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Geometry of Robinson consistency in Łukasiewicz logic

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Abstract

We establish the Robinson joint consistency theorem for the infinite-valued propositional logic of Łukasiewicz. As a corollary we easily obtain the amalgamation property for MV-algebras—the algebras of Łukasiewicz logic: all pre-existing proofs of this latter result make essential use of the Pierce amalgamation theorem for abelian lattice-ordered groups (with strong unit) *together with* the categorical equivalence Γ between these groups and MV-algebras. Our main tools are elementary and geometric.

Keywords: Łukasiewicz logic; Infinite-valued logic; Finite-valued logic; Robinson joint consistency; MV-algebra; Amalgamation; Free MV-algebra; McNaughton function; Unimodular triangulation; Prime ideal; Krull depth

1. Introduction

We assume familiarity with MV-algebras and Łukasiewicz propositional logic: we refer the reader to [1] for all unexplained notions. For X an arbitrary set of variables, L_X denotes the set of formulas ψ whose variables are in X. Any such ψ is said to be an L_X*-formula*. The definition is the same for boolean logic and for many-valued logic. A proper subset Θ of L_X is called a *theory* (or, an L_X-theory if necessary) if

(i) Θ contains all L_X-tautologies of Łukasiewicz infinite-valued propositional logic, and
(ii) Θ is closed under modus ponens.

An L_X-theory Θ is said to be *prime* (also called "complete" in Hájek's monograph [3, 2.4.1]) if for any L_X-formulas φ and ψ either $\varphi \to \psi$ or $\psi \to \varphi$ belongs to Θ.

Every prime theory Θ has a unique *maximally consistent* completion Θ'. In other words, $L_X \supseteq \Theta' \supseteq \Theta$ and there is no theory $\Theta'' \subseteq L_X$ properly extending Θ'. By contrast with boolean logic Θ' generally does not coincide with Θ.

The *Robinson consistency property* for boolean, as well as for Łukasiewicz logic, can be stated as follows:

> Suppose that Θ is a prime L_X-theory, and Ψ is a prime L_Y-theory. Let $Z = X \cap Y$ and $W = X \cup Y$. If $\Theta \cap L_Z = \Psi \cap L_Z$ then there is a prime L_W-theory Φ such that $\Theta = \Phi \cap L_X$ and $\Psi = \Phi \cap L_Y$.

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In Theorem 5.1 below we shall prove the Robinson consistency property for Łukasiewicz propositional logic. With a little more effort, we obtain an elementary proof of the amalgamation property for MV-algebras. We shall use the well-known one–one correspondence between theories (resp., prime theories, resp., maximally consistent theories) and ideals (resp., prime ideals, resp., maximal ideals) of free MV-algebras.

Throughout this paper we shall make constant use of the rich geometric machinery naturally arising from the theory of MV-algebras, such as McNaughton's representation of free MV-algebras in terms of [0, 1]-valued piecewise linear functions, their underlying unimodular triangulations of the n-cube, and the spectral theory of free MV-algebras.

As a warm up we prove a weak form of joint consistency for infinite-valued Łukasiewicz propositional logic:

Proposition 1.1. *Infinite-valued Łukasiewicz logic has the* Robinson property for maximally consistent theories. *In other words, whenever Θ and Ψ are maximally consistent theories, respectively in the language L_X and L_Y, and $\Theta \cap L_{X\cap Y} = \Psi \cap L_{X\cap Y}$ then there is a maximally consistent $L_{X\cup Y}$-theory Φ such that $\Theta = \Phi \cap L_X$ and $\Psi = \Phi \cap L_Y$.*

Proof. Let $Z = X \cap Y$ and $\Xi = \Theta \cap L_Z = \Psi \cap L_Z$. We denote by $Free_Z$ (resp., $Free_X$) the free MV-algebra over the free generating set Z (resp., over X). By [1, 3.1.8, 9.1.5] $Free_Z$ and $Free_X$ are algebras of McNaughton functions. We canonically identify $Free_Z$ with the subalgebra of $Free_X$ given by those McNaughton functions that only depend on the variables of Z. Let us similarly write $Free_Z \subseteq Free_Y$. Let x be the only valuation that satisfies every formula of Θ. (Ref. [1] uses the terminology "[0, 1]-valuation" and "[0, 1]-satisfies".) Following [1, p. 80] x is uniquely determined by its values over X, and hence x can be identified with a point in the cube $[0, 1]^X$. Similarly, let $y \in [0, 1]^Y$ be the only valuation that satisfies all formulas of Ψ. From our hypotheses it follows that Ξ is a maximally consistent theory and the only valuation $z \in [0, 1]^Z$ that satisfies all formulas of Ξ is given by $z = x \restriction Z = y \restriction Z$. Let w be the only point of $[0, 1]^{X\cup Y}$ whose X-coordinates are those of x and whose Y-coordinates are those of y, in symbols $w = x \cup y$. The point w is well defined because x and y agree on their common coordinates. Let Φ be the set of $L_{X\cup Y}$-formulas that are satisfied by the valuation w. Then the maximally consistent theory Φ has the desired properties. □

Corollary 1.2. *All finite-valued Łukasiewicz logics have the Robinson property for maximally consistent theories. The latter coincide with prime theories.*

Proof. Let $\mathbf{L}_n \subseteq [0, 1]$ be the n-element Łukasiewicz chain [1, p. 8]. We recall that an MV$_n$-algebra is an element of the variety generated by the MV-algebra $\mathbf{L}_n$. MV$_n$-algebras are the algebras of the n-valued Łukasiewicz logic. Let X be a set of variables. For each $n = 2, 3, \ldots$ let $F_X^{(n)}$ denote the free MV$_n$-algebra over the free generating set X. As is well known [1, 8.5, 8.6], $F_X^{(n)}$ is given by restricting to the product space $\mathbf{L}_n^X \subseteq [0, 1]^X$ all McNaughton functions of the free MV-algebra $Free_X$. Maximally consistent theories of $F_X^{(n)}$ canonically correspond to points in $\mathbf{L}_n^X$ via the map sending any such point x into the set of L_X-formulas that are satisfied by the valuation x. In finite-valued Łukasiewicz logic, maximally consistent theories are the same as prime theories, because $F_X^{(n)}$ is hyperarchimedean. (See [1, 6.3.1, 6.3.2, 8.5.1] for details.) The same argument of the foregoing proof now settles the present corollary as well. One simply notes that, whenever $x \in \mathbf{L}_n^X$ and $y \in \mathbf{L}_n^Y$ and $x \restriction Z = y \restriction Z$ then $x \cup y$ belongs to $\mathbf{L}_n^{X\cup Y}$. □

2. Classification of prime ideals in the free MV-algebra $Free_n$

To prepare the proof of Theorem 5.1, in this and in the next two sections we will embark on a geometrical investigation of prime ideals in finitely generated free MV-algebras. Let $Free_n$ denote the free n-generated MV-algebra. Since $Free_n$ consists of continuous piecewise linear [0, 1]-valued functions over the n-cube, (equipped with the usual topology) our main tools will be given by the affine (piecewise) linear geometry of $\mathbb{R}^n$. Our standard reference will be [1]. For general background and notation concerning simplicial complexes and related topics we also refer to the introductory pages of [5]. In particular, for any set $S = \{a_1, a_2, \ldots, a_s\}$ of elements of $\mathbb{R}^n$, we let $\mathrm{conv}\{a_1, a_2, \ldots, a_s\}$ denote the convex hull of S. Every simplex T considered in this paper shall be contained in some n-cube $[0, 1]^n$; accordingly, the interior int T and the relative interior relint T shall always be taken with respect to $[0, 1]^n$.

Definition 2.1. Let $n \in \mathbb{N}$ and $0 \le t \le n$. By an *index* $\mathbf{u} = (u_0, u_1, \ldots, u_t)$ we understand a $(t + 1)$-tuple of vectors in $\mathbb{R}^n$ such that $u_1, \ldots, u_t$ are linearly independent and for some $\epsilon_1, \ldots, \epsilon_t > 0$ the simplex

$$T = \mathrm{conv}\{u_0, u_0 + \epsilon_1 u_1, u_0 + \epsilon_1 u_1 + \epsilon_2 u_2, \ldots, u_0 + \epsilon_1 u_1 + \cdots + \epsilon_t u_t\}$$

is contained in $[0, 1]^n$. Any such T is said to be a **u**-*simplex*. The set $J_{\mathbf{u}} \subseteq Free_n$ is defined by

$$f \in J_{\mathbf{u}} \quad \text{iff} \quad \text{the zeroset } f^{-1}(0) \text{ of } f \text{ contains some } \mathbf{u}\text{-simplex}. \tag{1}$$

For each $j = 0, 1, \ldots t$, let us write u^j as an abbreviation of $(u_0, \ldots, u_j)$. Since u^j is an index, u^j-simplexes and J_{u^j} are well defined.

Proposition 2.2. *If T_1 and T_2 are* **u**-*simplexes then $T_1 \cap T_2$ contains a* **u**-*simplex.*

Proof. Induction on t. The cases $t = 0, 1$ are trivial.

For the induction step, assume without loss of generality $u_0 = 0$. Let

$$T_1 = \mathrm{conv}\{0, \epsilon_1 u_1, \epsilon_1 u_1 + \epsilon_2 u_2, \ldots, \epsilon_1 u_1 + \cdots + \epsilon_t u_t\},$$
$$T_2 = \mathrm{conv}\{0, \eta_1 u_1, \eta_1 u_1 + \eta_2 u_2, \ldots, \eta_1 u_1 + \cdots + \eta_t u_t\},$$
$$T_1' = \mathrm{conv}\{0, \epsilon_1 u_1, \epsilon_1 u_1 + \epsilon_2 u_2, \ldots, \epsilon_1 u_1 + \cdots + \epsilon_{t-1} u_{t-1}\},$$
$$T_2' = \mathrm{conv}\{0, \eta_1 u_1, \eta_1 u_1 + \eta_2 u_2, \ldots, \eta_1 u_1 + \cdots + \eta_{t-1} u_{t-1}\}.$$

By induction hypothesis, $T_1' \cap T_2'$ contains some u^{t-1}-simplex T', say

$$T' = \mathrm{conv}\{0, \omega_1 u_1, \omega_1 u_1 + \omega_2 u_2, \ldots, \omega_1 u_1 + \cdots + \omega_{t-1} u_{t-1}\}.$$

Since T_1 and T_2 are convex sets, for each $x \in \mathrm{relint}\, T_1' \cap \mathrm{relint}\, T_2'$ there are $0 < \delta_1, \delta_2$ such that $x + \delta_1 u_t \in T_1$ and $x + \delta_2 u_t \in T_2$ whence, letting $\delta = \min\{\delta_1, \delta_2\}$, we have $x + \delta u_t \in T_1 \cap T_2$. The point $c = \frac{\omega_1}{2} u_1 + \cdots + \frac{\omega_{t-1}}{2} u_{t-1}$ lies in relint T'. Since $\mathrm{relint}\, T' \subseteq \mathrm{relint}\, T_1' \cap \mathrm{relint}\, T_2'$, there exists $0 < \omega$ such that $c + \omega u_t \in T_1 \cap T_2$. Therefore the **u**-simplex $T = \mathrm{conv}\{0, \frac{\omega_1}{2} u_1, \frac{\omega_1}{2} u_1 + \frac{\omega_2}{2} u_2, \ldots, c, c + \omega u_t\}$ satisfies $T \subseteq T_1 \cap T_2$, as desired. □

Proposition 2.3. *$J_{\mathbf{u}}$ is an ideal of $Free_n$.*

Proof. Trivially $J_{\mathbf{u}}$ is closed under minorants. If $f, g \in J_{\mathbf{u}}$ then by definition we have **u**-simplexes T' and T'' such that, writing $\mathrm{Z}f$ for the zeroset of f, $\mathrm{Z}f \supseteq T'$ and $\mathrm{Z}g \supseteq T''$. By Proposition 2.2 we have a **u**-simplex with $T \subseteq T' \cap T''$. Now $\mathrm{Z}(f \oplus g) = \mathrm{Z}f \cap \mathrm{Z}g \supseteq T' \cap T'' \supseteq T$, and $f \oplus g \in J_{\mathbf{u}}$. □

As we shall see in Proposition 2.8 below, $J_{\mathbf{u}}$ is in fact a prime ideal. Conversely, in Corollary 2.18 we shall see that every prime ideal J of $Free_n$ has the form $J = J_{\mathbf{u}}$ for some index **u**.

Notation and terminology. Unless otherwise specified, every affine hyperplane H considered in this paper shall be *rational*. In other words, $H = \{x \in \mathbb{R}^n \mid \sum_{i=1}^n m_i x_i = m_0\}$ for suitable integers $m_0, m_1, \ldots, m_n$, where not all of $m_1, \ldots, m_n$ are zero. Throughout this paper the symbol H will denote a rational affine hyperplane in some euclidean space $\mathbb{R}^n$. As usual, the two closed half-spaces defined by H will be denoted by H^+ and H^- respectively. In more detail,

$$H^+ = \left\{x \in \mathbb{R}^n \mid \sum_{i=1}^n m_i x_i \geq m_0\right\} \quad \text{and} \quad H^- = \left\{x \in \mathbb{R}^n \mid \sum_{i=1}^n m_i x_i \leq m_0\right\}.$$

Every triangulation $\mathcal{T}$ considered in this paper shall be *unimodular* [1, 9.1.1]. The union of the simplexes of $\mathcal{T}$ shall always coincide with an n-cube $[0, 1]^n$, for some $n = 1, 2, \ldots$. Accordingly, we shall omit the specifications "unimodular" and "of the n-cube". By a *refinement* $\mathcal{U}$ of $\mathcal{T}$ we mean a unimodular triangulation $\mathcal{U}$ such that each simplex of $\mathcal{T}$ is the union of simplexes of $\mathcal{U}$. A triangulation $\mathcal{T}$ will be said to *respect* a hyperplane H if each simplex of $\mathcal{T}$ is either contained in H^+ or in H^-.

