10), (11) or equivalently (15), (16)) are simply reflected in the fact that for all times $t \geqslant 0$, $b_1(t)$ must remain in $[\ker(C)]^\perp$ and must be equal to $P_1^Bz(t)$. The former condition is included directly by defining $\mathcal{H}_b$ to be $[\ker(C)]^\perp$ instead of $\mathbb{R}^6$, while the latter condition is incorporated into $\mathrm{dom}(\mathcal{A})$.

4. A state-space formulation

Our state-space formulation relies on some important properties of the elastic operator $\mathcal{A}$ defined in Eq. (26).

Theorem 2. *The elastic operator $\mathcal{A} : \mathrm{dom}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is strictly positive and self-adjoint.*

Remark 3. At first sight it may seem counterintuitive that the operator $\mathcal{A}$ is self-adjoint since in its definition (26) it does not even "look" symmetric. However, recall that the symmetry (or not) of an operator depends strongly on its domain. Here the beams are clamped at the left end and satisfy a "nontypical" boundary condition at the right end. This translates into a "nonsymmetric" domain for $\mathcal{A}$. It is precisely this lack of symmetry in the definition of the domain that, in the end, makes the "asymmetric looking" operator $\mathcal{A}$, in fact, self-adjoint.

Proof. As noted earlier, $\mathrm{dom}(\mathcal{A})$ is dense in $\mathcal{H}$. We show first that $\mathcal{A}$ is symmetric. For that, let $X_2 = (y, a)^T \in \mathrm{dom}(\mathcal{A})$, where $y = (\xi^1, \xi^2, v^1, v^2)^T$ and $P_1^By = a \in [\ker(C)]^\perp$. Then for any $X_1 = (z, b)^T \in \mathrm{dom}(\mathcal{A})$ with $z = (w^1, w^2, u^1, u^2)^T$ one has $P_1^Bz = b \in [\ker(C)]^\perp$ and (unless otherwise indicated, the inner products are in $L^2(0, L_\iota)$),

$$
\begin{aligned}
\langle \mathcal{A}X_1, X_2\rangle_{\mathcal{H}} &= \langle \mathcal{A}_1 z, y\rangle_{\mathcal{H}_z} + \big\langle -C^T M^{-1} C E P_2^B z, a\big\rangle_{\mathcal{H}_b} \\
&= \left\langle \left(\frac{E_1 I_1}{\rho_1 A_1} w^1_{ssss}, \frac{E_2 I_2}{\rho_2 A_2} w^2_{ssss}, -\frac{E_1}{\rho_1} u^1_{ss}, -\frac{E_2}{\rho_2} u^2_{ss} \right)^T, \big(\xi^1, \xi^2, v^1, v^2\big)^T \right\rangle_{\mathcal{H}_z} \\
&\quad - \big\langle C^T M^{-1} C E P_2^B z, \big(C^T M^{-1} C\big)^{\dagger} a\big\rangle_{\mathbb{R}^6} \\
&= \sum_{i=1}^{2} E_i \big[I_i \big\langle w^i_{ssss}, \xi^i\big\rangle - A_i \big\langle u^i_{ss}, v^i\big\rangle\big] - \big\langle E P_2^B z, a\big\rangle_{\mathbb{R}^6} \\
&= \sum_{i=1}^{2} \big\{ E_i I_i \big[\big(w^i_{sss}\xi^i - w^i_{ss}\xi^i_s + w^i_s \xi^i_{ss} - w^i \xi^i_{sss}\big)(L_i) + \big\langle w^i, \xi^i_{ssss}\big\rangle\big] \\
&\quad - E_i A_i \big[\big(u^i_s v^i - u^i v^i_s\big)(L_i) + \big\langle u^i, v^i_{ss}\big\rangle\big]\big\} \\
&\quad - \big\langle E P_2^B z, a\big\rangle_{\mathbb{R}^6} \quad \text{(integrate by parts and use the b.c. at } s = 0) \\
&= \langle z, \mathcal{A}_1 y\rangle_{\mathcal{H}_z} - \big\langle b, E P_2^B y\big\rangle_{\mathbb{R}^6} + \big\langle E P_2^B z, a\big\rangle_{\mathbb{R}^6} - \big\langle E P_2^B z, a\big\rangle_{\mathbb{R}^6} \quad \big(\text{use } b = P_1^B z \text{ and } a = P_1^B y\big) \\
&= \langle z, \mathcal{A}_1 y\rangle_{\mathcal{H}_z} - \big\langle b, \widetilde{Q}_1 E P_2^B y\big\rangle_{\mathbb{R}^6} \quad \big(\text{since } b \in \big[\ker(C)\big]^{\perp}\big) \\
&= \langle z, \mathcal{A}_1 y\rangle_{\mathcal{H}_z} - \big\langle b, C^T M^{-1} C E P_2^B y\big\rangle_{\mathcal{H}_b} \quad \big(\text{since } T^{\dagger} T = \widetilde{Q}_1\big) \\
&= \left\langle \begin{pmatrix} z \\ b \end{pmatrix} \begin{pmatrix} y \\ q \end{pmatrix} \right\rangle_{\mathcal{H}} = \langle X_1, \mathcal{A}X_2\rangle_{\mathcal{H}}.
\end{aligned}
$$

Hence, $X_2 \in \operatorname{dom}(\mathcal{A}^*)$ and $\mathcal{A}^* X_2 = \mathcal{A}X_2$, i.e. $\mathcal{A}$ is a symmetric operator.

We prove now that $\mathcal{A}$ is in fact self-adjoint. It suffices to show that $\operatorname{dom}(\mathcal{A}^*) \subset \operatorname{dom}(\mathcal{A})$. Let $X_1 = (z, b)^T = (w^1, w^2, u^1, u^2, b)^T \in \operatorname{dom}(\mathcal{A}^*)$. Then, there exists $X_2 = (y, a)^T = (\xi^1, \xi^2, v^1, v^2, a)^T \in \mathcal{H}$ such that for all $W = (f, c)^T = (\alpha^1, \alpha^2, p^1, p^2, c)^T \in \operatorname{dom}(\mathcal{A})$ one has

