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Riesz transforms related to Schrödinger operators acting on *BMO* type spaces ☆

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ABSTRACT

In this work we obtain boundedness on suitable weighted *BMO* type spaces of Riesz transforms, and their adjoints, associated to the Schrödinger operator $-\Delta + V$, where V satisfies a reverse Hölder inequality. Our results are new even in the unweighted case.

1. Introduction

As it is well known, classical Riesz transforms map $L^p(w)$, $1 < p < \infty$, into itself as long as w belongs to the Muckenhoupt class A_p, i.e. weights satisfying

$$\left(\int_B w\right)\left(\int_B w^{-\frac{1}{p-1}}\right)^{p-1} \leqslant C|B|^p, \tag{1}$$

where B denotes any ball in $\mathbb{R}^d$. However they fail to be bounded for $p = \infty$. In the unweighted case the substitute result is that L^∞ is mapped into a larger space, the *BMO* space of John and Nirenberg. Moreover, it turns to be true that *BMO* itself is applied continuously into *BMO* under the Riesz transforms. This result has been generalized to the more general spaces $BMO^\beta(w)$, $0 \leqslant \beta < 1$, for certain classes of weights (see [10,11]). More precisely, for w belonging to $A_\infty = \bigcup_{p=1}^\infty A_p$ and satisfying

$$|B|^{\frac{1-\beta}{d}} \int_{B^c} \frac{w(y)}{|x_B - y|^{d+1-\beta}} \leqslant C\frac{w(B)}{|B|}, \tag{2}$$

each Riesz transform maps continuously $BMO^\beta(w)$ into itself, $0 \leqslant \beta < 1$, where

$$BMO^\beta(w) = \left\{ f \in L^1_{\text{loc}}\colon \sup_B \frac{1}{|B|^{\beta/d} w(B)} \int_B \left| f(x) - f_B \right| dx < \infty \right\},$$

with the supremum taken over all balls B and f_B denoting the average of f over B.

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Classical Riesz transforms are associated to the Laplacian operator by

$$\mathbf{R}_i = \frac{\partial}{\partial x_i}(-\Delta)^{-1/2}, \quad i = 1, 2, \ldots, d.$$

If we make a perturbation of the Laplace operator we obtain a Schrödinger operator

$$\mathfrak{L} = -\Delta + V,$$

where V is a no-negative function. Correspondingly, we may associate to the differential operator $\mathfrak{L}$ the Riesz transforms

$$\mathcal{R}_i = \frac{\partial}{\partial x_i}(-\Delta + V)^{-1/2}, \quad i = 1, 2, \ldots, d.$$

These operators have been considered in [12], where the author shows that they are also Calderón–Zygmund singular integrals as long as the potential V belongs to a reverse-Hölder class RH_q for some exponent $q \geqslant d \geqslant 3$, i.e. there exists a constant C such that

$$\left(\frac{1}{|B|}\int_B V(y)^q\,dy\right)^{1/q} \leqslant \frac{C}{|B|}\int_B V(y)\,dy, \tag{3}$$

for every ball $B \subset \mathbb{R}^d$.

As a consequence $\mathcal{R}_i$, $i = 1, 2, \ldots, d$, are bounded on $L^p(w)$, for $1 < p < \infty$ and $w \in A_p$, and of weak type on $L^1(w)$, for $w \in A_1$. Moreover, Shen shows that if V satisfies (3) with $\frac{d}{2} \leqslant q < d$ and $w \equiv 1$, then $\mathcal{R}_i$ are bounded only on a finite range of p, namely for $1 < p \leqslant p_0$ with $\frac{1}{p_0} = \frac{1}{d} - \frac{1}{q}$, which he proves to be optimal. Consequently, assuming (3) for $q \geqslant d/2$ we will have L^p boundedness of the adjoints $\mathcal{R}_i^*$, near $p = \infty$. In fact it will hold for $p_0' \leqslant p < \infty$ when $d/2 \leqslant q < d$ or $1 < p < \infty$ when $q \geqslant d$.

Also, regarding these operators, in [4] the authors introduced an appropriate version of the Hardy space H_1 which turns out to be invariant by $\mathcal{R}_i$, under the assumption $q > d/2$. Further related results can be found in [5] and [6].

In connection with boundedness of other operators associated to $\mathfrak{L}$, in [3] appears an appropriate version of the *BMO* space of John–Nirenberg, for potentials V satisfying (3), for some $q > \frac{d}{2}$, and $d \geqslant 3$. Such space is defined through the following function associated to V already used in [4–6,12]. Given $x \in \mathbb{R}^d$ we set

$$\rho(x) = \sup\left\{r > 0\colon \frac{1}{r^{d-2}}\int_{B(x,r)} V \leqslant 1\right\}, \quad x \in \mathbb{R}^d. \tag{4}$$

With this notation the space $BMO_{\mathfrak{L}}$ is defined as the set of functions f in L^1_{loc} satisfying

$$\int_B |f - f_B| \leqslant C|B|, \quad \text{with } f_B = \frac{1}{|B|}\int_B f,$$

for every ball $B \subset \mathbb{R}^d$, and

$$\int_B |f| \leqslant C|B|,$$

for every ball $B = B(x, R)$, with $R \geqslant \rho(x)$.

Clearly $BMO_{\mathfrak{L}}$ is a subspace of *BMO* and contains L^∞. In [3] it is proved that $BMO_{\mathfrak{L}}$ is the dual of the Hardy type space $H^1_{\mathfrak{L}}$ introduced in [4].

In [1] we defined the more general space $BMO^\beta_{\mathfrak{L}}(w)$ for an exponent $0 \leqslant \beta < 1$ and a weight w as the set of functions f in L^1_{loc} satisfying

$$\int_B |f - f_B| \leqslant C w(B)|B|^{\beta/d}, \tag{5}$$

for every ball $B \subset \mathbb{R}^d$, and

$$\int_B |f| \leqslant C w(B)|B|^{\beta/d}, \tag{6}$$

for every ball $B = B(x, R)$, with $R \geqslant \rho(x)$.

A norm in the space $BMO^\beta_{\mathfrak{L}}(w)$ can be given by the maximum of the two infima of the constants that satisfy (5) and (6) respectively. This norm will be denoted by $\|\cdot\|_{BMO^\beta_{\mathfrak{L}}(w)}$.

The aim of this paper is to explore boundedness properties of the Riesz transforms $\mathcal{R}_i$ and their adjoints $\mathcal{R}_i^*$ on the spaces $BMO_{\mathfrak{L}}^{\beta}(w)$. To our knowledge there were not results in this direction even in the simplest case $w \equiv 1$ and $\beta = 0$. However, during the revision of this article, the referee communicated us that in [2] the authors have proved the $BMO_{\mathfrak{L}}$-boundedness of $\mathcal{R}_i$, for $q > d$. Also, observe that due to the lack of symmetry of the problem, $\mathcal{R}_i$ and $\mathcal{R}_i^*$ may have different properties.

In order to give the precise statements we consider the following class of weights. For $\eta \geqslant 1$ we say that $w \in D_\eta$ if there exists a constant C such that

$$w(tB) \leqslant Ct^{d\eta} w(B), \tag{7}$$

for every ball $B \subset \mathbb{R}^d$ and $t > 1$. Here, as usual, tB denotes the ball with the same center as B and t times its radius. We remind that a weight w satisfies the doubling property

$$\int_{2B} w \leqslant C \int_B w, \tag{8}$$

for every ball $B \subset \mathbb{R}^d$, if and only if $w \in D_\eta$ for some $\eta \geqslant 1$.

Let us notice that our assumption (3) on V implies that V belongs to some A_p class and thus satisfies (8) and hence (7) for some $\mu \geqslant 1$.

Before stating the main theorems we introduce the definition of the reverse Hölder index of V as $q_0 = \sup\{q\colon V \in RH_q\}$. Observe that since $V \in RH_q$ implies $V \in RH_{q+\epsilon}$, under the assumption $V \in RH_d$ we may conclude $q_0 > d$.

Theorem 1. *Let $V \in RH_d$ and $w \in A_\infty \cap D_\eta$. Then*

(a) *For any $0 \leqslant \beta < 1 - d/q_0$ and $1 \leqslant \eta < 1 + \frac{1-d/q_0-\beta}{d}$, the operators $\mathcal{R}_i$, $1 \leqslant i \leqslant d$, are bounded on $BMO_{\mathfrak{L}}^{\beta}(w)$.*
(b) *For any $0 \leqslant \beta < 1$ and $1 \leqslant \eta < 1 + \frac{1-\beta}{d}$, the operators $\mathcal{R}_i^*$, $1 \leqslant j \leqslant d$, are bounded on $BMO_{\mathfrak{L}}^{\beta}(w)$.*

Theorem 2. *Let $V \in RH_{d/2}$ such that $q_0 \leqslant d$, $0 \leqslant \beta < 2 - \frac{d}{q_0}$, and $w \in D_\eta \cap \bigcup_{s > p_0'} (A_{p_0/s'} \cap RH_s)$ where $\frac{1}{p_0} = \frac{1}{q_0} - \frac{1}{d}$ and $1 \leqslant \eta < 1 + \frac{2-d/q_0-\beta}{d}$. Then the operators $\mathcal{R}_i^*$, $1 \leqslant i \leqslant d$, are bounded on $BMO_{\mathfrak{L}}^{\beta}(w)$.*

Remark 1. For $\mathcal{R}_i$ the condition $V \in RH_d$ in Theorem 1 can not be relaxed to $V \in RH_{d/2}$ as it is the case for $\mathcal{R}_i^*$. In fact, for $w \equiv 1$ and $V \in RH_q$ with $d/2 < q < d$, since $L^\infty \subset BMO_{\mathfrak{L}} \subset BMO$ we would have that $\mathcal{R}_i$, $i = 1, \ldots, d$, are bounded from L^∞ into the classical BMO. Besides, by [12, Theorem 0.5] they are also bounded on L^p, $\frac{1}{p} = \frac{1}{q} - \frac{1}{d}$. Therefore by interpolation $\mathcal{R}_i$, $i = 1, \ldots, d$, would be bounded on any L^r, $p < r < \infty$, leading to a contradiction since as we mentioned, the range given in [12] is optimal. This is also the reason why even in the case $V \in RH_d$ we obtain a wider class of weights for $\mathcal{R}_i^*$.

