Arbitrary divergence speed of the least-squares method in infinite-dimensional inverse ill-posed problems

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Abstract

A standard engineering procedure for approximating the solutions of an infinite-dimensional inverse problem of the form $Ax = y$, where A is a given compact linear operator on a Hilbert space X and y is the given data, is to find a sequence $\{X_N\}$ of finite-dimensional approximating subspaces of X whose union is dense in X and to construct the sequence $\{x_N\}$ of least-squares solutions of the problem in X_N. In 1980, Seidman showed that if the problem is ill-posed, then, without any additional assumptions on the exact solution or on the sequence of approximating subspaces X_N, it cannot be guaranteed that the sequence $\{x_N\}$ will converge to the exact solution. In this paper, this result is extended in the following sense: it is shown that if X is separable, then for any $y \in X$, $y \neq 0$ and for any arbitrarily given function $s : \mathbb{N} \to \mathbb{R}^+$ there exists an injective, compact linear operator A and an increasing sequence of finite-dimensional subspaces $X_N \subset X$ such that $\|x_N - A^{-1}y\| \geqslant s(N)$ for all $N \in \mathbb{N}$, where x_N is the least-squares solution of $Ax = y$ in X_N.

1. Introduction

A wide variety of applied problems lead to equations of the form

$$Ax = y \tag{1}$$

where x is an unknown element of a certain infinite-dimensional space, y is the data, supposed to be known up to a certain degree of noise and A is a given operator used for modelling the system under consideration. Very often, $A^\dagger$, the Moore–Penrose generalized inverse of A, is

unbounded and therefore the inverse problem (1) is ill-posed. It is well known that the best approximate solution, i.e. the least-squares solution of minimum norm of (1), is $x^{\dagger} = A^{\dagger} y$ (see [2]). In the application of the least-squares method to any concrete infinite-dimensional problem, discretization usually starts by choosing an increasing sequence of finite-dimensional subspaces X_N of X whose union is dense in X, and then the sequence $\{x_N\}_{N=1}^{\infty}$, with $x_N \in X_N$ minimizing $\|Ax - y\|^2$ over X_N, is constructed.

Luecke and Hickey [3] characterized the strong convergence to $x^{\dagger}$ of the sequence $\{x_N\}_{N=1}^{\infty}$ of least-squares solutions of (1). They proved that if A is a continuous linear operator and $y \in \mathcal{D}(A^{\dagger})$, then $x_N \to x^{\dagger}$ if and only if $\limsup_{N\to\infty} \|x_N\| \leqslant \|x^{\dagger}\|$. They also found a sufficient condition for convergence which does not require any *a priori* knowledge about the norm of the true solution. More precisely, they showed that if $y \in \mathcal{D}(A^{\dagger})$ then $x_N \to x^{\dagger}$ provided that $\sup_N \|(A_N^{\dagger})^* x_N\| < \infty$ (or equivalently $\sup_N \|(A_N^*)^{\dagger} x_N\| < \infty$), where $A_N \doteq A P_{X_N}$ with P_{X_N} being the orthogonal projection of X onto X_N. Moreover, they proved that if A is compact and it has infinite rank, then this condition is not necessary for convergence. As a corollary, they established that the uniform boundedness of the sequence $\{\|A_N^{\dagger}\|\}_{N=1}^{\infty}$ is also a sufficient condition for the strong convergence of x_N to $x^{\dagger}$.

On the other hand, Luecke and Hickey [3] also showed that the boundedness of the sequence $\{\|x_N\|\}_{N=1}^{\infty}$ is not sufficient to guarantee strong convergence. However, Groetsch and Neubauer [4] proved that if $y \in \mathcal{D}(A^{\dagger})$ then that condition is sufficient and, moreover, necessary, for weak convergence. As a corollary of this result, they were also able to derive in a much more comprehensive manner the necessary and sufficient condition for strong convergence established earlier by Luecke and Hickey.