$$
\begin{aligned}
0 &= \langle \mathcal{A}W, X_1\rangle_{\mathcal{H}} - \langle W, X_2\rangle_{\mathcal{H}} \\
&= \left\langle \begin{pmatrix} \mathcal{A}_1 f \\ -C^T M^{-1} C E P_2^B f \end{pmatrix}, \begin{pmatrix} z \\ b \end{pmatrix} \right\rangle_{\mathcal{H}} - \left\langle \begin{pmatrix} f \\ c \end{pmatrix}, \begin{pmatrix} y \\ a \end{pmatrix} \right\rangle_{\mathcal{H}} \\
&= \langle \mathcal{A}_1 f, z\rangle_{\mathcal{H}_z} - \langle f, y\rangle_{\mathcal{H}_z} + \big\langle -C^T M^{-1} C E P_2^B f, b\big\rangle_{\mathcal{H}_b} - \langle c, a\rangle_{\mathcal{H}_b} \\
&= \sum_{i=1}^{2} \big[E_i I_i \big\langle \alpha^i_{ssss}, w^i\big\rangle - E_i A_i \big\langle p^i_{ss}, u^i\big\rangle - \rho_i A_i \langle \alpha_i, \xi_i\rangle - \rho_i A_i \big\langle p_i, v^i\big\rangle\big] \\
&\quad - \big\langle E P_2^B f, b\big\rangle_{\mathbb{R}^6} - \big\langle c, \big(C^T M^{-1} C\big)^{\dagger} a\big\rangle_{\mathbb{R}^6} \\
&= \sum_{i=1}^{2} \left[\int_0^{L_i} \big(E_i I_i \alpha^i_{ssss} w^i - \rho_i A_i \alpha^i \xi^i\big)\, ds - \int_0^{L_i} \big(E_i A_i p^i_{ss} u^i + \rho_i A_i p^i v^i\big)\, ds \right] \\
&\quad - \big\langle E P_2^B f, b\big\rangle_{\mathbb{R}^6} - \big\langle c, \big(C^T M^{-1} C\big)^{\dagger} a\big\rangle_{\mathbb{R}^6}.
\end{aligned} \tag{28}
$$

Since (28) must hold for all $W = (f, c)^T = (\alpha^1, \alpha^2, p^1, p^2, c)^T \in \operatorname{dom}(\mathcal{A})$, it must hold in particular for all $\alpha^1 \in H_0^4(0, L_1)$, $\alpha^2 \in H_0^4(0, L_2)$, $p^1 \in H_0^2(0, L_1)$, $p^2 \in H_0^2(0, L_2)$ (in which case we have $c = P_1^B f = 0$ and also $P_2^B f = 0$). Hence, it follows that

$$
\int_0^{L_i} \big(E_i I_i \alpha^i_{ssss} w^i - \rho_i A_i \alpha^i \xi^i\big)\, ds = 0, \quad \text{for all } \alpha^i \in H_0^4(0, L_i),\ i = 1, 2,
$$

and also

$$\int_0^{L_i} \left(E_i A_i p^i_{ss} u^i + \rho_i A_i p^i v^i\right) ds = 0, \quad \text{for all } p^i \in H^2_0(0, L_i),\ i = 1, 2.$$

The Fundamental Lemma of the Calculus of Variations (see [5, pp. 31–32]) implies that there exist constants β^i_1, β^i_2, β^i_3, β^i_4, δ^i_1, δ^i_2, $i = 1, 2$, such that

$$\frac{E_i I_i}{\rho_i A_i} w^i = \beta^i_1 + \beta^i_2 s + \beta^i_3 s^2 + \beta^i_4 s^3 + \int_0^s \int_0^{\tau_1} \int_0^{\tau_2} \int_0^{\tau_3} \xi^i(\sigma)\, d\sigma\, d\tau_3\, d\tau_2\, d\tau_1,$$

and

$$\frac{E_i}{\rho_i} u^i = \delta^i_1 + \delta^i_2 s - \int_0^s \int_0^{\tau} v^i(\sigma)\, d\sigma\, d\tau,$$

for $s \in [0, L_i]$, $i = 1, 2$. Thus,

$$w^i \in H^4(0, L_i), \quad u^i \in H^2(0, L_i), \quad i = 1, 2, \tag{29}$$

and when we take derivatives

$$\xi^i(s) = \frac{E_i I_i}{\rho_i A_i} w^i_{ssss}(s), \quad v^i(s) = -\frac{E_i}{\rho_i} u^i_{ss}(s), \quad i = 1, 2,\ s \in [0, L_i]. \tag{30}$$

Substitution of (30) into (28) implies that for each $W = (f, c)^T = (\alpha^1, \alpha^2, p^1, p^2, c)^T$ with $W \in \mathrm{dom}(\mathcal{A})$ one must have

$$\begin{aligned}
0 &= \sum_{i=1}^2 \left[E_i I_i \int_0^{L_i} \left(\alpha^i_{ssss} w^i - \alpha^i w^i_{ssss}\right) ds - E_i A_i \int_0^{L_i} \left(p^i_{ss} u^i - p^i u^i_{ss}\right) ds \right] \\
&\quad - \left\langle E P^B_2 f, b\right\rangle_{\mathbb{R}^6} - \left\langle P^B_1 f, \left(C^T M^{-1} C\right)^\dagger a\right\rangle_{\mathbb{R}^6} \\
&= \sum_{i=1}^2 E_i I_i \left[\left(\alpha^i_{sss} w^i - \alpha^i_{ss} w^i_s + \alpha^i_s w^i_{ss} - \alpha^i w^i_{sss}\right)_0^{L_i} - E_i A_i \left(p^i_s u^i - p^i u^i_s\right)_0^{L_i}\right] \\
&\quad - \left\langle E P^B_2 f, b\right\rangle_{\mathbb{R}^6} - \left\langle P^B_1 f, \left(C^T M^{-1} C\right)^\dagger a\right\rangle_{\mathbb{R}^6} \quad \text{(integration by parts)} \\
&= \begin{pmatrix} E_1 I_1 \alpha^1_{ss}(L_1) \\ E_1 I_1 \alpha^1_{sss}(L_1) \\ E_2 I_2 \alpha^2_{ss}(L_2) \\ E_2 I_2 \alpha^2_{sss}(L_2) \\ A_1 E_1 p^1_s(L_1) \\ A_2 E_2 p^2_s(L_2) \end{pmatrix} \bullet \begin{pmatrix} -w^1_s(L_1) \\ w^1(L_1) \\ -w^2_s(L_2) \\ w^2(L_2) \\ -u^1(L_1) \\ -u_2(L_2) \end{pmatrix} - \begin{pmatrix} -\alpha^1_s(L_1) \\ \alpha^1(L_1) \\ -\alpha^2_s(L_2) \\ \alpha^2(L_2) \\ -p^1(L_1) \\ -p_2(L_2) \end{pmatrix} \bullet \begin{pmatrix} E_1 I_1 w^1_{ss}(L_1) \\ E_1 I_1 w^1_{sss}(L_1) \\ E_2 I_2 w^2_{ss}(L_2) \\ E_2 I_2 w^2_{sss}(L_2) \\ E_1 A_1 u^1_s(L_1) \\ E_2 A_2 u^2_s(L_2) \end{pmatrix} \\
&\quad - \sum_{i=1}^2 E_i I_i \left[\alpha^i_{sss} w^i - \alpha^i_{ss} w^i_s\right](0) + \sum_{i=1}^2 E_i A_i p^i_s(0) u^i(0) \\
&\quad - \left\langle E P^B_2 f, b\right\rangle_{\mathbb{R}^6} - \left\langle P^B_1 f, \left(C^T M^{-1} C\right)^\dagger a\right\rangle_{\mathbb{R}^6} \quad \left(\text{use } \alpha^i(0) = \alpha^i_s(0) = p^i(0) = 0\right) \\
&= \left\langle E P^B_2 f, P^B_1 z\right\rangle_{\mathbb{R}^6} - \left\langle P^B_1 f, E P^B_2 z\right\rangle_{\mathbb{R}^6} - \sum_{i=1}^2 E_i I_i \left[\alpha^i_{sss} w^i - \alpha^i_{ss} w^i_s\right](0) + \sum_{i=1}^2 E_i A_i p^i_s(0) u^i(0) \\
&\quad - \left\langle E P^B_2 f, b\right\rangle_{\mathbb{R}^6} - \left\langle P^B_1 f, \left(C^T M^{-1} C\right)^\dagger a\right\rangle_{\mathbb{R}^6}
\end{aligned}$$