Remark 2. We point out that any non-negative polynomial gives an example of a potential V satisfying the assumption of Theorem 1. In fact, those potentials satisfy (3) for any $q > 1$. In particular it applies to $V(x) = |x|^2$ which gives the Hermite operator. In this situation it can be seen that the weights given by Theorem 1 in part (a) and (b) and those associated to the classical Riesz transforms coincide (see Proposition 4 below).

As a corollary of Theorem 1 we have the following application.

Corollary 1. *Let $V \in RH_d$, $w \in A_\infty \cap D_\eta$ and $1 \leqslant \eta < 1 + \frac{1-d/q_0-\beta}{d}$. If u is a solution of*

$$-\Delta u + Vu = \operatorname{div} \bar{g},$$

then

$$\|u\|_{BMO_{\mathfrak{L}}^{\beta}(w)} \leqslant C \|\bar{g}\|_{BMO_{\mathfrak{L}}^{\beta}(w)}.$$

Proof. Since $\nabla u = \mathcal{R}(\mathcal{R}^* \cdot \bar{g})$, the result follows applying Theorem 1. □

The paper is organized as follows. In Section 2 we present some estimates related to the potential V and properties regarding the spaces and weights under consideration. Section 3 is due to estimates on the size and smoothness of the kernels. Finally, in Section 4 we prove our main results.

2. Some preliminary results

We start stating some properties of the function ρ defined in (4) that we will use frequently.

Proposition 1. *(See [12].) Let $V \in RH_{d/2}$. For the associated function ρ there exist C and $k_0 \geqslant 1$ such that*

$$C^{-1}\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{-k_0} \leqslant \rho(y) \leqslant C\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{k_0}{k_0+1}}, \tag{9}$$

for all $x, y \in \mathbb{R}^d$.

Lemma 1. *Let $V \in RH_q$ with $q > d/2$ and $\epsilon > \frac{d}{q}$. Then for any constant C_1 there exists a constant C_2 such that*

$$\int\limits_{B(x,C_1 r)} \frac{V(u)}{|u-x|^{d-\epsilon}}\,du \leqslant C_2 r^{\epsilon-2}\left(\frac{r}{\rho(x)}\right)^{2-d/q}, \tag{10}$$

if $0 < r \leqslant \rho(x)$, and

$$\int\limits_{B(x,C_1 r)} \frac{V(u)}{|u-x|^{d-\epsilon}}\,du \leqslant C_2 r^{\epsilon-2}\left(\frac{r}{\rho(x)}\right)^{2+(\mu-1)d},$$

if $r > \rho(x)$, where μ is such that $V \in D_\mu$.

Proof. Clearly we may assume $C_1 \geqslant 1$. Since $\epsilon > \frac{d}{q}$, by Hölder's inequality,

$$\int\limits_{B(x,C_1 r)} \frac{V(u)}{|u-x|^{d-\epsilon}}\,du \leqslant C r^{\epsilon-d/q}\left(\int\limits_{B(x,C_1 r)} V^q\right)^{1/q}.$$

If $0 < r \leqslant \rho(x)$, using (3), the doubling property (8) and the definition of ρ, the last factor can be bounded by

$$\left(\int\limits_{B(x,C_1\rho(x))} V^q\right)^{1/q} \leqslant C\rho(x)^{\frac{d}{q}-d}\int\limits_{B(x,\rho(x))} V \leqslant C\rho(x)^{\frac{d}{q}-2}.$$

In the case $r > \rho(x)$, we use (3) and $V \in D_\mu$ to obtain the bound

$$Cr^{\epsilon-d}\int\limits_{B(x,C_1 r)} V \leqslant Cr^{\epsilon-d}\left(\frac{r}{\rho(x)}\right)^{\mu d}\int\limits_{B(x,\rho(x))} V \leqslant Cr^{\epsilon-d}\left(\frac{r}{\rho(x)}\right)^{\mu d}\rho(x)^{d-2}. \qquad \square$$

Next we present some special properties of the spaces $BMO^\beta_{\mathfrak{L}}(w)$.

Proposition 2. *Let $0 \leqslant \beta < 1$ and a weight $w \in D_\eta$ for some $\eta \geqslant 1$. A function f belongs to $BMO^\beta_{\mathfrak{L}}(w)$ if and only if condition (5) is satisfied for every ball $B = B(x, R)$ with $R < \rho(x)$, and*

$$\int\limits_{B(x,\rho(x))} |f| \leqslant Cw\big(B\big(x,\rho(x)\big)\big)\big|\rho(x)\big|^\beta, \tag{11}$$

for all $x \in \mathbb{R}^d$.

A proof of this result can be found in [3] for the case $w \equiv 1$ and $\beta = 0$, and in [1] for the general case.

Recall that functions belonging to the classical *BMO* space satisfy the John–Nirenberg estimate (see [9]). An extension of this result to the weighted case was given by Muckenhoupt and Wheeden in [11] and a general version that includes $BMO^\beta(w)$, $0 \leqslant \beta < 1$, appears in [10]. Even though the proofs are worked out in $d = 1$, they can be easily carried out in higher dimension as well.

Weighted John–Nirenberg inequalities have an important consequence, namely that equivalent norms can be obtained taking appropriate *r-averages* for the oscillations as long as $1 \leqslant r \leqslant p'$. More precisely, a function $f \in BMO^\beta(w)$ if and only if

$$\sup_B \frac{1}{|B|^{\beta/d}}\left(\frac{1}{w(B)|B|^{\beta/d}}\int\limits_B |f-f_B|^r w^{1-r}\right)^{1/r} < \infty, \tag{12}$$

and, moreover, this quantity gives an equivalent norm.

An extension of such results for $BMO^\beta_{\mathfrak{L}}(w)$ spaces is contained in the following proposition.

Proposition 3. *Let $0 \leqslant \beta < 1$, $w \in A_p$ and $1 \leqslant r \leqslant p'$, $r < \infty$. Then $f \in BMO^{\beta}_{\mathfrak{L}}(w)$ if and only if*

$$\sup_B \frac{1}{|B|^{\beta/d}} \left(\frac{1}{w(B)} \int_B |f - f_B|^r w^{1-r} \right)^{1/r} < \infty, \tag{13}$$

and

$$\sup_{B \in \mathcal{B}_\rho} \frac{1}{|B|^{\beta/d}} \left(\frac{1}{w(B)} \int_B |f|^r w^{1-r} \right)^{1/r} < \infty, \tag{14}$$

where $\mathcal{B}_\rho$ is the set of balls $B = B(x, R)$ with $R \geqslant \rho(x)$. Moreover, the maximum of the two suprema gives an equivalent norm.

Proof. First, if (13) and (14) are satisfied, Hölder's inequality implies that $f \in BMO^{\beta}_{\mathfrak{L}}(w)$ with the norm is controlled by the sum of the two suprema. On the other hand, by the continuous inclusion $BMO^{\beta}_{\mathfrak{L}}(w) \subset BMO^{\beta}(w)$ we only have to prove that the left-hand side of (14) is dominated by $\|f\|_{BMO^{\beta}_{\mathfrak{L}}(w)}$. Since $A_p \subset A_{r'}$, for every ball $B \in \mathcal{B}_\rho$ we have

$$\begin{aligned}
\left(\frac{1}{w(B)} \int_B |f|^r w^{1-r} \right)^{1/r} &\leqslant \left(\frac{1}{w(B)} \int_B |f - f_B|^r w^{1-r} \right)^{1/r} + |f_B| \left(\frac{w^{1-r}(B)}{w(B)} \right)^{1/r} \\
&\leqslant \|f\|_{BMO^{\beta}_{\mathfrak{L}}(w)} |B|^{\beta/d} \left(1 + \frac{w(B)^{1/r'} (w^{1-r}(B))^{1/r}}{|B|} \right) \\
&\leqslant C \|f\|_{BMO^{\beta}_{\mathfrak{L}}(w)} |B|^{\beta/d}. \qquad \square
\end{aligned} \tag{15}$$

Before finishing this section we state the following lemma, providing a very useful property for the functions in $BMO^{\beta}_{\mathfrak{L}}(w)$. A proof for the case $\nu = 1$ was given in [1].

Lemma 2. *Let $w \in A_t \cap D_\eta$ with $t \geqslant 1$, $\eta \geqslant 1$ and $f \in BMO^{\beta}_{\mathfrak{L}}(w)$. Then, for every ball $B = B(x, r)$ and any finite $\nu \leqslant t'$, we have*

$$\left(\int_B |f|^\nu w^{1-\nu} \right)^{1/\nu} \leqslant C \|f\|_{BMO^{\beta}_{L}(w)} w(B)^{1/\nu} |B|^{\beta/d} \max\left\{ 1, \left(\frac{\rho(x)}{r} \right)^{d\eta - d + \beta} \right\},$$

if $\eta > 1$ or $\beta > 0$, and

$$\left(\int_B |f|^\nu w^{1-\nu} \right)^{1/\nu} \leqslant C \|f\|_{BMO_{\mathfrak{L}}(w)} w(B)^{1/\nu} \max\left\{ 1, 1 + \log\left(\frac{\rho(x)}{r} \right) \right\},$$

if $\eta = 1$ and $\beta = 0$.

