

THE RIESZ POTENTIAL AS A MULTILINEAR OPERATOR INTO GENERAL BMO_β SPACES

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Given $\alpha > 0$ and a space of homogeneous type X , n -normal, with $n \in \mathbb{R}^+$, we consider an extension of the standard multilinear fractional integral on $L^{p_1} \times \cdots \times L^{p_k}$ for the range of $1/p_i = 1/p_1 + \cdots + 1/p_k - \alpha/n \leq 0$. We show that the target space is an adequate space BMO_β defined through mean oscillations. For general spaces of homogeneous type this is a Banach space of classes of functions modulii constants and the range of β is $[0, 1)$. However, if $X = \mathbb{R}^n$ ($n \in \mathbb{N}$), we can extend the result to $\beta > 0$ taking in account that BMO_β is a space of classes modulii polynomials of order lower than or equal to $[\beta]$. Bibliography: 15 titles.

1 Introduction

Let $X := (X, \delta, \mu)$ be an n -normal space of homogeneous type of order θ (cf. Section 2). For $0 < \alpha < kn$ we consider a multilinear fractional integral defined as the restriction to the diagonal of X^k of the usual Riesz integral operator applied to the tensor product $\vec{f}(\vec{y}) = f_1(y_1) \dots f_k(y_k)$ with $(f_1, \dots, f_k) \in L^{p_1} \times \cdots \times L^{p_k}$, $1 \leq p_i \leq \infty$ for $1 \leq i \leq k$, i.e.,

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$$I_{\alpha,k} \vec{f}(x) = \int_{X^k} \frac{f_1(y_1) \dots f_k(y_k)}{\left(\sum_{1 \leq i \leq k} \delta(x, y_i) \right)^{(kn-\alpha)}} d\vec{\mu}(\vec{y}), \quad x \in X, \quad (1.1)$$

where $\vec{\mu}$ is the product measure on X^k .

We write $1/p = 1/p_1 + \dots + 1/p_k$ for the harmonic mean of $p_i \geq 1$, $1 \leq i \leq k$. In the case $X = \mathbb{R}^n$, it is well known that if $1/p > \alpha/n$ and $p_i > 1$ for all i , then $I_{\alpha,k}$ maps $L^{p_1} \times \dots \times L^{p_k}$ into L^q and, if $p_i = 1$ for some i , the image is the Lorentz space $L^{q,\infty}$, where $1/q = 1/p - \alpha/n$ (cf. [1, 2]). Weighted inequalities with this operator were studied by many authors (cf., for example, [3]–[6]).

In the opposite situation $1/p \leq \alpha/n$, it is reasonable to expect that the spaces defined through mean oscillations appear in the image of the multilinear operator, as is the linear case. In this contexts, results close to those of this paper were obtained in different settings (cf., for example, [7]–[10] and also [11]).

In the multilinear setting, the situation differs from the linear case, for example, this concerns the continuity of the operator for some $p < 1$ with sufficiently small α . On the other hand, since the quasimetric δ on a general space of homogeneous type X is assumed to be Lipschitz of order θ with $0 < \theta \leq 1$ (cf. (2.1)) and the convergence of the mean oscillations of the fractional integral of Lebesgue functions far off the center of a ball is directly related to that regularity, we see that Theorem 2.1 yields the adequate target space provided that $0 \leq \alpha - n/p < \theta$.

More exactly, for $0 < \beta \leq 1$ the space $BMO_\beta(X)$ is the space of locally integrable functions f defined on X such that for some nonnegative constant C and any ball $B = B(x_0, R) \subset X$ there is a constant a_B such that

$$\frac{1}{\mu(B)} \int_B |f(x) - a_B| d\mu(x) \leq CR^\beta.$$

The seminorm $\|f\|_\beta$ given by the infimum of the above constants C defines a norm on the quotient space of classes modulo constants providing it with the Banach structure. These spaces were first studied by John and Nirenberg [12], Spanne [13], and Campanato [14]. The space $BMO_0 = BMO$ is the well known space of bounded mean oscillation.

Theorem 2.1 below establishes that, in the case of a general n -normal space of homogeneous type, the adequate extension of the multilinear fractional integral operator defined in (1.1) maps the product of Lebesgue spaces $\{L^{p_i}(X)\}_{1 \leq i \leq k}$ into $BMO_\beta(X)$ if $0 \leq \beta = \alpha - n/p < \theta$.

If $X = \mathbb{R}^n$, $n \in \mathbb{N}$, and δ is the Euclidean metric, the interval admitted by Theorem 2.1 is $0 \leq \beta = \alpha - n/p < 1$. However, the technique used for proving the theorem on spaces of homogeneous type allows us to cover the case $\beta = \alpha - n/p \geq 1$ in the Euclidean setting and find the required space of the multilinear fractional integral for these values. These spaces are Campanato spaces of classes modulo polynomials, i.e., $BMO_\beta(\mathbb{R}^n) = \mathcal{L}_{[\beta]}^{1,n+\beta}(\mathbb{R}^n)$ (cf. [14]).

More precisely, for any $\beta \geq 0$ the space $BMO_\beta(\mathbb{R}^n)$ consists of locally integrable functions f on \mathbb{R}^n such that for some nonnegative constant C and any ball $B = B(x_0, R)$ there is a polynomial $P_B(f)$, of degree at most the integer part $[\beta]$ of β and such that

$$\frac{1}{\mu(B)} \int_B |f(x) - P_B(f)(x)| d\mu(x) \leq CR^\beta.$$

It is a Banach space of classes modulo polynomial of order $[\beta]$, where the norm $\|f\|_\beta$ is defined as the infimum of the above constants.