$$= \langle EP_2^B f, P_1^B z - b\rangle_{\mathbb{R}^6} - \langle P_1^B f, EP_2^B z + (C^T M^{-1} C)^\dagger a\rangle_{\mathbb{R}^6}$$
$$- \sum_{i=1}^{2} E_i I_i\left[\alpha^i_{sss} w^i - \alpha^i_{ss} w^i_s\right](0) + \sum_{i=1}^{2} E_i A_i p^i_s u^i(0).$$

Since this must hold for all $W = (f, c)^T = (\alpha^1, \alpha^2, p^1, p^2, c)^T \in \mathrm{dom}(\mathcal{A})$ the above equation implies that $P_1^B z - b = 0$, $EP_2^B z + (C^T M^{-1} C)^\dagger a = 0$ and $w^i(0) = w^i_s(0) = u^i(0) = 0$ which, together with (29) implies that $X_1 = (z, b)^T = (w^1, w^2, u^1, u^2, b)^T \in \mathrm{dom}(\mathcal{A})$. This concludes the proof of the self-adjointness of $\mathcal{A}$.

It remains to be shown that $\mathcal{A}$ is strictly positive. For this, note that for any $X = (z, b)^T = (w^1, w^2, u^1, u^2, b)^T \in \mathrm{dom}(\mathcal{A})$, after integration by parts and following similar steps as above, we have that

$$\begin{aligned}
\langle \mathcal{A}X, X\rangle_{\mathcal{H}} &= \langle \mathcal{A}_1 z, z\rangle_{\mathcal{H}_z} - \langle b, EP_2^B z\rangle_{\mathbb{R}^6} \\
&= \sum_{i=1}^{2} E_i\left[I_i\langle w^i_{ssss}, w^i\rangle - A_i\langle u^i_{ss}, u^i\rangle\right] - \langle b, EP_2^B z\rangle_{\mathbb{R}^6} \\
&= \sum_{i=1}^{2} E_i I_i\left[\left(w^i_{sss} w^i - w^i_{ss} w^i_s\right)(L_i) + \|w^i_{ss}\|^2_{L^2(0,L_i)}\right] \\
&\quad - \sum_{i=1}^{2} E_i A_i\left[u^i_s u^i(L_i) - \|u^i_s\|^2_{L^2(0,L_i)}\right] - \langle b, EP_2^B z\rangle_{\mathbb{R}^6} \\
&= \sum_{i=1}^{2}\left[E_i I_i \|w^i_{ss}\|^2_{L^2(0,L_i)} + E_i A_i \|u^i_s\|^2_{L^2(0,L_i)}\right] \\
&\quad + \begin{pmatrix} -w^1_s(L_1) \\ w^1(L_1) \\ -w^2_s(L_2) \\ w^2(L_2) \\ -u^1(L_1) \\ -u^2(L_2) \end{pmatrix} \bullet \begin{pmatrix} E_1 I_1 w^1_{ss}(L_1) \\ E_1 I_1 w^1_{sss}(L_1) \\ E_2 I_2 w^2_{ss}(L_2) \\ E_2 I_2 w^2_{sss}(L_2) \\ E_1 A_1 u^1_s(L_1) \\ E_2 A_2 u^1_s(L_2) \end{pmatrix} - \langle b, EP_2^B z\rangle_{\mathbb{R}^6} \\
&= \sum_{i=1}^{2}\left[E_i I_i \|w^i_{ss}\|^2_{L^2(0,L_i)} + E_i A_i \|u^i_s\|^2_{L^2(0,L_i)}\right] \geqslant 0.
\end{aligned}$$

This last expression constitutes precisely the strain energy of the beams in our system. Hence $\mathcal{A}$ is nonnegative. Moreover, if for some $X = (z, b)^T \in \mathrm{dom}(\mathcal{A})$ one has $\langle \mathcal{A}X, X\rangle = 0$, then the above implies that w^i is linear and u^i is constant, $i = 1, 2$. Since $w^i(0) = w^i_s(0) = 0$ and $u^i(0) = 0$ one concludes that $w^i \equiv 0$ and $u^i \equiv 0$ which implies that $z = 0$, $b = P_2^B z = 0$ and therefore $X = 0$. Hence $\mathcal{A}$ is a strictly positive, self-adjoint operator on $\mathcal{H}$. □

Since $\mathcal{A}$ is strictly positive and self-adjoint, it possesses a unique strictly positive self-adjoint square root, $\mathcal{A}^{\frac{1}{2}}$ (see, for instance, [8,12]). Moreover, any fractional power $\mathcal{A}^\alpha$ of $\mathcal{A}$, $\alpha > 0$, is well defined, strictly positive and self-adjoint. We finally arrive at the definition of the state-space:

$$\mathcal{Y} \doteq \mathrm{dom}\big(\mathcal{A}^{\frac{1}{2}}\big) \times \mathcal{H}, \quad \text{with the inner product,}$$
$$\left\langle \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \right\rangle_{\mathcal{Y}} \doteq \langle \mathcal{A}^{\frac{1}{2}} X_1, \mathcal{A}^{\frac{1}{2}} W_1\rangle_{\mathcal{H}} + \langle X_2, W_2\rangle_{\mathcal{H}}. \tag{31}$$