Proof. The proof follows the same lines as in [1] for the case $\nu = 1$. For the sake of completeness we include it here. We write $f = f - f_B + (\sum_{j=1}^{j_0 - 1} f_{2^j B} - f_{2^{j+1} B}) + f_{2^{j_0} B}$, where $2^{j_0 - 1} < \frac{\rho(x)}{r} \leqslant 2^{j_0}$. Then,

$$\left(\int_B |f|^\nu w^{1-\nu} \right)^{1/\nu} \leqslant I_1 + I_2 + I_3,$$

with $I_1 = (\int_B |f - f_B|^\nu w^{1-\nu})^{1/\nu}$, $I_2 = (w^{1-\nu}(B))^{1/\nu} \sum_{j=1}^{j_0 - 1} |f_{2^j B} - f_{2^{j+1} B}|$ and $I_3 = (w^{1-\nu}(B))^{1/\nu} |f|_{2^{j_0} B}$.

For the first term we just use Proposition 3. For I_2 and I_3 we bound the oscillation and the average using the definition of the norm, and

$$\left(w^{1-\nu}(B) \right)^{1/\nu} \leqslant C \frac{|B|}{w(B)^{1/\nu'}},$$

since $w \in A_{\nu'}$.

Combining these estimates we obtain

$$I_2 + I_3 \leqslant C \|f\|_{BMO_{\mathfrak{L}}(w)} w(B)^{1/\nu} |B|^{\beta/d} \sum_{j=1}^{j_0} 2^{j(d\eta - d + \beta)}.$$

Evaluating the sum according to the cases $d\eta - d + \beta = 0$ and $d\eta - d + \beta > 0$ we arrive to the desired result. $\square$

We finish this section making some remarks about the weights appearing in Theorems 1 and 2.

The weights for classical Riesz transforms are given by an integral condition (2) while our classes are stated through a doubling condition. Nevertheless, all the classes can be described in both ways as the following proposition shows.

Proposition 4. *Let $\gamma > 0$ and $s \geqslant 1$. Then, $w \in RH_s \cap D_\eta$ with $\eta < 1 + \gamma/d$ if and only if*

$$|B|^{\frac{\gamma}{d}} \left(\int_{B^c} \frac{w(y)^s}{|x-y|^{d+\gamma s}} \right)^{1/s} \leqslant C \frac{w(B)}{|B|}, \tag{16}$$

for every ball $B = B(x,r)$.

Proof. If we suppose $w \in RH_s \cap D_\eta$, denoting $B_k = B(x, 2^k r)$,

$$\begin{aligned} \int_{B^c} \frac{w(y)^s}{|x-y|^{d+\gamma s}} &\leqslant \sum_{k=1}^{\infty} \frac{1}{(2^k r)^{d+s\gamma}} \int_{B_k} w^s \\ &\leqslant C \sum_{k=1}^{\infty} \left(\frac{w(B_k)}{(2^k r)^{\gamma+d}} \right)^s \\ &\leqslant C \left(\frac{w(B)}{|B|^{1+\frac{\gamma}{d}}} \sum_{k=1}^{\infty} 2^{k(d\eta-\gamma-d)} \right)^s, \end{aligned}$$

where the last series converges since $\eta < 1 + \gamma/d$, obtaining (16).

On the other hand, if we suppose (16), by Hölder's inequality we have

$$|B|^{\frac{\gamma s}{d}} \int_{B^c} \frac{w(y)^s}{|x-y|^{d+\gamma s}} \leqslant C \frac{w^s(B)}{|B|}, \tag{17}$$

and this implies

$$w^s(2B \setminus B) \leqslant C w^s(B)$$

which in turn gives the doubling condition for w^s. Therefore, with standard arguments we obtain

$$w^s(B) \leqslant C w^s(2B \setminus B).$$

Now it is easy to see that (16) implies $w \in RH_s$.

Next we check that the function $\psi(t) = w^s(B(x,t))$ satisfies

$$\int_t^{\infty} \frac{\psi(s)}{s^{d+\gamma s+1}}\, ds \leqslant C \frac{\psi(t)}{t^{d+\gamma s}}.$$

This follows from (17) splitting the integral dyadically and using the doubling condition for w^s.

Therefore, applying [8, Lemma (3.3)] there exists there exists $\epsilon > 0$ such that

$$w^s(tB) \leqslant C t^{d+\gamma s-\epsilon} w^s(B),$$

for every ball B and $t \geqslant 1$. Finally, as a consequence of Hölder's inequality and $w \in RH_s$ we obtain that $w \in D_\eta$ with $\eta < 1 + \frac{\gamma}{d}$. □

Remark 3. In view of this proposition the class of weights appearing in Theorem 1 are those A_∞ weights satisfying (16) with $s = 1$, and $\gamma = 1 - \beta - \frac{d}{q_0}$ for the part (a) and $\gamma = 1 - \beta$ for the part (b).

Regarding Theorem 2 we obtain the weights satisfying (16) with $s > p_0'$ and $\gamma = 2 - \beta - d/q_0$, which also belong to $A_{p_0/s'}$.

Remark 4. Clearly, the class of weights mentioned in the introduction regarding the classical Riesz transforms coincide with that of Theorem 1 part (b) and contains those of Theorem 1 part (a) and Theorem 2.

Examples of power weights satisfying the assumptions of the previous results are $w(x) = |x|^\alpha$, with $-d < \alpha < 1 - \beta - d/q_0$ for Theorem 1 part (a), and $-d < \alpha < 1 - \beta$ for part (b), while for Theorem 2, the exponent α should be in the range

$$-d + \frac{d}{q_0} - 1 < \alpha < 1 - \beta - \left(\frac{d}{q_0} - 1 \right).$$

3. Some estimates for the kernels

We shall denote by $\mathcal{R}$ and $\mathcal{R}^*$ the vectors whose components are the Riesz transforms $\mathcal{R}_i$ and $\mathcal{R}_i^*$ respectively, i.e.,

$$\mathcal{R} = \nabla(-\Delta + V)^{-1/2}, \qquad \mathcal{R}^* = (-\Delta + V)^{-1/2}\nabla.$$

According to [12], under the assumption that $V \in RH_q$ with $q > d$, $\mathcal{R}$ is a Calderón–Zygmund operator. In particular he shows that its $\mathbb{R}^d$ vector valued kernel $\mathcal{K}$ satisfies for any $0 < \delta < 1 - d/q$ the smoothness condition

$$\left|\mathcal{K}(x,z) - \mathcal{K}(y,z)\right| + \left|\mathcal{K}(z,x) - \mathcal{K}(z,y)\right| \leqslant C\frac{|x-y|^\delta}{|x-z|^{d+\delta}}, \tag{18}$$

whenever $|x - z| > 2|x - y|$.

However, Calderón–Zygmund estimates are not enough to obtain our results. We shall need some sharper estimates for the kernel and its difference with the corresponding to the classical Riesz operator. That is the content of the next lemma which is basically contained in [12].

Lemma 3. *If $V \in RH_q$ with $q > d$, then we have*:

(a) *For every k there exists a constant C_k such that*

$$\left|\mathcal{K}(x,z)\right| \leqslant \frac{C_k}{(1+\frac{|x-z|}{\rho(x)})^k}\frac{1}{|x-z|^d}. \tag{19}$$

(b) *If $\mathbf{K}$ denotes the $\mathbb{R}^d$ vector valued kernel of the classical Riesz operator $\mathbf{R}$, then*

$$\left|\mathcal{K}(x,z) - \mathbf{K}(x,z)\right| \leqslant \frac{C}{|x-z|^d}\left(\frac{|x-z|}{\rho(x)}\right)^{2-d/q}. \tag{20}$$

Proof. For part (a) we refer to [12, inequality (6.5)]. To deal with (b) we first observe that if $|x - z| \geqslant \rho(x)$ the result is true since both are Calderón–Zygmund kernels. The case $|x - z| < \rho(x)$ is a consequence of the estimate (valid for $q > d/2$)

$$\left|\mathcal{K}(x,z) - \mathbf{K}(x,z)\right| \leqslant \frac{C}{|x-z|^{d-1}}\left(\int\limits_{B(x,|x-z|/4)} \frac{V(u)}{|u-x|^{d-1}}du + \frac{1}{|x-z|}\left(\frac{|x-z|}{\rho(x)}\right)^{2-d/q}\right)$$

appearing in the same paper as inequality (5.9). In fact if $q > d$, we may use Lemma 1 with $\epsilon = 1$ and we bound the first term in the sum by the second one. □

In order to control the operator $\mathcal{R}$ acting on functions in $BMO^\beta_{\mathfrak{L}}(w)$ we need a new estimate concerning the smoothness of the difference $\mathcal{K} - \mathbf{K}$.