The main result of the paper (Theorem 2.2) generalizes Theorem 2.1 in the context of the underlying structure of \mathbb{R}^n by setting the action of the multilinear fractional integral operator on product of Lebesgue spaces $\{L^{p_i}\}_{1 \leq i \leq k}$ with $1/p_1 + \cdots + 1/p_k = 1/p$ on the entire range of p and α such that $0 \leq 1/p \leq \alpha/n < k$.

2 Preliminaries and the Main Results

We denote by (X, δ, μ) a space of homogeneous type, where δ is a quasimetric on X , i.e., a nonnegative function such that $\delta(x, y) = 0$ if and only if $x = y$, $\delta(x, y) = \delta(y, x)$, and $\delta(x, y) \leq \kappa(\delta(x, z) + \delta(z, y))$ for some $\kappa > 0$.

We assume that δ is of order θ , i.e., there exists a finite constant C and a number $0 < \theta \leq 1$ such that

$$|\delta(x, y) - \delta(x', y)| \leq C\delta(x, x')^\theta(\delta(x, y) + \delta(x', y))^{1-\theta}. \quad (2.1)$$

Note that the δ -balls $B(x, r) = \{y \in X : \delta(x, y) < r\}$ are open sets in the topology induced by δ on X (cf. [15]). Denote by μ a Borel measure satisfying the doubling condition

$$0 < \mu(2B) \leq A\mu(B) < \infty$$

if $0 < r(B) < \mu(X)$. We also assume that X is an n -normal space for some $n \in \mathbb{R}^+$, i.e.,

$$cr(B)^n \leq \mu(B) \leq Cr(B)^n,$$

where c and C are constants and B is an ball.

On the space X^k , $k \geq 2$, we define a quasimetric by the formula $\delta(\vec{x}, \vec{y}) = \delta(x_1, y_1) + \cdots + \delta(x_k, y_k)$, where $\vec{x} = (x_1, \dots, x_k)$ and $\vec{y} = (y_1, \dots, y_k)$. We write $\delta(x, \vec{y}) = \delta(x, y_1) + \cdots + \delta(x, y_k)$ if $\vec{x} = (x, \dots, x)$. We denote by B^k the “cube” $\{\vec{y} \in X^k : y_i \in B, 1 \leq i \leq k\}$.

In order to guarantee the convergence of the integrals when $0 \leq \alpha - n/p < 1$, we consider the multilinear fractional integral operator

$$\mathcal{I}_{\alpha, k} \vec{f}(x) = \int_{X^k} \vec{f}(\vec{y}) (\delta(x, \vec{y})^{-(kn-\alpha)} - (1 - \chi_{B(z_0, 1)^k}(\vec{y})) \delta(z_0, \vec{y})^{-(kn-\alpha)}) d\vec{\mu}(\vec{y}), \quad (2.2)$$

where $B(z_0, 1)$ is a fixed δ -ball with center $z_0 \in X$ and radius 1. It is a natural extension of the operator in (1.1) which follows from the fact that if (f_1, \dots, f_k) is of compact support included, say, in $B(z_0, R)^k$, $R \geq 1$, then the difference

$$\mathcal{I}_{\alpha, k} \vec{f}(x) - I_{\alpha, k} \vec{f}(x) = - \int_{1 \leq \max_{1 \leq i \leq k} \delta(z_0, y_i) \leq R} \vec{f}(\vec{y}) \delta(z_0, \vec{y})^{-(kn-\alpha)} d\vec{\mu}(\vec{y})$$

is a constant. Hence if either $\mathcal{I}_{\alpha, k} \vec{f}(x)$ or $I_{\alpha, k} \vec{f}(x)$ belongs to BMO_β , $0 \leq \beta < 1$, then the same is true for the other.

In the case of spaces of homogeneous type, we prove the following assertion.

Theorem 2.1. Let (X, δ, μ) be an n -normal space of homogeneous type of order θ , where $0 < \theta \leq 1$ and $n \in \mathbb{R}^+$. Assume that $1 \leq p_i \leq \infty$, $1 \leq i \leq k$, $k \geq 2$, $1/p = 1/p_1 + \cdots + 1/p_k$, and $0 < \alpha < kn$. If $0 \leq \alpha - n/p < \theta$, then the multilinear fractional integral $\mathcal{J}_{\alpha, k}$ defined in (2.2) is bounded as an operator from $L^{p_1}(X) \times \cdots \times L^{p_k}(X)$ to $BMO_{\alpha-n/p}(X)$, i.e., for some nonnegative constant C , all $(f_1, \dots, f_k) \in L^{p_1}(X) \times \cdots \times L^{p_k}(X)$ and any ball B of radius R in X there is a constant $a_{B, \mathcal{J}_{\alpha, k}} \vec{f}$ such that

$$\frac{1}{\mu(B)} \int_B |\mathcal{J}_{\alpha, k} \vec{f}(x) - a_{B, \mathcal{J}_{\alpha, k}} \vec{f}| d\mu(x) \leq CR^{\alpha - \frac{n}{p}} \prod_{i=1}^k \|f_i\|_{L^{p_i}}. \quad (2.3)$$

