REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Volumen 50, Número 2, 2009, Páginas 105–114

POINTWISE ESTIMATES FOR GRADIENTS OF TEMPERATURES IN TERMS OF MAXIMAL FUNCTIONS

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ABSTRACT. We give a detailed proof, in the case of one space dimension, of a pointwise upper estimate for the space gradient of a temperature. The operators involved are a one-sided Hardy-Littlewood maximal in time and the Calderón sharp maximal operator in space.

1. INTRODUCTION

We shall say that the function $u(x,t)$ defined on $\mathbb{R} \times \mathbb{R}^+$, is a temperature if it satisfies the one dimensional heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$. The fundamental solution of the heat equation on $\mathbb{R}^2$ is given by the Weierstrass kernel $W_t(x) = (4\pi t)^{-\frac{1}{2}} e^{-\frac{x^2}{4t}}$, if $t > 0$. The level sets of the two variable function $W(x,t) = W_t(x)$ define a family of "heat balls". This family shares with the Euclidean balls, for the elliptic case, the property of being the right shapes that support a reproducing mean value kernel for temperatures.

There are well established parabolic mean value formulae, with nonsmooth kernels, which can be found for example in [4]. Starting from these type of identities, it is not difficult to produce convolution formulas with kernels which are smooth in the space variable.

Once such smooth reproducing formulae are available, we may study the behavior of the space derivative of a temperature $u(x,t)$ as a convolution. Since, even when the reproducing kernel is in L^1, its space derivative is no longer integrable, and this convolution has to be understood in the distribution sense. In particular the singularity of the space derivative of the reproducing kernel, can be avoided by subtracting a constant to the temperature.

This representation formula for $\frac{\partial u}{\partial x}$ gives us the central identity to start our search for the maximal estimates. Two elementary facts of this formula for $\frac{\partial u}{\partial x}$ are leading us: the support of the distributional kernel is biased in time and a constant value of the temperature is subtracted to the function u in the representation. The first fact suggests the control by one-sided Hardy-Littlewood operators and the second suggests the control by sharp Calderón maximal operators.

2000 *Mathematics Subject Classification.* Primary 42B25, 35K05, 35B65.

Key words and phrases. Maximal operators; Gradient estimates; Heat equation.
The research was supported in part by CONICET, CAI+D (UNL) and ANCPyT.

The result presented here, see Theorem 6, is contained in [1], where a more general situation is considered. Nevertheless we shall give here a more detailed version for the case of dimension one and for first order derivatives. This basic case has all the ingredients of the general one.

The boundedness properties of the two maximal operators involved are giving us integral inequalities. The use of these inequalities and their local versions are described in [1]. The results concerning Besov regularity of temperatures are corollaries of the general case of Theorem 6 on domains and can be found in [2].

The paper is organized in two parts. The first one is devoted to obtain a distribution representation of $\frac{\partial u}{\partial x}$ and the second contains the main estimate.

2. Representation of the space derivative of a temperature

The Weierstrass kernel W_t in one space dimension is given by $W_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$ for $t > 0$ and $x \in I\!R$. The "heat ball with radius r" at the point (x,t) in space-time is given by $E(x,t;r) = \left\{(y,s) \in I\!R^2 : s \leq t, W_{t-s}(x-y) \geq \frac{1}{r}\right\}$. In Figure 1, four of these "balls" are represented at the origin $(0,0)$. It has to be observed that the point $(0,0)$ is actually a boundary point of each one of these "balls". On the other hand each heat ball at any point (x,t) in space-time can be obtained as a translation of one at the origin.

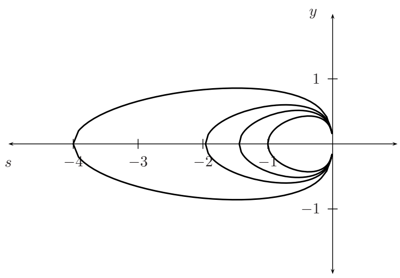

Figure 1. Four heat balls at the origin.