Lemma 4. *Let $V \in RH_q$ with $q > d$ and $0 < \delta < 1 - \frac{d}{q}$. Then, there exists a constant C such that*

$$\left|\left[\mathcal{K}(x,z) - \mathbf{K}(x,z)\right] - \left[\mathcal{K}(y,z) - \mathbf{K}(y,z)\right]\right| \leqslant \frac{C|x-y|^\delta}{|x-z|^{d+\delta}}\left(\frac{|x-z|}{\rho(x)}\right)^{2-d/q}, \tag{21}$$

whenever $|x - z| \geqslant 2|x - y|$.

Proof. Inequality (21) certainly holds when $|x-z| \geqslant \rho(x)$ since both kernels $\mathbf{K}$ and $\mathcal{K}$ satisfy the Calderón-Zygmund smoothness estimate (18) for $\delta < 1 - d/q$. Now suppose $|x - z| < \rho(x)$. Let $\Lambda(x,z,\tau)$ and $\Gamma(x,z,\tau)$ be the fundamental solutions of $(-\Delta + V + i\tau)$ and $(-\Delta + i\tau)$ respectively. It is well known (see [12, p. 529]) that for any positive k there exists a constant C_k such that

$$\left|\nabla_1\Gamma(x,z,\tau)\right| \leqslant \frac{C_k}{(1+|\tau|^{1/2}|x-z|)^k}\frac{1}{|x-z|^{d-1}} \tag{22}$$

and

$$\left|(\nabla_1)^2\Gamma(x,z,\tau)\right| \leqslant \frac{C_k}{(1+|\tau|^{1/2}|x-z|)^k}\frac{1}{|x-z|^{d}}, \tag{23}$$

for all $x, z \in \mathbb{R}^d$, where ∇_1 means that we are taking all the partial derivatives with respect to the first variable. Also, from [12, Theorem 2.7], we have

$$\left|\Lambda(x,z,\tau)\right| \leqslant \frac{C_k}{[1+|\tau|^{1/2}|x-z|]^k[1+|x-z|/\rho(x)]^k|x-z|^{d-2}} \tag{24}$$

for all $x, z \in \mathbb{R}^d$. Notice that since $\Lambda(x,z,\tau) = \Lambda(z,x,-\tau)$ we may replace $\rho(x)$ by $\rho(z)$ in the previous inequality.

With this notation, following [12, p. 538] the difference of the kernels can be written as

$$\mathcal{K}(x,z)-\mathbf{K}(x,z)=-\frac{1}{2\pi}\int_{\mathbb{R}}(-i\tau)^{-1/2}\big[\nabla_1\Lambda(x,z,\tau)-\nabla_1\Gamma(x,z,\tau)\big]\,d\tau.$$

On the other hand since $u=\Lambda-\Gamma$, as a function of the first variable, satisfies the equation $-\Delta u+i\tau u=-V\Lambda$, we obtain

$$\Lambda-\Gamma=-\int_{\mathbb{R}^d}\Gamma V\Lambda.$$

Then

$$\mathcal{K}(x,z)-\mathbf{K}(x,z)=-\frac{1}{2\pi}\int_{\mathbb{R}}(-i\tau)^{-1/2}\int_{\mathbb{R}^d}\nabla_1\Gamma(x,u,\tau)V(u)\Lambda(u,z,\tau)\,du\,d\tau. \tag{25}$$

Consequently,

$$\begin{aligned}&\big[\mathcal{K}(x,z)-\mathbf{K}(x,z)\big]-\big[\mathcal{K}(y,z)-\mathbf{K}(y,z)\big]\\&\quad=-\frac{1}{2\pi}\int_{\mathbb{R}}(-i\tau)^{-1/2}\int_{\mathbb{R}^d}\big[\nabla_1\Gamma(x,u,\tau)-\nabla_1\Gamma(y,u,\tau)\big]V(u)\Lambda(u,z,\tau)\,du\,d\tau.\end{aligned}$$

We will deal first with the absolute value of the inner integral before performing the integration in τ. To this end we consider four regions covering $\mathbb{R}^d$:

$$\begin{aligned}E_1&=\Big\{u\colon\ |u-x|<\frac{3}{2}|x-y|\Big\};\\E_2&=\Big\{u\colon\ \frac{3}{2}|x-y|\leqslant|u-x|<\frac{1}{2}|x-z|\Big\};\\E_3&=\Big\{u\colon\ \frac{1}{2}|x-z|\leqslant|u-x|<2|x-z|\Big\};\\E_4&=\big\{u\colon\ |u-x|\geqslant 2|x-z|\big\}.\end{aligned}$$

After taking absolute value inside, we call I_j, $j=1,2,3,4$, the corresponding integrals and we proceed to estimate them.

For I_1, we majorize by the sum of the gradients and estimate each integral separately. Since both are similar we work out one of them. Due to the assumption $|x-z|>2|x-y|$, for $u\in E_1$ we have $|u-z|\geqslant\frac{1}{4}|x-z|$, and by (22) and (24), we get

$$\begin{aligned}\int_{E_1}\big|\nabla_1\Gamma(x,u,\tau)\Lambda(u,z,\tau)\big|V(u)\,du&\leqslant C_k\int_{E_1}\frac{V(u)}{(1+|\tau|^{1/2}|u-z|)^k|x-u|^{d-1}|u-z|^{d-2}}\,du\\&\leqslant\frac{C_k}{(1+|\tau|^{1/2}|x-z|)^k|x-z|^{d-2}}\int_{B(x,2|x-y|)}\frac{V(u)}{|x-u|^{d-1}}\,du\\&\leqslant\frac{C_k\,|x-y|^{\delta}}{(1+|\tau|^{1/2}|x-z|)^k|x-z|^{d-1+\delta}}\Big(\frac{|x-z|}{\rho(x)}\Big)^{2-d/q},\end{aligned}\tag{26}$$

where in the last inequality we have used Lemma 1 with $\epsilon=1$ and $r=|x-y|<\frac{1}{2}|x-z|$, and $\delta<1-\frac{d}{q}$.

Next, to take care of the integrals on the remaining regions, we observe that for $|u-x|\geqslant\frac{3}{2}|x-y|$ the Mean Value Theorem together with (23) and (24) give

$$\begin{aligned}&\big|\big[\nabla_1\Gamma(x,u,\tau)-\nabla_1\Gamma(y,u,\tau)\big]V(u)\Lambda(u,z,\tau)\big|\\&\quad\leqslant\frac{C_k|x-y|V(u)}{(1+|\tau|^{1/2}|u-z|)^k(1+\frac{|u-z|}{\rho(z)})^k|u-z|^{d-2}(1+|\tau|^{1/2}|x-u|)^k|x-u|^d}.\end{aligned}\tag{27}$$

Then, since $u\in E_2$ implies $|u-z|\geqslant|x-z|-|u-x|>\frac{1}{2}|x-z|$, we obtain

$$\begin{aligned}I_2&\leqslant\frac{C_k|x-y|}{(1+|\tau|^{1/2}|x-z|)^k|x-z|^{d-2}}\int_{E_2}\frac{V(u)}{|x-u|^d}\,du\\&\leqslant\frac{C_k|x-y|^{\delta}}{(1+|\tau|^{1/2}|x-z|)^k|x-z|^{d-2}}\int_{B(x,|x-z|)}\frac{V(u)}{|x-u|^{d-1+\delta}}\,du.\end{aligned}$$

By Lemma 1 with $\epsilon = 1-\delta$ and $r = |x-z|$, we arrive to

$$I_2 \leqslant \frac{C_k|x-y|^\delta}{(1+|\tau|^{1/2}|x-z|)^k|x-z|^{d-1+\delta}}\left(\frac{|x-z|}{\rho(x)}\right)^{2-d/q}. \tag{28}$$

By (27) and using that $u \in E_3$ implies $|x-z| \sim |u-x|$,

$$I_3 \leqslant \frac{C_k|x-y|}{(1+|\tau|^{1/2}|x-z|)^k|x-z|^d}\int\limits_{E_3}\frac{V(u)}{|u-z|^{d-2}}\,du$$

and since $E_3 \subset B(z, 3|x-z|)$ we may use Lemma 1 with $\epsilon = 2$ and $r = |x-z|$ to obtain

$$I_3 \leqslant \frac{C_k|x-y|}{(1+|\tau|^{1/2}|x-z|)^k|x-z|^d}\left(\frac{|x-z|}{\rho(x)}\right)^{2-d/q}, \tag{29}$$

where we have use also that $\rho(z) \sim \rho(x)$.

Finally, to deal with I_4 we use again (27). Noticing that for $u \in E_4$ $|u-x| \sim |u-z|$ and $\rho(x) \sim \rho(z)$ we get

$$I_4 \leqslant \frac{C_k|x-y|}{(1+|\tau|^{1/2}|x-z|)^k}\int\limits_{E_4}\frac{V(u)}{|x-u|^{2d-2}(1+\frac{|u-x|}{\rho(x)})^k}\,du. \tag{30}$$

We split the integral above into $E_4 \cap B(x, \rho(x))$ and $E_4 \cap B(x, \rho(x))^c$.

For the first part, we have

$$\begin{aligned}\int\limits_{2|x-z|<|u-x|<\rho(x)}\frac{V(u)}{|x-u|^{2d-2}(1+\frac{|u-x|}{\rho(x)})^k}\,du &\leqslant \left(\int\limits_{B(x,2|x-z|)^c}\frac{1}{|u-x|^{(2d-2)q'}}\,du\right)^{1/q'}\left(\int\limits_{B(x,\rho(x))}V^q\right)^{1/q}\\ &\leqslant \frac{C}{|x-z|^d}\left(\frac{|x-z|}{\rho(x)}\right)^{2-d/q},\end{aligned} \tag{31}$$

where we have used (3) and the definition of ρ.