To include the situation $\alpha - n/p \geq 1$ in the setting of \mathbb{R}^n , we consider the extension of the multilinear fractional operator given by the formula

$$\mathcal{J}_{\alpha, k}^m \vec{f}(x) = \int_{\mathbb{R}^{nk}} \vec{f}(\vec{y}) (K(x, \vec{y}) - (1 - \chi_{B(0,1)^k}(\vec{y})) (T_0^m K)(x, \vec{y})) d\vec{y}, \quad (2.4)$$

where $K(x, \vec{y}) = \delta(x, \vec{y})^{\alpha - kn}$ and

$$(T_0^m K)(x, \vec{y}) = \sum_{|\gamma| \leq m} \frac{1}{|\gamma|!} (D_x^\gamma K)(x_0, y) \cdot (x - x_0)^\gamma$$

is the Taylor polynomial of $K(x, \vec{y})$ of degree $m \in \mathbb{Z}^+$ with respect to the x -variable centered in x_0 for each fixed \vec{y} . Here, we used the notation: $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{N}_0^n$, $|\gamma| = \gamma_1 + \cdots + \gamma_n$, $D_x^\gamma = \left(\frac{\partial}{\partial x_1} \right)^{\gamma_1} \cdots \left(\frac{\partial}{\partial x_n} \right)^{\gamma_n}$, and $x^\gamma = x_1^{\gamma_1} \cdots x_k^{\gamma_k}$.

As above, one can show that (2.4) is a natural extension of the operator (1.1) since for compactly supported functions (f_1, \dots, f_k) the difference $\mathcal{J}_{\alpha, k}^m \vec{f}(x) - I_{\alpha, k} \vec{f}(x)$ is a polynomial in the x -variable of degree at most m . Note that when $m = 0$ we recover the definition (2.2).

In the context of \mathbb{R}^n , the following theorem sets the action of the multilinear fractional integral on the product of Lebesgue spaces on the entire range of α and p such that $0 \leq \alpha - n/p < kn$. In this setting, it is a generalization of 2.1.

Theorem 2.2. Suppose that $1 \leq p_i \leq \infty$, $1 \leq i \leq k$, $k \geq 2$, $1/p = 1/p_1 + \cdots + 1/p_k$, $0 < \alpha < kn$, and $m \in \mathbb{N} \cup \{0\}$. If $m \leq \alpha - n/p < m + 1$, then the multilinear fractional integral $\mathcal{J}_{\alpha, k}^m$ defined in (2.4) is bounded as an operator from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_k}(\mathbb{R}^n)$ to $BMO_{\alpha/n-1/p}(\mathbb{R}^n)$, i.e., for some constant C all $(f_1, \dots, f_k) \in L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_k}(\mathbb{R}^n)$ and any ball $B = B(x_0, R)$ in \mathbb{R}^n there is a polynomial $P_B(\mathcal{J}_{\alpha, k}^m \vec{f})(x)$ such that

$$\frac{1}{\mu(B)} \int_B |\mathcal{J}_{\alpha, k}^m \vec{f}(x) - P_B(\mathcal{J}_{\alpha, k}^m \vec{f})(x)| d\mu(x) \leq CR^{\alpha - \frac{n}{p}} \prod_{i=1}^k \|f_i\|_{L^{p_i}}. \quad (2.5)$$

3 Auxiliaries

In the sequel, we assume that $1 \leq p_i \leq \infty$ for all $1 \leq i \leq k$ and $1/p = 1/p_1 + \cdots + 1/p_k$. Denote $\hat{x} = (x, \dots, x)$ and $tB = B(x_0, tR)$ for any $t > 0$.

The proof of Theorems 2.1 and 2.1 is based on the following lemmas, where the local behavior of the multilinear fractional integral is analyzed in the case $\alpha/n - 1/p > 0$ (Lemma 3.1) and in the case $\alpha/n = 1/p$ (Lemma 3.2).

Lemma 3.1. *Let (X, δ, μ) be an n -normal space of homogeneous type of order θ , with $0 < \theta \leq 1$ and $n \in \mathbb{R}^+$. Assume that $1 \leq p_i \leq \infty$, $(1 \leq i \leq k, k \geq 2, 1/p = 1/p_1 + \dots + q/p_k)$, and $0 < \alpha < kn$. If $\alpha/n - 1/p > 0$, then*

$$\int_{(2\kappa B)^k} \frac{|\vec{f}(\vec{y})|}{\left(\sum_{1 \leq i \leq k} \delta(x, y_i)\right)^{kn-\alpha}} d\vec{\mu}(\vec{y}) \leq Cr(B)^{\alpha-\frac{n}{p}} \prod_{1 \leq i \leq k} \|f_i\|_{p_i}$$

for some constant C , all $\vec{f} \in L^{p_1}(X) \times \dots \times L^{p_k}(X)$, and all $x \in B$.

Proof. We split $\alpha = \sum_{i=1}^k \alpha_i$, with $0 \leq \alpha_i \leq n$ by setting

$$\alpha_i = \frac{n}{p_i} + \epsilon \frac{n}{p'_i}, \quad \epsilon = \frac{\alpha - \frac{n}{p}}{kn - n/p} \in (0, 1), \quad 1 \leq i \leq k.$$

Note that $p_i > 1$ for at least one i since $n/p < \alpha < kn$. Then the kernel is estimated as follows:

$$\left(\sum_{1 \leq i \leq k} \delta(x, y_i) \right)^{\alpha-kn} \leq \prod_{i=1}^k \delta(x, y_i)^{\alpha_i-n};$$

moreover, since $(\alpha_i - n)p'_i + n = \epsilon n > 0$ if $1 < p_i \leq \infty$ and $\alpha_i - n = 0$ if $p_i = 1$, we have $\delta(x, y_i)^{\alpha_i-n} \in L_{loc}^{p'_i}$ for $1 \leq i \leq k$ and for all $x \in B$

$$\left(\int_{2\kappa B} \delta(x, y_i)^{(\alpha_i-n)p'_i} d\mu(y_i) \right)^{1/p'_i} \leq Cr(B)^{\epsilon \frac{n}{p'_i}}, \quad 1 < p_i \leq \infty, \quad (3.1)$$

$$\sup_{y_i \in 2\kappa B} \delta(x, y_i)^{\alpha_i-n} = 1, \quad p_i = 1. \quad (3.2)$$