We shall make frequent use of the following simple result:

Lemma 2.4. *Let $\mathcal{T}$ be a triangulation. Let $H \subseteq \mathbb{R}^n$ be a hyperplane. Then there exists a refinement $\mathcal{U}$ of $\mathcal{T}$ such that $\mathcal{U}$ respects H. Further, any two triangulations have a joint refinement that respects H.*

Proof. This is an immediate consequence of the De Concini–Procesi lemma [4] on elimination of points of indeterminacy in toric varieties. The rationality assumption for H is essential. See [12, Lemma 2.2] for an elementary proof using only MV-algebraic machinery. □

A standard tool to construct McNaughton functions out of triangulations is given by the following:

Lemma 2.5. *Let $\mathcal{T}$ be a triangulation and μ be a $\{0, 1\}$-valued map defined over the set of vertices of simplexes in $\mathcal{T}$. Let $f : [0, 1]^n \to [0, 1]$ be the only function that is linear over each simplex of $\mathcal{T}$ and also satisfies $f(x) = \mu(x)$. Then $f \in Free_n$.*

Proof. This follows from the assumed unimodularity of $\mathcal{T}$. See [1, 9.1.4] for details. □

An ideal J of an MV-algebra A is said to be *prime* iff the quotient MV-algebra A/J is ($\neq \{0\}$ and) totally ordered. The following characterization is folklore [1] and shall be used without explicit mention throughout this paper:

Lemma 2.6. *For every MV-algebra A and ideal $J \neq A$ of A the following conditions are equivalent:*

(1) *J is prime;*
(2) *whenever $x, y \in A$ and $x \wedge y = 0$ then $x \in J$ or $y \in J$;*
(3) *whenever $x, y \in A$ and $x \wedge y \in J$ then $x \in J$ or $y \in J$;*
(4) *if P and Q are ideals of A and $P \cap Q \subseteq J$ then $P \subseteq J$ or $Q \subseteq J$;*
(5) *if P and Q are ideals of A and $P \cap Q = J$ then $P = J$ or $Q = J$;*
(6) *if P and Q are ideals of A containing J then $P \subseteq Q$ or $Q \subseteq P$;*
(7) *for all $x, y \in A$ either $x \to y \in \neg J$ or $y \to x \in \neg J$; here, as usual, $x \to y$ is short for $\neg x \oplus y$, and the dual ideal (also known as* filter*) $\neg J$ is given by $\neg J = \{\neg z \mid z \in J\}$;*
(8) *for all $x, y \in A$ either $x \ominus y \in J$ or $y \ominus x \in J$, (where $x \ominus y$ is short for $x \odot \neg y$).*

Definition 2.7. For every triangulation $\mathcal{T}$ and index $\mathbf{u}$ we let

$$\mathcal{T}^{\mathbf{u}} = \bigcap\{F \mid F \text{ is a simplex of } \mathcal{T} \text{ and } F \text{ contains some } \mathbf{u}\text{-simplex}\}. \tag{2}$$

As an immediate consequence of Proposition 2.2, one sees that $\mathcal{T}^{\mathbf{u}}$ is a simplex of $\mathcal{T}$ containing a $\mathbf{u}$-simplex. Recalling the notation u^j for the index $(u_0, \ldots u_j)$, it follows that $\mathcal{T}^{u^j}$ is well defined for each $j = 0, 1, \ldots, t$.

Proposition 2.8. *For any index $\mathbf{u} = (u_0, u_1, \ldots, u_t)$, $J_{\mathbf{u}}$ is a prime ideal of $Free_n$.*

Proof. We have already seen in Proposition 2.3 that $J_{\mathbf{u}}$ is an ideal of $Free_n$. To see that $J_{\mathbf{u}}$ is prime, suppose that $f \notin J_{\mathbf{u}}$, $g \notin J_{\mathbf{u}}$, with the intent of proving $f \wedge g \neq 0$. Using Lemma 2.4 let $\mathcal{T}$ be such that each of $f, g, f \wedge g$ is linear over each simplex of $\mathcal{T}$. It follows that $f(x) > 0$ for some $x \in \mathcal{T}^{\mathbf{u}}$ (otherwise f would vanish over $\mathcal{T}^{\mathbf{u}} \supseteq T$ for some $\mathbf{u}$-simplex T, whence $f \in J_{\mathbf{u}}$, which is impossible.) Similarly, $g(y) > 0$ for some $y \in \mathcal{T}^{\mathbf{u}}$. Our assumption about $\mathcal{T}$ ensures that both f and g must be > 0 over relint $\mathcal{T}^{\mathbf{u}}$, and that $f \wedge g$ is linear over $\mathcal{T}^{\mathbf{u}}$. Therefore, over $\mathcal{T}^{\mathbf{u}}$ we must either have $f \geq g$ or $g \geq f$. In either case, $f \wedge g \neq 0$ as desired. □

Definition 2.9. For every index $\mathbf{u} = (u_0, \ldots, u_t)$ we set $\zeta(u^0) = \bigcap\{H \mid u_0 \in H\}$ and for each $i = 1, \ldots, t$,

$$\zeta(u^i) = \bigcap\{H \mid \mathrm{conv}\{u_0, u_0 + u_1, \ldots, u_0 + u_1 + \cdots + u_i\} \subseteq H\}. \tag{3}$$

Equivalently,

$$\zeta(u^i) = \bigcap\{H \mid T \subseteq H \text{ for some } u^i\text{-simplex } T\}.$$

In the terminology of [5, p. 3], $\zeta(u^i)$ is a *flat* space, or an affine variety. For each $0 \leq i \leq t$, translation by $-u_0$ of $\zeta(u^i)$ yields its associated linear space $\lambda(u^i)$; in symbols,

$$\lambda(u^i) = \zeta(u^i) - u_0 = \{x \in \mathbb{R}^n \mid (u_0 + x) \in \zeta(u^i)\}. \tag{4}$$

We often write $\zeta(\mathbf{u})$ instead of $\zeta(u^t)$, and $\lambda(\mathbf{u})$ instead of $\lambda(u^t)$.

Let $\mathcal{T}$ be a triangulation and $\mathbf{u} = (u_0, \ldots, u_t)$ an index. Then by Definition 2.7 we must have

$$\dim \mathcal{T}^{u^j} \geq \dim \zeta(u^j) \tag{5}$$

for all $j \leq t$: indeed by the assumed unimodularity of $\mathcal{T}$ every simplex of $\mathcal{T}$ of codimension 1 is contained in a rational hyperplane.

Definition 2.10. We say that $\mathcal{T}$ is **u**-*good* if

$$\dim \mathcal{T}^{u^j} = \dim \zeta(u^j) \quad \text{for each } j = 0, 1, \ldots, t. \tag{6}$$

Further, for any $f \in Free_n$ we say that $\mathcal{T}$ is f-*good* if f is linear (in the affine sense) over each simplex $T \in \mathcal{T}$. More generally, $\mathcal{T}$ is said to be $\mathbf{u}f$-*good* if it is f-good and **u**-good. Given finitely many indexes **v**, **w**, ... and functions $g, h, \ldots \in Free_n$, one similarly defines $\mathbf{vw}g$-good, $\mathbf{vw}gh$-good, and the like.

Lemma 2.11. *Let* $\mathbf{u} = (u_0, \ldots, u_t)$ *be an index.*

(i) *For every triangulation* $\mathcal{T}$, $\mathcal{T}^{u^j}$ *is a face of* $\mathcal{T}^{u^{j+1}}$, *in symbols,* $\mathcal{T}^{u^j} \preceq \mathcal{T}^{u^{j+1}}$.
(ii) *Every triangulation* $\mathcal{T}$ *can be refined to a* **u**-*good triangulation.*
(iii) *If* $\mathcal{W}$ *is a refinement of a* **u**-*good triangulation* $\mathcal{T}$, *then* $\mathcal{W}^{\mathbf{u}} \subseteq \mathcal{T}^{\mathbf{u}}$. *Specifically,* $\mathcal{T}^{\mathbf{u}}$ *is the smallest simplex of* $\mathcal{T}$ *containing* $\mathcal{W}^{\mathbf{u}}$.
(iv) *Every refinement of a* **u**-*good (resp.,* $\mathbf{u}f$-*good, ...) triangulation is* **u**-*good (resp.,* $\mathbf{u}f$-*good, ...).*
(v) *We have the identity*

$$J_{\mathbf{u}} = \{f \in Free_n \mid \textit{for } \text{some } \mathbf{u}f\textit{-good triangulation } \mathcal{T},\ f \restriction \mathcal{T}^{\mathbf{u}} = 0\}.$$

(vi) *If* $f \in J_{\mathbf{u}}$ *then* $f \restriction \mathcal{U}^{\mathbf{u}} = 0$ *for* every $\mathbf{u}f$-*good triangulation* $\mathcal{U}$.

Proof. (i) is an immediate consequence of the definition. (ii) follows from Lemma 2.4.

(iii) Let T be the smallest simplex of $\mathcal{T}$ containing $\mathcal{W}^{\mathbf{u}}$. We *claim* that $\mathcal{T}^{\mathbf{u}} = T$. By way of contradiction assume $\mathcal{T}^{\mathbf{u}} \neq T$. By construction, $\mathcal{T}^{\mathbf{u}} \cap T$ is a simplex of $\mathcal{T}$ which contains some **u**-simplex (Proposition 2.2). By minimality of $\mathcal{T}^{\mathbf{u}}$, T strictly contains $\mathcal{T}^{\mathbf{u}}$. By minimality of T, $\mathcal{T}^{\mathbf{u}}$ does not contain $\mathcal{W}^{\mathbf{u}}$. Let $S = \mathcal{T}^{\mathbf{u}} \cap \mathcal{W}^{\mathbf{u}}$. Since $\mathcal{W}$ refines $\mathcal{T}$ then S is a simplex of $\mathcal{W}$ and $\emptyset \neq S \subsetneqq \mathcal{W}^{\mathbf{u}}$. Because both $\mathcal{T}^{\mathbf{u}}$ and $\mathcal{W}^{\mathbf{u}}$ contain some **u**-simplex, again by Proposition 2.2, S contains a **u**-simplex R. This contradicts the minimality of $\mathcal{W}^{\mathbf{u}}$. Our claim is settled and $\mathcal{T}^{\mathbf{u}} = T \supseteq \mathcal{W}^{\mathbf{u}}$, as required to complete the proof.

(iv) Let $\mathcal{U}$ be a refinement of a **u**-good triangulation $\mathcal{T}$. By (iii), $\mathcal{U}^{u^i} \subseteq \mathcal{T}^{u^i}$ for each $i = 0, \ldots, t$, whence $\dim \mathcal{U}^{u^i} \leq \dim \mathcal{T}^{u^i} = \dim \zeta(u^i)$. Conversely, from (5) we also have $\dim \mathcal{U}^{u^i} \geq \dim \zeta(u^i)$, as desired.

(v) The nontrivial inclusion follows from (ii).

(vi) By (v) there exists at least one $\mathbf{u}f$-good triangulation $\mathcal{T}$ such that $f \restriction \mathcal{T}^{\mathbf{u}} = 0$. Let $\mathcal{U}$ be any arbitrary $\mathbf{u}f$-good triangulation. Let $\mathcal{V}$ be a joint refinement of $\mathcal{T}$ and $\mathcal{U}$ as given by Lemma 2.4. Then by (iii) and (iv) the simplex $\mathcal{V}^{\mathbf{u}}$ is a subset of $\mathcal{U}^{\mathbf{u}} \cap \mathcal{T}^{\mathbf{u}}$ having the same dimension as $\mathcal{U}^{\mathbf{u}}$. We have $f \restriction \mathcal{V}^{\mathbf{u}} = 0$. Since f is linear over $\mathcal{U}^{\mathbf{u}}$ then $f \restriction \mathcal{U}^{\mathbf{u}} = 0$. □

Definition 2.12. Let $\mathbf{u} = (u_0, u_1, \ldots, u_t)$ and $\mathbf{v} = (v_0, v_1, \ldots, v_r)$ be indexes with $t \leq r$. If $v_i = u_i$ for all $i = 0, \ldots, t$, then **v** is called an *extension* of **u**. If in addition $\zeta(u^t) \subsetneqq \zeta(v^r)$ we say that **v** is a *proper* extension of **u**.

Lemma 2.13. *If* **v** *is an extension of* **u** *then* $J_{\mathbf{v}} \subseteq J_{\mathbf{u}}$.

Proof. Suppose $f \in J_{\mathbf{v}}$. By Lemma 2.11(vi), for any $\mathbf{v}f$-good triangulation $\mathcal{T}$, the function f vanishes over $\mathcal{T}^{\mathbf{v}}$. Trivially, $\mathcal{T}$ is $\mathbf{u}f$-good and $\mathcal{T}^{\mathbf{u}} = \mathcal{T}^{v^t} \subseteq \mathcal{T}^{\mathbf{v}}$. Thus f vanishes over $\mathcal{T}^{\mathbf{u}}$, whence by Lemma 2.11(v) $f \in J_{\mathbf{u}}$. □

Remark. Given $\mathbf{u} = (u_0, \ldots, u_t)$ and $n > t$ it may happen that $t < \dim \zeta(\mathbf{u}) \leq n$. (For example, if $\mathbf{u} = u_0$ and $u_0 \notin ([0, 1] \cap \mathbb{Q})^n$, then $\dim \zeta(\mathbf{u}) > 0$). In this case there is $v \in \lambda(\mathbf{u})$ such that the vectors $u_1, \ldots, u_t, v$ form a linearly independent set and $\zeta(u_0, \ldots, u_t, v) = \zeta(\mathbf{u})$. Thus $(u_0, \ldots, u_t, v)$ is not a proper extension of **u**.

Definition 2.14. Following [5, p. 40], for any triangulation $\mathcal{T}$ of the n-cube and simplex $F \in \mathcal{T}$ the *star* $\mathrm{st}(F; \mathcal{T})$ of F in $\mathcal{T}$ is the smallest subcomplex of $\mathcal{T}$ containing all the members of $\mathcal{T}$ that contain F. The point-set-theoretical union of $\mathrm{st}(F; \mathcal{T})$ is called the *closed star* of F in $\mathcal{T}$ and it is denoted by $\mathrm{clstar}(F; \mathcal{T})$. (The notation $\mathrm{set\,st}(F; \mathcal{T})$ is used in [5]). The interior of $\mathrm{clstar}(F; \mathcal{T})$ relative to the n-cube, is called the *open star* of F in $\mathcal{T}$, denoted $\mathrm{ostar}(F; \mathcal{T})$. It follows that

$$\mathrm{ostar}(F; \mathcal{T}) = \mathrm{int}\{x \in [0, 1]^n \mid \exists\, n\text{-dimensional } T \in \mathcal{T} \text{ with } x \in T \supseteq F\}. \tag{7}$$

When $\mathcal{T}$ is clear from the context we simply write $\mathrm{clstar}(F)$ and $\mathrm{ostar}(F)$.

For every prime ideal J the *germinal ideal* $\mathrm{germ}(J)$ is the intersection of all prime ideals contained in J. Germinal ideals have the following characterization:

Theorem 2.15. *Let* $\mathbf{u} = (u_0, u_1, \ldots, u_t)$ *be an index and* $f \in \mathit{Free}_n$*. Then the following conditions are equivalent:*

(i) $f \upharpoonright \mathrm{ostar}(\mathcal{T}^{\mathbf{u}}) = 0$ *for some* $\mathbf{u}f$*-good triangulation* $\mathcal{T}$*;*
(ii) $f \upharpoonright \mathrm{ostar}(\mathcal{T}^{\mathbf{u}}) = 0$ *for every* $\mathbf{u}f$*-good triangulation* $\mathcal{T}$*;*
(iii) $f \in \mathrm{germ}(J_{\mathbf{u}})$.

Proof. (ii) → (i) is trivial, because at least one $\mathbf{u}f$-good triangulation $\mathcal{T}$ exists.