In order to guarantee convergence, it is necessary that the subspaces X_N be carefully chosen. For example, if A is compact and the X_N are the eigenspaces associated with the singular value decomposition of A, Luecke and Hickey [3] showed that one always has that $x_N \to x^{\dagger}$. They also proved that this convergence is guaranteed if A is bounded and the subspaces X_N are chosen so that they all reduce A (in the sense that $A(X_N) \subset X_N$ and $A(X_N^{\perp}) \subset X_N^{\perp}$ for every N).

In contrast, if the subspaces X_N are carelessly chosen, e.g. when the X_N are poorly or inadequately associated with the operator A, then almost anything can happen. For instance in [1] Seidman proved that if problem (1) is ill-posed, then without any additional assumptions on $x^{\dagger}$ it cannot be guaranteed that $x_N \to x^{\dagger}$ and that it is possible for $\|x_N - x^{\dagger}\|$ to increase without bound. In this paper, we further show that if the subspaces X_N are not carefully chosen, then things might not only go wrong with the application of the least-squares method, but also they can go arbitrarily bad. In fact, it will be shown that it is possible for $\|x_N - x^{\dagger}\|$ to go to infinity at any arbitrary rate.

2. Preliminaries

The following result can be immediately derived from example 3.1 in [1]:

Theorem 2.1. *Let X be an infinite-dimensional separable Hilbert space, $B \doteq \{e_n\}_{n=1}^{\infty}$ an orthonormal basis of X and $X_N \doteq \mathrm{span}\{e_1, \ldots, e_N\}$. Then, there exist a linear, injective, compact, self-adjoint operator $A : X \to X$ whose range is dense in X and $b \in \mathcal{R}(A)$ with $b = Ax^*$ for some $x^* \in X$, such that if x_N is the least-squares solution of $Ax = b$ in X_N, then $\|x_N - x^*\| \to \infty$.*

Remark 2.2. The operator $A_0 : X \to X$ of example 3.1 in [1] is not necessarily self-adjoint. If $\{(\sigma_n; v_n, u_n)\}$ denotes the singular system associated with the compact operator A_0 and

$U : X \rightarrow X$ is the linear unitary operator defined by $U v_n = u_n$, then the linear operator $A \doteq U^{-1} A_0$, besides being compact, injective and having dense range, is self-adjoint and $\|A_0 x - y\| = \|Ax - U^{-1} y\|$ for every $y \in X$.

Corollary 2.3. *Let X and Y be infinite-dimensional Hilbert spaces, X separable, $B \doteq \{e_n\}_{n=1}^{\infty}$ an orthonormal basis of X and $X_N \doteq \text{span}\{e_1, \ldots, e_N\}$. Then, there exist a linear, injective, compact operator $A : X \rightarrow Y$ whose range is dense in Y and an element $b \in \mathcal{R}(A), b = Ax^*$ for some $x^* \in X$, such that if x_N is the least-squares solution of $Ax = b$ in X_N, then $\|x_N - x^*\| \rightarrow \infty$.*

Proof. From theorem 2.1 there exist $A_0 : X \rightarrow X$ and $b_0 \in \mathcal{R}(A_0), b_0 = A_0 x^*$ for some $x^* \in X$, such that $\|x_N^0 - x^*\| \rightarrow \infty$, where x_N^0 is the least-squares solution of $A_0 x = b_0$ in X_N. Let $V : X \rightarrow Y$ be an arbitrary unitary operator and let us define $A \doteq V A_0$ and $b \doteq V b_0$. Then, it follows immediately that the operator $A : X \rightarrow Y$ is linear, compact, injective, with range dense in Y and $b \in \mathcal{R}(A)$. Let x_N be the least-squares solution of $Ax = b$ in X_N. Since $\|Ax - b\|_Y = \|A_0 x - b_0\|_X$, then it turns out that x_N^0 is also the least-squares solution of $Ax = b$ in X_N, that is, $x_N^0 = x_N$. Therefore, $\|x_N - x^*\| \rightarrow \infty$. $\qquad \square$