For the other term, splitting into dyadic annuli and choosing k big enough, we obtain

$$\begin{aligned}\int\limits_{|u-x|>\rho(x)}\frac{V(u)}{|x-u|^{2d-2}(1+\frac{|u-x|}{\rho(x)})^k}\,du &\leqslant \rho(x)^k\int\limits_{|u-x|>\rho(x)}\frac{V(u)}{|x-u|^{k+2d-2}}\,du\\ &\leqslant \frac{C}{\rho(x)^{2d-2}}\sum_j\frac{1}{2^{j(k+2d-2)}}\int\limits_{|u-x|<2^{j+1}\rho(x)}V\\ &\leqslant \frac{C}{\rho(x)^{2d-2}}\left(\int\limits_{|u-x|<\rho(x)}V\right)\sum_j\frac{1}{2^{j(k+2d-2-\mu)}}\\ &\leqslant \frac{C}{\rho(x)^d} \leqslant \frac{C}{|x-z|^d}\left(\frac{|x-z|}{\rho(x)}\right)^{2-d/q},\end{aligned} \tag{32}$$

where in the third inequality we have use that V belongs to D_μ for some $\mu \geqslant 1$.

From (30), (31), (32), we obtain

$$I_4 \leqslant \frac{C_k|x-y|}{(1+|\tau|^{1/2}|x-z|)^k}\frac{C}{|x-z|^d}\left(\frac{|x-z|}{\rho(x)}\right)^{2-d/q}. \tag{33}$$

Now from (26), (28), (29) and (33), integrating on τ we get the desired estimate and we finish the proof of the lemma. □

Regarding $\mathcal{R}^*$ we will work under a milder condition on V, that is V satisfies (3) with $q > d/2$. Under this hypothesis $\mathcal{R}^*$ is not necessarily a Calderón–Zygmund operator. However, by [12] it is bounded "near" L^∞. We state in the next two lemmas properties of $\mathcal{K}^*$ that replace (18) and inequalities of Lemma 3.

Lemma 5. *If $V \in RH_q$ with $d/2 < q < d$, then we have:*

(a) *For every k there exists a constant C such that*

$$\left|\mathcal{K}^*(x,z)\right| \leqslant \frac{C}{(1+\frac{|x-z|}{\rho(x)})^k}\frac{1}{|x-z|^{d-1}}\left(\int\limits_{B(z,|x-z|/4)}\frac{V(u)}{|u-z|^{d-1}}du + \frac{1}{|x-z|}\right). \tag{34}$$

Moreover, the last inequality also holds with $\rho(x)$ replaced by $\rho(z)$.

(b) *For every k and $0<\delta<2-d/q$ there exists a constant C such that*

$$\left|\mathcal{K}^*(x,z)-\mathcal{K}^*(y,z)\right| \leqslant \frac{C}{(1+\frac{|x-z|}{\rho(x)})^k}\frac{|x-y|^\delta}{|x-z|^{d-1+\delta}}\left(\int\limits_{B(z,|x-z|/4)}\frac{V(u)}{|u-z|^{d-1}}du+\frac{1}{|x-z|}\right), \tag{35}$$

whenever $|x-y|<\frac{2}{3}|x-z|$. Moreover, the last inequality also holds with $\rho(x)$ replaced by $\rho(z)$.

(c) *If $\mathbf{K}^*$ denotes the $\mathbb{R}^d$ vector valued kernel of the adjoint of the classical Riesz operator, then*

$$\left|\mathcal{K}^*(x,z)-\mathbf{K}^*(x,z)\right| \leqslant \frac{C}{|x-z|^{d-1}}\left(\int\limits_{B(z,|x-z|/4)}\frac{V(u)}{|u-z|^{d-1}}du+\frac{1}{|x-z|}\left(\frac{|x-z|}{\rho(x)}\right)^{2-d/q}\right). \tag{36}$$

Proof. Inequalities (34) and (36) can be found in [12], pages 538 and 540 respectively. We point out that inequality (36) is proved only for $|x-z|<\rho(x)$ but using the size of $\mathcal{K}^*$ and $\mathbf{K}^*$ this restriction is not necessary. Estimate (35) appears in [7, Lemma 4] for $|x-y|<\frac{1}{16}|x-z|$. However, it is possible to change the factor $1/16$ for any positive constant less than one. In order to see that both estimates (34) and (35) still hold with $\rho(z)$, it is enough to consider the case $\rho(z)<|x-z|$, since otherwise $\rho(x)\sim\rho(z)$. In that case, using Proposition 1 we have

$$\left(1+\frac{|x-z|}{\rho(x)}\right)^{-k} \leqslant C\left(1+\frac{|x-z|}{\rho(z)}\right)^{-(1-\sigma)k} \tag{37}$$

where $0<\sigma<1$. □

Lemma 6. *If $V\in RH_q$ with $q>d$, then we have:*

(a) *For every k there exists a constant C such that*

$$\left|\mathcal{K}^*(x,z)\right| \leqslant \frac{C}{(1+\frac{|x-z|}{\rho(x)})^k}\frac{1}{|x-z|^d}. \tag{38}$$

Moreover, the last inequality also holds with $\rho(x)$ replaced by $\rho(z)$.

(2) *For every k and $0<\delta<1$ there exists a constant C such that*

$$\left|\mathcal{K}^*(x,z)-\mathcal{K}^*(y,z)\right| \leqslant \frac{C}{(1+\frac{|x-z|}{\rho(x)})^k}\frac{|x-y|^\delta}{|x-z|^{d+\delta}}, \tag{39}$$

whenever $|x-y|<\frac{2}{3}|x-z|$. Moreover, the last inequality also holds with $\rho(x)$ replaced by $\rho(z)$.

(c) *If $\mathbf{K}^*$ denotes the $\mathbb{R}^d$ vector-valued kernel of the adjoint of the classical Riesz operator, then*

$$\left|\mathcal{K}^*(x,z)-\mathbf{K}^*(x,z)\right| \leqslant \frac{C}{|x-z|^d}\left(\frac{|x-z|}{\rho(x)}\right)^{2-d/q}. \tag{40}$$

Proof. Since $\mathcal{K}^*(x,z)=\mathcal{K}(z,x)$, inequality (38) is a consequence of (19) and (37).

In order to see (39), given $0<\delta<1$, we consider $d/2<s<d$ and such that $0<\delta<2-d/s$. Since V satisfies (3) for every $s<q$, inequality (35) holds, in particular with $\rho(z)$. Now, if $|x-z|<\rho(z)$ we use the first inequality in Lemma 1 to see

$$\int\limits_{B(z,|x-z|/4)}\frac{V(u)}{|u-z|^{d-1}}\,du \leqslant C\frac{1}{|x-z|}.$$

In the case $|x-z|\geqslant\rho(z)$, using the second inequality in Lemma 1 we get

$$\int\limits_{B(z,|x-z|/4)}\frac{V(u)}{|u-z|^{d-1}}\,du \leqslant C\frac{1}{|x-z|}\left(1+\frac{|x-z|}{\rho(z)}\right)^{2+(\mu-1)d}.$$

Finally, by (37) we may replace $\rho(z)$ with $\rho(x)$ and (38) holds.

To check (40), if $|x-z|<\rho(x)$ the result follows from (20) since $\rho(x)\sim\rho(z)$. In the case $|x-z|\geqslant\rho(x)$ we use that the size of each kernel is like $\frac{1}{|x-z|^d}$ and that $2-d/q>0$. □

The following result gives an appropriate version of Lemma 4 for $\mathcal{R}^*$ under the weaker assumption $V\in RH_{d/2}$.

Lemma 7. *Let $V \in RH_q$ with $q > d/2$ and $0 < \delta < \min\{1, 2 - d/q\}$. Then, there exists a constant C such that*

$$\begin{aligned} &\left|\mathcal{K}^*(x,z) - \mathbf{K}^*(x,z) - \left[\mathcal{K}^*(y,z) - \mathbf{K}^*(y,z)\right]\right| \\ &\quad \leqslant \frac{C|x-y|^\delta}{|x-z|^{d-1+\delta}} \left(\int\limits_{B(z,|x-z|/4)} \frac{V(u)}{|u-z|^{d-1}} du + \frac{1}{|x-z|} \left(\frac{|x-z|}{\rho(x)}\right)^{2-d/q} \right) \end{aligned} \tag{41}$$

whenever $|x-z| \geqslant 2|x-y|$. Moreover, in the case $q > d$,

$$\left|\mathcal{K}^*(x,z) - \mathbf{K}^*(x,z) - \left[\mathcal{K}^*(y,z) - \mathbf{K}^*(y,z)\right]\right| \leqslant \frac{C|x-y|^\delta}{|x-z|^{d+\delta}} \left(\frac{|x-z|}{\rho(x)}\right)^{2-d/q}, \tag{42}$$

whenever $|x-z| \geqslant 2|x-y|$.

Proof. First observe that for $|x-z| \geqslant \rho(x)$, estimates (41) and (42) can be derived using the smoothness of each kernel (see (35) and (39) for $\mathcal{K}^*$).