In order to prove (iii) → (ii) suppose $f \in \mathrm{germ}(J_{\mathbf{u}}) \subseteq J_{\mathbf{u}}$ and let $\mathcal{T}$ be an arbitrary $\mathbf{u}f$-good triangulation. By Lemma 2.11(vi), $f \upharpoonright \mathcal{T}^{\mathbf{u}} = 0$; by definition of $\mathcal{T}^{\mathbf{u}}$ there exist real numbers $\epsilon_1, \ldots, \epsilon_t > 0$, such that $\mathrm{conv}\{u_0, u_0 + \epsilon_1 u_1, \ldots, u_0 + \epsilon_1 u_1 + \cdots + \epsilon_t u_t\} \subseteq \mathcal{T}^{\mathbf{u}}$. By way of contradiction, suppose $f(x) > 0$ for some $x \in \mathrm{ostar}(\mathcal{T}^{\mathbf{u}})$. Then there is a vector v orthogonal to $\lambda(\mathbf{u})$ such that for all suitably small $\delta > 0$ the function f is linear and not constantly zero over the set

$$R = \mathrm{conv}\{u_0, u_0 + \epsilon_1 u_1, \ldots, u_0 + \epsilon_1 u_1 + \cdots + \epsilon_t u_t, u_0 + \epsilon_1 u_1 + \cdots + \epsilon_t u_t + \delta v\}.$$

Thus $f > 0$ over relint R. Let us write $(\mathbf{u}, v)$ instead of $(u_0, u_1, \ldots, u_t, v)$. It follows that $f \notin J_{(\mathbf{u},v)}$. (For otherwise, f vanishes over some $(\mathbf{u}, v)$-simplex S; by Proposition 2.2, S may be assumed to satisfy $S \subseteq R$, which contradicts $f > 0$ over relint R). By Lemma 2.13, $J_{(\mathbf{u},v)} \subseteq J_{\mathbf{u}}$ and by Proposition 2.8 $J_{(\mathbf{u},v)}$ is prime. Thus, by definition of germinal ideal, $f \notin \mathrm{germ}(J_{\mathbf{u}})$, a contradiction.

(i) → (iii) Assume $f \upharpoonright \mathrm{ostar}(\mathcal{T}^{\mathbf{u}}) = 0$ for some $\mathbf{u}f$-good triangulation $\mathcal{T}$. Then $f \upharpoonright \mathcal{T}^{\mathbf{u}} = 0$ and $f \in J_{\mathbf{u}}$. Let J be a prime ideal of Free_n such that $J \subseteq J_{\mathbf{u}}$ and $f \notin J$ (absurdum hypothesis). Let $b \in ([0, 1] \cap \mathbb{Q})^n$ be the *Farey mediant* of (the vertices of) $\mathcal{T}^{\mathbf{u}}$: b is obtained by writing each vertex $(v_1/v, \ldots, v_n/v)$ of $\mathcal{T}^{\mathbf{u}}$ in homogeneous integer coordinates as $(v_1, \ldots, v_n, v)$, then taking the sum $(s_1, \ldots, s_n, s)$ of all these vectors in $\mathbb{Z}^{n+1}$, and finally letting $b = (s_1/s, \ldots, s_n/s)$. See [1, p. 56] or [9, 2.2] for details. The resulting refinement $\mathcal{W}$ of $\mathcal{T}$ whose only new vertex is b is said to be obtained via *starring* $\mathcal{T}$ at the mediant of $\mathcal{T}^{\mathbf{u}}$. As is well known, $\mathcal{W}$ is automatically unimodular, $\mathbf{u}$-good and $b \in \mathrm{relint}\, \mathcal{T}^{\mathbf{u}}$. In the light of Lemma 2.5, let the function $g \in \mathit{Free}_n$ be defined by specifying its values at each vertex of $\mathcal{W}$ (with g linear over each simplex of $\mathcal{W}$) as follows:

$$g(x) = \begin{cases} 1 & \text{if } x = b, \\ 0 & \text{if } x \text{ is any other vertex of } \mathcal{W}. \end{cases}$$

Then by Lemma 2.11(vi), $g \notin J_{\mathbf{u}}$, whence $g \notin J$. By construction, g identically vanishes over the complement of $\mathrm{ostar}(\mathcal{T}^{\mathbf{u}})$ in $[0, 1]^n$. Therefore, $f \wedge g = 0 \in J$, thus contradicting the primeness of J. □

Proposition 2.16. *Let* $\mathbf{u} = (u_0, u_1, \ldots, u_t)$ *be an index such that* $\dim \zeta(\mathbf{u}) < n$*. Suppose* J *is a prime ideal such that* $J \subseteq J_{\mathbf{u}}$*. Suppose there does not exist a* proper *extension* $\mathbf{v}$ *of* $\mathbf{u}$ *such that* J *is contained in* $J_{\mathbf{v}}$*. Then there is a function* $f \in J$ *and a* $\mathbf{u}f$*-good triangulation* $\mathcal{T}$ *such that*

(i) $f \upharpoonright \mathcal{T}^{\mathbf{u}} = 0$ *and*
(ii) $f(x) > 0$ *for all* $x \in \mathrm{clstar}(\mathcal{T}^{\mathbf{u}}) \setminus \mathcal{T}^{\mathbf{u}}$.

Proof. Let $\lambda^{\perp}$ denote the orthogonal complement of $\lambda(\mathbf{u})$ in $\mathbb{R}^n$. Let $\zeta^{\perp} = \lambda^{\perp} + u_0$ denote the affine space given by u_0-translation of $\lambda^{\perp}$. The dimension d of $\zeta^{\perp}$ satisfies $d = n - \dim \zeta(\mathbf{u}) > 0$. Let $\mathcal{S}$ be the $(d-1)$-dimensional sphere of radius one, centered at u_0, and lying in $\zeta^{\perp}$, in symbols,

$$\mathcal{S} = \{z \in \zeta^{\perp} \mid \mathrm{distance}(z, u_0) = 1\}.$$

Fix an arbitrary unit vector $v \in \lambda^{\perp}$. Then the index $(\mathbf{u}, v)$ is a proper extension of $\mathbf{u}$. Since by hypothesis, $J \not\subseteq J_{(\mathbf{u},v)}$ let $f_v \in J \setminus J_{(\mathbf{u},v)}$, whence $f_v \in J_{\mathbf{u}} \supseteq J$. Let $\mathcal{T}_v$ be a $(\mathbf{u}, v) f_v$-good triangulation. Then by Lemma 2.11(v)–(vi) we have

$$f_v \upharpoonright \mathcal{T}_v^{\mathbf{u}} = 0, \quad \text{and} \quad f_v(x) > 0 \quad \text{for all } x \in \mathrm{relint}\, \mathcal{T}_v^{(\mathbf{u},v)}. \tag{8}$$

Letting $O_v = \mathrm{ostar}(\mathcal{T}_v^{(\mathbf{u},v)}; \mathcal{T}_v)$ it follows that

$$f_v(x) > 0 \quad \text{for all } x \in O_v. \tag{9}$$

As a matter of fact, f_v is linear over each n-simplex of the star of $\mathcal{T}_v^{(\mathbf{u},v)}$ in $\mathcal{T}_v$, and is > 0 over $\mathrm{relint}\mathcal{T}_v^{(\mathbf{u},v)} \subseteq O_v$. Let O'_v be the projection of O_v into $\zeta^{\perp}$. Since O_v is open then O'_v is relatively open in $\zeta^{\perp}$. For each $y \in O'_v$ let $\tilde{y}$ be the intersection of $\mathcal{S}$ with the half-line originating in u_0 and passing through y. Then the set

$$\tilde{O}_v = \{\tilde{y} \mid y \in O'_v\}$$

is relatively open in the sphere $\mathcal{S}$. Letting now v range over all unit vectors of $\lambda^{\perp}$, we define the family $\mathcal{O}$ by

$$\mathcal{O} = \{\tilde{O}_v \mid v \in \lambda^{\perp}\}.$$

Then $\mathcal{O}$ is an open cover of $\mathcal{S}$. The compactness of $\mathcal{S}$ yields a finite subfamily $\{\tilde{O}_{v(1)}, \tilde{O}_{v(2)}, \ldots, \tilde{O}_{v(k)}\}$ of $\mathcal{O}$ still covering $\mathcal{S}$. Each $v(i)$ comes together with a function $f_i = f_{v(i)} \in J \setminus J_{(\mathbf{u},v(i))}$ and some f_i-good triangulation $\mathcal{T}_i = \mathcal{T}_{v(i)}$ which is also $(\mathbf{u}, v(i))$-good.

Claim 1. *For each nonzero vector $w \in \lambda^{\perp}$ there is $i \in \{1, \ldots, k\}$ such that the closed star of $\mathcal{T}_i^{(\mathbf{u},v(i))}$ in $\mathcal{T}_i$ contains some $(\mathbf{u}, w)$-simplex*

$$\mathrm{conv}\{u_0, u_0 + \epsilon_1 u_1, \ldots, u_0 + \epsilon_1 u_1 + \cdots + \epsilon w\}$$

whose vertex $u_0 + \epsilon_1 u_1 + \cdots + \epsilon w$ lies in $O_{v(i)}$.

As a matter of fact, let $x = u_0 + w \in \zeta^{\perp}$. Let $\tilde{x}$ be the intersection of $\mathcal{S}$ with the half-line originating in u_0 and passing through x. Since $\{\tilde{O}_{v(1)}, \tilde{O}_{v(2)}, \ldots, \tilde{O}_{v(k)}\}$ is an open cover of $\mathcal{S}$, there exists a $v(i)$ together with a y in the projection $O'_{v(i)}$ such that y coincides with $u_0 + \delta w$ for some $\delta > 0$. Thus, there is a point $z = u_0 + \epsilon_1 u_1 + \cdots \epsilon_t u_t + \epsilon w \in O_{v(i)}$ whose projection into $\zeta^{\perp}$ coincides with y. By definition of $O_{v(i)}$ there is an n-simplex R in the star of $\mathcal{T}_i^{(\mathbf{u},v(i))}$ such that $z \in R$. Since R is convex and $\mathcal{T}_i^{\mathbf{u}}$ is a proper face of R we have

$$\mathrm{conv}\{u_0, u_0 + \epsilon_1 u_1, \ldots, u_0 + \epsilon_1 u_1 + \ldots \epsilon_t u_t + \epsilon w\} \subseteq R \subseteq \mathrm{clstar}(\mathcal{T}_i^{(\mathbf{u},v(i))}; \mathcal{T}_i),$$

and the claim follows.

Let the function $f \in J$ be defined by

$$f = f_1 \vee f_2 \vee \cdots \vee f_k. \tag{10}$$

In the light of Lemma 2.4, let $\mathcal{T}$ be an f-good triangulation that jointly refines each triangulation $\mathcal{T}_1, \ldots, \mathcal{T}_k$. By (8) and Lemma 2.11(iii), $f_i \restriction \mathcal{T}^{\mathbf{u}} = 0$ for each $i = 1, \ldots, k$. Thus,

$$f \restriction \mathcal{T}^{\mathbf{u}} = 0. \tag{11}$$

Claim 2. $f(x) > 0$ *for each $x \in \mathrm{clstar}(\mathcal{T}^{\mathbf{u}}) \setminus \mathcal{T}^{\mathbf{u}}$.*

Write for short Q instead of $\mathrm{ostar}(\mathcal{T}^{\mathbf{u}}) \setminus \mathcal{T}^{\mathbf{u}}$. We first assume $x \in Q$. Then x belongs to relint T, for a uniquely determined *smallest* simplex T in the star of $\mathcal{T}^{\mathbf{u}}$. Further, $\mathcal{T}^{\mathbf{u}}$ is a proper face of T, whence $\dim T > \dim \mathcal{T}^{\mathbf{u}}$. The vector $x - u_0$ can be uniquely written as $x - u_0 = l + v$ where $l \in \lambda$, $v \in \lambda^{\perp}$. We have $v \neq 0$ because $x \notin \mathcal{T}^{\mathbf{u}}$. It follows that T contains some $(\mathbf{u}, v)$-simplex. For any subset O of $[0, 1]^n$ let $\overline{O}$ denote the closure of O. By Claim 1, for some $i = 1, \ldots, k$ the closed star $\overline{O_{v(i)}}$ of $\mathcal{T}_i^{(\mathbf{u},v(i))}$ in $\mathcal{T}_i$ contains some $(\mathbf{u}, v)$-simplex, whence so does $T \cap \overline{O_{v(i)}}$ by Proposition 2.2. So let $T \cap \overline{O_{v(i)}} \supseteq T' = \mathrm{conv}\{u_0, u_0 + \omega_1 u_1, \ldots, u_0 + \omega_1 u_1 + \cdots + \omega v\}$ for suitable $\omega_i > 0$. Let $c \in \mathrm{relint}\, T'$. Then $c \in O_{v(i)}$ and from (9) we obtain $f_i(c) > 0$. Since $f \geq f_i > 0$ over $O_{v(i)}$ we have $f(c) > 0$. From $c \in \mathrm{relint}\, T$ it follows that $f > 0$ over relint T, because $T \in \mathcal{T}$ and $\mathcal{T}$ is f-good. Thus $f(x) > 0$ for all $x \in \mathrm{relint}\, T$, and our claim is settled in the special case when $x \in Q$. Assume now $x \in \mathrm{clstar}(\mathcal{T}^{\mathbf{u}}) \setminus \mathcal{T}^{\mathbf{u}}$. Then there is a point $y \in \mathrm{relint}\, \mathcal{T}^{\mathbf{u}}$ (e.g., $y =$ Farey mediant of the vertices of $\mathcal{T}^{\mathbf{u}}$) such that the segment $[x, y]$ contains some point $z \in \mathrm{ostar}(\mathcal{T}^{\mathbf{u}}) \setminus \mathcal{T}^{\mathbf{u}}$. The segment $[x, y]$ is contained in some simplex of the star of $\mathcal{T}^{\mathbf{u}}$, and f is linear over such simplex. By (11), $f(y) = 0$ and by our previous discussion, $f(z) > 0$. Then $f(x) > 0$, and our claim is settled.

The proof is complete. $\square$

Theorem 2.17. *Let* $\mathbf{u} = (u_0, u_1, \ldots, u_t)$ *be an index and* J *a prime ideal with* $J \subseteq J_{\mathbf{u}}$*. If there does not exist a* proper *extension* $\mathbf{v}$ *of* $\mathbf{u}$ *such that* J *is contained in* $J_{\mathbf{v}}$ *then* $J = J_{\mathbf{u}}$.

Proof. Case 1. $\dim \zeta(\mathbf{u}) = n$.

This is equivalent to saying that for any possible choice of a vector $v \in \mathbb{R}^n$, the index $(\mathbf{u}, v)$ is not a proper extension of $\mathbf{u}$. Suppose that $J \subsetneqq J_{\mathbf{u}}$ (absurdum hypothesis). Let $f \in J_{\mathbf{u}} \setminus J$ and let $\mathcal{T}$ be a $\mathbf{u}f$-good triangulation. Then $\dim \mathcal{T}^{\mathbf{u}} = n$ and $f \restriction \mathcal{T}^{\mathbf{u}} = 0$. As in the proof of Theorem 2.15, let $\mathcal{W}$ be the refinement of $\mathcal{T}$ obtained by starring $\mathcal{T}$ at the mediant b of $\mathcal{T}^{\mathbf{u}}$. Then $b \in \operatorname{relint} \mathcal{T}^{\mathbf{u}}$. In the light of Lemma 2.5 let the McNaughton function $g \in Free_n$ be uniquely determined by specifying its value at each vertex of $\mathcal{W}$ as follows:

$$g(x) = \begin{cases} 1 & \text{if } x = b, \\ 0 & \text{if } x \text{ is any other vertex of } \mathcal{W}, \end{cases}$$

with g linear over each simplex of $\mathcal{W}$. Since $f \restriction \mathcal{T}^{\mathbf{u}} = 0$ we can write $g \wedge f = 0$, whence $f \wedge g \in J$. By construction, $g \notin J_{\mathbf{u}}$, whence $g \notin J$; since $f \notin J$ we conclude that J is not prime, a contradiction.