The most important result concerning this family of heat balls is the mean value Watson formula for temperatures ([8]). If $u = u(x,t)$ is a temperature on the open 2-dimensional set Ω, then

$$u(x,t) = \frac{1}{4r} \iint_{E(x,t;r)} u(y,s) \frac{|x-y|^2}{(t-s)^2} \, dyds \tag{2.1}$$

for every $(x,t) \in \Omega$ and every $r > 0$ such that $E(x,t;r) \subset \Omega$. For a nice proof of this formula see [4].

Formula (2.1) gives the temperature at a point as convolution of a kernel with the temperature itself. Notice that the support properties of the kernel imply that the temperature at a point x at time t depends on the past values of the temperature around x. The kernel has two types of singularities. The basic singularity reflecting the reproducing effect of the kernel is the one contained in the expression $\frac{(x-y)^2}{(t-s)^2}$. On the other hand the kernel is not smooth on the boundaries of the heat balls. This second type of singularity in the kernel can be avoided by changing the indicator function of $E(x,t;r)$ by a smooth function in space variable whose level curves are in the family of heat balls at (x,t). The following result is a consequence of (2.1) and the details for the d-dimensional case can be found in [1].

Lemma 1. *Let η be a nonnegative $\mathcal{C}^\infty$ function supported on $[0,1]$ such that $\int_0^1 \eta(r)dr = 1$. Then, for every temperature u in $\mathbb{R}^2$ we have that*

$$u(x,t) = \iint_{\mathbb{R}^2} K_\delta(x-y,t-s)u(y,s)\,dyds \tag{2.2}$$

where $K_\delta(x,t) = \frac{1}{\delta^3}K\left(\frac{x}{\delta},\frac{t}{\delta^2}\right)$, $\delta > 0$ and $K(x,t) = \eta\Big((4\pi\, t)^{\frac{1}{2}}\, e^{\frac{|x|^2}{4\,t}}\Big)\frac{|x|^2}{t^2}$ for $t > 0$ and $K(x,t) \equiv 0$ for $t \leq 0$.

Notice that (2.2) with $u \equiv 1$, shows at a glance that $K \in L^1(\mathbb{R}^2)$ and that $\iint_{\mathbb{R}^2} K_\delta\, dyds = 1$ for each $\delta > 0$.

We may look at $K(x,t)$ as a smooth function of x for each $t \in \mathbb{R}$ or as a distribution in $(x,t) \in \mathbb{R}^2$. Hence we define $N = \frac{\partial K}{\partial x}$, in the classical sense, since for each $t \in \mathbb{R}$, as a function of $x \in \mathbb{R}$, $K(x,t)$ is $\mathcal{C}^\infty$. The fact that we shall need to deal with the distribution derivative of K, with respect to x, instead of N is reflected by the result contained in the next statement.

Proposition 2. *The function N does not belong to $L^1(\mathbb{R}^2)$.*

Proof. For simplicity we shall use the notation $\nu(x,t) = (4\pi t)^{\frac{1}{2}} e^{\frac{x^2}{4t}}$. Computing the x derivative of $K(x,t)$ for a fixed $t > 0$ we have that

$$\begin{aligned} N(x,t) &= \frac{1}{2}\frac{x^3}{t^3}\nu(x,t)\eta'(\nu(x,t)) + \frac{2x}{t^2}\eta(\nu(x,t)) \\ &= \frac{x}{t^2}\left[\frac{1}{2}\frac{x^2}{t}\nu(x,t)\eta'(\nu(x,t)) + 2\eta(\nu(x,t))\right]. \end{aligned} \tag{2.3}$$

Since η is smooth, nontrivial, nonnegative and supported in $[0,1]$, we can take two numbers a and b with $0 < a < b < 1$ in such a way that $\eta'(s) > 0$ and $\eta(s) \geq c > 0$ for every $s \in [a,b]$. Hence

$$\iint_{\mathbb{R}^2} |N|\; dxdt \geq 2c \iint_{\{(x,t): a<\nu(x,t)<b;\, x>0\}} \frac{x}{t^2} dxdt.$$