Thus, for all $x \in B$, by the Hölder inequality and the inequality (3.1) if $1 < p_i \leq \infty$ or the inequality (3.2) if $p_i = 1$, we have

$$\begin{aligned} \int_{(2\kappa B)^k} \frac{|\vec{f}(\vec{y})|}{\left(\sum_{1 \leq i \leq k} \delta(x, y_i)\right)^{kn-\alpha}} d\vec{\mu}(\vec{y}) &\leq \prod_{i=1}^k \int_{(2\kappa B)} \frac{f_i(y_i)}{\delta(x, y_i)^{n-\alpha_i}} d\mu(y_i) \leq C \prod_{i=1}^k \|f_i\|_{p_i} R^{\epsilon \frac{n}{p'_i}} \\ &\leq CR^{\epsilon(kn-\frac{n}{p})} \prod_{i=1}^k \|f_i\|_{p_i} = CR^{\alpha-\frac{n}{p}} \prod_{i=1}^k \|f_i\|_{p_i}, \end{aligned}$$

where $R = r(B)$ is the radius of B . □

Lemma 3.2. Let (X, δ, μ) be an n -normal space of homogeneous type of order θ , with $0 < \theta \leq 1$ and $n \in \mathbb{R}^+$. Assume that $1 \leq p_i \leq \infty$, $1 \leq i \leq k$, $k \geq 2$, $1/p = 1/p_1 + \dots + 1/p_k$, and $0 < \alpha < kn$. If $1/p = \alpha/n$, then

$$\begin{aligned} \frac{1}{\mu(B)} \int_B |I_{\alpha, k}(\vec{f}\chi_{(2\kappa B)^k})(x)| d\mu(x) &\leq \frac{1}{\mu(B)} \int_B \int_{(2\kappa B)^k} \frac{|\vec{f}(\vec{y})|}{\left(\sum_{1 \leq i \leq k} \delta(x, y_i)\right)^{kn-\alpha}} d\vec{\mu}(\vec{y}) d\mu(x) \\ &\leq C \prod_{1 \leq i \leq k} \|f_i\|_{p_i} \end{aligned}$$

for some constant C , all $(f_1, \dots, f_k) \in L^{p_1}(X) \times \dots \times L^{p_k}(X)$, and all balls $B \subset X$.

Proof. Since $n/p = \alpha > 0$, there is at least one i such that $p_i < \infty$ and we can assume that $i = 1$. Choosing

$$\alpha_i = \frac{n}{p_i} + \epsilon \frac{n}{p'_i}, \quad 2 \leq i \leq k,$$

and some $0 \leq \epsilon < 1$ close enough to 0, we have $0 \leq \alpha_i \leq n$; moreover, the inequality (3.1) holds if $p_i > 1$ and the inequality (3.2) holds if $p_i = 1$.

On the other hand, since

$$\sum_{i=2}^k \alpha_i = \sum_{i=2}^k \frac{n}{p_i} + \epsilon \sum_{i=2}^k \frac{n}{p'_i} = \alpha - \frac{n}{p_1} + \epsilon \left(kn - \alpha - \frac{n}{p'_1} \right),$$

we have

$$\alpha_1 = \alpha - \sum_{i=2}^k \alpha_i = \frac{n}{p_1} - \epsilon \left(kn - \alpha - \frac{n}{p'_1} \right),$$

where $0 \leq \alpha_1 \leq n$ and, consequently,

$$\left(\sum_{1 \leq i \leq k} \delta(x, y_i) \right)^{\alpha-kn} \leq \prod_{i=1}^k \delta(x, y_i)^{\alpha_i - n}.$$

By the Hölder inequality and the Fubini theorem, we get

$$\begin{aligned} \frac{1}{\mu(B)} \int_B \int_{(2\kappa B)^k} \frac{|\vec{f}(\vec{y})|}{\left(\sum_{1 \leq i \leq k} \delta(x, y_i)\right)^{kn-\alpha}} d\vec{\mu}(\vec{y}) d\mu(x) \\ &\leq \frac{1}{\mu(B)} \int_B \left(\int_{(2\kappa B)} \frac{|f_1(y_1)|}{\delta(x, y_1)^{n-\alpha_1}} d\mu(y_1) \right) \prod_{i=2}^k \int_{(2\kappa B)} \frac{|f_i(y_i)|}{\delta(x, y_i)^{n-\alpha_i}} d\mu(y_i) d\mu(x) \\ &\leq C \frac{1}{\mu(B)} \int_B |f_1(y_1)| \left(\int_{B(x, 4\kappa^2 R)} \frac{1}{\delta(x, y_1)^{n-\alpha_1}} d\mu(x) \right) d\mu(y_1) R^{\epsilon \frac{n}{p'_1}} \|f_1\|_{p_1} \\ &\leq C \frac{1}{\mu(B)} \int_{(2\kappa B)} |f_1(y_1)| \left(\int_{B(x, 4\kappa^2 R)} \frac{1}{\delta(x, y_1)^{n-\alpha_1}} d\mu(x) \right) d\mu(y_1) R^{\epsilon(kn-\alpha-\frac{n}{p'_1})} \prod_{i=2}^k \|f_i\|_{p_i} \end{aligned}$$

$$\begin{aligned}
&\leq C \frac{1}{\mu(B)} \int_{(2\kappa B)^k} |f_1(y_1)| d\mu(y_1) R^{\alpha_1 + \epsilon(kn - \alpha - \frac{n}{p_1})} \prod_{i=2}^k \|f_i\|_{p_i} \\
&\leq C \frac{1}{R^{\frac{n}{p_1}}} R^{\frac{n}{p_1}} \prod_{i=1}^k \|f_i\|_{p_i} \leq C \prod_{i=1}^k \|f_i\|_{p_i}
\end{aligned}$$