Case 2. $\dim \zeta(\mathbf{u}) < n$.

Let g be an arbitrary function in $J_{\mathbf{u}}$. An application of Proposition 2.16 yields a function $f \in J$ and a $\mathbf{u}f$-good triangulation $\mathcal{T}$ satisfying conditions (i)–(ii) therein. We shall construct a function $h \in J$ such that g is in the ideal generated by $f \oplus h$, thus showing that $J_{\mathbf{u}} = J$. To this purpose, let $\mathcal{V}$ be a $\mathbf{u}fg$-good triangulation refining $\mathcal{T}$. By Lemma 2.11, $g \restriction \mathcal{V}^{\mathbf{u}} = 0$. As an application of Lemma 2.5, let the McNaughton function $h \in Free_n$ be given by

$$h(x) = \begin{cases} 0 & \text{if } x \text{ is a vertex of some simplex in the star of } \mathcal{V}^{\mathbf{u}}, \\ 1 & \text{if } x \text{ is any other vertex of } \mathcal{V}, \end{cases}$$

with h being assumed linear over each simplex of $\mathcal{V}$. By Theorem 2.15, $h \in \operatorname{germ}(J_{\mathbf{u}})$. Since J is prime and $J \subseteq J_{\mathbf{u}}$, it follows that $h \in J$. As an effect of Proposition 2.16 together with the inclusion $\mathcal{V}^{\mathbf{u}} \subseteq \mathcal{T}^{\mathbf{u}}$ given by Lemma 2.11(iii), the only points in the closed star of $\mathcal{V}^{\mathbf{u}}$ where f vanishes are those of $\mathcal{V}^{\mathbf{u}}$. Therefore, $(f \oplus h)(x) = 0$ iff $x \in \mathcal{V}^{\mathbf{u}}$. Since $g \restriction \mathcal{V}^{\mathbf{u}} = 0$ and $f \oplus h \in J$, by an application of [1, Lemma 3.4.8] we conclude that $g \in J$, as desired. □

Corollary 2.18. *Every prime ideal* J *of* $Free_n$ *has the form* $J = J_{\mathbf{u}}$ *form some index* $\mathbf{u}$.

Proof. Every prime ideal of $Free_n$ is contained in exactly one maximal ideal [1, Corollary 1.2.12]. By [1, 3.4.7], maximal ideals of $Free_n$ are exactly those of the form $J_x = \{f \in Free_n \mid f(x) = 0\}$ for some $x \in [0, 1]^n$. So let $\mathbf{u} = (u_0, \ldots, u_t)$ be and index such that $J_{\mathbf{u}} \supseteq J$, and for no proper extension $\mathbf{v}$ of $\mathbf{u}$ it is the case that $J_{\mathbf{v}} \supseteq J$. An application of Theorem 2.17 shows that $J = J_{\mathbf{u}}$. □

Let $\mathbf{u} = (u_0, \ldots u_t)$ be an index. If the vector u_{i+1} belongs to the linear space $\lambda(u^i)$, then $\lambda(u^{i+1}) = \lambda(u^i)$, and in a sense that will be made precise in the next proposition, u_{i+1} is redundant in $\mathbf{u}$. An index $\mathbf{u}$ is said to be *reduced* if for all $i = 0, \ldots, t-1$, $\dim \lambda(u^i) < \dim \lambda(u^{i+1})$. Equivalently, $\mathbf{u}$ is reduced iff for every $\mathbf{u}$-good triangulation $\mathcal{T}$ we have $\mathcal{T}^{u^0} \prec \mathcal{T}^{u^1} \prec \cdots \prec \mathcal{T}^{u^{t-1}} \prec \mathcal{T}^{u^t}$, where $\prec$ denotes proper subface.

The following strengthening of Corollary 2.18 shall find several applications in the rest of our paper: [1]

Proposition 2.19. *For every prime ideal* J *of* $Free_n$ *there exists a reduced index* $\mathbf{u}$ *such that* $J = J_{\mathbf{u}}$.

Proof. Corollary 2.18 yields an index $\mathbf{u} = (u_0, \ldots, u_t)$ such that $J = J_{\mathbf{u}}$. If $\mathbf{u}$ is reduced we are done. Otherwise, we define the function κ by

$$\kappa(1) = \min\{s > 0 \mid u_s \notin \lambda(u^0)\}, \quad \text{and for } 2 \le j \le r$$
$$\kappa(j) = \min\{s > \kappa(j-1) \mid u_s \notin \lambda(u_0, \ldots, u_{\kappa(j-1)})\}.$$

[1] This result was first proved by Panti in [13, Corollary 4.9] using the Baker–Beynon representation of free vector lattices and free abelian lattice-ordered groups, together with the categorical equivalence Γ of [7, 3.9] between MV-algebras and abelian lattice-ordered groups with strong order-unit. Our shorter proof here only uses elementary MV-algebraic machinery.

Let $\mathbf{v} = (u_0, u_{\kappa(1)}, \ldots, u_{\kappa(r)})$ and observe that $r < t$. Direct verification shows that $\mathbf{v}$ is a reduced index. From the definition of $\mathbf{v}$ we see that $\zeta(\mathbf{u}) = \zeta(\mathbf{v})$. For any $\mathbf{uv}$-good triangulation $\mathcal{T}$ we also have $\mathcal{T}^{\mathbf{u}} = \mathcal{T}^{\mathbf{v}}$. As a matter of fact, by definition of $\mathbf{v}$, any $\mathbf{v}$-simplex is contained in some $\mathbf{u}$-simplex and any $\mathbf{u}$-simplex contains some $\mathbf{v}$-simplex. Thus $\mathcal{T}^{\mathbf{u}} \subseteq \mathcal{T}^{\mathbf{v}}$. Since $\dim \mathcal{T}^{\mathbf{u}} = \dim \mathcal{T}^{\mathbf{v}}$, then $\mathcal{T}^{\mathbf{u}} = \mathcal{T}^{\mathbf{v}}$, as desired. Finally, by Lemma 2.11(v), $J_{\mathbf{u}} = J_{\mathbf{v}}$. □

3. Equal indexes for the same prime ideal

The principal topic of this section is the introduction of necessary and sufficient conditions for two reduced indexes to represent the same prime ideal.

Proposition 3.1. *Let $\mathbf{u} = (u_0, u_1, \ldots, u_t)$ and $\mathbf{v} = (v_0, v_1, \ldots, v_t)$ be two reduced indexes with $\zeta(u^{t-1}) = \zeta(v^{t-1})$. Let $\mathcal{T}$ be a $\mathbf{uv}$-good triangulation, and suppose that $\mathcal{T}^{\mathbf{u}}$ is a* proper *subface of $\mathcal{T}^{\mathbf{v}}$, in symbols,*

$$\mathcal{T}^{\mathbf{u}} \prec \mathcal{T}^{\mathbf{v}}. \tag{12}$$

Then there exists a refinement $\mathcal{W}$ of $\mathcal{T}$ such that $\mathcal{W}^{u^{t-1}} = \mathcal{W}^{v^{t-1}}$, $\mathcal{W}^{\mathbf{u}} \not\subseteq \mathcal{W}^{\mathbf{v}}$ and $\mathcal{W}^{\mathbf{v}} \not\subseteq \mathcal{W}^{\mathbf{u}}$.

Proof. One first verifies that

$$\mathcal{T}^{u^{t-1}} = \mathcal{T}^{v^{t-1}}. \tag{13}$$

As a matter of fact, $\mathcal{T}^{u^{t-1}} \prec \mathcal{T}^{\mathbf{u}} \prec \mathcal{T}^{\mathbf{v}}$, $\mathcal{T}^{v^{t-1}} \prec \mathcal{T}^{\mathbf{v}}$ and both $\mathcal{T}^{v^{t-1}}$ and $\mathcal{T}^{u^{t-1}}$ lie on the space $\zeta(u^{t-1}) = \zeta(v^{t-1})$. This proves (13).

By hypothesis, since $\mathcal{T}^{\mathbf{u}}$ is a proper face of $\mathcal{T}^{\mathbf{v}}$, we have that $\dim(\zeta(\mathbf{u})) < \dim(\zeta(\mathbf{v}))$. By definition of $\zeta(\mathbf{u})$ we can write

$$\mathrm{conv}\{v_0, v_0 + v_1, \ldots, v_0 + v_1 + \cdots + v_t\} \not\subseteq \zeta(\mathbf{u}),$$

while $\mathrm{conv}\{u_0, u_0 + u_1, \ldots, u_0 + u_1 + \cdots + u_t\} \subseteq \zeta(\mathbf{u})$. Since

$$\mathrm{conv}\{u_0, u_0 + u_1, \ldots, u_0 + \cdots + u_{t-1}\} \subseteq \zeta(\mathbf{u}) \supseteq \mathrm{conv}\{v_0, v_0 + v_1, \ldots, v_0 + \cdots + v_{t-1}\}$$

we conclude that the points $v = v_0 + v_1 + \cdots + v_t$ and $u = u_0 + u_1 + \cdots + u_t$ must be distinct.

Claim. *There exists a hyperplane H satisfying:*

(1) $\mathcal{T}^{u^{t-1}} = \mathcal{T}^{v^{t-1}} \subseteq H$;
(2) $v \in \mathrm{int}\, H^{+}$;
(3) $u \in \mathrm{int}\, H^{-}$.

As a matter of fact, from our assumption that $\mathbf{v}$ and $\mathbf{u}$ are reduced and $\mathcal{T}^{\mathbf{u}} \prec \mathcal{T}^{\mathbf{v}}$ it follows that codim $\zeta(u^{t-1}) \geq 2$. Let z be a point in the interior of the line segment between u and v. The simplex

$$T = \mathrm{conv}\{v_0, v_0 + v_1, \ldots, v_0 + v_1 + \cdots + v_{t-1}, v_0 + v_1 + \cdots + v_{t-1} + z\}$$

satisfies $\dim T \leq n-1$. For simplicity let us first assume that $\dim T = n-1$. Let K be the hyperplane determined by T. Then K satisfies conditions (1)–(3). Let $a_0, a_1, \ldots, a_n$ be the coefficients of K. By slightly perturbing – if necessary – the coefficients a_i, we may find a hyperplane H such that H still satisfies conditions (1)–(3), and the coefficients of H are all rational. Indeed, continuity ensures that conditions (2) and (3) are preserved under small perturbations; and condition (1) is fulfilled by H as a consequence of the rationality of $\zeta(u^{t-1})$. The case $\dim T < n - 1$ is proved in a similar way. This settles our claim.

Letting now $\mathcal{W}$ be a refinement of $\mathcal{T}$ that respects H, as given by Lemma 2.4, a moment's reflection shows that $\mathcal{W}^{\mathbf{u}}$ and $\mathcal{W}^{\mathbf{v}}$ have the desired properties. □

Theorem 3.2. *Let $\mathbf{u} = (u_0, u_1, \ldots, u_t)$ and $\mathbf{v} = (v_0, v_1, \ldots, v_r)$ be two reduced indexes such that $J_{\mathbf{u}} = J_{\mathbf{v}}$. Then $t = r$, $u_0 = v_0$ and for each $0 \leq j \leq t$, $J_{u^j} = J_{v^j}$.*

Proof. For definiteness assume $t \le r$. If (absurdum hypothesis) $u_0 \neq v_0$ then by [1, Proposition 3.4.7] the maximal ideals J_{u_0} and J_{v_0} are different. By Lemma 2.13, $J_{\mathbf{u}} \subseteq J_{u_0}$ and $J_{\mathbf{u}} = J_{\mathbf{v}} \subseteq J_{v_0}$, thus contradicting the fact that a prime ideal is contained in precisely one maximal ideal [1, Corollary 1.2.12]. This shows that $u_0 = v_0$ and $J_{u_0} = J_{v_0}$.

Next we prove the identity

$$J_{u^i} = J_{v^i}, \quad 1 \le i \le t. \tag{14}$$

By way of contradiction let $j = \min\{i \mid J_{u^i} \neq J_{v^i}\}$. Since $j \ge 1$ and $J_{u^{j-1}} = J_{v^{j-1}}$, by Lemma 2.11 we can write

$$\mathcal{U}^{u^{j-1}} = \mathcal{U}^{v^{j-1}} \tag{15}$$

for every **uv**-good triangulation $\mathcal{U}$. As a matter of fact, if $\mathcal{U}$ is a counterexample and, say, x is a vertex in $\mathcal{U}$ with $x \in \mathcal{U}^{v^{j-1}} \setminus \mathcal{U}^{u^{j-1}}$ then let the function $f \in Free_n$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is a vertex of } \mathcal{U}^{u^{j-1}}, \\ 1 & \text{if } x \text{ is any other vertex of } \mathcal{U}, \end{cases}$$

with f linear over each simplex of $\mathcal{U}$. By Lemma 2.5 f is well defined. It follows that $f \in J_{u^{j-1}}$ and $f \notin J_{v^{j-1}}$, which is impossible. This settles (15).

In order to prove (14) we consider the three possible cases for our absurdum hypothesis $J_{u^j} \neq J_{v^j}$:

Case 1. $J_{u^j} \not\subseteq J_{v^j}$ and $J_{v^j} \not\subseteq J_{u^j}$.

There must be a **uv**-good triangulation $\mathcal{U}$ such that $\mathcal{U}^{v^j} \not\subseteq \mathcal{U}^{u^j}$; for otherwise, by Lemma 2.11(iv), every function $f \in J^{u^j}$ would also be in J^{v^j}, against our standing hypothesis. Symmetrically, there is a **uv**-good triangulation $\mathcal{V}$ such that $\mathcal{V}^{u^j} \not\subseteq \mathcal{V}^{v^j}$. Let $\mathcal{T}$ be a refinement of both $\mathcal{U}$ and $\mathcal{V}$. Then $\mathcal{T}$ is a **uv**-good triangulation with the following properties:

(i) $\mathcal{T}^{u^{j-1}} = \mathcal{T}^{v^{j-1}}$,
(ii) $\mathcal{T}^{u^j} \not\subseteq \mathcal{T}^{v^j}$, and
(iii) $\mathcal{T}^{v^j} \not\subseteq \mathcal{T}^{u^j}$

where the first equality follows from (15). Therefore, there exists a vertex $x \in \mathcal{T}^{u^j} \setminus \mathcal{T}^{v^j}$, and a vertex $y \in \mathcal{T}^{v^j} \setminus \mathcal{T}^{u^j}$. If $\dim \mathcal{T}^{u^{j-1}} = n-1$, letting H be the hyperplane of $\mathcal{T}^{u^{j-1}}$ in n-space we immediately see that $x \in H^+ \setminus H$ and $y \in H^- \setminus H$. If $\dim \mathcal{T}^{u^{j-1}} < n-1$, then, again, x and y can be separated by a hyperplane $H \supseteq \mathcal{T}^{u^{j-1}}$. In the light of Lemma 2.4 let $\mathcal{W}$ be a refinement of $\mathcal{T}$ that respects H. We can write

$$\mathcal{W}^{u^{j-1}} \subseteq H, \qquad \mathcal{W}^{u^t} \subseteq H^+ \quad \text{and} \quad \mathcal{W}^{v^r} \subseteq H^-.$$

Let U be the set of vertices of $\mathcal{W}^{u^t}$. Using Lemma 2.5, let the McNaughton function $f \in Free_n$ be uniquely defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in U, \\ 1 & \text{if } x \text{ is any other vertex of } \mathcal{W}, \end{cases}$$

with f linear over each simplex of $\mathcal{W}$. Then $f \in J_{\mathbf{u}}$ and since there is at least one vertex in $\mathcal{W}^{v^r} \setminus \mathcal{W}^{u^t}$ we have $f \notin J_{\mathbf{v}}$, a contradiction.