The operator $\mathcal{A}_B$ is defined on $\mathcal{Y}$ by

$$\mathrm{dom}(\mathcal{A}_B) \doteq \left\{(X_1, X_2)^T \in \mathcal{Y} \;\middle|\; X_2 \in \mathrm{dom}\big(\mathcal{A}^{\frac{1}{2}}\big),\ SX_2 + X_1 \in \mathrm{dom}(\mathcal{A})\right\},$$
$$\mathcal{A}_B \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \doteq \begin{pmatrix} X_2 \\ -\mathcal{A}(SX_2 + X_1) \end{pmatrix}. \tag{32}$$

It is clear that $\mathrm{dom}(\mathcal{A}) \times \mathrm{dom}(\mathcal{A}) \subset \mathrm{dom}(\mathcal{A}_B)$, and $\mathrm{dom}(\mathcal{A}) \times \mathrm{dom}(\mathcal{A})$ is dense in $\mathcal{Y}$ (because $\mathrm{dom}(\mathcal{A})$ is dense in both $\mathrm{dom}(\mathcal{A}^{\frac{1}{2}})$ and in $\mathcal{H}$). Hence, it follows that $\mathrm{dom}(\mathcal{A}_B)$ is dense in $\mathcal{Y}$.

With this notation, the system (27) can be written in first-order form as

$$\dot{Y}(t) = \mathcal{A}_B Y(t) \quad \text{on } \mathcal{Y}, \tag{33}$$

where $Y(t) \doteq \binom{X(t)}{\dot{X}(t)}$ is the state-space variable.

Theorem 4 *(Well-posedness). $\mathcal{A}_B : \mathcal{Y} \to \mathcal{Y}$ as defined in* (32) *is the infinitesimal generator of a strongly continuous semigroup of contractions $\mathcal{S}(t)$ on $\mathcal{Y}$. Hence for any initial condition $Y_0 = Y(0) = \binom{X(0)}{\dot{X}(0)} \in \mathrm{dom}(\mathcal{A}_B)$, system* (33) *has a unique global solution $Y(t)$ given by $Y(t) = \mathcal{S}(t)Y_0$.*

Proof. Let $Y = (X_1, X_2)^T \in \mathrm{dom}(\mathcal{A}_B)$, then $X_1, X_2 \in \mathrm{dom}(\mathcal{A}^{\frac{1}{2}})$, $X_1 + SX_2 \in \mathrm{dom}(\mathcal{A}) \subset \mathrm{dom}(\mathcal{A}^{\frac{1}{2}})$ and therefore $SX_2 \in \mathrm{dom}(\mathcal{A}^{\frac{1}{2}})$,

$$\begin{aligned}\langle \mathcal{A}_B Y, Y\rangle_{\mathcal{Y}} &= \left\langle \begin{pmatrix} X_2 \\ -\mathcal{A}(X_1 + SX_2)\end{pmatrix}, \begin{pmatrix} X_1 \\ X_2\end{pmatrix}\right\rangle_{\mathcal{Y}} \\ &= \left\langle \mathcal{A}^{\frac{1}{2}} X_2, \mathcal{A}^{\frac{1}{2}} X_1\right\rangle_{\mathcal{H}} - \left\langle \mathcal{A}^{\frac{1}{2}} X_1 + \mathcal{A}^{\frac{1}{2}} SX_2, \mathcal{A}^{\frac{1}{2}} X_2\right\rangle_{\mathcal{H}} \\ &= -\left\langle \mathcal{A}^{\frac{1}{2}} SX_2, \mathcal{A}^{\frac{1}{2}} X_2\right\rangle_{\mathcal{H}}.\end{aligned}$$

Now use the fact that $\mathrm{dom}(\mathcal{A})$ is a core for $\mathcal{A}^{\frac{1}{2}}$ (see [8, Lemma 3.38]), so that given $SX_2 \in \mathrm{dom}(\mathcal{A}^{\frac{1}{2}})$ there exists a sequence $\{W^n\}$ with $W^n = (z^n, b^n)^T \in \mathrm{dom}(\mathcal{A})$ such that $W^n \to SX_2$ and $\mathcal{A}^{\frac{1}{2}} W^n \to \mathcal{A}^{\frac{1}{2}} SX_2$.

Write $X_2 = (z_2, b_2)^T$ and use the definition of S and of $\mathcal{A}$ to find that

$$SX_2 = \begin{pmatrix} S_1 z_2 \\ E^{-1} G b_2\end{pmatrix}^T \quad \text{and} \quad \mathcal{A}^{\frac{1}{2}} = \begin{pmatrix} \mathcal{A}_1^{\frac{1}{2}} & 0 \\ -C^T M^{-1} C E P_2^B \mathcal{A}_1^{-\frac{1}{2}} & 0\end{pmatrix}.$$

The limit sequences above can be written $(z^n, b^n)^T \to (S_1 z_2, E^{-1} G b_2)^T$ and $(\mathcal{A}_1^{\frac{1}{2}} z^n, -C^T M^{-1} C E P_2^B \mathcal{A}_1^{-\frac{1}{2}} z^n) \to (\mathcal{A}_1^{\frac{1}{2}} S_1 z_2, -C^T M^{-1} C E P_2^B \mathcal{A}_1^{-\frac{1}{2}} S_1 z_2)$. From these it follows that $z^n \to S_1 z_2$ and $\mathcal{A}_1^{\frac{1}{2}} z^n \to \mathcal{A}_1^{\frac{1}{2}} S_1 z_2$ (in $\mathcal{H}_z$) and $b^n \to E^{-1} G b_2$ (in $\mathcal{H}_b$).

Since S_1 is diagonal and positive definite, so is its inverse. Both $\mathcal{A}_1$ and $\mathcal{A}_1^{\frac{1}{2}}$ are diagonal operators so these commute with S_1^{-1}. Applying these observations it can be shown that $\mathcal{A}^{\frac{1}{2}}(S_1^{-1} z^n, P_1^B S_1^{-1} z^n)^T \to \mathcal{A}^{\frac{1}{2}} X_2$.