For the rest of the proof we assume $|x-z| < \rho(x)$. From (25) and the fact that $\Lambda(u,x,\tau) = \Lambda(x,u,-\tau)$ we obtain

$$\begin{aligned} &\mathcal{K}^*(x,z) - \mathbf{K}^*(x,z) - \left[\mathcal{K}^*(y,z) - \mathbf{K}^*(y,z)\right] \\ &\quad = -\frac{1}{2\pi} \int\limits_{\mathbb{R}} (-i\tau)^{-1/2} \int\limits_{\mathbb{R}^d} \nabla_1 \Gamma(z,u,\tau) V(u) \left[\Lambda(x,u,-\tau) - \Lambda(y,u,-\tau)\right] du\, d\tau. \end{aligned}$$

We call I the absolute value of the inner integral in the above expression, and we split $\mathbb{R}^d$ into the same regions E_j, $j = 1,2,3,4$ as in Lemma 4. We denote by I_j, the integral over E_j, $j = 1,2,3,4$ after taking absolute value inside.

For I_1, we majorize the absolute value of the difference related to Λ by the sum of the absolute values of each term and estimate each integral separately. Since both are similar we work out one of them. First we notice that $|x-z| > 2|x-y|$ implies $|z-u| > \frac{1}{4}|x-z|$ for $u \in E_1$. Then, using (24) and (22), we have

$$\begin{aligned} \int\limits_{E_1} \left|\nabla_1 \Gamma(z,u,\tau) \Lambda(x,u,-\tau)\right| V(u)\, du &\leqslant \frac{C_k}{(1+|\tau|^{1/2}|x-z|)^k |x-z|^{d-1}} \int\limits_{B(x,2|x-y|)} \frac{V(u)}{|x-u|^{d-2}} du \\ &\leqslant \frac{C_k |x-y|^\delta}{(1+|\tau|^{1/2}|x-z|)^k |x-z|^{d-1+\delta}} \left(\frac{|x-z|}{\rho(x)}\right)^{2-d/q}, \end{aligned} \tag{43}$$

where in the last inequality we have used Lemma 1 with $\epsilon = 2$ and $r = |x-y| < 2\rho(x)$, and that $\delta \leqslant 2 - d/q$.

For the remaining regions we will use the following estimate taken from [7, p. 427],

$$\left|\Lambda(x,u,-\tau) - \Lambda(y,u,-\tau)\right| \leqslant C \frac{|x-y|^\delta}{|x-u|^{d-2+\delta}} \left[\left(1+|\tau|^{1/2}|x-u|\right)\left(1+\frac{|x-u|}{\rho(u)}\right)\right]^{-k}, \tag{44}$$

for $|x-y| < \frac{2}{3}|x-u|$ and $0 < \delta < \min\{1, 2-d/q\}$. In fact, in [7] the inequality is proved for $q < d$. However, for $q \geqslant d$ since V belongs to RH_s for every $s \leqslant q$, the above inequality holds for any $0 < \delta < 1$.

To estimate I_2 we use (44) and (22) to get

$$\begin{aligned} I_2 &\leqslant \frac{C_k |x-y|^\delta}{(1+|\tau|^{1/2}|x-z|)^k |x-z|^{d-1}} \int\limits_{B(x,\frac{1}{2}|x-z|)} \frac{V(u)}{|u-x|^{d-2+\delta}} du \\ &\leqslant \frac{C_k |x-y|^\delta}{(1+|\tau|^{1/2}|x-z|)^k |x-z|^{d-1+\delta}} \left(\frac{|x-z|}{\rho(x)}\right)^{2-d/q}, \end{aligned} \tag{45}$$

where in the last inequality we have used Lemma 1 with $r = \frac{1}{2}|x-z|$ and $\epsilon = 2-\delta$.

To deal with I_3 we notice $E_3 \subset B(z, 3|x-z|)$. Using again (44) and (22) we arrive to

$$I_3 \leqslant \frac{C_k |x-y|^\delta}{(1+|\tau|^{1/2}|x-z|)^k |x-z|^{d-2+\delta}} \int\limits_{B(z,3|x-z|)} \frac{V(u)}{|u-z|^{d-1}} du. \tag{46}$$

Finally, for $u \in E_4$ we have $|u-x| \sim |u-z|$ and hence, using (44) and (22),

$$I_4 \leqslant \frac{C_k |x-y|^\delta}{(1+|\tau|^{1/2}|x-z|)^k} \int\limits_{E_4} \frac{V(u)}{|u-x|^{2d-3+\delta}} \left[1+\left(\frac{|u-x|}{\rho(u)}\right)\right]^{-k} du.$$

We set $E_4 = E_4^1 \cup E_4^2$, where $E_4^1 = \{u\colon\ 2|x-z| \leqslant |u-x| \leqslant \rho(x)\}$. Applying Hölder's inequality the above integral over E_4^1 is bounded by

$$\left(\int_{B(x,\rho(x))} V^q\right)^{1/q}\left(\int_{|u-x|>2|x-z|} \frac{1}{|u-x|^{(2d-3+\delta)q'}}\,du\right)^{1/q'} \leqslant \frac{C}{|x-z|^{d-1+\delta}}\left(\frac{|x-z|}{\rho(z)}\right)^{2-d/q},$$

where in the last inequality we have used the reverse Hölder condition on V and the definition of ρ.

To estimate the integral on E_4^2, by Proposition 1 we have

$$\rho(u) \leqslant C\rho(x)^{1-\sigma}|u-x|^{\sigma}, \tag{47}$$

with $0 < \sigma < 1$. Therefore, we set $N = k(1-\sigma)$ to get

$$\int_{|u-x|>\rho(x)} \frac{V(u)}{|u-x|^{2d-3+\delta}}\left(\frac{\rho(x)}{|u-x|}\right)^N du \leqslant \rho(x)^{-2d+3-\delta}\sum_{j=1}^{\infty} 2^{-j(2d-3+\delta+N)} \int_{|u-x|<2^j\rho(x)} V.$$

Since V satisfies a doubling condition and we can choose k large enough, proceeding as in (32) the last expression is bounded by a constant times

$$\rho(x)^{-d+1-\delta} \leqslant \frac{C}{|x-z|^{d-1+\delta}}\left(\frac{|x-z|}{\rho(z)}\right)^{2-d/q},$$

since $d-1+\delta \geqslant 2-d/q$.

Now using the estimates in E_4^1 and E_4^2 reminding that $|x-z| \leqslant \rho(x)$, we obtain

$$I_4 \leqslant \frac{C_k|x-y|^{\delta}}{(1+|\tau|^{1/2}|x-z|)^k|x-z|^{d-1+\delta}}\left(\frac{|x-z|}{\rho(x)}\right)^{2-d/q}. \tag{48}$$

From (43), (45), (46) and (48), performing the integration on τ we get (41). It remains to check (42) for $|x-z| < \rho(x)$. For $q > d$, this is a consequence of Lemma 1 and the fact that $\rho(x) \sim \rho(z)$. In the case $q = d$ we use that V belongs to $RH_{q+\eta}$ for some $\eta > 0$. □

4. Proofs of the main results

Proof of Theorem 1. First we will prove (a). Notice that by our assumptions if we fix β and η we may choose $q > d$ and $\beta < \delta < 1 - d/q$ such that $V \in RH_q$ and

$$1 \leqslant \eta < 1 + \frac{\delta-\beta}{d}. \tag{49}$$

According to Proposition 2 we only need to check that

$$\int_B |\mathcal{R}f| \leqslant C\|f\|_{BMO^{\beta}_{\mathfrak{L}}(w)} w(B)|B|^{\beta/d}, \tag{50}$$

for all $B = B(x_0, \rho(x_0))$, $x_0 \in \mathbb{R}^d$, and

$$\int_B \left|\mathcal{R}f - (\mathcal{R}f)_B\right| \leqslant C\|f\|_{BMO^{\beta}_{\mathfrak{L}}(w)} w(B)|B|^{\beta/d}, \tag{51}$$

with $B = B(x_0, r)$, $r < \rho(x_0)$.

We start with (50). For $B = B(x_0, \rho(x_0))$ we write $f = f_1 + f_2$, with $f_1 = f\chi_{2B}$.

Since $w \in A_\infty$, $w \in A_p$ for some $1 < p < \infty$ and hence $w^{1-p'} \in A_{p'}$. Using that under our assumptions $\mathcal{R}$ is a Calderón–Zygmund operator we have

$$\begin{aligned}
\int_B |\mathcal{R}f_1| &\leqslant w(B)^{1/p}\left(\int_B |\mathcal{R}f_1|^{p'} w^{1-p'}\right)^{1/p'} \\
&\leqslant C w(B)^{1/p}\left(\int_{2B} |f|^{p'} w^{1-p'}\right)^{1/p'} \\
&\leqslant C\|f\|_{BMO^{\beta}_{\mathfrak{L}}(w)} w(B)\,|B|^{\beta/d},
\end{aligned}$$

where in the last inequality we apply Proposition 3 and the doubling property of the weight w.

On the other hand, an application of Lemma 3 gives

$$\begin{aligned}\int_B |\mathcal{R} f_2| &\leqslant \int_B \int_{(2B)^c} \left|\mathcal{K}(x,z) f(z)\right| dz\, dx \\ &\leqslant C_k \int_B \int_{(2B)^c} \left(\frac{\rho(x)}{|x-z|}\right)^k \frac{1}{|x-z|^d} \left|f(z)\right| dz\, dx \\ &\leqslant C_k \rho(x_0)^{k+d} \int_{(2B)^c} \frac{|f(z)|}{|x_0-z|^{k+d}}\, dz,\end{aligned}$$

where we have use that $\rho(x) \sim \rho(x_0)$ (Proposition 1) and $|x_0 - z| \sim |x - z|$.