Case 2. J_{u^j} is strictly contained in J_{v^j}.

As a matter of fact, for every **uv**-good triangulation $\mathcal{V}$ we necessarily have $\mathcal{V}^{v^j} \subseteq \mathcal{V}^{u^j}$ (for otherwise, arguing as in the proof of (15) we would obtain a function $f \in J_{u^j} \setminus J_{v^j}$, which is impossible). Moreover, the proper inclusion of J_{u^j} in J_{v^j} yields a **uv**-good triangulation $\mathcal{U}$ such that $\mathcal{U}^{v^j}$ is a proper subface of $\mathcal{U}^{u^j}$. As a consequence of (15), the triangulation $\mathcal{U}$ also satisfies $\mathcal{U}^{u^{j-1}} = \mathcal{U}^{v^{j-1}}$. Therefore, $\zeta(u^{j-1}) = \zeta(v^{j-1})$, whence by Proposition 3.1 there exists a refinement $\mathcal{T}$ of $\mathcal{U}$ such that $\mathcal{T}^{u^{j-1}} = \mathcal{T}^{v^{j-1}}$, $\mathcal{T}^{u^j} \not\subseteq \mathcal{T}^{v^j}$ and $\mathcal{T}^{v^j} \not\subseteq \mathcal{T}^{u^j}$. The same argument as in the previous case again yields a contradiction.

Case 3. J_{v^j} is strictly contained in J_{u^j}.

This is similar to Case 2.

We have just proved (14). To conclude the proof, by way of contradiction, suppose $t < r$ and let $\mathcal{T}$ be a **uv**-good triangulation. Since the indexes **u** and **v** are reduced then $\mathcal{T}^{u^t}$ is a proper subface of $\mathcal{T}^{v^r}$. Let U be the set of vertices in $\mathcal{T}^{u^t}$. Using Lemma 2.5, let the McNaughton function $g \in Free_n$ be defined by

$$g(x) = \begin{cases} 0 & \text{if } x \in U, \\ 1 & \text{if } x \text{ is any other vertex of } \mathcal{T}, \end{cases}$$

with g linear over each simplex of $\mathcal{T}$. Then Lemma 2.11(v)–(vi) shows that $g \in J_{\mathbf{u}}$ and $g \notin J_{\mathbf{v}}$, again contradicting $J_{\mathbf{u}} = J_{\mathbf{v}}$. In conclusion, $t = r$, as required to complete the proof of the theorem. □

Remark. In the light of Proposition 2.19 we can now assign to every prime ideal J of $Free_n$ a uniquely determined integer $r = r_J \geq 0$, where

$$r_J + 1 = \text{ number of elements of any reduced index of } J. \tag{16}$$

Our results show that r_J is the length of the maximal path of prime ideals

$$J_0 \supseteq J_1 \supseteq \cdots \supseteq J_{r_J} = J$$

leading to J from the maximal ideal above J. It is natural to say that r_J is the *(Krull) depth* of J.

Corollary 3.3. *Let* $\mathbf{u} = (u_0, u_1, \ldots, u_t)$ *and* $\mathbf{v} = (v_0, v_1, \ldots, v_k)$ *be two reduced indexes and let* $j \leq \min(t, k)$. *If* $\zeta(u^{j-1}) = \zeta(v^{j-1})$ *and* $\mathcal{T}^{u^j} \subsetneqq \mathcal{T}^{v^j}$ *for some* **uv**-*good triangulation* $\mathcal{T}$, *then* $J_{\mathbf{u}} \neq J_{\mathbf{v}}$.

Proof. First note that $\mathcal{T}^{u^j}$ is a proper subface of $\mathcal{T}^{v^j}$. By Proposition 3.1 there exists a refinement $\mathcal{W}$ of $\mathcal{T}$ such that $\mathcal{W}^{u^j} \nsubseteq \mathcal{W}^{v^j}$ and $\mathcal{W}^{v^j} \nsubseteq \mathcal{W}^{u^j}$. A routine variant of the argument given in the proof of Case 1 of Theorem 3.2, yields the desired conclusion. □

Definition 3.4. Given a reduced index $\mathbf{u} = (u_0, u_1, \ldots, u_t)$, for each $0 < j \leq t$ we define the set $\theta(u^j)$, by the stipulation

$$\theta(u^j) = \{x \in \mathbb{R}^n \mid x = y + \beta u_j \text{ for some } y \in \lambda(u^{j-1}) \text{ and } 0 < \beta \in \mathbb{R}\}. \tag{17}$$

Theorem 3.5. *Let* $\mathbf{u} = (u_0, u_1, \ldots, u_t)$ *and* $\mathbf{v} = (v_0, v_1, \ldots, v_t)$ *be two reduced indexes. Then the following conditions are equivalent:*

(i) $J_{\mathbf{u}} = J_{\mathbf{v}}$;
(ii) *For every* **uv**-*good triangulation* $\mathcal{T}$, *the sequence* $(\mathcal{T}^{u^0}, \ldots, \mathcal{T}^{u^t})$ *coincides with* $(\mathcal{T}^{v^0}, \ldots, \mathcal{T}^{v^t})$;
(iii) $u_0 = v_0$ *and* $u_j \in \theta(v^j)$ *for all* $j = 1, \ldots, t$.

Proof. (ii) → (i) is an easy consequence of the Lemma 2.11(v)–(vi). The converse implication (i) → (ii) follows from Corollary 3.3.

(iii) → (ii). Let $\mathcal{T}$ be a **uv**-good triangulation. Then trivially, $\mathcal{T}^{u^0} = \mathcal{T}^{v^0}$, $\zeta(u^0) = \zeta(v^0)$ and $\lambda(u^0) = \lambda(v^0)$. Turning attention to $j = 1$, the hypothesis $u_1 \in \theta(v^1)$ yields an element $y \in \lambda(v^0)$ such that $u_1 = y + \beta v_1$ for some $0 < \beta \in \mathbb{R}$. Let $\epsilon_0 > 0$ be such that $u_0 + \epsilon_0 y \in \text{relint}\,(\mathcal{T}^{u^0})$. By definition of $\mathcal{T}^{v^1}$ for each $z \in \text{relint}\,\mathcal{T}^{u^0} = \text{relint}\,\mathcal{T}^{v^0}$, there exists $0 < \epsilon_z$ such that $\text{conv}\{z, z+\epsilon_z v_1\} \subseteq \mathcal{T}^{v^1}$. Thus there exists ϵ_1 such that $\text{conv}\{u_0, u_0+\epsilon_1(y+\beta v_1)\} \subseteq \mathcal{T}^{v^1}$, that is, $\text{conv}\{u_0, u_0 + \epsilon_1 u_1\} \subseteq \mathcal{T}^{v^1}$. By definition of $\mathcal{T}^{u^1}$ we can write $\mathcal{T}^{u^1} \subseteq \mathcal{T}^{v^1}$. On the other hand, from $\lambda(u^0) = \lambda(v^0)$, it follows that $v_1 = -\frac{1}{\beta} y + \frac{1}{\beta} u_1$, whence $v_1 \in \theta(u^1)$. Symmetrically, $\mathcal{T}^{u^1} \supseteq \mathcal{T}^{v^1}$, whence $\mathcal{T}^{u^1} = \mathcal{T}^{v^1}$, $\zeta(u^1) = \zeta(v^1)$ and $\lambda(u^1) = \lambda(v^1)$.

Proceeding now by induction, assume $\mathcal{T}^{u^i} = \mathcal{T}^{v^i}$, $\zeta(u^i) = \zeta(v^i)$ and $\lambda(u^i) = \lambda(v^i)$ for all $i < j$. Let $y \in \lambda(v^{j-1})$ be such that $u_j = y + \beta v_j$ for some $0 < \beta \in \mathbb{R}$. Since $\mathcal{T}^{u^{j-1}} = \mathcal{T}^{v^{j-1}}$ there exist $\epsilon_1, \ldots, \epsilon_{j-1} > 0$ such that

$$\text{relint}(\text{conv}\{u_0, u_0 + \epsilon_1 u_1, \ldots, u_0 + \epsilon_1 u_1 + \cdots + \epsilon_{j-1} u_{j-1}\}) \subseteq \text{relint}(\mathcal{T}^{v^{j-1}}).$$

Since $y \in \lambda(v^{j-1})$, there is $\delta > 0$ such that

$$\text{relint}(\text{conv}\{u_0, u_0 + \epsilon_1 u_1, \ldots, u_0 + \epsilon_1 u_1 + \cdots + \epsilon_{j-1} u_{j-1} + \delta y\}) \subseteq \text{relint}(\mathcal{T}^{v^{j-1}}).$$

Arguing as for the case $j = 1$, by definition of $\mathcal{T}^{v^j}$ we have real $0 < \epsilon_j$ such that

$$\text{conv}\{u_0, \ldots, u_0 + \epsilon_1 u_1 + \cdots + \epsilon_{j-1} u_{j-1}, u_0 + \epsilon_1 u_1 + \cdots + \epsilon_{j-1} u_{j-1} + \epsilon_j (y + \beta v_j)\} \subseteq \mathcal{T}^{v^j}.$$

We conclude that $\mathcal{T}^{u^j} \subseteq \mathcal{T}^{v^j}$. For the converse inclusion one similarly notes that $v_j = -\frac{1}{\beta} y + \frac{1}{\beta} u_j$ for some $y \in \lambda(u^{j-1})$, i.e., $v_j \in \theta(u^j)$. Thus, $\mathcal{T}^{u^j} \supseteq \mathcal{T}^{v^j}$. The rest is clear.

(ii) → (iii). Suppose that $\mathcal{T}^{u^j} = \mathcal{T}^{v^j}$ for every **uv**-good triangulation $\mathcal{T}$ and all $j = 0, 1, \ldots, t$. Then $\zeta(u^j) = \zeta(v^j)$ and $\lambda(u^j) = \lambda(v^j)$. If $u_0 \neq v_0$ (absurdum hypothesis) then there exists a hyperplane H such that $u_0 \in H^+ \setminus H$ and $v_0 \in H^- \setminus H$. Let $\mathcal{T}$ be a **uv**-good triangulation that respects H. Then, trivially, $\mathcal{T}^{u^0} \neq \mathcal{T}^{v^0}$, against our hypothesis.

Having thus proved that $u_0 = v_0$, by way of contradiction let

$$j = \min\{i \mid u_i \notin \theta(v^i)\} \geq 1. \tag{18}$$

It follows that

$$u_j \in \lambda(v^j), \tag{19}$$

for otherwise $\zeta(v^j) \neq \zeta(u^j)$, whence for every **uv**-good triangulation $\mathcal{T}$ it would follow that $\mathcal{T}^{u^j} \neq \mathcal{T}^{v^j}$, which is impossible. Also notice that

$$u_j \notin \lambda(v^{j-1}), \tag{20}$$

because $\lambda(v^{j-1}) = \lambda(u^{j-1})$ and $\mathbf{u}$ is reduced. Let

$$a_1, \ldots, a_r, v_j, w_1, \ldots, w_s$$

be a basis for $\lambda(v^j)$ such that $a_1, \ldots, a_r$ form a basis of $\lambda(v^{j-1})$. By (19) there are uniquely determined coefficients such that

$$u_j = \gamma_1 a_1 + \cdots + \gamma_r a_r + \beta v_j + \alpha_1 w_1 + \cdots + \alpha_s w_s.$$

As a consequence of our assumption that $u_j \notin \theta(v^j)$, it is not the case that $\beta > 0$ and all α's are zero. By (20), not all of β and the α's are zero. We shall see that the two remaining possibilities lead to contradiction.

Case 1. Some α is nonzero.

Then some hyperplane $H \supseteq \lambda(v^{j-1})$ *separates* the points $u = u_0 + u_1 + \cdots + u_j$ and $v = v_0 + v_1 + \cdots + v_j$, in the sense that $u \in H^+$ and $v \in H^-$. H is constructed as in Proposition 3.1. Any triangulation $\mathcal{T}$ which is **uv**-good and respects H will satisfy the condition $\mathcal{T}^{v^j} \neq \mathcal{T}^{u^j}$, against our hypothesis.

Case 2. All α's are zero, and $\beta \leq 0$.

Then necessarily $\beta < 0$ by (19)–(20). Again, some hyperplane $H \supseteq \lambda(v^{j-1})$ separates $u = u_0 + u_1 + \cdots + u_j$ and $v = v_0 + v_1 + \cdots + v_j$. Any triangulation $\mathcal{T}$ which is **uv**-good and respects H will contradict our hypothesis.

This completes the proof of the theorem. □

4. Projections and prime ideal extensions

Suppose I is a prime ideal in $Free_n$ and $m \leq n$. Then $K = I \cap Free_m$ is a prime ideal in $Free_m$. We shall be concerned with the relationships between indexes of K and indexes of I, as given by Proposition 2.19. Whenever $m \leq n$ we shall canonically identify $Free_m$ with the subalgebra of $Free_n$ given by all functions in $Free_n$ that only depend on the first m variables, in symbols,

$$Free_m \subseteq Free_n. \tag{21}$$

To avoid any danger of confusion, any index $\mathbf{u}$ where each u_i is in $\mathbb{R}^n$ (resp., each u_i is in $\mathbb{R}^m$) shall be called an *index for n-space* (resp., index for m-space). In Theorem 4.10 below we shall prove a result to the effect that, whatever reduced index $\mathbf{w}$ we choose for K we can always find for I a reduced index $\mathbf{u}$ whose projection into m-space is precisely equal to $\mathbf{w}$, once all zero vectors are deleted. This is the crucial step for proving the Robinson consistency property for Łukasiewicz logic.

Notation. For integers $0 < m \leq n$ and $x \in \mathbb{R}^n$ we let for short $\mathbb{P}_m(x)$ denote the projection of x into $\mathbb{R}^m$. For each set $S \subseteq \mathbb{R}^n$, we let $\mathbb{P}_m(S)$ be defined by

$$\mathbb{P}_m(S) = \{y \in \mathbb{R}^m \mid y = \mathbb{P}_m(x) \text{ for some } x \in S\}.$$

When the context is clear we shall simply write $\mathbb{P}$ instead of $\mathbb{P}_m$.

Throughout this section M will denote an arbitrary (but always rational and affine) hyperplane in m-space; H will still denote a hyperplane in n-space.