Finally, then

$$\begin{aligned}\left\langle \mathcal{A}^{\frac{1}{2}} SX_2, \mathcal{A}^{\frac{1}{2}} X_2\right\rangle_{\mathcal{H}} &= \lim_{n\to\infty} \left\langle \mathcal{A}^{\frac{1}{2}} W^n, \mathcal{A}^{\frac{1}{2}} \left(S_1^{-1} z^n, P_1^B S_1^{-1} z^n\right)^T\right\rangle_{\mathcal{H}} \\ &= \lim_{n\to\infty} \left\langle W^n, \mathcal{A}\left(S_1^{-1} z^n, P_1^B S_1^{-1} z^n\right)^T\right\rangle_{\mathcal{H}} \\ &= \lim_{n\to\infty} \left[\left\langle z^n, \mathcal{A}_1 S_1^{-1} z^n\right\rangle_{\mathcal{H}_z} - \left\langle b^n, C^T M^{-1} C P_2^B S_1^{-1} z^n\right\rangle_{\mathcal{H}_b}\right] \\ &= \sum_{i=1}^{2} \left[\mu_i \left\|u_s^i\right\|^2 + \gamma_i \left\|w_{ss}^i\right\|^2\right] \geqslant 0.\end{aligned}$$

The last step follows from the definitions of the operators and direct calculations as in the proof of Theorem 2. Thus, $\mathcal{A}_B$ is dissipative.

It can be easily checked that the adjoint of the operator $\mathcal{A}_B$ is given by

$$\mathrm{dom}\left(\mathcal{A}_B{}^*\right) = \left\{(X_1, X_2)^T \in \mathcal{Y} \,\middle|\, X_2 \in \mathrm{dom}\left(\mathcal{A}^{\frac{1}{2}}\right),\ X_1 - SX_2 \in \mathrm{dom}(\mathcal{A})\right\}, \quad \mathcal{A}_B{}^* \begin{pmatrix} X_1 \\ X_2\end{pmatrix} = \begin{pmatrix} -X_2 \\ \mathcal{A}(X_1 - SX_2)\end{pmatrix}.$$

If $Y = (X_1, X_2)^T \in \mathrm{dom}(\mathcal{A}_B{}^*)$, then we have

$$\begin{aligned}\langle \mathcal{A}_B{}^*Y, Y\rangle_{\mathcal{Y}} &= \left\langle \begin{pmatrix} -X_2 \\ \mathcal{A}(X_1 - SX_2)\end{pmatrix}, \begin{pmatrix} X_1 \\ X_2\end{pmatrix}\right\rangle_{\mathcal{Y}} \\ &= -\langle \mathcal{A}^{\frac12}X_2, \mathcal{A}^{\frac12}X_1\rangle_{\mathcal{H}} + \langle \mathcal{A}^{\frac12}X_1 + \mathcal{A}^{\frac12}SX_2, \mathcal{A}^{\frac12}X_2\rangle_{\mathcal{H}} \\ &= -\langle \mathcal{A}^{\frac12}SX_2, \mathcal{A}^{\frac12}X_2\rangle_{\mathcal{H}} \leqslant 0.\end{aligned}$$

Therefore, $\mathcal{A}_B{}^*$ is also dissipative. The claim now follows from the Lumer–Phillips Theorem (see, for instance, [11, Corollary 4.4]). □

Theorem 5 *(Analyticity and exponential stability). Let $\mathcal{A}_B : \mathcal{Y} \to \mathcal{Y}$ be as defined in* (32). *If γ_1, γ_2, μ_1, μ_2 are all positive, then the strongly continuous semigroup $\mathcal{S}(t)$ generated by $\mathcal{A}_B$ is analytic and exponentially stable.*

Proof. Assume that $\gamma_1 > 0$, $\gamma_2 > 0$, $\mu_1 > 0$, $\mu_2 > 0$. The analyticity of $S(t)$ follows from Theorem 2.1 in [1] or Theorem 4.1 in [10] with $\alpha = 1$. Moreover, it is easy to check that the intersection of the spectrum of $\mathcal{A}_B$ with the imaginary axis is empty. In fact, suppose that there exist $i\beta \in \sigma(\mathcal{A}_B)$, for some $\beta \in \mathbb{R}$. Then, there exists a sequence $Y_n = \binom{X_{1,n}}{X_{2,n}} \in \operatorname{dom}(\mathcal{A}_B)$ with $\|Y_n\|_{\mathcal{Y}} = 1$ such that

$$\lim_{n\to\infty} \left\|(i\beta I - \mathcal{A}_B)Y_n\right\|_{\mathcal{Y}} = 0, \tag{34}$$

i.e.

$$i\beta X_{1,n} - X_{2,n} \to 0 \quad \text{in } \operatorname{dom}\big(\mathcal{A}^{\frac12}\big), \tag{35}$$

$$i\beta X_{2,n} - \mathcal{A}(X_{1,n} + SX_{2,n}) \to 0 \quad \text{in } \mathcal{H}. \tag{36}$$

Equation (34) implies that

$$\operatorname{Re}\langle \mathcal{A}_B Y_n, Y_n\rangle = -\left\|\mathcal{A}^{\frac12}S^{\frac12}X_{2,n}\right\|^2_{\mathcal{H}} \to 0. \tag{37}$$

If $\beta = 0$, then (36) yields $\|\mathcal{A}^{\frac12}X_{1,n}\|_{\mathcal{H}} \to 0$. If $\beta \neq 0$, then (35) and (37) imply $\|\mathcal{A}^{\frac12}X_{1,n}\|_{\mathcal{H}} \to 0$. In any case, this contradicts the fact that $\|Y_n\|_{\mathcal{Y}} = 1$. Thus, the strongly continuous semigroup of contractions $\mathcal{S}(t)$ generated by $\mathcal{A}_B$ on $\mathcal{Y}$ is exponentially stable (see [3]), i.e. $\|\mathcal{S}(t)\|_{\mathcal{L}(Y)} \leqslant e^{-\delta t}$, where $\delta = -\sup \operatorname{Re}\sigma(\mathcal{A}_B) > 0$. □

Remark 6. In the case where $d, k > 0$, changes are required in the $\mathcal{H}_b$ (finite-dimensional) parts of the operator $\mathcal{A}_B$. Since this is a bounded perturbation, the so-modified $\mathcal{A}_B$ is the infinitesimal generator of an analytic semigroup [11, Corollary 2.2, p. 81].