The lemma is proved. \square

Lemma 3.3. Let (X, δ, μ) be an n -normal space of homogeneous type of order θ , with $0 < \theta \leq 1$ and $n \in \mathbb{R}^+$. Assume that $1 \leq p_i \leq \infty$, $1 \leq i \leq k$, $k \geq 2$, $1/p = 1/p_1 + \dots + 1/p_k$, and $0 < \alpha < kn$. If $0 \leq \alpha - n/p < \theta$, then

$$|\mathcal{J}_{\alpha, k}(\vec{f} \chi_{X^k \setminus (2\kappa B)^k})(x) - \mathcal{J}_{\alpha, k}(\vec{f} \chi_{X^k \setminus (2\kappa B)^k})(x_0)| \leq CR^{\alpha - \frac{n}{p}} \prod_{1 \leq i \leq k} \|f_i\|_{p_i} \quad (3.3)$$

for some constant C , all $\vec{f} \in L^{p_1}(X) \times \dots \times L^{p_k}(X)$, and all x such that $\delta(x, x_0) < R$.

Proof. In fact, we prove the following stronger inequality:

$$\mathcal{J}(x) := \int_{X^k \setminus (2\kappa B)^k} |\delta(x, \vec{y})^{\alpha - kn} - \delta(x_0, \vec{y})^{\alpha - kn}| |\vec{f}(\vec{y})| d\vec{\mu}(\vec{y}) \leq CR^{\alpha - \frac{n}{p}} \prod_{1 \leq i \leq k} \|f_i\|_{p_i} \quad (3.4)$$

for some constant C , all $\vec{f} \in L^{p_1} \times \dots \times L^{p_k}$, and all $x \in B$. By the mean value theorem and the Lipschitz condition (2.1) for δ ,

$$\sum_{1 \leq i \leq k} \delta(x, y_i) > 2\kappa k \delta(x, x_0)$$

implies

$$|\delta(x, \vec{y})^{\alpha - kn} - \delta(x_0, \vec{y})^{\alpha - kn}| \leq C \delta(x, x_0)^\theta \left(\sum_{1 \leq i \leq k} \delta(x_0, y_i) \right)^{\alpha - kn - \theta}. \quad (3.5)$$

Let split $f_i = f_i^1 + f_i^2$, where $f_i^1 = f_i \chi_{2B}$, $1 \leq i \leq k$. Denote by $\vec{j} > (1, \dots, 1) = \vec{1}$ any k -uple $(j_1, \dots, j_k) \in \{1, 2\}^k$ with at least one coordinate $j_i > 1$. Then $\{i \leq k : j_i = 2\} \neq \emptyset$ and

$$|\vec{f} \chi_{X^k \setminus (2\kappa B)^k}(\vec{y})| = \sum_{\vec{j} > \vec{1}} |f_1^{j_1}(y_1) \dots f_k^{j_k}(y_k)|$$

and

$$\begin{aligned}
\mathcal{J}(x) &\leq C \delta(x, x_0)^\theta \int_{X^k \setminus (2\kappa B)^k} \delta(x_0, \vec{y})^{\alpha - kn - \theta} |\vec{f}(\vec{y})| d\vec{\mu}(\vec{y}) \\
&\leq CR^\theta \sum_{\vec{j} > \vec{1}} \int_{X^k} \left(\sum_{1 \leq i \leq k} \delta(x_0, y_i) \right)^{\alpha - kn - \theta} \prod_{1 \leq i \leq k} |f_i^{j_i}(y_i)| d\vec{\mu}(\vec{y}) \\
&= CR^\theta \sum_{\vec{j} > \vec{1}} I_{\vec{j}}.
\end{aligned} \quad (3.6)$$

Fix \vec{j} and assume that $\{i \leq k : j_i = 2\} = \{1, 2, \dots, t\}$, where $1 \leq t \leq k$.

We begin with the case $t < k$. Then

$$I_{\vec{j}} \leq C \prod_{1 \leq i \leq t} \int_{X \setminus 2B} \frac{|f_i(y_i)|}{\delta(x_0, y_i)^{n-\alpha_i + \frac{\theta}{t}}} d\mu(y_i) \prod_{t+1 \leq i \leq k} \int_{2B} \frac{|f_i(y_i)|}{\delta(x_0, y_i)^{n-\alpha_i}} d\mu(y_i), \quad (3.7)$$

where for a positive number δ close enough to 0 we define

$$\alpha_i = \frac{n}{p_i} + \frac{1}{t} \left(\alpha - \frac{n}{p} - \delta \sum_{t < i \leq k} \frac{n}{p'_i} \right), \quad 1 \leq i \leq t,$$

and

$$\alpha_i = \frac{n}{p_i} + \delta \frac{n}{p'_i}, \quad t < i \leq k.$$

It is clear that $\sum_{i=1}^k \alpha_i = \alpha$. Since $\alpha - n/p < \theta$, we have

$$n - \alpha_i + \frac{\theta}{t} = \frac{n}{p'_i} + \frac{1}{t} \left(\theta - \left(\alpha - \frac{n}{p} \right) + \delta \sum_{t < i \leq k} \frac{n}{p'_i} \right) > 0, \quad 1 \leq i \leq t,$$

$$n - \alpha_i = (1 - \delta) \frac{n}{p'_i} \geq 0, \quad t < i \leq k;$$

moreover,

$$\begin{aligned} n - \alpha_i + \frac{\theta}{t} - \frac{n}{p'_i} &> 0, \quad 1 \leq i \leq t, \\ \frac{n}{p'_i} - (n - \alpha_i) &= \delta \frac{n}{p'_i} > 0, \quad p_i = 1, \\ n - \alpha_i &= 0 \quad \text{if } p_i > 1, \quad t < i \leq k. \end{aligned}$$