For later use we record here the following trivial fact:

Lemma 4.1. *Let H be a hyperplane in $\mathbb{R}^n$ and let $1 \leq m < n$. Then either $\mathbb{P}(H) = \mathbb{R}^m$ or $\mathbb{P}(H)$ is a rational and affine hyperplane in $\mathbb{R}^m$.*

For any hyperplane M in $\mathbb{R}^m$ we denote by $\mathrm{cyl}(M)$ the *cylindrification* of M in $\mathbb{R}^n$, i.e.,

$$\mathrm{cyl}(M) = \{x \in \mathbb{R}^n \mid \mathbb{P}(x) \in M\}.$$

Then $\mathrm{cyl}(M)$ is a (rational and affine) hyperplane in $\mathbb{R}^n$ and $\mathbb{P}(\mathrm{cyl}(M)) = M$. Further, for every hyperplane H in $\mathbb{R}^n$ we have $\mathbb{P}(H) \neq \mathbb{R}^m$ iff $H = \mathrm{cyl}(M)$ for some (necessarily unique) hyperplane M in $\mathbb{R}^m$.

For the sake of completeness we give a proof of the following elementary fact:

Lemma 4.2. *For each simplex $T \subseteq \mathbb{R}^n$ and each hyperplane $M \subseteq \mathbb{R}^m$ we have*

$$T \subseteq \mathrm{cyl}(M) \quad \textit{iff} \quad \mathbb{P}(T) \subseteq M.$$

Proof. If $T \subseteq \mathrm{cyl}(M)$, then $\mathbb{P}(T) \subseteq M$. On the other hand, if $T \nsubseteq \mathrm{cyl}(M)$, let $x \in T \setminus \mathrm{cyl}(M)$. Since the defining equation of $\mathrm{cyl}(M)$ depends only on the first m variables we have $\mathbb{P}(x) \notin M$ and $\mathbb{P}(x) \in \mathbb{P}(T)$. Then $\mathbb{P}(T) \nsubseteq M$. □

We also record the following

Lemma 4.3. *Let $T \subseteq \mathbb{R}^n$ be a simplex. Then we have*

$$\mathbb{P}\left(\bigcap\{H \mid T \subseteq H\}\right) = \mathbb{P}\left(\bigcap\{H \mid T \subseteq H \textit{ and } H = \mathrm{cyl}(M) \textit{ for some } M\}\right)$$
$$= \bigcap\{\mathbb{P}(H) \mid T \subseteq H \textit{ and } H = \mathrm{cyl}(M) \textit{ for some } M\} = \bigcap\{M \mid T \subseteq \mathrm{cyl}(M)\}.$$

Proof. The first identity is a direct consequence of Lemma 4.1. The second follows from the fact that every hyperplane M in $\mathbb{R}^m$ has the same defining equation as $\mathrm{cyl}(M)$. The last identity is trivial. □

Given an index $\mathbf{u} = (u_0, u_1, \ldots, u_t)$ for n-space the tuple $(\mathbb{P}(u_0), \mathbb{P}(u_1), \ldots, \mathbb{P}(u_t))$ need not be an index for m-space, because the vectors $\mathbb{P}(u_1), \ldots, \mathbb{P}(u_t)$ may fail to be independent. However, for a uniquely determined integer $r \geq 0$ (given by the Krull depth of $J_{\mathbf{u}}$ as in (16)) we can give the following

Definition 4.4. We define $\pi(\mathbf{u})$ by $\pi(\mathbf{u}) = (\mathbb{P}(u_0), \mathbb{P}(u_{\iota(1)}), \ldots, \mathbb{P}(u_{\iota(r)}))$, where

$$\iota(1) = \min\{s > 0 \mid \mathbb{P}(u_s) \notin \lambda(\mathbb{P}(u_0))\}, \quad \text{and for } 2 \leq j \leq r$$
$$\iota(j) = \min\{s > \iota(j-1) \mid \mathbb{P}(u_s) \notin \lambda((\mathbb{P}(u_0), \ldots, \mathbb{P}(u_{\iota(j-1)})))\}.$$

Then $\pi(\mathbf{u})$ is automatically a *reduced* index for m-space. The following is an immediate consequence of the definition:

Lemma 4.5. *Let* $\mathbf{u} = (u_0, u_1, \ldots, u_t)$ *and* $\pi(\mathbf{u}) = (\mathbb{P}(u_0), \mathbb{P}(u_{\iota(1)}), \ldots, \mathbb{P}(u_{\iota(r)}))$. *Let the simplexes* $T_{\pi(\mathbf{u})}$, $T_{\mathbb{P}(\mathbf{u})}$ *and* T *be defined by*

$$T_{\pi(\mathbf{u})} = \text{conv}\{\mathbb{P}(u_0), \mathbb{P}(u_0) + \mathbb{P}(u_{\iota(1)}), \ldots, \mathbb{P}(u_0) + \mathbb{P}(u_{\iota(1)}) + \cdots + \mathbb{P}(u_{\iota(r)})\},$$
$$T_{\mathbb{P}(\mathbf{u})} = \text{conv}\{\mathbb{P}(u_0), \mathbb{P}(u_0) + \mathbb{P}(u_1), \ldots, \mathbb{P}(u_0) + \mathbb{P}(u_1) + \cdots + \mathbb{P}(u_t)\},$$
$$T = \text{conv}\{u_0, u_0 + u_1, \ldots, u_0 + u_1 + \cdots + u_t\}.$$

It follows that

(i) $\mathbb{P}(T) = T_{\mathbb{P}(\mathbf{u})}$;
(ii) *for any hyperplane* $M \subseteq \mathbb{R}^m$, $T_{\pi(\mathbf{u})} \subseteq M$ *iff* $T_{\mathbb{P}(\mathbf{u})} \subseteq M$.

Proposition 4.6. *For every index* $\mathbf{u}$ *for* n*-space we have*

$$\mathbb{P}(\zeta(\mathbf{u})) = \zeta(\pi(\mathbf{u})).$$

Proof. By Lemmas 4.2 and 4.5 we can write $\zeta(\pi(\mathbf{u})) = \bigcap\{M \mid T_{\pi(\mathbf{u})} \subseteq M\} = \bigcap\{M \mid T_{\mathbb{P}(\mathbf{u})} \subseteq M\} = \bigcap\{M \mid T \subseteq \text{cyl}(M)\}$. Using the identity $\mathbb{P}(\zeta(\mathbf{u})) = \mathbb{P}(\bigcap\{H \mid T \subseteq H\})$ and applying Lemma 4.3 we obtain the desired conclusion. □

Theorem 4.7. *Let* $1 \le m \le n$. *Then for every index* $\mathbf{u}$ *for* n*-space we have*

$$J_{\mathbf{u}} \cap \mathit{Free}_m = J_{\pi(\mathbf{u})}.$$

Proof. For some $t \le n$ let us write $\mathbf{u} = (u_0, u_1, \ldots u_t)$. Let $f \in J_{\mathbf{u}} \cap \mathit{Free}_m$ and let $\mathcal{T}$ be a $\mathbf{u}f$-good triangulation of $[0, 1]^n$. Then $f\restriction\mathcal{T}^{\mathbf{u}} = 0$. Let $\mathcal{U}$ be a triangulation of $[0, 1]^m$ that $\mathbf{u}$-*reflects* the triangulation $\mathcal{T}$, in the sense that $\mathcal{U}$ is $\pi(\mathbf{u})f$-good and $\mathcal{U}^{\pi(\mathbf{u})} \subseteq \mathbb{P}(\mathcal{T}^{\mathbf{u}})$. The existence of such triangulation is guaranteed by Lemma 4.5 together with Proposition 4.6, because

$$\dim \mathcal{U}^{\pi(\mathbf{u})} = \dim \zeta(\pi(\mathbf{u})) = \dim \mathbb{P}(\zeta(\mathbf{u})) = \dim \mathbb{P}(\mathcal{T}^{\mathbf{u}}).$$

Since $f \in \mathit{Free}_m$ and $f\restriction\mathcal{U}^{\pi(\mathbf{u})} = 0$, we conclude that $f \in J_{\pi(\mathbf{u})}$.

Conversely, let $f \in J_{\pi(\mathbf{u})}$ and let $\mathcal{U}$ be a $\pi(\mathbf{u})f$-good triangulation of $[0, 1]^m$. Then $f \in \mathit{Free}_m$ and $f\restriction\mathcal{U}^{\pi(\mathbf{u})} = 0$. Let $\mathcal{T}$ be a $\mathbf{u}f$-good triangulation of $[0, 1]^n$ that $\pi(\mathbf{u})$-reflects $\mathcal{U}$. Stated otherwise, $\mathbb{P}(\mathcal{T}^{\mathbf{u}}) \subseteq \mathcal{U}^{\pi(\mathbf{u})}$. Again, the existence of $\mathcal{T}$ is guaranteed by Proposition 4.6 and Lemma 4.5. Now let $x \in \mathcal{T}^{\mathbf{u}}$. Since $\mathcal{T}$ $\pi(\mathbf{u})$-reflects $\mathcal{U}$, then $\mathbb{P}(x) \in \mathbb{P}(\mathcal{T}^{\mathbf{u}}) \subseteq \mathcal{U}^{\pi(\mathbf{u})}$. From the fact that $f \in \mathit{Free}_m$ we have $f(x) = f(\mathbb{P}(x)) = 0$, whence $f \in J_{\mathbf{u}}$ as desired. □

Definition 4.8. Given any arbitrary index $\mathbf{u} = (u_0, u_1, \ldots, u_t)$ for n-space, let $\pi(\mathbf{u}) = (\mathbb{P}(u_0), \mathbb{P}(u_{\iota(1)}), \ldots, \mathbb{P}(u_{\iota(k)}))$ be as in Definition 4.4. Let further $\mathbf{w} = (w_0, w_1, \ldots, w_k)$ be a reduced index for m-space. We say that $\pi(\mathbf{u})$ *adheres* to $\mathbf{w}$ if $\mathbb{P}(u_{\iota(j)}) = w_j$ for each $j = 0, 1, \ldots, k$ and $\mathbb{P}(u_s) = 0$ for each $s \in \{1, \ldots, t\}$ not belonging to the range of ι.

Proposition 4.9. *Let* $b \in \lambda(\pi(\mathbf{u}))$. *Then there exists* $t \in \lambda(\mathbf{u})$ *such that* $\mathbb{P}(t) = b$.

Proof. Immediate from Proposition 4.6. □

Our main tool to prove the Robinson consistency property for Łukasiewicz logic is given by the following:

Theorem 4.10. *Let* J *be a prime ideal in* Free_n, *and* $1 \le m \le n$. *Write* $\mathit{Free}_m \subseteq \mathit{Free}_n$ *as in* (21). *Let* $K = J \cap \mathit{Free}_m$. *Then* K *is a prime ideal of* Free_m. *In the light of Proposition* 2.19 *write* $K = J_{\mathbf{w}}$ *for some reduced index for* m*-space. Then there is a reduced index* $\mathbf{u}$ *for* n*-space such that*

(i) $J = J_{\mathbf{u}}$ *and*
(ii) $\pi(\mathbf{u})$ *adheres to* $\mathbf{w}$.

Proof (*Preliminaries*). It is easy to prove that K is a prime ideal in Free_m. Using Proposition 2.19 let $\mathbf{w} = (w_0, w_1, \ldots, w_k)$ be a reduced index for m-space such that $J_{\mathbf{w}} = K$. Also let $\mathbf{v} = (v_0, v_1, \ldots, v_t)$ be a reduced index for n-space such that $J = J_{\mathbf{v}}$. From Theorem 4.7 it follows that $J_{\mathbf{w}} = J_{\mathbf{v}} \cap \mathit{Free}_m = J_{\pi(\mathbf{v})}$. Since $\mathbf{w}$ and $\pi(\mathbf{v})$ are reduced indexes for the same ideal in m-space, by Theorem 3.2, $\mathbf{w}$ and $\pi(\mathbf{v})$ have the same length. We can write

$$\pi(\mathbf{v}) = (\mathbb{P}(v_0), \mathbb{P}(v_{\iota(1)}), \ldots, \mathbb{P}(v_{\iota(k)}),$$

where the map ι is as given in Definition 4.4. By Theorem 3.5, $\mathbb{P}(v_0) = w_0$ and for each $j = 1, \ldots, k$, $\mathbb{P}(v_{\iota(j)}) \in \theta(w^j)$, i.e.,

$$\mathbb{P}(v_{\iota(j)}) = y_j + \beta_j w_j \tag{22}$$

for some $y_j \in \lambda(w^{j-1})$ and $0 < \beta_j \in \mathbb{R}$. Let $B_0 = (b_0^1, \ldots, b_0^{s_0})$ be a basis for $\lambda(w^0) = \lambda(\mathbb{P}(v_0))$. For each $j = 1, \ldots, k$, let the basis B_j of $\lambda(w^j)$ be defined by:

(1) B_j extends B_{j-1}, i.e., if $b \in B_{j-1}$, then $b \in B_j$;
(2) all vectors $(b_j^1, b_j^2 \ldots b_j^{s_j}) \in B_j \setminus B_{j-1}$ satisfy the identity $b_j^1 = w_j$.

We now begin our construction of a reduced index $\mathbf{u}$ such that $J = J_{\mathbf{v}} = J_{\mathbf{u}}$, $\mathbb{P}(u_{\iota(j)}) = w_j$, and $\mathbb{P}(u_s) = 0$ for each s not belonging to the range of ι.

Construction of u_0 and $u_{\iota(1)}$. First of all, upon defining

$$u_0 = v_0,$$

we immediately obtain the desired identity $\mathbb{P}(u_0) = \mathbb{P}(v_0) = w_0$. By Proposition 4.9 there exists a set $T_0 = (t_0^1, \ldots, t_0^{s_0}) \subseteq \lambda(v^0) = \lambda(u^0)$ such that $\mathbb{P}(t_0^i) = b_0^i \in B_0$ for each $i = 1, \ldots, s_0$.

Next we construct $u_{\iota(1)}$. Since $\mathbb{P}(v_{\iota(1)}) \in \theta(w^1)$, we can write $y_1 = \sum_{j=1}^{s_0} \alpha_j b_0^j \in \lambda(w^0)$ for suitable $\alpha_j \in \mathbb{R}$ in such a way that

$$\mathbb{P}(v_{\iota(1)}) = y_1 + \beta_1 w_1.$$

Since $\beta_1 > 0$, we have

$$w_1 = -\frac{1}{\beta_1} y_1 + \frac{1}{\beta_1} \mathbb{P}(v_{\iota(1)}).$$

Letting $z_1 = \sum_{j=1}^{s_0} \alpha_j t_0^j \in \lambda(v^0) = \lambda(u^0)$, define

$$u_{\iota(1)} = -\frac{1}{\beta_1} z_1 + \frac{1}{\beta_1} v_{\iota(1)}.$$

Then

$$\mathbb{P}(u_{\iota(1)}) = -\frac{1}{\beta_1} \mathbb{P}(z_1) + \frac{1}{\beta_1} \mathbb{P}(v_{\iota(1)}) = w_1.$$

From $\lambda(v^0) \subseteq \lambda(v^{\iota(1)-1})$, it follows that $u_{\iota(1)} \in \theta(v^{\iota(1)})$. By Proposition 4.6, $\mathbb{P}(\zeta((u_0, u_{\iota(1)}))) = \zeta(w^1)$. Then by Proposition 4.9 for each $j = 2, \ldots, s_1$, there is a vector t_1^j in $\lambda((u_0, u_{\iota(1)}))$ such that $\mathbb{P}(t_1^j) = b_1^j \in B_1$. Let $T_1 = T_0 \cup \{u_{\iota(1)}, t_1^2, \ldots, t_1^{s_1}\}$. Since $u_0 = v_0$ and $u_{\iota(1)} \in \theta(v^{\iota(1)})$ it follows that

$$T_1 \subseteq \lambda((u_0, u_{\iota(1)})) \subseteq \lambda(v^{\iota(1)}).$$

Construction of $u_{\iota(j)}$ for $j \geq 2$. Proceeding by induction, suppose that we have already constructed $u_{\iota(i)}$ and T_i for every $0 < i < j$, in such a way that:

(1) $u_{\iota(i)} \in \theta(v^{\iota(i)})$;
(2) $\mathbb{P}(u_{\iota(i)}) = w_i$;
(3) $T_i \subseteq \lambda((u_0, u_{\iota(1)}, \ldots, u_{\iota(i)})) \subseteq \lambda(v^{\iota(i)})$, where $T_i = T_{i-1} \cup \{u_{\iota(i)}, t_i^2, \ldots t_i^{s_i})$ and $\mathbb{P}(t_i^r) = b_i^r \in B_i$ for all $1 < r \leq s_i$.