Let us prove that this integral is $+\infty$. In fact, the domain of the last double integral contains the set $A = \left\{(x,t) : 0 < t < \frac{a^2}{4\pi} \text{ and } \sqrt{2t\ln\frac{a^2}{4\pi t}} < x < \sqrt{2t\ln\frac{b^2}{4\pi t}}\right\}$. Hence

$$\iint_{\mathbb{R}^2} |N|\, dxdt \geq 2c \int_0^{\frac{a^2}{4\pi}} \frac{1}{t^2} \left(\int_{\sqrt{2t \ln \frac{a^2}{4\pi t}}}^{\sqrt{2t \ln \frac{b^2}{4\pi t}}} x dx \right) dt = 4c \ln\left(\frac{b}{a}\right) \int_0^{\frac{a^2}{4\pi}} \frac{dt}{t}.$$

□

We shall use DK to denote the weak derivative with respect to x of the distribution K. The aim of this section is to identify DK in terms of N, as far as possible. The result is the following.

Theorem 3. *For each $v \in \mathcal{C}^\infty(\mathbb{R}^2)$ and each $\delta > 0$ we have that*

$$D\left(K_\delta * v\right)(x,t) = \frac{1}{\delta} \iint_{\mathbb{R}^2} N_\delta\left(x-y, t-s\right)\left[v(y,s) - v(x,s)\right] dyds \tag{2.4}$$

where $N_\delta\left(x,t\right) = \frac{1}{\delta^3} N\left(\frac{x}{\delta}, \frac{t}{\delta^2}\right)$.

Proof. Since K_δ is a compactly supported Schwartz distribution, for $v \in \mathcal{C}^\infty(\mathbb{R}^2)$ we have that $D(K_\delta * v) = (DK_\delta) * v$. Hence (2.4) will be a consequence of

$$\langle DK_\delta, \varphi\rangle = \frac{1}{\delta} \iint_{\mathbb{R}^2} N_\delta(y,s)[\varphi(y,s) - \varphi(0,s)]\, dyds \tag{2.5}$$

for $\varphi \in \mathcal{C}^\infty(\mathbb{R}^2)$. It is also easy to see that it is enough to prove (2.5) for $\delta = 1$. Notice that since $K = K_1$ belongs to $L^1(\mathbb{R}^2)$, we have that

$$\begin{aligned}\langle DK, \varphi\rangle &= -\left\langle K, \frac{\partial \varphi}{\partial x}\right\rangle \\ &= -\iint_{\mathbb{R}^2} K(x,t) \frac{\partial \varphi}{\partial x}(x,t) dxdt,\end{aligned}$$

hence, from Fubini's theorem

$$\langle DK, \varphi\rangle = -\int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} K(x,t) \frac{\partial \varphi}{\partial x}(x,t) dx \right\} dt.$$

For each fixed $t \in \mathbb{R}$ both functions of x in the inner integral are smooth and K has compact support. So that we can integrate by parts in order to obtain a new representation for DK,

$$\langle DK, \varphi\rangle = \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} N(x,t)\varphi(x,t) dx \right\} dt.$$

Of course, the last integral has only the sense of an iteration but, from Proposition 2, it is certainly not a double integral. In order to recover a double integral representation for DK we observe that, being $N(x,t)$ for t fixed the derivative with respect to x of a compactly supported smooth function, $\int_{\mathbb{R}} N(x,t)dx = 0$ for every t. So that, we may also write

$$\langle DK, \varphi\rangle = \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} N(x,t)\left[\varphi(x,t) - \varphi(0,t)\right] dx \right\} dt.$$

Now the last integral is absolutely convergent. Hence

$$\langle DK, \varphi \rangle = \iint_{I\!R^2} N(x,t)\,[\varphi(x,t) - \varphi(0,t)]\,dxdt$$

where the right hand side is now an absolutely convergent integral in the plane. And the desired formula is proved. Let us check that $N(x,t)[\varphi(x,t) - \varphi(0,t)] \in L^1(I\!R^2)$. From the Lagrange mean value theorem and the formula for N obtained in the proof of Proposition 2, with $\eta_1(s) = s\eta'(s)$ we have that

$$\begin{aligned}|N(x,t)\,[\varphi(x,t) - \varphi(0,t)]| &= \left|\frac{1}{2}\frac{x^3}{t^3}\eta_1(\nu(x,t)) + 2\frac{x}{t^2}\eta(\nu(x,t))\right| |\varphi(x,t) - \varphi(0,t)| \\ &\leq c\,|x| \left|\frac{1}{2}\frac{x^3}{t^3}\eta_1(\nu(x,t)) + 2\frac{x}{t^2}\eta(\nu(x,t))\right|.\end{aligned}$$