Hence

$$\begin{aligned} I_{\vec{j}} &\leq CR^{-\left(\theta - \left(\alpha - \frac{n}{p}\right) + \delta \sum_{t < i \leq k} \frac{n}{p'_i}\right)} \prod_{1 \leq i \leq t} \|f_i\|_{p_i} R^{\delta \sum_{t < i \leq k} \frac{n}{p'_i}} \prod_{t < i \leq k} \|f_i\|_{p_i} \\ &= CR^{\left(\alpha - \frac{n}{p}\right) - \theta} \prod_{1 \leq i \leq k} \|f_i\|_{p_i}. \end{aligned} \quad (3.8)$$

On the other hand, in the case $t = k$, we get $\vec{j} = (2, \dots, 2) = \vec{2}$ and

$$I_{\vec{2}} \leq \prod_{1 \leq i \leq k} \int_{X \setminus 2B} \frac{|f_i(y_i)|}{\delta(x_0, y_i)^{n-\alpha_i + \frac{\theta}{k}}} d\mu(y_i), \quad (3.9)$$

where

$$\alpha_i = \frac{n}{p_i} + \frac{1}{k} \left(\alpha - \frac{n}{p} \right), \quad 1 \leq i \leq k.$$

Hence

$$n - \alpha_i + \frac{\theta}{k} - \frac{n}{p'_i} = \frac{1}{k} \left(\theta - \left(\alpha - \frac{n}{p} \right) \right) > 0, \quad 1 \leq i \leq k.$$

Therefore, $I_{\vec{2}}$ also satisfies (3.8). Replacing this inequality in (3.6), we obtain (3.4). \square

Remark 3.4. If θ in (3.6) is replaced by any positive number Θ such that $\alpha - n/p < \Theta$, the proof remains unchanged, and we obtain the estimate

$$\int_{X^k \setminus (2\kappa B)^k} \left(\sum_{1 \leq i \leq k} \delta(x_0, y_i) \right)^{\alpha-kn-\Theta} |\vec{f}(\vec{y})| d\vec{\mu}(\vec{y}) \leq CR^{(\alpha-\frac{n}{p})-\Theta} \prod_{1 \leq i \leq k} \|f_i\|_{p_i}. \quad (3.10)$$

4 Proof of the Main Theorems

Without loss of generality we assume that $f_i \geq 0$ for all $1 \leq i \leq k$.

Proof of Theorem 2.1. For any ball $B = B(x_0, R)$ and a fixed ball $B_0 = B_0(z_0, 1)$ we introduce

$$a_{B, \mathcal{I}_{\alpha, k} \vec{f}} = \int \vec{f}(\vec{y}) ((1 - \chi_{(2\kappa B)^k}(\vec{y})) \delta(x_0, \vec{y})^{\alpha-kn} - (1 - \chi_{B_0^k}(\vec{y})) \delta(z_0, \vec{y})^{\alpha-kn}) d\vec{\mu}(\vec{y}). \quad (4.1)$$

Assume, for a moment, that $a_{B, \mathcal{I}_{\alpha, k} \vec{f}}$ is finite. If we show that

$$\frac{1}{\mu(B)} \int_B |\mathcal{I}_{\alpha, k} \vec{f}(x) - a_{B, \mathcal{I}_{\alpha, k} \vec{f}}| d\mu(x) \leq CR^{\alpha-\frac{n}{p}} \prod_{i=1}^k \|f_i\|_{L^{p_i}}, \quad (4.2)$$

then $|\mathcal{I}_{\alpha, k} \vec{f}(x) - a_{B, \mathcal{I}_{\alpha, k} \vec{f}}|$ will be finite for almost every $x \in B$ and every ball B . Therefore,

$$|\mathcal{I}_{\alpha, k} \vec{f}(x)| \leq |\mathcal{I}_{\alpha, k} \vec{f}(x) - a_{B, \mathcal{I}_{\alpha, k} \vec{f}}| + |a_{B, \mathcal{I}_{\alpha, k} \vec{f}}|$$

will be finite for almost every $x \in X$ and locally integrable. Together with (4.2), this means that $\mathcal{I}_{\alpha, k} \vec{f}$ is an element of BMO_β .

Postponing the proof of the finiteness of $a_{B, \mathcal{I}_{\alpha, k} \vec{f}}$ we begin by proving (4.2). To this purpose, we split

$$\begin{aligned} & \mathcal{I}_{\alpha, k} \vec{f}(x) - a_{B, \mathcal{I}_{\alpha, k} \vec{f}} \\ &= \int \delta(x_0, \vec{y})^{\alpha-kn} |\vec{f} \chi_{(2\kappa B)^k}(y)| d\vec{\mu}(\vec{y}) + (\mathcal{I}_{\alpha, k}(\vec{f} \chi_{X^k \setminus (2\kappa B)^k})(x) - \mathcal{I}_{\alpha, k}(\vec{f} \chi_{X^k \setminus (2\kappa B)^k})(x_0)) \\ &= \mathcal{J}(x) + \mathcal{I}(x). \end{aligned}$$

Lemma 3.1 for $\alpha - n/p > 0$ and Lemma 3.2 for $\alpha - n/p = 0$ show that

$$\frac{1}{\mu(B)} \int_B |\mathcal{I}(x)| d\mu(x) \leq CR^{\alpha-\frac{n}{p}} \prod_{1 \leq i \leq k} \|f_i\|_{p_i}. \quad (4.3)$$

On the other hand, since $\alpha - n/p < \theta$, from Lemma 3.3 it follows that

$$\frac{1}{\mu(B)} \int_B |\mathcal{J}(x)| d\mu(x) \leq CR^{\alpha-\frac{n}{p}} \prod_{1 \leq i \leq k} \|f_i\|_{p_i}$$

which, together with (4.3), implies (4.2).