Splitting the integral into dyadic annuli and using the doubling property, the above expression is bounded by

$$\begin{aligned}C_k \sum_{j=2}^{\infty} \frac{1}{2^{j(k+d)}} \int_{2^j B} \left|f(z)\right| dz &\leqslant C_k \|f\|_{BMO^{\beta}_{\mathfrak{L}}(w)} \rho(x_0)^{\beta} \sum_{j=2}^{\infty} \frac{w(2^j B)}{2^{j(k+d-\beta)}} \\ &\leqslant C_k \|f\|_{BMO^{\beta}_{\mathfrak{L}}(w)} \rho(x_0)^{\beta} w(B) \sum_{j=2}^{\infty} \frac{1}{2^{j(k+d-\beta-d\eta)}},\end{aligned}$$

and the last sum is finite choosing k big enough. This completes the proof of (50).

In order to check (51) we consider the ball $B = B(x_0, r)$, $r < \rho(x_0)$.

$$\begin{aligned}\int_B \left|\mathcal{R} f(x) - (\mathcal{R} f)_B\right| dx &\leqslant \int_B \left|(\mathcal{R} - \mathbf{R}) f(x) - \left[(\mathcal{R} - \mathbf{R}) f\right]_B\right| dx + \int_B \left|\mathbf{R} f(x) - (\mathbf{R} f)_B\right| dx \\ &= I + II. \end{aligned} \tag{52}$$

Since $BMO^{\beta}_{\mathfrak{L}}(w) \subset BMO^{\beta}(w)$ and the weight w satisfies (2) (see Remark 4), the classical Riesz transform preserves $BMO^{\beta}(w)$ and thus

$$II \leqslant C \|f\|_{BMO^{\beta}_{\mathfrak{L}}(w)} w(B) |B|^{\beta/d}.$$

It remains to take care of I. We set $f = f_1 + f_2 + f_3$ with $f_1 = f\chi_{5B}$ and $f_3 = f\chi_{B_0^c}$ with $B_0 = B(x_0, 5\rho(x_0))$. Then $I \leqslant I_1 + I_2 + I_3$ where I_j is the integral that defines I with f_j instead of f.

To estimate I_1 we use Lemma 3 obtaining

$$\begin{aligned}I_1 &\leqslant 2 \int_B \left|(\mathcal{R} - \mathbf{R}) f(x)\right| dx \\ &\leqslant C \int_B \int_{5B} \frac{|f(z)|}{|x-z|^d} \left(\frac{|x-z|}{\rho(x)}\right)^{2-d/q} dz\, dx \\ &\leqslant C \rho(x_0)^{d/q-2} \int_{5B} \int_B \frac{1}{|x-z|^{d+d/q-2}}\, dx \left|f(z)\right| dz \\ &\leqslant C \left(\frac{r}{\rho(x_0)}\right)^{2-d/q} \int_{5B} \left|f(z)\right| dz.\end{aligned}$$

By Lemma 2 in the case $\beta > 0$ or $\eta > 1$, the last expression is bounded by

$$\left(\frac{r}{\rho(x_0)}\right)^{2-d/q-d\eta+d-\beta} r^{\beta} w(B) \|f\|_{BMO^{\beta}_{\mathfrak{L}}(w)} \leqslant C r^{\beta} w(B) \|f\|_{BMO^{\beta}_{\mathfrak{L}}(w)},$$

since by assumption the exponent $2 - d/q - d\eta + d - \beta$ is non-negative. The case $\beta = 0$ and $\eta = 1$ follows in the same way.

To deal with I_2 we clearly have

$$I_2 \leqslant \frac{1}{|B|} \int_B \int_B \int_{B_0 \setminus 5B} \left|\left[\mathcal{K}(x,z) - \mathbf{K}(x,z)\right] - \left[\mathcal{K}(y,z) - \mathbf{K}(y,z)\right]\right| \left|f(z)\right| dz\, dx\, dy.$$

Now, since $x, y \in B$ and $z \in (5B)^c$ it follows $|x - z| \geqslant 2|x - y|$, and therefore we may apply Lemma 4 for δ chosen as above to get

$$I_2 \leqslant \frac{C}{|B|}\int_B\int_B\int_{B_0\setminus 5B}\frac{|x-y|^\delta}{|x-z|^{d+\delta}}\left(\frac{|x-z|}{\rho(x)}\right)^{2-d/q}\left|f(z)\right|dz\,dx\,dy$$
$$\leqslant C\frac{r^{d+\delta}}{\rho(x_0)^{2-d/q}}\int_{B_0\setminus 5B}\frac{|f(z)|}{|x_0-z|^{d+\delta-2+d/q}}\,dz,$$

since $\rho(x_0)\sim\rho(x)$ and $|x-z|\sim|x_0-z|$.

Splitting the integral, using Lemma 2 for $\beta>0$ or $\eta>1$, and the doubling condition we obtain for j_0 the integer part of $\log(\rho(x_0)/5r)$,

$$I_2 \leqslant C\left(\frac{r}{\rho(x)}\right)^{2-d/q}\sum_{j=2}^{j_0}\frac{1}{2^{j(d+\delta-2+d/q)}}\int_{2^{j+1}B\setminus 2^jB}\left|f(z)\right|dz$$
$$\leqslant C\left(\frac{r}{\rho(x_0)}\right)^{2-d/q-d\eta+d-\beta} r^\beta w(B)\|f\|_{BMO^\beta_{\mathfrak{L}}(w)}\sum_{j=2}^{j_0}2^{j(2-\delta-d/q)}$$
$$\leqslant C\left(\frac{r}{\rho(x_0)}\right)^{\delta-d\eta+d-\beta} r^\beta w(B)\|f\|_{BMO^\beta_{\mathfrak{L}}(w)},$$

and since $r<\rho(x_0)$ and (49) implies $0<\delta-d\eta+d-\beta$, we arrive to the desired estimate. The case $\beta=0$ and $\eta=1$ follows in the same way majorizing the log function by an appropriate positive power.

Finally, for I_3 we use that both kernels $\mathbf{K}$ and $\mathcal{K}$ satisfy the Calderón–Zygmund smoothness estimate (18) for $\delta<1-d/q$. Therefore proceeding as with I_2 we obtain

$$I_3 \leqslant Cr^{d+\delta}\int_{B_0^c}\frac{|f(z)|}{|x_0-z|^{d+\delta}}\,dz$$
$$\leqslant C\sum_{j=j_0}^{\infty}\frac{1}{2^{j(d+\delta)}}\int_{2^{j+1}B}\left|f(z)\right|dz$$
$$\leqslant Cr^\beta\|f\|_{BMO^\beta_{\mathfrak{L}}(w)}\sum_{j=j_0}^{\infty}\frac{1}{2^{j(d+\delta-\beta)}}w\left(2^jB\right).$$

Applying the doubling condition our choice of δ implies that the last series converges and we obtain the desired result.

In order to prove (b), we may proceed as before, this time choosing $q>d$ and $\beta<\delta<1$ such that $V\in RH_q$ and (49) holds, and using Lemma 6 and Lemma 7 instead of Lemma 3 an Lemma 4 respectively. □

Before the proof of Theorem 2 we need the following technical lemma. In what follows we denote by $\mathcal{I}_1=(-\Delta)^{-1/2}$ the classical fractional integral of order 1.

Lemma 8. *Let $V\in RH_q$ with $d/2<q<d$ and $w\in RH_s\cap A_{p/s'}$ for some $s<p'$ where $\frac{1}{p}=\frac{1}{q}-\frac{1}{d}$. Then for any $f\in BMO^\beta_{\mathfrak{L}}(w)$, $0\leqslant\beta<1$, and any ball $B=B(x,r)$,*

$$\int_B\left|f(z)\right|\mathcal{I}_1(V\chi_{2B})(z)\,dz\leqslant C\|f\|_{BMO^\beta_{\mathfrak{L}}(w)}w(B)r^{\beta-1}\Phi_{\beta,\eta}\left(\frac{r}{\rho(x)}\right), \tag{53}$$

where

$$\Phi_{\beta,\eta}(t)=\begin{cases} t^{2+\mu d-d} & \text{if } t\geqslant 1,\\ t^{d-d\eta-\beta+2-d/q} & \text{if } t<1,\ \text{and either } \beta>0 \text{ or } \eta>1,\\ [1+\log(1/t)]t^{2-d/q} & \text{if } t<1,\ \eta=1 \text{ and } \beta=0,\end{cases}$$

for η and μ being the exponent of the doubling property satisfied by w and V respectively.

Proof. We first apply Hölder's inequality to estimate the right-hand side of (53) by

$$\|f\chi_B\|_{p'}\left\|\mathcal{I}_1(V\chi_{2B})\right\|_p.$$

To bound the first factor we apply again Hölder's inequality with exponent σ such that $\sigma p'=(p/s')'=\nu$ to the functions $|f|^{p'}w^{\frac{1}{\sigma}-p'}$ and $w^{p'-\frac{1}{\sigma}}$. It is easy to check that $(p'-\frac{1}{\sigma})\sigma'=s$ and $\frac{1}{\sigma'p'}=\frac{s'}{sp}$. Therefore,

$$\|f\chi_B\|_{p'} \leqslant \left(\int_B w^s\right)^{s'/sp}\left(\int_B |f|^\nu w^{1-\nu}\right)^{1/\nu}$$
$$\leqslant C\frac{w(B)^{s'/p}}{|B|^{1/p}}\left(\int_B |f|^\nu w^{1-\nu}\right)^{1/\nu}. \tag{54}$$

On the other hand, due to the boundedness of $\mathcal{I}_1$ and the doubling property of V we have

$$\left\|\mathcal{I}_1(V\chi_{2B})\right\|_p \leqslant C\|V\chi_{2B}\|_q \leqslant C\frac{V(B)}{|B|^{1/q'}}. \tag{55}$$

In the case $r \geqslant \rho(x_0)$, since $w \in A_{p/s'}$, an application of Proposition 3 gives us

$$\|f\chi_B\|_{p'} \leqslant \|f\|_{BMO^\beta_{\mathfrak{L}}} w(B) r^{\beta-d/p}. \tag{56}$$

Now we apply the second part of Lemma 1 to estimate the right-hand side of (55) by

$$r^{d-2-d/q'}\left(\frac{r}{\rho(x)}\right)^{2+(\mu-1)d}.$$

Combining the above estimates we arrive to (53).