So that, in order to prove the integrability of $N(x,t)[\varphi(x,t) - \varphi(0,t)]$ we only have to check the convergence of the two following integrals

$$\mathrm{I} = \iint_{I\!R^2} \left|\frac{x^4}{t^3}\eta_1(\nu(x,t))\right| dxdt$$

and

$$\mathrm{II} = \iint_{I\!R^2} \left|\frac{x^2}{t^2}\eta(\nu(x,t))\right| dxdt.$$

The second is bounded by $\iint_{E(0,0;1)} \frac{x^2}{t^2}\,dxdt$ which, from the mean value formula, equals 4 since it is the mean value of the temperature $u \equiv 1$. On the other hand, since η_1, as η, is continuous and has compact support in $[0,1]$ we have for I the estimate

$$\begin{aligned}\mathrm{I} &\leq c \iint_{\{(x,t):\nu(x,t)<1\}} \frac{x^4}{t^3}\,dxdt \\ &= c\int_0^{\frac{1}{4\pi}} \frac{1}{t^3}\left\{\int_{|x|<\sqrt{2t\ln\frac{1}{4\pi t}}} x^4 dx\right\} dt \\ &\leq c\int_0^{\frac{1}{4\pi}} \frac{1}{t^3}\left(\sqrt{2t\ln\tfrac{1}{4\pi t}}\right)^4 2\sqrt{2t\ln\tfrac{1}{4\pi t}}\,dt \\ &= c\int_0^{\frac{1}{4\pi}} t^{-\frac{1}{2}}\left(\ln\tfrac{1}{4\pi t}\right)^{\frac{5}{2}} dt\end{aligned}$$

where the last integral is finite. □

3. Maximal function estimates

The result proved in Section 2 contained in Theorem 3 allows us to prove the main statement of this note. Let us start by the definition of the three maximal operators involved.

A way of measuring the stability of the λ regularity $(0 < \lambda < 1)$ of $v(x,t)$ as a function of x, even when v itself is smooth, is to look at the maximal operator defined as the supremum for $\delta > 0$ of the quantities $\frac{|(\delta D K_\delta)*v|}{\delta^\lambda}$. Precisely, for smooth $v(x,t)$ define

$$\mathcal{M}^\lambda v(x,t) = \sup_{\delta>0} \delta^{1-\lambda} \left| D(K_\delta * v)(x,t) \right|.$$

This maximal operator is sublinear and the basic problem is the analysis of classes of functions v for which it is finite almost everywhere, and its boundedness properties. Let us say that the above definition is inspired in the elliptic results proved by Jerison and Kenig in [5].

In this note we show that $\mathcal{M}^\lambda$ is bounded by an iteration of two well known classical operators in harmonic analysis. Let us introduce these two operators. The simplest one is the one-sided Hardy-Littlewood maximal operator in one dimension (time). Let g be a locally integrable function of the real variable t, define

$$M^- g(t) = \sup_{h>0} \frac{1}{h} \int_{t-h}^{t} |g(s)| \, ds.$$

The L^p boundedness properties of this operator are well known and are a consequence of the classical Hardy-Littlewood theorem. More recent results characterizing the weights w for which the boundedness holds in $L^p(w)$ are contained in [7] and [6]. The operator M^- gives a pointwise upper estimates for convolution operators with kernels supported in $[0,\infty)$. The next result and specially its corollary shall be important in the proof of our main result.

Lemma 4. *Let κ be a nonnegative, nonincreasing, integrable kernel supported in $\mathbb{R}^+$. Set*

$$\kappa^* g(t) = \sup_{\varepsilon>0} \left| \frac{1}{\varepsilon} \int_{\mathbb{R}} \kappa\left(\frac{s}{\varepsilon}\right) g(t-s)\, ds \right|. \tag{3.1}$$

Then, there exists a positive constant C such that the inequality

$$\kappa^* g(t) \le C M^- g(t)$$

holds for every measurable function g and every $t \in \mathbb{R}$.