5. Characterization of the spectrum of $\mathcal{A}_B$

Suppose that λ is an eigenvalue of $\mathcal{A}_B$ corresponding to the eigenvector $Y = (X_1, X_2)^T \in \operatorname{dom}(\mathcal{A}_B)$, where $X_1 = (z, b)^T = (w^1, w^2, u^1, u^2, b)^T$. Then $\mathcal{A}_B Y = \lambda Y$, which implies

$$X_2 = \lambda X_1, \qquad -\mathcal{A}(SX_2 + X_1) = \lambda X_2,$$

and therefore

$$\mathcal{A}(I + \lambda S)X_1 = -\lambda^2 X_1. \tag{38}$$

Use the definitions of $\mathcal{A}$ and S, write $X_1 = (z, b)^T$ and Eq. (38) takes the form

$$\begin{pmatrix} \mathcal{A}_1 & 0 \\ -C^T M^{-1} C E P_2^B & 0\end{pmatrix}\begin{pmatrix} I + \lambda S_1 & 0 \\ 0 & I + \lambda E^{-1}G\end{pmatrix}\begin{pmatrix} z \\ b\end{pmatrix} = -\lambda^2\begin{pmatrix} z \\ b\end{pmatrix}.$$

From this equation, and the fact that $Y \in \operatorname{dom}(\mathcal{A}_B)$, we obtain

$$\begin{aligned}
&\left(1+\frac{\lambda\gamma_1}{E_1I_1}\right)\frac{E_1I_1}{\rho_1A_1}w^1_{ssss}=-\lambda^2w^1, \quad w^1(0)=w^1_s(0)=0,\\
&\left(1+\frac{\lambda\gamma_2}{E_2I_2}\right)\frac{E_2I_2}{\rho_2A_2}w^2_{ssss}=-\lambda^2w^2, \quad w^2(0)=w^2_s(0)=0,\\
&\left(1+\frac{\lambda\mu_1}{E_1A_1}\right)\frac{E_1}{\rho_1}u^1_{ss}=\lambda^2u^1, \quad u^1(0)=0,\\
&\left(1+\frac{\lambda\mu_2}{E_2A_2}\right)\frac{E_2}{\rho_2}u^2_{ss}=\lambda^2u^2, \quad u^2(0)=0,
\end{aligned} \tag{39}$$

and

$$C^TM^{-1}CEP^B_2(I+\lambda S_1)z=\lambda^2b, \qquad P^B_1z=b\in\left[\ker(C)\right]^{\perp}. \tag{40}$$

Now let β_i, $i=1,2,3,4$, be the columns of C^T (Eq. (20)). Since $[\ker(C)]^{\perp}=\operatorname{span}\{\beta_i,\ i=1,2,3,4\}$, it follows that b must be a linear combination of the β_i's, i.e., there exist four constants a_1, a_2, a_3, a_4 such that

$$b=a_1\beta_1+a_2\beta_2+a_3\beta_3+a_4\beta_4.$$

Since $b=P^B_1z$, the above equation takes the form

$$\begin{pmatrix}-w^1_s(L_1)\\ w^1(L_1)\\ -w^2_s(L_2)\\ w^2(L_2)\\ -u^1(L_1)\\ -u^2(L_2)\end{pmatrix}=\begin{pmatrix}a_3\\ -a_1\cos\varphi_1+a_2\sin\varphi_1+a_3l_1\\ a_4\\ a_1\cos\varphi_2+a_2\sin\varphi_2+a_4l_2\\ a_1\sin\varphi_1+a_2\cos\varphi_1\\ a_1\sin\varphi_2-a_2\cos\varphi_2\end{pmatrix}.$$

Substitution of these boundary conditions back into (39) yields the following four two-point boundary value problems for w^1, w^2, u^1 and u^2:

$$\begin{aligned}
&w^1_{ssss}(s)=c_1(\lambda)w^1(s), \quad w^1(0)=w^1_s(0)=0, \quad w^1_s(L_1)=-a_3, \quad w^1(L_1)=-a_1\cos\varphi_1+a_2\sin\varphi_1+a_3l_1,\\
&w^2_{ssss}(s)=c_2(\lambda)w^2(s), \quad w^2(0)=w^2_s(0)=0, \quad w^2_s(L_2)=-a_4, \quad w^2(L_2)=a_1\cos\varphi_2+a_2\sin\varphi_2+a_4l_2,\\
&u^1_{ss}(s)=d_1(\lambda)u^1(s), \quad u^1(0)=0, \quad u_1(L_1)=-a_1\sin\varphi_1-a_2\cos\varphi_1,\\
&u^2_{ss}(s)=d_2(\lambda)u^2(s), \quad u^2(0)=0, \quad u^2(L_2)=-a_1\sin\varphi_2+a_2\cos\varphi_2,
\end{aligned}$$

where

$$c_i(\lambda)\doteq-\frac{\lambda^2\rho_iA_i}{E_iI_i+\lambda\gamma_i}, \quad d_i(\lambda)\doteq\frac{\lambda^2\rho_iA_i}{E_iA_i+\lambda\mu_i}, \quad i=1,2.$$

The general solution of $w^1_{ssss}(s)=c_1(\lambda)w^1(s)$ is

$$w^1(s)=k_1\sin\alpha_1s+k_2\cos\alpha_1s+q\sinh\alpha_1s+r\cosh\alpha_1s,$$

where k_1, k_2, q and r are arbitrary constants and $\alpha_1=\alpha_1(\lambda)$ is such that $(\alpha_1)^4=c_1(\lambda)$. Substitution of the boundary conditions at $s=0$ implies that $q=-k_1$ and $r=-k_2$. Thus

$$w^1(s)=k_1(\sin\alpha_1s-\sinh\alpha_1s)+k_2(\cos\alpha_1s-\cosh\alpha_1s). \tag{41}$$

Applying the two boundary conditions at $s=L_1$ we obtain the following system of equations:

$$\begin{cases}k_1(\sin\alpha_1L_1-\sinh\alpha_1L_1)+k_2(\cos\alpha_1L_1-\cosh\alpha_1L_1)=-a_1\cos\varphi_1+a_2\sin\varphi_1+a_3l_1,\\ \alpha_1\left[k_1(\cos\alpha_1L_1-\cosh\alpha_1L_1)-k_2(\sin\alpha_1L_1+\sinh\alpha_1L_1)\right]=-a_3.\end{cases} \tag{42}$$

A similar analysis for $w^2(s)$ gives

$$w^2(s)=k_3(\sin\alpha_2s-\sinh\alpha_2s)+k_4(\cos\alpha_2s-\cosh\alpha_2s), \tag{43}$$

where $\alpha_2 = \alpha_2(\lambda)$ satisfies $(\alpha_2)^4 = c_2(\lambda)$ and k_3, k_4 are constants satisfying the system of equations

$$\begin{cases} k_3(\sin\alpha_2 L_2 - \sinh\alpha_2 L_2) + k_4(\cos\alpha_2 L_2 - \cosh\alpha_2 L_2) = a_1\cos\varphi_2 + a_2\sin\varphi_2 + a_4 l_2, \\ \alpha_2\big[k_3(\cos\alpha_2 L_2 - \cosh\alpha_2 L_2) - k_4(\sin\alpha_2 L_2 + \sinh\alpha_2 L_2)\big] = -a_4. \end{cases} \tag{44}$$