The case $r < \rho(x)$, is handled similarly, using Lemma 2 and the first part of Lemma 1 to bound (54) and (55) respectively. $\square$

Proof of Theorem 2. Let $s > p_0'$ such that $w \in A_{p_0/s'} \cap RH_s$. We choose q satisfying $d/2 < q < q_0 \leqslant d$, $V \in RH_q$, $0 \leqslant \beta < 2 - \frac{d}{q}$, $1 \leqslant \eta < 1 + \frac{2-d/q-\beta}{d}$ and such that $w \in A_{p/s'}$ for $\frac{1}{p} = \frac{1}{q} - \frac{1}{d}$.

As in the proof of Theorem 1 we only need to check (50) and (51) with $\mathcal{R}^*$ instead of $\mathcal{R}$. To obtain these estimates, we follow the same steps as for the previous theorem. Let us notice that there we used estimates of the kernel given by Lemma 6 and Lemma 7 for $q > d$. This time we have to take care of an additional term involving V.

Let $x_0 \in \mathbb{R}^d$ and $B = B(x_0, \rho(x_0))$, and set $f = f_1 + f_2$ with $f_1 = \chi_{2B} f$. Since $\mathcal{R}^*$ is bounded in $L^{p'}$ (see [12, Theorem 0.5]) and using (56) we have

$$\int_B |\mathcal{R}^* f_1| \leqslant |B|^{1/p}\left(\int_B |\mathcal{R}^* f_1|^{p'}\right)^{1/p'} \leqslant C|B|^{1/p}\left(\int_B |f|^{p'}\right)^{1/p'}$$
$$\leqslant C\|f\|_{BMO^\beta_L}|B|^{\beta/d} w(B). \tag{57}$$

For f_2 we estimate the size of K^* using Lemma 5. We only have to take care of the term with V. The other is the same as in Theorem 1.

Now, using that for $x \in B$ and $z \in \mathbb{R}^d \setminus 2B$, $\rho(x) \sim \rho(x_0)$, $|x-z| \sim |x_0 - z|$, $B(z, \frac{|z-x|}{4}) \subset B(x_0, 2|x_0 - z|)$, we have that

$$\int_B \int_{\mathbb{R}^d\setminus 2B} \rho(x)^k \left(\int_{B(z,\frac{|x-z|}{4})} \frac{V(u)}{|u-z|^{d-1}}\,du\right)\frac{|f(z)|}{|x-z|^{k+d-1}}\,dz\,dx$$

is bounded by a constant times

$$\rho(x_0)\sum_{j=1}^\infty \frac{1}{2^{j(k+d-1)}}\int_{2^{j+1}B\setminus 2^j B}\left(\int_{2^{j+2}B}\frac{V(u)}{|u-z|^{d-1}}\,du\right)\left|f(z)\right|dz.$$

Noticing that $\int_{2^{j+2}B}\frac{V(u)}{|u-z|^{d-1}}du = \mathcal{I}_1(\chi_{2^{j+2}B}V)(z)$, we may use Lemma 8 and $w \in D_\eta$, to obtain the bound

$$C\|f\|_{BMO^\beta_{\mathfrak{L}}(w)} w(B)\rho(x_0)^\beta \sum_{j=1}^\infty \frac{1}{2^{j(k+2d-\beta-2-\mu d-\eta d)}}.$$

Choosing k large enough to make the series convergent we arrive to the desired estimate.

Now we take care of the oscillation of $\mathcal{R}^*$ on a ball $B = B(x_0, r)$ with $r < \rho(x_0)$.

First, we use the same estimate as in (52) with $\mathcal{R}$ and **R** replaced by their adjoints and we again call I and II to the corresponding terms. For II, the same argument is valid since w satisfies (2) (see Remark 4). For I we set I_j, $j = 1, 2, 3$ as in there.

To estimate I_1 we use part (c) of Lemma 5. The term without V can be carried out in the same way. For the term involving V we notice that $B(z, \frac{1}{4}|z-x|) \subset 8B$ for $x \in B$ and $z \in 5B$. Therefore it can be bounded by

$$\int_B \int_{5B} \frac{|f(z)|}{|x-z|^{d-1}} \mathcal{I}_1(V\chi_{8B})(z)\,dz\,dx = Cr \int_{5B} |f(z)| \mathcal{I}_1(V\chi_{8B})(z)\,dz.$$

An application of Lemma 8 yields to the bound

$$\|f\|_{BMO^\beta_{\mathfrak{L}}(w)} w(B) r^\beta \left(\frac{r}{\rho(x_0)}\right)^{d-\eta d-\beta+2-d/q},$$

when $\beta > 0$ or $\eta > 1$, or

$$\|f\|_{BMO^\beta_{\mathfrak{L}}(w)} w(B) \left(1 + \log\frac{\rho(x_0)}{r}\right)\left(\frac{r}{\rho(x_0)}\right)^{2-d/q},$$

when $\beta = 0$ and $\eta = 1$. Due to the assumptions on η and q we obtain the desired result.

Now we proceed to estimate I_2. Notice that we may assume $5r < \rho(x_0)$, otherwise $I_2 = 0$. Making use of Lemma 7 we obtain two terms. One is the same as in Theorem 1 and can be handled in a similar way, this time choosing δ close enough to $2 - d/q$. For the term containing V we use that for $x \in B$ and $z \in \mathbb{R}^d \setminus 2B$, $\rho(x) \sim \rho(x_0)$, $|x-z| \sim |x_0 - z|$, $B(z, \frac{|z-x|}{4}) \subset B(x_0, 2|x_0 - z|)$. Then we need to estimate

$$r^{\delta+d} \int_{B(x_0,\rho(x_0))\setminus 5B} \frac{|f(z)|}{|x_0-z|^{d+\delta-1}} \int_{B(x_0,2|x_0-z|)} \frac{V(u)}{|u-z|^{d-1}}\,du\,dz.$$

Breaking the integral in z dyadically and setting j_0 such that $2^{j_0-1}r \leqslant \rho(x_0) \leqslant 2^{j_0}r$, the last expression is bounded by

$$r\sum_{j=3}^{j_0} \frac{1}{2^{j(d+\delta-1)}} \int_{2^jB} |f(z)| \mathcal{I}_1(\chi_{2^{j+1}B}V)(z)\,dz.$$

Applying Lemma 8, we obtain for the case $\beta > 0$ or $\eta > 1$ the bound

$$\begin{aligned} r^\beta w(B)\|f\|_{BMO^\beta_{\mathfrak{L}}} \left(\frac{r}{\rho(x_0)}\right)^{d-\eta d-\beta+2-d/q} \sum_{j=3}^{j_0} 2^{j(2-d/q-\delta)} &\leqslant Cr^\beta w(B)\|f\|_{BMO^\beta_{\mathfrak{L}}} \left(\frac{r}{\rho(x_0)}\right)^{d-\eta d-\beta+\delta} \\ &\leqslant Cr^\beta w(B)\|f\|_{BMO^\beta_{\mathfrak{L}}}, \end{aligned}$$

choosing δ close enough to $2 - d/q$. The case $\beta = 0$ and $\eta = 1$ follows in the same way.

Now we take care of I_3. Here, as in Theorem 1, we use the smoothness of each kernel separately. For $\mathbf{R}^*$ we use Calderón–Zygmund condition and for $\mathcal{R}^*$ we use Lemma 5 with δ as above. Again we only have to deal with the term with V, which can be bounded by

$$\begin{aligned} &C\rho(x_0)^k r^{\delta+d} \int_{\mathbb{R}^d\setminus B(x_0,\rho(x_0))} \frac{|f(z)|}{|x_0-z|^{k+d+\delta-1}} \int_{B(x_0,2|x_0-z|)} \frac{V(u)}{|u-z|^{d-1}}\,du\,dz \\ &\quad\leqslant C\rho(x_0)^k r^{1-k} \sum_{j=j_0}^{\infty} \frac{1}{2^{j(k+d+\delta-1)}} \int_{2^jB} |f(z)| \mathcal{I}_1(\chi_{2^{j+1}B}V)(z)\,dz, \end{aligned}$$

and applying again Lemma 8 this time we obtain the bound

$$C\|f\|_{BMO^\beta_{\mathfrak{L}}} w(B) r^\beta \left(\frac{r}{\rho(x_0)}\right)^{2+(\mu-1)d-k} \sum_{j=j_0}^{\infty} \frac{1}{2^{j(k+2d-2+\delta-\beta-d\mu-d\eta)}}.$$

Choosing k large enough to make the series convergent we get

$$C\|f\|_{BMO^\beta_{\mathfrak{L}}} w(B) r^\beta \left(\frac{r}{\rho(x_0)}\right)^{d+\delta-\beta-d\eta},$$

and the last factor is bounded since $r < \rho(x_0)$ and the exponent is positive according to our assumptions and the choice of δ. $\square$

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