Proof. Since $\kappa \geq 0$ and nonincreasing, we have, for each positive ε that

$$\begin{aligned} \left| \frac{1}{\varepsilon} \int_{\mathbb{R}} \kappa\left(\frac{s}{\varepsilon}\right) g(t-s)\, ds \right| &\le \sum_{j\in\mathbb{Z}} \frac{1}{\varepsilon} \int_{\varepsilon 2^j \le s < \varepsilon 2^{j+1}} \kappa\left(\frac{s}{\varepsilon}\right) |g(t-s)| \, ds \\ &\le \sum_{j\in\mathbb{Z}} \frac{1}{\varepsilon} \kappa(2^j) \int_{0 \le s \le \varepsilon 2^{j+1}} |g(t-s)| \, ds \\ &= 2 \sum_{j\in\mathbb{Z}} 2^j \kappa(2^j) \left(\frac{1}{\varepsilon 2^{j+1}} \int_{0 \le s \le \varepsilon 2^{j+1}} |g(t-s)| \, ds \right) \\ &\le 2 \left(\sum_{j\in\mathbb{Z}} 2^j \kappa(2^j) \right) M^- g(t). \end{aligned}$$

On the other hand, since

$$\int_{\mathbb{R}^+} \kappa(s)\,ds = \sum_{j\in\mathbb{Z}} \int_{2^{j-1}}^{2^j} \kappa(s)\,ds \geq \sum_{j\in\mathbb{Z}} \kappa(2^j)(2^j - 2^{j-1}) = \frac{1}{2}\sum_{j\in\mathbb{Z}} 2^j \kappa(2^j)$$

the lemma is proved. □

Corollary 5. *Let $\kappa(t) = t^{\vartheta}\left(\ln \frac{1}{t}\right)^{\theta} \mathcal{X}_{(0,1)}(t)$ with $-1 < \vartheta < 0 < \theta$. Then, there exists a constant C depending only on ϑ and θ such that $\kappa^* g(t) \leq C M^- g(t)$ for every measurable function g defined on $\mathbb{R}$.*

Proof. It is easy to see that κ is nonincreasing on $(0,1)$. Its integrability follows from the fact that

$$\kappa(t) = t^{\vartheta-\varepsilon} \cdot t^{\varepsilon}\left(\ln \frac{1}{t}\right)^{\theta} \mathcal{X}_{(0,1)}(t)$$

for $\varepsilon > 0$ such that $\vartheta - \varepsilon > -1$, since $t^{\varepsilon}\left(\ln \frac{1}{t}\right)^{\theta}$ is bounded on $(0,1)$. □

Let us next introduce the remaining operator: the Calderón sharp maximal operator. For a given λ in $(0,1)$ and a given continuous function f on $\mathbb{R}$, define

$$M^{\#,\lambda} f(x) = \sup_{r>0} \frac{1}{r^{1+\lambda}} \int_{x-r}^{x+r} |f(y) - f(x)|\,dy.$$

Notice that $M^{\#,\lambda}$, as M^-, is also a sublinear operator, but its finiteness requires some regularity of f, since $M^{\#,\lambda}$ applied to the indicator function of an interval is equals $+\infty$ at the endpoints of the interval. An extensive and deep analysis of the relationship of the classical spaces of regularity and the boundedness properties of $M^{\#,\lambda}$ can be found in [3].

We are in position to state and prove the main result.

Theorem 6. *For $0 < \lambda < 1$ there exists $C = C(\lambda)$ such that the inequality*

$$\mathcal{M}^{\lambda} v(x,t) \leq C M^-[M^{\#,\lambda} v](x,t) \tag{3.2}$$

holds for every smooth function v defined on $\mathbb{R}^2$.

The right hand side in (3.2) is the iteration of the operators $M^{\#,\lambda}$ acting in the space variable x and M^- in the time variable t. In other words

$$\begin{aligned} M^-[M^{\#,\lambda} v](x,t) &= \sup_{h>0} \frac{1}{h} \int_{t-h}^{t} \left(\sup_{r>0} \frac{1}{r^{1+\lambda}} \int_{x-r}^{x+r} |v(y,s) - v(x,s)|\,dy \right) ds \\ &= \sup_{h>0} \frac{1}{h} \int_{t-h}^{t} \left(\sup_{r>0} \frac{2^{1+\lambda}}{|I(x,r)|} \int_{I(x,r)} |v(y,s) - v(x,s)|\,dy \right) ds \end{aligned}$$

where $I(x,t)$ is the interval $(x-r, x+r)$.