To show that $a_{B,\mathcal{J}_{\alpha,k}\vec{f}}$ is finite, we set $M = k \max(\delta(x_0, z_0), R, 1)$. If $\delta(x_0, \vec{y}) \leq 2\kappa M$, then $\delta(z_0, \vec{y}) \leq 3\kappa^2 M$. Hence

$$\begin{aligned} |a_{B,\mathcal{J}_{\alpha,k}\vec{f}}| &\leqslant \int_{\{\delta(x_0, \vec{y}) \leq 2\kappa M\}} \delta(x_0, \vec{y})^{\alpha-kn} \vec{f}(\vec{y})(1 - \chi_{(2\kappa B)^k}(\vec{y})) d\vec{\mu}(\vec{y}) \\ &+ \int_{\{\delta(z_0, \vec{y}) \leq 3\kappa^2 M\}} \delta(z_0, \vec{y})^{\alpha-kn} |\vec{f}(\vec{y})|(1 - \chi_{B(z_0, 1)^k}(\vec{y})) d\vec{\mu}(\vec{y}) \\ &+ \int_{\{\delta(x_0, \vec{y}) > 2\kappa M\}} \vec{f}(\vec{y}) \left| \delta(z_0, \vec{y})^{\alpha-kn} - \delta(x_0, \vec{y})^{\alpha-kn} \right| d\vec{\mu}(\vec{y}) \\ &= A_1 + A_2 + A_3. \end{aligned}$$

Since A_2 and A_1 have similar behavior, it suffices to estimate one of them. Since

$$\sum \frac{n}{p_i'} = (kn - \alpha) + (\alpha - \frac{n}{p}),$$

we have

$$\begin{aligned} A_1 &\leqslant \int_{\{2\kappa R \leq \delta(x_0, \vec{y}) \leq 2\kappa M\}} \frac{|\vec{f}(\vec{y})|}{\delta(x_0, \vec{y})^{kn-\alpha}} d\vec{\mu}(\vec{y}) \leq CR^{-(kn-\alpha)} \int_{\{\delta(x_0, \vec{y}) \leq 2\kappa M\}} |\vec{f}(\vec{y})| d\vec{\mu}(\vec{y}) \\ &\leq CR^{-(kn-\alpha)} \prod_{1 \leq i \leq k} \|f\|_{p_i} M^{\frac{n}{p_i'}} \leq C \left(\frac{M}{R} \right)^{kn-\alpha} M^{\alpha - \frac{n}{p}} \prod_{1 \leq i \leq k} \|f\|_{p_i}. \end{aligned} \quad (4.4)$$

The term A_3 is estimated in the same way as $\mathcal{J}(x)$. Using Lemma 3.3, we get

$$A_3 \leq CM^{\alpha - \frac{n}{p}} \prod_{1 \leq i \leq k} \|f_i\|_{p_i}.$$

The theorem is proved. \square

Proof of Theorem 2.2. Let $m = [\alpha - n/p]$. The case $m = 0$ is covered by Theorem 2.1. Let $m \geq 1$. For a given $x \in B(x_0, R) \subset \mathbb{R}^n$ we consider the polynomial

$$P_B(\mathcal{J}_{\alpha,k}^m \vec{f})(x) = \int_{(\mathbb{R}^n)^k} \vec{f}(\vec{y}) ((1 - \chi_{(2\kappa B)^k}(\vec{y}))(T_{x_0}^m K)(x, \vec{y}) - (1 - \chi_{B_0^k}(\vec{y}))(T_0^m K)(x, \vec{y})) d\vec{y}.$$

Postponing the proof of the finiteness of $P_B(\mathcal{J}_{\alpha,k}^m \vec{f})(x)$, we now observe that

$$\begin{aligned} \mathcal{J}_{\alpha,k}^m \vec{f}(x) - P_B(\mathcal{J}_{\alpha,k}^m \vec{f})(x) &= \int_{\mathbb{R}^{nk}} \vec{f}(\vec{y}) (K(x, \vec{y}) - (1 - \chi_{(2B)^k}(\vec{y}))(T_{x_0}^m K)(x, \vec{y})) d\vec{y} \\ &= \int (\vec{f} \chi_{(2B)^k})(\vec{y}) K(x, \vec{y}) d\vec{y} + \int (\vec{f} (1 - \chi_{(2B)^k})(\vec{y})) (K(x, \vec{y}) - (T_{x_0}^m K)(x, \vec{y})) d\vec{y} = \mathcal{J}(x) + \mathcal{J}(x). \end{aligned}$$

Since $\alpha - n/p > 0$, we can use Lemma 3.1 to obtain

$$|\mathcal{J}(x)| \leq CR^{\alpha-\frac{n}{p}} \prod_{1 \leq i \leq k} \|f_i\|_{p_i}. \quad (4.5)$$