On the other hand, solving the two-point boundary-value problems for $u^1(s)$ and $u^2(s)$ gives

$$u^1(s) = r_1 \sin\delta_1 s, \tag{45}$$

where $\delta_1 = \delta_1(\lambda)$ satisfies $(\delta_1)^2 = -d_1(\lambda)$ and

$$r_1 \sin\delta_1 L_1 = -a_1\sin\varphi_1 - a_2\cos\varphi_1. \tag{46}$$

Also,

$$u^2(s) = r_2 \sin\delta_2 s, \tag{47}$$

where $\delta_2 = \delta_2(\lambda)$ satisfies $(\delta_2)^2 = -d_2(\lambda)$ and

$$r_2 \sin\delta_2 L_2 = -a_1\sin\varphi_2 + a_2\cos\varphi_2. \tag{48}$$

Note that Eqs. (42), (44), (46), (48), constitute a system of 6 linear equations in the 10 unknowns a_1, a_2, a_3, a_4, k_1, k_2, k_3, k_4, r_1, r_2. This system can be written in the form

$$B_1 q = C^T a, \tag{49}$$

where $a \doteq (a_1, a_2, a_3, a_4)^T$, $q \doteq (k_1, k_2, k_3, k_4, r_1, r_2)^T$, and B_1 is the following matrix:

$$B_1 \doteq \begin{pmatrix} \alpha_1(\cosh\alpha_1 L_1 - \cos\alpha_1 L_1) & \alpha_1(\sin\alpha_1 L_1 + \sinh\alpha_1 L_1) & 0 & 0 & 0 & 0 \\ \sin\alpha_1 L_1 - \sinh\alpha_1 L_1 & \cos\alpha_1 L_1 - \cosh\alpha_1 L_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2(\cosh\alpha_2 L_2 - \cos\alpha_2 L_2) & \alpha_2(\sin\alpha_2 L_2 + \sinh\alpha_2 L_2) & 0 & 0 \\ 0 & 0 & \sin\alpha_2 L_2 - \sinh\alpha_2 L_2 & \cos\alpha_2 L_2 - \cosh\alpha_2 L_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sin\delta_1 L_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sin\delta_2 L_2 \end{pmatrix}. \tag{50}$$

Four more equations are obtained from (40). In fact, note that the first equation in (40) can be written in the form

$$C^T M^{-1} C E P_2^B (I + \lambda S_1) z = \lambda^2 C^T \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}.$$

Multiply through by $M(CC^T)^{-1}C$ and use the definitions of P_2^B, E, and S_1 to obtain

$$C \begin{pmatrix} (E_1 I_1 + \lambda\gamma_1) w_{ss}^1(L_1) \\ (E_1 I_1 + \lambda\gamma_1) w_{sss}^1(L_1) \\ (E_2 I_2 + \lambda\gamma_2) w_{ss}^2(L_2) \\ (E_2 I_2 + \lambda\gamma_2) w_{sss}^2(L_2) \\ (E_1 A_1 + \lambda\mu_1) u_s^1(L_1) \\ (E_2 A_2 + \lambda\mu_2) u_s^2(L_2) \end{pmatrix} = \lambda^2 M \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}. \tag{51}$$

Use Eqs. (41), (43), (45), (47) to find that

$$\begin{aligned} w_{ss}^1(L_1) &= -\alpha_1^2\big[k_1(\sin\alpha_1 L_1 + \sinh\alpha_1 L_1) + k_2(\cos\alpha_1 L_1 + \cosh\alpha_1 L_1)\big], \\ w_{sss}^1(L_1) &= \alpha_1^3\big[-k_1(\cos\alpha_1 L_1 + \cosh\alpha_1 L_1) + k_2(\sin\alpha_1 L_1 - \sinh\alpha_1 L_1)\big], \\ w_{ss}^2(L_2) &= -\alpha_2^2\big[k_3(\sin\alpha_2 L_2 + \sinh\alpha_2 L_2) + k_3(\cos\alpha_2 L_2 + \cosh\alpha_2 L_2)\big], \\ w_{sss}^2(L_2) &= \alpha_2^3\big[-k_3(\cos\alpha_2 L_2 + \cosh\alpha_2 L_2) + k_3(\sin\alpha_2 L_2 - \sinh\alpha_2 L_2)\big], \\ u_s^1(L_1) &= \delta_1 r_1 \cos\delta_1 L_1, \\ u_s^2(L_2) &= \delta_2 r_2 \cos\delta_2 L_2. \end{aligned}$$

Substitution of these expressions into (51) yields

$$CRq = \lambda^2 Ma, \tag{52}$$

where

$$R \doteq \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix},$$

with

$$R_1 \doteq \begin{pmatrix} (E_1A_1 + \lambda\mu_1)\delta_1 \cos\delta_1 L_1 & 0 \\ 0 & (E_2A_2 + \lambda\mu_2)\delta_2 \cos\delta_2 L_2 \end{pmatrix}$$

and

$$R_2 \doteq \begin{pmatrix} -(E_1I_1+\lambda\gamma_1)\alpha_1^2(\sin\alpha_1L_1+\sinh\alpha_1L_1) & -(E_1I_1+\lambda\gamma_1)\alpha_1^2(\cos\alpha_1L_1+\cosh\alpha_1L_1) & 0 & 0 \\ -(E_1I_1+\lambda\gamma_1)\alpha_1^3(\cos\alpha_1L_1+\cosh\alpha_1L_1) & (E_1I_1+\lambda\gamma_1)\alpha_1^3(\sin\alpha_1L_1-\sinh\alpha_1L_1) & 0 & 0 \\ 0 & 0 & -(E_2I_2+\lambda\gamma_2)\alpha_2^2(\sin\alpha_2L_2+\sinh\alpha_2L_2) & -(E_2I_2+\lambda\gamma_2)\alpha_2^2(\cos\alpha_2L_2+\cosh\alpha_2L_2) \\ 0 & 0 & -(E_2I_2+\lambda\gamma_2)\alpha_2^3(\cos\alpha_2L_2+\cosh\alpha_2L_2) & (E_2I_2+\lambda\gamma_2)\alpha_2^3(\sin\alpha_2L_2-\sinh\alpha_2L_2) \end{pmatrix}.$$

Equation (52) implies that $a = \lambda^{-2}M^{-1}CRq$ and substitution of this expression into Eq. (49) produces $B_1q = \lambda^{-2}C^TM^{-1}CRq$, or equivalently, $(\lambda^2 I - B_1^{-1}C^TM^{-1}CR)q = 0$.