3. Main results

In this section, we will prove some results that are more general than those of theorem 2.1 and corollary 2.3. More precisely, we will show that for any infinite-dimensional separable Hilbert space X, for any $b \in X, b \neq 0$, and for any arbitrarily given function $s : \mathbb{N} \rightarrow \mathbb{R}^+$, it is possible to find an injective operator A and a sequence of subspaces $X_N \subset X$ such that $\|x_N - A^{-1} b\| \geqslant s(N)$ for all $N \in \mathbb{N}$, where x_N is the least-squares solution of $Ax = b$ in X_N. The following definitions will be needed.

Definition 3.1. *Let X be an infinite-dimensional separable Hilbert space and $B \doteq \{e_n\}_{n=1}^{\infty}$ an orthonormal basis of X. An element $x \in X$ is said to be 'degenerate with respect to B' if x can be written as a finite linear combination of elements of B. Otherwise we say that x is 'non-degenerate with respect to B'. In particular, if $\langle x, e_n \rangle \neq 0 \; \forall n \in \mathbb{N}$ then we say that x is 'strongly non-degenerate with respect to B'. If $\langle x, e_n \rangle \neq 0$ for infinitely many n, then we say that x is 'weakly non-degenerate with respect to B'. Clearly, if an element is strongly non-degenerate with respect to a basis, then it is also weakly non-degenerate with respect to the same basis.*

The following results will also be needed later.

Lemma 3.2. *Let X be an infinite-dimensional separable Hilbert space and $b \in X, b \neq 0$. Then there exists an orthonormal basis B of X such that b is strongly non-degenerate with respect to B.*

Proof. Given $b \in X, b \neq 0$, let $\{g_n\}_{n=1}^{\infty}$ be an orthonormal basis of $[\text{span}\{b\}]^{\perp}$ and define $f_n \doteq b + g_n, n \in \mathbb{N}$. Suppose that for a certain sequence of scalars $\alpha_1, \alpha_2, \ldots$, one has

$$0 = \sum_n \alpha_n f_n = \left(\sum_n \alpha_n \right) b + \sum_n \alpha_n g_n.$$

Since $\{b\} \cup \{g_n\}_{n=1}^{\infty}$ is an orthogonal system, it is linearly independent and, therefore, the above equation implies $\alpha_n = 0$ for every $n \in \mathbb{N}$. Hence, f_n are linearly independent.

We will now use the Gram–Schmidt process (see [5]) to construct an orthonormal system starting from f_n (note that $\langle f_i, f_j \rangle = \|b\|^2 + \delta_{i,j}$ for $i, j \in \mathbb{N}$). For that, let us define

$$e_1 \doteq \frac{f_1}{\|f_1\|},$$

and for $n \geqslant 2$, define e_n recursively by

$$e_n \doteq \left(f_n - \sum_{j=1}^{n-1} \langle f_n, e_j \rangle e_j \right) \left\| f_n - \sum_{j=1}^{n-1} \langle f_n, e_j \rangle e_j \right\|^{-1}.$$

Then, it turns out that

$$\text{span}\,\{f_n\}_{n=1}^{\infty} = \text{span}\,\{e_n\}_{n=1}^{\infty} \tag{2}$$

and $\{e_n\}_{n=1}^{\infty}$ is an orthonormal system in X (see [5]). Moreover, this system is complete in X. In fact, suppose there exists $y_1 \in X$ such that $\langle y_1, e_n \rangle = 0$ for every $n \in \mathbb{N}$. This implies that $\langle y_1, f_n \rangle = 0$ for every $n \in \mathbb{N}$ by virtue of (2) and therefore

$$\langle y_1, g_n \rangle = -\langle y_1, b \rangle \qquad \text{for every} \quad n \in \mathbb{N}. \tag{3}$$

But, since $\{g_n\}_{n=1}^{\infty}$ is an orthonormal system in X, the sequence $\{\langle y_1, g_n \rangle\}_{n=1}^{\infty}$ must be in ℓ^2. Thus, since by (3) that sequence is constant, we must have $\langle y_1, g_n \rangle = 0$ for every $n \in \mathbb{N}$ which, again by virtue of (3), also implies that $\langle y_1, b \rangle = 0$. Thus, $y_1 = 0$ since $\{b, g_1, g_2, \ldots\}$ is complete in X. Hence, $\{e_n\}_{n=1}^{\infty}$ is a complete orthonormal system in X.