To estimate $\mathcal{J}(x)$, we note that, by the Taylor theorem and triangular inequality, $\max_{1 \leq i \leq k} |x_0 - y_i| > 2R$ and $|x_0 - x| < R$, we have

$$\begin{aligned} |K(x, \vec{y}) - (T_{x_0}^m K)(x, \vec{y})| &\leq C \max_{|x_0 - \xi| \leq |x_0 - x|, |\gamma|=m+1} |D_x^\gamma K(\xi, \vec{y})| |x_0 - x|^{m+1} \\ &\leq C \frac{|x_0 - x|^{m+1}}{\left(\sum_{1 \leq i \leq k} |x_0 - y_i| \right)^{kn-\alpha+m+1}}. \end{aligned}$$

Thus,

$$|\mathcal{J}(x)| \leq CR^{m+1} \int_{(R^n)^k \setminus (2B)^k} \frac{|\vec{f}(\vec{y})|}{\left(\sum_{1 \leq i \leq k} |x_0 - y_i| \right)^{kn-\alpha+m+1}} dy_1 \dots dy_k. \quad (4.6)$$

According to Remark 3.4, we can replace Θ with $m+1$ in the inequality (3.10). Then

$$|\mathcal{J}(x)| \leq CR^{\alpha-\frac{n}{p}} \prod_{1 \leq i \leq k} \|f_i\|_{p_i}. \quad (4.7)$$

The inequality (2.5) follows from (4.7) and (4.5).

To prove that $P_B(\mathcal{J}_{\alpha,k}^m \vec{f})(x)$ is finite for a.e. $x \in B$, we set $M = \max(|x_0|, R, 1)$. Since $\sum_i |x_0 - y_i| \leq 2kM$ implies $\sum_i |y_i| \leq 3kM$, we have

$$\begin{aligned} |P_B(\mathcal{J}_{\alpha,k}^m \vec{f})(x)| &\leq \int_{2kR \leq \sum_i |x_0 - y_i| \leq 2kM} |\vec{f}(\vec{y})| (T_{x_0}^m K)(x, \vec{y}) d\vec{y} \\ &\quad + \int_{2k \leq \sum_i |y_i| \leq 3kM} |\vec{f}(\vec{y})| (T_0^m K)(x, \vec{y}) d\vec{y} \\ &\quad + \int_{2kM \leq \sum_i |x_0 - y_i|} |\vec{f}(\vec{y})| |(T_{x_0}^m K)(x, \vec{y}) - (T_0^m K)(x, \vec{y})| d\vec{y} \\ &= A_1(x) + A_2(x) + A_3(x). \end{aligned}$$

It suffices to estimate A_1 since the case of A_2 is similar. For $|x_0 - x| \leq R$ we get

$$\begin{aligned} A_1(x) &\leq \sum_{|\gamma| \leq m} \frac{|x_0 - x|^{\gamma|}}{|\gamma|!} \int_{2kR \leq \sum_i |x_0 - y_i| \leq 2kM} \frac{|\vec{f}(\vec{y})| d\vec{\mu}(\vec{y})}{\left(\sum_i |x_0 - y_i| \right)^{kn-\alpha+|\gamma|}} \\ &\leq C \sum_{|\gamma| \leq m} \frac{R^{|\gamma|}}{|\gamma|!} (2kR)^{-(kn-\alpha+|\gamma|)} \int_{\sum_i |x_0 - y_i| \leq 2kM} |\vec{f}(\vec{y})| d\vec{y} \end{aligned}$$

$$\leq C(2kR)^{-(kn-\alpha)} \prod_{i=1}^k \|f_i\|_{p_i} (2kM)^{n/p'_i} \leq C(2kR)^{-(kn-\alpha)} (2kM)^{kn-\alpha+\alpha-\frac{n}{p}} \prod_{i=1}^k \|f_i\|_{p_i}.$$

If $\sum_i |x_0 - y_i| > 2kM$, $|x_0| \leq M$, and $|x_0 - x| \leq M$, then $\sum_i |y_i| > 2^{-1} \sum_i |x_0 - y_i|$. As in the case of (4.7), we get

$$\begin{aligned} & |(T_{x_0}^m K)(x, \vec{y}) - (T_0^m K)(x, \vec{y})| \\ & \leq |(T_{x_0}^m K)(x, \vec{y}) - K(x, y)| + |K(x, y) - (T_0^m K)(x, \vec{y})| \\ & \leq C \max_{\substack{\{|x_0 - \xi| \leq |x_0 - x| \\ |\gamma|=m+1\}}} |(D_x^\gamma K)(\xi, \vec{y})| |x - x_0|^{m+1} + C \max_{\substack{\{|\xi| \leq |x_0|, \\ |\gamma|=m+1\}}} |(D_x^\gamma K)(\xi, \vec{y})| |x|^{m+1} \\ & \leq CM^{m+1} \left(\frac{1}{\left(\sum_i |x_0 - y_i|\right)^{kn-\alpha+m+1}} + \frac{1}{\left(\sum_{i=1}^k |y_i|\right)^{kn-\alpha+m+1}} \right) \\ & \leq CM^{m+1} \frac{1}{\left(\sum_i |x_0 - y_i|\right)^{kn-\alpha+m+1}}. \end{aligned}$$

By the estimate (4.7), we find

$$\begin{aligned} A_3(x) & \leq CM^{m+1} \int_{2kM \leq \sum_i |x_0 - y_i|} \frac{\vec{f}(\vec{y})}{\left(\sum_i |x_0 - y_i|\right)^{kn-\alpha+m+1}} d\vec{y} \\ & \leq CM^{m+1} (2kM)^{\alpha - \frac{n}{p} - m - 1} \prod_{1 \leq i \leq k} \|f_i\|_{p_i} \leq CM^{\alpha - \frac{n}{p}} \prod_{1 \leq i \leq k} \|f_i\|_{p_i}. \end{aligned}$$

Thus, the theorem is proved. \square

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