Let $U \doteq B_1^{-1}C^TM^{-1}CR$ and note that the matrices M and C depend only on the model parameters, while B_1 and R also depend on α_1, α_2, δ_1, δ_2, and λ. We denote this dependency by $U = U(\lambda, \alpha_1, \alpha_2, \delta_1, \delta_2)$. Since λ is an eigenvalue of $\mathcal{A}_B$, it follows that

$$\det\bigl(\lambda^2 I - U(\lambda, \alpha_1, \alpha_2, \delta_1, \delta_2)\bigr) = 0.$$

Now, $\alpha_1, \alpha_2, \delta_1, \delta_2$ satisfy $\alpha_1(\lambda)^4 = c_1(\lambda)$, $\alpha_2(\lambda)^4 = c_2(\lambda)$, $\delta_1(\lambda)^2 = -d_1(\lambda)$, $\delta_2(\lambda)^2 = -d_2(\lambda)$. Thus, the eigenvalues of $\mathcal{A}_B$ are the solutions of the equation

$$\det\Bigl(\lambda^2 I - U\bigl(\lambda, c_1(\lambda)^{\frac{1}{4}}, c_2(\lambda)^{\frac{1}{4}}, \bigl[-d_1(\lambda)\bigr]^{\frac{1}{2}}, \bigl[-d_2(\lambda)\bigr]^{\frac{1}{2}}\bigr)\Bigr) = 0.$$

Here, "$f^{\frac{1}{n}}$" must be understood to be any of the n branches of the corresponding complex-valued function.

Remark 7. In the case of no damping one has $\mu_i = \gamma_i = 0$, so that the operators S_1 and S are identically zero. In this case, if λ is an eigenvalue of $\mathcal{A}_B$ corresponding to the eigenvector $(X_1, X_2)^T$, then by (38), $-(\lambda)^2$ is an eigenvalue of $\mathcal{A}$ corresponding to the eigenvector X_1. Since by Theorem 2 the operator $\mathcal{A}$ is strictly positive definite and self adjoint, one must have $-(\lambda)^2 \in \mathbb{R} > 0$. Thus, λ must be purely imaginary, so that $\alpha_1^4 = -\lambda^2 \frac{\rho_1 A_1}{E_1 I_1} > 0$ and therefore $\tilde{\alpha}_1 \triangleq +\sqrt[4]{(-\lambda^2 \frac{\rho_1 A_1}{E_1 I_1})} > 0$. Other values of α_1 are obtained by multiplying $\tilde{\alpha}_1$ by the other fourth-roots of unity, but these introduce no new solutions to (41). Therefore, there are no additional mode shapes. Similarly, $\alpha_2 = (-\lambda^2 \frac{\rho_2 A_2}{E_2 I_2})^{\frac{1}{4}} > 0$, $\delta_1 = (-\lambda^2 \frac{\rho_1}{E_1})^{\frac{1}{2}} > 0$ and $\delta_2 = (-\lambda^2 \frac{\rho_2}{E_2})^{\frac{1}{2}} > 0$. Hence, B_1 and R, being matrix functions of α_1, α_2, γ_1, γ_2 can now be written as functions of λ^2 only. Thus, in this case the eigenvalues are the solutions of the equation $\det(\lambda^2 B_1(\lambda^2)[R(\lambda^2)]^{-1} - C^TM^{-1}C) = 0$. If we denote by $W(\sigma)$ the 6×6 matrix $W(\sigma) \doteq \sigma B_1(\sigma)[R(\sigma)]^{-1}$, then the eigenvalues are the solutions of the equation $\det(W(\lambda^2) - C^TM^{-1}C) = 0$.

6. Conclusions and future directions

In this paper we developed a state space model for the dynamics of two beams with Kelvin–Voigt damping, coupled to a joint through two legs. The model was based on the four PDEs describing the transverse and longitudinal motions of both beams, with homogeneous boundary conditions at one end and dynamic boundary conditions at the other end, reflecting the dynamic and geometric compatibility conditions at the beam-leg interfaces. These dynamic boundary conditions translate into an ODE for certain beam boundary projection terms. The complete system was written a second-order, linear IVP in an appropriate Hilbert space $\mathcal{H}$. After recasting as a first-order system, semigroup theory was then used to prove well-posedness and exponential energy decay. A characterization of the spectrum of the associated infinitesimal generator was also given.

As noted in the introduction, a primary motivation for this effort is to improve understanding of damping and to investigate the effects of specific joint models typical of these inflatable/rigidizable space structures. We are using the abstract framework developed in this paper to construct rigorous numerical schemes for identification, design and control of such systems. Note that the results in this paper allow us to employ both frequency domain and time domain techniques. In particular, in the frequency domain setting one may use the spectral characterization from Section 5, while in a time domain setting one requires numerical approximation for the abstract Cauchy problem (33) [2]. In either case, it is of interest to generalize the mathematical model for the internal joint-moment (9).

References

[1] J.A. Burns, B.B. King, A note on the mathematical modelling of damped second order systems, J. Math. Systems Estimation Control 8 (2) (1998) 189–192.

[2] J.A. Burns, E.M. Cliff, Z. Liu, R.D. Spies, Results on transversal and axial motions of a system of two beams coupled to a joint through two legs, ICAM Report 20070213-1; also J. Comput. Appl. Math., submitted for publication.

[3] G. Chen, D.L. Russell, A mathematical model for linear elastic systems with structural damping, Quart. Appl. Math. (January 1982) 433–454.

[4] H.W. Engl, M. Hanke, A. Neubauer, Regularization of Inverse Problems, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1996.

[5] G.M. Ewing, Calculus of Variations with Applications, first ed., W.W. Norton & Company Inc., New York, 1969.

[6] K. Guidanean, D. Lichodziejewski, An inflatable rigidizable truss structure based on new Sub-Tg polyurethane composites, AIAA Paper 02-1593, AIAA Publications, 2002.

[7] C.H.M. Jenkins (Ed.), Gossamer Spacecraft: Membrane and Inflatable Structures Technology for Space Applications, Progr. Aeronautics Astronautics Ser., vol. 191, AIAA Publications, 2001.

[8] T. Kato, Perturbation Theory for Linear Operators, second ed., Springer-Verlag, New York, 1980.

[9] J.E. Lagnese, G. Leugering, E.J.P.G. Schmidt, Modeling, Analysis and Control of Dynamic Elastic Multi-Link Structures, Birkhäuser, Boston, 1994.

[10] K. Liu, Z. Liu, Analyticity and differentiability of semigroups associated with elastic systems with damping and gyroscopic forces, J. Differential Equations 141 (2) (1997) 340–355.

[11] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, second ed., Springer-Verlag, New York, 1983.

[12] J. Weidmann, Linear Operators in Hilbert Spaces, Springer-Verlag, New York, 1980.