Proof of Theorem 6. In the proof of Proposition 2, formula (2.3) gives the following decomposition of N when $t > 0$

$$N(x,t) = \frac{1}{2}\frac{x^3}{t^3}\eta_1(\nu(x,t)) + \frac{2x}{t^2}\eta(\nu(x,t)), \tag{3.3}$$

where $\eta_1(s) = s\eta'(s)$ and η' is the derivative of η. For $t \le 0$, $N(x,t) \equiv 0$. Let us notice that the seemingly different terms in the right hand side of this decomposition of N share the following common structure,

$$\widetilde{N}(x,t) = \mathcal{Q}(x,t)\widetilde{\eta}(\nu(x,t)),$$

where $\widetilde{\eta}$ has the same smoothness and support properties of η, and $\mathcal{Q}$ is parabolically homogeneous of degree -3. In fact, for both terms $\mathcal{Q}(\mu x, \mu^2 t) = \mu^{-3}\mathcal{Q}(x,t)$. In particular $\mathcal{Q}_\delta(x,t) = \mathcal{Q}(x,t)$. Moreover, notice that for fixed $t > 0$ we have that each $|\mathcal{Q}(x,t)|$ is increasing as a function of $|x|$. Hence it would be enough to obtain a pointwise estimate as (3.2) for a maximal operator of the form

$$\widetilde{\mathcal{M}}^\lambda v(x,t) = \sup_{\delta>0} \delta^{1-\lambda}\widetilde{\mathcal{M}}_\delta^\lambda v(x,t)$$

where $\widetilde{\mathcal{M}}_\delta^\lambda v(x,t) = \left|\frac{1}{\delta}\iint_{I\!R^2} \widetilde{N}_\delta(x-y,t-s)[v(y,s)-v(x,s)]\,dyds\right|$ and $\widetilde{N}$ has the above described structure and v is a smooth function on $I\!R^2$.

Let us fix $v(x,t) \in \mathcal{C}^\infty(I\!R^2)$ and $\delta > 0$. Then

$$\delta^{1-\lambda}\widetilde{\mathcal{M}}_\delta^\lambda v(x,t) \le \delta^{-\lambda}\iint_{I\!R^2}\left|\widetilde{N}_\delta(x-y,t-s)\right| |v(y,s)-v(x,s)|\,dyds.$$

From Fubini-Tonelli theorem the last integral is bounded by

$$C\delta^{-\lambda}\int_{t-\frac{\delta^2}{4\pi}}^{t} |I(x,R_\delta(t-s))|^{1+\lambda} \cdot\left\{\frac{1}{|I(x,R_\delta(t-s))|^{1+\lambda}}\int_{I(x,R_\delta(t-s))}|\mathcal{Q}(x-y,t-s)|\,|v(y,s)-v(x,s)|\,dy\right\}ds$$

where $R_\delta(t-s) = \sqrt{2(t-s)\ln\frac{\delta^2}{4\pi(t-s)}}$. Since $|\mathcal{Q}(x-y,t-s)|$ is increasing as a function of $|x-y|$ and, in the domain of the inner integral, $|x-y| < R_\delta(t-s)$, we have that $|\mathcal{Q}(x-y,t-s)| \le |\mathcal{Q}(R_\delta(t-s),t-s)|$. Hence

$$\begin{aligned}
&\delta^{1-\lambda}\widetilde{\mathcal{M}}_\delta^\lambda v(x,t)\\
&\quad\le C\delta^{-\lambda}\int_{t-\frac{\delta^2}{4\pi}}^{t}(R_\delta(t-s))^{1+\lambda}\,|\mathcal{Q}(R_\delta(t-s),t-s)|\\
&\qquad\qquad\cdot\left\{\frac{1}{|I(x,R_\delta(t-s))|^{1+\lambda}}\int_{I(x,R_\delta(t-s))}|v(y,s)-v(x,s)|\,dy\right\}ds\\
&\quad\le C\frac{4\pi}{\delta^2}\int_{t-\frac{\delta^2}{4\pi}}^{t}\delta^{2-\lambda}\,(R_\delta(t-s))^{1+\lambda}\,|\mathcal{Q}(R_\delta(t-s),t-s)|\,M^{\#,\lambda}v(\cdot,s)(x)\,ds.
\end{aligned}$$