Since $\mathbb{P}(v_{\iota(j)}) \in \theta(w^j)$ let

$$y_j = \sum_{d=0}^{j-1} \left(\sum_{l=1}^{s_d} \alpha_l^d b_d^l \right)$$

be an element of $\lambda(w^{j-1})$ with $\alpha_l^d \in \mathbb{R}$ and $b_d^l \in B_d$, (for each $d = 0, 1, \ldots, j-1$) satisfying the condition $\mathbb{P}(v_{\iota(j)}) = y_j + \beta_j w_j$. From $\beta_j > 0$ we obtain $w_j = -\frac{1}{\beta_j} y_j + \frac{1}{\beta_j}\mathbb{P}(v_{\iota(j)})$. Letting now

$$z_j = \sum_{d=0}^{j-1}\left(\sum_{l=1}^{s_d} \alpha_l^d t_d^l\right),$$

we have $z_j \in \lambda(v^{\iota(j-1)})$. Upon defining $u_{\iota(j)} = -\frac{1}{\beta_j} z_j + \frac{1}{\beta_j} v_{\iota(j)}$, we have

$$\mathbb{P}(u_{\iota(j)}) = -\frac{1}{\beta_j}\mathbb{P}(z_j) + \frac{1}{\beta_j}\mathbb{P}(v_{\iota(j)}) = w_j \quad \text{and} \quad u_{\iota(j)} \in \theta(v^{\iota(j)}).$$

Finally, from Proposition 4.6 we have $\mathbb{P}(\zeta((u_0, \ldots, u_{\iota(j)}))) = \zeta(w^j)$. Hence Proposition 4.9 ensures the existence of vectors $t_j^l \in \lambda((u_0, \ldots, u_{\iota(j)}))$ for $l = 2, \ldots s_j$ such that $\mathbb{P}(t_j^l) = b_j^l \in B_j$. Let $T_j = T_{j-1} \cup \{u_{\iota(j)}, t_j^2, \ldots, t_j^{s_j}\}$. From $u_{\iota(i)} \in \theta(v^{\iota(i)})$ for each $i \le j$ we get

$$T_j \subseteq \lambda(v^{\iota(j)}). \tag{23}$$

The remaining elements of **u**. To conclude, we shall construct the vectors u_s's for each s not belonging to the range of ι. In other words, $s \in \{0, 1, \ldots, t\}$ is such that $\mathbb{P}(v_s)$ lies in $\lambda(\mathbb{P}(v_0), \ldots, \mathbb{P}(v_{\iota(j)})) = \lambda(w^j)$ for some j with $\iota(j) < s$. We can write

$$\mathbb{P}(v_s) = \sum_{d=0}^{j}\left(\sum_{l=1}^{s_d} \alpha_l^d b_d^l\right),$$

with $b_d^l \in B_j$ and $\alpha_l^d \in \mathbb{R}$. Let

$$u_s = v_s - \sum_{d=0}^{j}\left(\sum_{l=1}^{s_d} \alpha_l^d t_d^l\right),$$

with $t_d^l \in T_j$ such that $\mathbb{P}(t_d^l) = b_d^l$. From (23) we obtain

$$\sum_{d=0}^{j}\left(\sum_{l=1}^{s_d} \alpha_l^d t_d^l\right) \in \lambda(v^{\iota(j)}),$$

We also have $u_s \in \theta(v^s)$ and $\mathbb{P}(u_s) = 0$.

Letting $\mathbf{u} = (u_0, u_1, \ldots u_t)$ we conclude that u_i lies in $\theta(v^i)$ for each $i = 1, 2, \ldots, t$. From Theorem 3.5(ii) and (i) it follows that **u** is a reduced index for n-space and $J_{\mathbf{u}} = J_{\mathbf{v}}$. The construction of **u** also ensures that $\mathbb{P}(\mathbf{u})$ adheres to **w**, as desired. □

5. Robinson consistency in Łukasiewicz logic

Recall from the Introduction the appropriate definitions. We shall now derive the central result of our paper: the Robinson consistency property for infinite-valued Łukasiewicz propositional logic:

Theorem 5.1. *Suppose Θ is a prime L_X-theory, and Ψ is a prime L_Y-theory. Let $Z = X \cap Y$ and $W = X \cup Y$. If $\Theta \cap L_Z = \Psi \cap L_Z$ then there is a prime L_W-theory Φ such that $\Theta = \Phi \cap L_X$ and $\Psi = \Phi \cap L_Y$.*

The proof will immediately follow from Theorem 5.2 below, via the familiar correspondence [1, 4.2.7, 4.6.3] between L_V-theories, implicative filters in the free MV-algebra $Free_V$ over the free generating set V, and ideals of $Free_V$, for any set V of variables.

Theorem 5.2. *Let X, Y and Z be sets of free variables, with $Z = X \cap Y$. Let I and J be prime ideals of $Free_X$ and $Free_Y$ respectively. Suppose $I \cap Free_Z = J \cap Free_Z$. Then there is a prime ideal $A = A_{XY}$ of $Free_{X\cup Y}$ such that*

$$A_{XY} \cap Free_X = I \quad \textit{and} \quad A_{XY} \cap Free_Y = J. \tag{24}$$

We first settle the case of finitely many variables[2]:

Lemma 5.3. *Theorem 5.2 holds in case X and Y are* finite *sets.*

Proof. Let $W = X \cup Y$. Let $K = I \cap Free_Z = J \cap Free_Z$. For some integers $1 \leq m \leq n', n''$ the free MV-algebras $Free_Z$, $Free_X$ and $Free_Y$ consist of all McNaughton functions defined on the m -, n'-, and n''-cube respectively. These cubes live in Z-, X-, and Y-space respectively. By hypothesis, the dimension n of W-space satisfies the identity $n = n' + n'' - m$. For definiteness let us assume that the set of these n dimensions is equipped with a total order, and that the first m dimensions pertain to Z-space, followed by $n' - m$ dimensions pertaining to $(X \setminus Z)$-space, and finally $n'' - m$ dimensions pertaining to $(Y \setminus Z)$-space. Proposition 2.19 allows us to write $K = J_{\mathbf{w}}$ for some reduced index $\mathbf{w} = (w_0, \dots w_r)$ for Z-space and some $r \leq m$.

By Theorem 4.10 we have reduced indexes $\mathbf{u} = (u_0, \dots, u_{s'})$ and $\mathbf{v} = (v_0, \dots, v_{s''})$, respectively for n'-space and n''-space, such that $I = J_{\mathbf{u}}$, $J = J_{\mathbf{v}}$ and both $\pi(\mathbf{u})$ and $\pi(\mathbf{v})$ adhere to $\mathbf{w}$. Thus, we have maps $\iota', \iota'' \colon \{0, 1, \dots, r\} \to \mathbb{N}$, as given by Definition 4.4, such that

$$\mathbb{P}_Z(u_{\iota'(j)}) = \mathbb{P}_Z(v_{\iota''(j)}) = w_j \quad \text{for each } j = 0, \dots, r, \tag{25}$$

$$\mathbb{P}_Z(u_i) = 0 \quad \text{for all } i \in \{0, \dots, s'\} \text{ not in the range of } \iota', \tag{26}$$

$$\mathbb{P}_Z(v_j) = 0 \quad \text{for all } j \in \{0, \dots, s''\} \text{ not in the range of } \iota''. \tag{27}$$

Let $s = s' + s'' - r$.

We shall build an index $\mathbf{e} = (e_0, \dots, e_s)$ for n-space such that the ideal $A = J_{\mathbf{e}}$ (is a prime ideal of $Free_W$ and) satisfies $J_{\mathbf{e}} \cap Free_X = J_{\mathbf{u}}$ and $J_{\mathbf{e}} \cap Free_Y = J_{\mathbf{v}}$. From our construction it will follow in particular that $s \leq n$.

We define the function $\iota \colon \{0, 1, \dots, r\} \to \mathbb{N}$ by

$$\iota(0) = 0; \qquad \iota(j) = \iota'(j) + \iota''(j) - 1, \quad (j = 1, \dots, r). \tag{28}$$

Construction of $e_{\iota(j)}$ $(j = 0, \dots, r)$. By (25) the two vectors $u_{\iota'(j)}$ and $v_{\iota''(j)}$ agree (with w_j) in their first m coordinates. Since these are the only common coordinates, the set-theoretical union $u_{\iota'(j)} \cup v_{\iota''(j)}$ yields an n-dimensional vector in W-space, whose projections into X- and Y-space respectively coincide with $u_{\iota'(j)}$ and $v_{\iota''(j)}$. Upon defining $e_{\iota(j)} = u_{\iota'(j)} \cup v_{\iota''(j)}$ we have

$$\mathbb{P}_X(e_{\iota(j)}) = u_{\iota'(j)} \quad \text{and} \quad \mathbb{P}_Y(e_{\iota(j)}) = v_{\iota''(j)}. \tag{29}$$

This completes the construction of $e_{\iota(0)}, \dots, e_{\iota(r)}$.

As a preliminary step for the construction of the remaining vectors e_i, let $\xi \colon \{0, \dots, r\} \to \mathbb{N}$ stand for any of the maps ι', ι'', ι. Correspondingly let R_ξ denote the set $\{0, \dots, s'\}, \{0, \dots, s''\}, \{0, \dots, s\}$. The set of elements $j \in R_\xi$ not belonging to the range of ξ shall be partitioned into $r + 1$ (possibly empty) *bands* as follows:

the 0th band $band_\xi(0)$ is the set of integers i with $0 < i < \xi(1)$,
the first band $band_\xi(1)$ is the set of i with $\xi(1) < i < \xi(2)$,
...
the rth band $band_\xi(r)$ is the set of those $i \in R_\xi$ such that $\xi(r) < i$.

For each $j = 0, \dots, r$ letting $\#band_\xi(j)$ denote the cardinality of the jth band of ξ, by (28) we have

$$\#band_\iota(j) = \#band_{\iota'}(j) + \#band_{\iota''}(j).$$

There certainly exists an order-preserving one–one map μ_j from the disjoint union $band_{\iota'}(j) \bigcup band_{\iota''}(j)$ onto $band_\iota(j)$. To fix ideas, let μ_j first accommodate – in their order – all elements of $band_{\iota'}(j)$ and then all elements of $band_{\iota''}(j)$. Let $\mu = \bigcup_j \mu_j$.

Construction of the remaining e_j: adding batches of zeros.

[2] In [10] it is proved that Gödel incompleteness cannot occur for prime theories over finitely many variables, but may occur otherwise. Thus, certain properties of prime theories over finitely many variables need not automatically hold for prime theories over infinitely many variables. Fortunately, as proved in this section, Lemma 5.3 extends to Theorem 5.2.

For every index $i \in band_{\iota'}(j)$ for some $j = 0, \dots r$, let $\mu(i)$ be its corresponding index in $\{0, \dots, s\}$. By construction, $\mu(i)$ does not belong to the range of ι. Define the vector $e_{\mu(i)}$ as the vector obtained by adding to u_i a batch of zero coordinates for all $(Y \setminus X)$-dimensions, (and agreeing with u_i in each X-dimension). Thus

$$\mathbb{P}_X(e_{\mu(i)}) = u_i. \tag{30}$$

Since by (26) all coordinates of u_i for the Z-dimensions are zeros, we also have

$$\mathbb{P}_Y(e_{\mu(i)}) = 0. \tag{31}$$

In a similar way, for every index $i \in band_{\iota''(j)}$ for some $j = 0, \dots r$, let $\mu(i)$ be its corresponding index in $\{0, \dots, s\} \setminus range(\iota)$. Let $e_{\mu(i)}$ be obtained from v_i by adding a batch of zero coordinates for all $(X \setminus Y)$-dimensions, and agreeing with v_i otherwise. Thus

$$\mathbb{P}_Y(e_{\mu(i)}) = v_i \tag{32}$$

and from (27),

$$\mathbb{P}_X(e_{\mu(i)}) = 0. \tag{33}$$

Claim. *The tuple $(e_1, \dots, e_s)$ forms an independent set of vectors in W-space.*

Suppose $\sum_{l=1}^{s} \alpha_l e_l = 0$ for suitable real coefficients α_l. An application of the projection operator $\mathbb{P}_X$ yields $\sum_{l=1}^{s} \alpha_l \mathbb{P}_X(e_l) = 0$. By construction of $\mathbf{e}$, from (29), (30) and (33) we either have $\mathbb{P}_X(e_l) = 0$, or else $\mathbb{P}_X(e_l)$ is one of the vectors occurring in the tuple $(u_1, \dots, u_{s'})$. We also have that for each $i \in \{0, \dots, s'\}$, $\mathbb{P}_X(e_l) = u_i$ for at most one l in $\{0, \dots, s\}$. Since the vectors $u_1, \dots, u_{s'}$ are independent, it follows that $\alpha_l = 0$ for each dimension l pertaining to X. Similarly, for each $1 \le l \le s$ from (29), (31) and (32) we either have $\mathbb{P}_Y(e_l) = 0$ or else $\mathbb{P}_Y(e_l)$ occurs in the tuple $(v_1, \dots, v_{s''})$. Again for each $i \in \{0, \dots, s''\}$, $\mathbb{P}_Y(e_l) = v_i$ for at most one l in $\{0, \dots, s\}$. Hence, it follows that $\alpha_l = 0$ for each dimension l pertaining to Y. In conclusion, all α_l vanish, as required to settle our claim.

Thus $\mathbf{e}$ is an index for W-space, and by Proposition 2.8, $J_{\mathbf{e}}$ is a prime ideal in $Free_W$. By (29)–(33) $\pi_X(\mathbf{e}) = \mathbf{u}$ and $\pi_Y(\mathbf{e}) = \mathbf{v}$. Moreover, $\pi_X(\mathbf{e})$ adheres to $\mathbf{u}$ and $\pi_Y(\mathbf{e})$ adheres to $\mathbf{v}$. From Theorem 4.7 we have $J_{\mathbf{e}} \cap Free_X = J_{\mathbf{u}}$ and $J_{\mathbf{e}} \cap Free_Y = J_{\mathbf{v}}$. The proof is complete. □

In order to extend this result and get a proof of Theorem 5.2 we give the following three lemmas.

Lemma 5.4. *Under the hypotheses and notation of Theorem 5.2, let $\langle I, J\rangle$ denote the ideal of $Free_{X\cup Y}$ generated by I and J. Then*

$$\langle I, J\rangle = \bigcup_{X',Y'} \left\{\langle I \cap Free_{X'}, J \cap Free_{Y'}\rangle \mid X', Y' \text{ finite, } X' \subseteq X, Y' \subseteq Y\right\}. \tag{34}$$

Proof. Since every element of $Free_X$ is a McNaughton function depending on finitely many variables [1, 3.1.8], the canonical inclusions $Free_{X'} \subseteq Free_X$ and $Free_{Y'} \subseteq Free_Y$ (whenever $X' \subseteq X$ and $Y' \subseteq Y$) are to the effect that

$$I = \bigcup_{X' \subseteq X} \{I \cap Free_{X'} \mid X' \text{ finite}\}. \tag{35}$$

Therefore, the right-hand term of (34) is contained in the left-hand term. For the converse inclusion, if $f \in \langle I, J\rangle$ then $f \le p \oplus q$, where $p \in I \cap Free_{X'}$ and $q \in J \cap Free_{Y'}$ for some finite $X' \subseteq X$ and $Y' \subseteq Y$. Thus $f \in \langle I \cap Free_{X'}, J \cap Free_{Y'}\rangle$. □

Lemma 5.5. *Under the hypotheses and notation of Theorem 5.2, we have the identities*

$$\langle I, J\rangle \cap Free_X = I \quad \text{and} \quad \langle I, J\rangle \cap Free_Y = J. \tag{36}$$

Proof. If X and Y are finite sets the result follows from Lemma 5.3. Otherwise, skipping all trivialities, assume $f \in \langle I, J\rangle \cap Free_X$. From Lemma 5.4, for some finite $X_f \subseteq X$ and finite $X' \subseteq X$ and $Y' \subseteq Y$ with $X' \supseteq X_f$ we can write $p \oplus q \ge f \in Free_{X_f}$, where $p \in I \cap Free_{X'}$ and $q \in J \cap Free_{Y'}$. Thus $f \in \langle I \cap Free_{X'}, J \cap Free_{Y'}\rangle$ and we are in the finite case. Then

$$f \in Free_{X'} \cap \langle I \cap Free_{X'}, J \cap Free_{Y'}\rangle = I \cap Free_{X'} \subseteq I.$$

The rest is obtained by symmetry. □

Lemma 5.6. *Under the hypotheses and notation of Theorem 5.2, there is a prime ideal* $A = A_{XY}$ *of* $Free_{X\cup Y}$ *satisfying* $A \supseteq \langle I, J\rangle$, $A \cap Free_X = I$ *and* $A \cap Free_Y = J$.

Proof. Using the foregoing lemma and the axiom of choice let A be an ideal of $Free_{X\cup Y}$ which is maximal for the following three properties:

$$A \supseteq \langle I, J\rangle, \qquad A \cap Free_X = I, \qquad A \cap Free_Y = J. \tag{37}$$

It is enough to prove that A is prime. For otherwise (absurdum hypothesis) there are $a, b \in Free_{X\cup Y}$ with

$$a \notin A, \qquad b \notin A, \qquad a \wedge b = 0. \tag{38}$$

It follows that

$$a \notin \langle I, J\rangle, \qquad b \notin \langle I, J\rangle. \tag{39}$$

Let $A' = \langle A, a\rangle$ and $A'' = \langle A, b\rangle$ respectively denote the ideals generated by A and a, and by A and b. By (37) we can write without loss of generality

$$A' \cap Free_X \supsetneqq I \quad \text{and} \quad \left[\text{either } A'' \cap Free_Y \supsetneqq J \text{ or } A'' \cap Free_X \supsetneqq I\right].$$

Case 1. $A' \cap Free_X \supsetneqq I$ and $A'' \cap Free_Y \supsetneqq J$.

Then there are McNaughton functions f and g, together with finite sets of variables $X_f \subseteq X$ and $Y_g \subseteq Y$, such that

$$f \in Free_{X_f} \cap A', \qquad g \in Free_{Y_g} \cap A'' \tag{40}$$

and

$$f \notin I, \qquad g \notin J, \qquad A' \cap Free_{X_f} \supsetneqq I \cap Free_{X_f}, \qquad A'' \cap Free_{Y_g} \supsetneqq J \cap Free_{Y_g}. \tag{41}$$

For all finite $\tilde{X} \supseteq X_f$ and $\tilde{Y} \supseteq Y_g$ with $\tilde{X} \subseteq X$ and $\tilde{Y} \subseteq Y$, letting $A_{\tilde{X},\tilde{Y}}$ be as given by Lemma 5.3, it follows that

$$f \notin A_{\tilde{X}\tilde{Y}}, \qquad g \notin A_{\tilde{X}\tilde{Y}}. \tag{42}$$

(Otherwise $f \in A_{\tilde{X}\tilde{Y}}$ implies $f \in A_{\tilde{X}\tilde{Y}} \cap Free_{X_f} = I \cap Free_{X_f}$ against (41).)

Recall from [1, p. 33] the notation $m.x$ for $x \oplus x \oplus \cdots \oplus x$ (m times). Then by [1, 1.1.8], from Eq. (40) it follows that for some $p, q, r \in \langle I, J\rangle$ and $m \in \mathbb{N} f \le m.a \oplus p$ and $g \le m.b \oplus q$, whence by (38), $f \wedge g \le m.(a \wedge b) \oplus r = r$ and $f \wedge g \in \langle I, J\rangle$. By Lemma 5.4 for some large finite $X^\circ \subseteq X$ and $Y^\circ \subseteq Y$ we can write $f \wedge g \in \langle I \cap Free_{X^\circ}, J \cap Free_{Y^\circ}\rangle$, whence by the construction (Lemma 5.3 and Theorem 4.10) of the prime ideal $A_{X^\circ Y^\circ} \supseteq \langle I \cap Free_{X^\circ}, J \cap Free_{Y^\circ}\rangle$ we get $f \wedge g \in A_{X^\circ Y^\circ}$. In view of (42) this contradicts the primeness of $A_{X^\circ Y^\circ}$. The present case is settled.

Case 2. $A' \cap Free_X \supsetneqq I$ and $A'' \cap Free_X \supsetneqq I$.

Then one similarly contradicts the primeness of some $I \cap Free_{X^\circ}$. Thus A has the desired properties, and the lemma is proved. □

The proof of this lemma completes the proof of Theorem 5.2, whence of Theorem 5.1, thus establishing the Robinson consistency property for the infinite-valued Łukasiewicz logic.

6. A consequence of Robinson consistency: Amalgamation

Under very general conditions it is known [6, 5.2] that whenever a model-theoretic logic L with Tarski semantics has the Robinson property, then L automatically satisfies the Craig interpolation theorem. This is no longer true of Łukasiewicz logic which, as we have seen, has the Robinson property but does not satisfy Craig's interpolation (for, the Kleene tautology $x \wedge \neg x \to y \vee \neg y$ has no interpolant.) However, the Robinson consistency property offers a quick dividend: it can be used to prove that MV-algebras and finite-valued MV-algebras have the amalgamation property:

Theorem 6.1. *MV-algebras have the* amalgamation property*: for any two embeddings* $\beta: M \to B$ *and* $\gamma: M \to C$ *there is an MV-algebra* N *together with embeddings* $\beta': B \to N$ *and* $\gamma': C \to N$ *such that* $\gamma' \circ \gamma = \beta' \circ \beta$.

For the proof we prepare:

Lemma 6.2. *Let X and Z be two sets of free variables, with $Z \subseteq X$. Let K and I be ideals in $Free_Z$ and $Free_X$ respectively, with $K = I \cap Free_Z$. Then for every ideal $K' \supseteq K$ of $Free_Z$ there is an ideal $I' \supseteq I$ of $Free_X$ such that $I' \cap Free_Z = K'$.*

Proof. Let σ_K and σ_I denote the canonical surjections

$$\sigma_K : a \in Free_Z \mapsto [a]_K = a/K \in Free_Z/K$$

and

$$\sigma_I : b \in Free_X \mapsto [b]_I = b/I \in Free_X/I.$$

Let $\eta : Free_Z/K \to Free_X/I$ be the embedding given by our hypothesis $K = I \cap Free_Z$. In symbols,

$$\eta : [a]_K \mapsto [a]_I \quad \forall a \in Free_Z. \tag{43}$$

Since $K' \supseteq K$ the set $N = \sigma_K(K') = \{[v]_K \mid v \in K'\}$ is an ideal of $Free_Z/K$. We are using the well-known correspondence between ideals of $Free_Z/K$ and ideals of $Free_Z$ containing K. Recalling (43), let the ideal M of $Free_X/I$ be defined by

$$M = \langle \eta(N) \rangle = \text{ ideal generated by } \eta(N) = \langle \{[v]_I \mid v \in K'\} \rangle. \tag{44}$$

The inverse σ_I-image of M is an ideal $I' \supseteq I$ of $Free_X$. In detail, for all $c \in Free_X$,

$$c \in I' \quad \text{iff } [c]_I \in M \quad \text{iff } [c]_I \leq [v]_I \quad \text{for some } v \in K'. \tag{45}$$

Claim. $I' \cap Free_Z = K'$.

As a matter of fact, if $a \in K'$ then $[a]_K \in \sigma_K(K')$, whence by (44) $[a]_I \in M$ and $a \in I' \cap Free_Z$. Conversely, if $a \in I' \cap Free_Z$ then by (45) $[a]_I \leq [v]_I$ for some $v \in K'$. The familiar correspondence between congruences and ideals [1, 1.2.6] and the definition of the implication operation are to the effect that the element $\neg(a \to v)$ belongs to I. Since $\neg(a \to v)$ also belongs to $Free_Z$ then by hypothesis it belongs to K, in symbols $[a]_K \leq [v]_K$. Thus $\neg(a \to v) \in K \subseteq K'$. Because $v \in K'$ and K' is closed under minorants, $a \in K'$. □

We next prove

Lemma 6.3. *Let X and Y be two sets of free variables, with $Z = X \cap Y$. Let I and J be proper ideals, respectively of $Free_X$ and of $Free_Y$, satisfying $I \cap Free_Z = J \cap Free_Z$. Then for every prime ideal $I' \supseteq I$ of $Free_X$ there is a prime ideal $J' \supseteq J$ of $Free_Y$ such that $I' \cap Free_Z = J' \cap Free_Z$.*

Proof. Write $K = I \cap Free_Z$ and $K' = I' \cap Free_Z$. Then both K and K' are ideals of $Free_Z$. Moreover, K' is a prime ideal. Let $\mathcal{F}$ be the set of ideals T of $Free_Y$ such that $T \supseteq J$ and $T \cap Free_Z = K'$. By the foregoing lemma $\mathcal{F}$ is non-empty. Further, the axiom of choice yields maximal elements in $\mathcal{F}$. So let $J' \in \mathcal{F}$ have the following properties:

$$J' \cap Free_Z = K' \quad \text{and} \quad J' \supseteq J \tag{46}$$

$$J' \text{ is maximal for the above two properties.} \tag{47}$$

Claim. *J' is a prime ideal of $Free_Y$.*

Otherwise (absurdum hypothesis) let $a, b \in Free_Y$ satisfy

$$a \wedge b = 0, \quad a \notin J', b \notin J'. \tag{48}$$

By (46)–(47), letting as usual $\langle J', a \rangle$ denote the ideal of $Free_Y$ generated by J' and a, we can write

$$\langle J', a \rangle \supsetneq J' \quad \text{and} \quad \langle J', b \rangle \supsetneq J'. \tag{49}$$

We also have

$$\langle J', a \rangle \cap \langle J', b \rangle = J'. \tag{50}$$

For, if $c \in \langle J', a\rangle \cap \langle J', b\rangle$ then for some $m \in \mathbb{N}$ and $q \in J'$ we can write $c \leq m.a \oplus q$, $c \leq m.b \oplus q$, whence $c \leq m.(a \wedge b) \oplus q = q$, thus settling (50). From (46)–(48) we conclude

$$K' \subsetneqq \langle J', a\rangle \cap Free_Z \quad \text{and} \quad K' \subsetneqq \langle J', b\rangle \cap Free_Z, \tag{51}$$

and since K' is a prime ideal,

$$K' \subsetneqq \langle J', a\rangle \cap Free_Z \cap \langle J', b\rangle = J' \cap Free_Z = K',$$

a contradiction that settles our claim and completes the proof of the lemma. □

Next we prove[3]:

Lemma 6.4. *Let X and Y be two sets of free variables, with $Z = X \cap Y$. Let P and Q be ideals in $Free_X$ and $Free_Y$ respectively. If $P \cap Free_Z = Q \cap Free_Z$ then the ideal $\langle P, Q\rangle$ of $Free_{X\cup Y}$ satisfies the identities*

$$\langle P, Q\rangle \cap Free_X = P \quad \textit{and} \quad \langle P, Q\rangle \cap Free_Y = Q. \tag{52}$$

Proof. By way of contradiction suppose

$$f \in (\langle P, Q\rangle \cap Free_X)\setminus P.$$

By [1, 1.2.14] there is a prime $I \supseteq P$ of $Free_X$ such that $f \notin I$. Let $K = I \cap Free_Z$. by Lemma 6.3 there is a prime $J \supseteq Q$ of $Free_Y$ such that $K = J \cap Free_Z$. Now we are in the hypotheses of Theorem 5.2, whence let A be a prime ideal in $Free_{X\cup Y}$ such that $A \cap Free_X = I$ and $A \cap Free_Y = J$. Observe that $A \supseteq \langle I, J\rangle \supseteq \langle P, Q\rangle$. Thus $f \in A$ whence $f \in I$, a contradiction. □

Proof of Theorem 6.1. For suitable sets X, Y of variables and ideals P of $Free_X$ and Q of $Free_Y$ we can write $B = Free_X/P$ and $C = Free_Y/Q$. Recalling that $Z = X \cap Y$, let $R' = P \cap Free_Z$ and $R'' = Q \cap Free_Z$. Our assumption about M allows the identification $M = Free_Z/R' = Free_Z/R''$. Let $N = Free_{X\cup Y}/\langle P, Q\rangle$. Then Lemma 6.4 yields the desired embeddings. The proof of Theorem 6.1 is complete.

We refer to [1, 8.5] for the variety of (Grigolia) MV_n-algebras, $n = 2, 3, \ldots$.

Corollary 6.5. *For each $n = 2, 3, \ldots$ the variety of MV_n-algebras has the amalgamation property.*

Proof. We have already proved in Corollary 1.2 that the Robinson consistency theorem trivially holds for MV_n-algebras, because in any free MV_n-algebra prime ideals coincide with maximal ideals. Then one can derive the amalgamation property arguing verbatim as for the foregoing proof of amalgamation for the variety of MV-algebras. □

Remark. Theorem 6.1 was first proved in [8, p. 91, Step 2] using the Γ functor of [7] and Pierce's amalgamation theorem for lattice-ordered abelian groups [14]. Corollary 6.5 was first proved in [2, Theorem 8], using the same tools.

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[3] The property stated in this lemma is a variant of a property first considered by Pigozzi [15, 1.2.7] and by Ono [11, p. 113].

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