Notice now that from the definition of R_δ and the homogeneity of $\mathcal{Q}$ we have that

$$\begin{aligned}\mathcal{Q}\left(R_\delta(t-s), t-s\right) &= \mathcal{Q}\left((t-s)^{\frac{1}{2}}\sqrt{2\ln\tfrac{\delta^2}{4\pi(t-s)}}, t-s\right)\\ &= \frac{1}{(t-s)^{\frac{3}{2}}}\mathcal{Q}\left(\sqrt{2\ln\tfrac{\delta^2}{4\pi(t-s)}}, 1\right).\end{aligned}$$

Since, from (3.3), $\mathcal{Q}(x,t)$ is $\frac{x^3}{2t^3}$ for the first term on the right hand side and $\frac{2x}{t^2}$ for the second, we may write both cases together in the form

$$\mathcal{Q}\left(R_\delta(t-s), t-s\right) = \frac{1}{(t-s)^{\frac{3}{2}}}\left(2\ln\tfrac{\delta^2}{4\pi(t-s)}\right)^{\xi_i}$$

for $i = 1, 2$, with $\xi_1 = \frac{3}{2}$ and $\xi_2 = \frac{1}{2}$.

So that

$$\begin{aligned}\delta^{1-\lambda}\widetilde{\mathcal{M}}_\delta^\lambda v(x,t) &\leq C\frac{4\pi}{\delta^2}\int_{t-\frac{\delta^2}{4\pi}}^{t}\left(\frac{t-s}{\delta^2}\right)^{\frac{1+\lambda}{2}}\left(2\ln\frac{1}{4\pi\left(\frac{t-s}{\delta^2}\right)}\right)^{\frac{1+\lambda}{2}}\\ &\qquad\cdot\frac{1}{\left(\frac{t-s}{\delta^2}\right)^{\frac{3}{2}}}\left(2\ln\frac{1}{4\pi\left(\frac{t-s}{\delta^2}\right)}\right)^{\xi_i} M^{\#,\lambda}v(\cdot,s)(x)\,ds\\ &= C\frac{4\pi}{\delta^2}\int\limits_{s\in\mathbb{R}}\left(\frac{4\pi(t-s)}{\delta^2}\right)^{\frac{\lambda}{2}-1}\mathcal{X}_{(0,1)}\left(\tfrac{4\pi(t-s)}{\delta^2}\right)\\ &\qquad\cdot\left(2\ln\frac{1}{4\pi\left(\frac{t-s}{\delta^2}\right)}\right)^{\frac{1+\lambda}{2}+\xi_i} M^{\#,\lambda}v(\cdot,s)(x)\,ds\\ &= C\int\limits_{s\in\mathbb{R}}\kappa^i_{\frac{\delta^2}{4\pi}}(t-s)\,M^{\#,\lambda}v(\cdot,s)(x)\,ds\end{aligned}$$

where $\kappa^i_\varepsilon(t) = \frac{1}{\varepsilon}\kappa^i(\frac{t}{\varepsilon})$ with $\kappa^i(t) = t^{\frac{\lambda}{2}-1}\left(2\ln\frac{1}{t}\right)^{\frac{1+\lambda}{2}+\xi_i}\mathcal{X}_{(0,1)}(t)$ for $i = 1, 2$. In the both cases, $i = 1, 2$, the kernel κ^i satisfies all the hypotheses of Corollary 5. Hence

$$\delta^{1-\lambda}\widetilde{\mathcal{M}}_\delta^\lambda v(x,t) \leq CM^-[M^{\#,\lambda}v](x,t)$$

for every positive δ. □

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Recibido: 1 de agosto de 2008
Aceptado: 25 de febrero de 2009