Next, we will show that b is strongly non-degenerate with respect to the basis $B \doteq \{e_n\}_{n=1}^{\infty}$. In contrast, suppose there exists $n^* \in \mathbb{N}$ such that $\langle e_{n^*}, b \rangle = 0$ and $\langle e_n, b \rangle \neq 0$ for all $n < n^*$. Since $\langle e_1, b \rangle = \|f_1\|^{-1} \|b\|^2 \neq 0$ we must have that $n^* \geqslant 2$. Let $F_n \doteq \text{span}\{f_1, \ldots, f_n\}$ and $E_n \doteq \text{span}\{e_1, \ldots, e_n\}$. Since $\{f_n\}_{n=1}^{\infty}$ is a linearly independent set by the Gram–Schmidt process, it turns out that $F_n = E_n$ for all $n \in \mathbb{N}$ (see [5]). Let now P_{F_n} and P_{E_n} denote the orthogonal projections of X onto F_n and E_n, respectively. Since $\langle e_{n^*}, b \rangle = 0$, it follows that

$$P_{F_{n^*}} b = P_{E_{n^*}} b = P_{E_{n^*-1}} b = P_{F_{n^*-1}} b,$$

and therefore $\langle f_{n^*}, b \rangle = 0$. But this contradicts the fact that $\langle f_{n^*}, b \rangle = \langle b, b \rangle + \langle g_{n^*}, b \rangle = \|b\|^2 \neq 0$. Thus, such an n^* does not exist and $\langle e_n, b \rangle \neq 0$ for every $n \in \mathbb{N}$. We therefore conclude that b is strongly non-degenerate with respect to the basis B of X. $\qquad \square$

The following lemma is based upon the ideas presented in example 3.1 in [1].

Lemma 3.3. *Let X be an infinite-dimensional separable Hilbert space, $B \doteq \{e_n\}_{n=1}^{\infty}$ an orthonormal basis of X and let $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ be two sequences in ℓ^2 satisfying*

$$\alpha_n \neq 0 \quad \forall n \in \mathbb{N}, \qquad \beta_1 = 0, \qquad \beta_n \neq 0 \quad \forall n \geqslant 2. \tag{4}$$

Then, given $x = \sum_{n=1}^{\infty} \xi_n e_n \in X$, the linear operator $A : X \to X$ defined by

$$Ax \doteq \sum_{n=1}^{\infty} (\alpha_n \xi_n + \beta_n \xi_1) e_n \tag{5}$$

is compact, injective and its range is dense in X.

Proof. We will first show that this operator is compact. For each $N \in \mathbb{N}$, let $\tilde{A}_N : X \to X_N$ be the operator defined by $\tilde{A}_N x \doteq \sum_{i=1}^{N} (\alpha_i \xi_i + \beta_i \xi_1) e_i$, where $x = \sum_{i=1}^{\infty} \xi_i e_i \in X$ and

$X_N \doteq \text{span}\{e_1, \ldots, e_N\}$. For each $N \in \mathbb{N}$, $\tilde{A}_N$ is compact since it is clearly a finite rank operator. Moreover,

$$
\begin{aligned}
\|Ax - \tilde{A}_N x\|^2 &= \left\| \sum_{i=N+1}^{\infty} (\alpha_i \xi_i + \beta_i \xi_1) e_i \right\|^2 \\
&= \sum_{i=N+1}^{\infty} (\alpha_i \xi_i + \beta_i \xi_1)^2 \\
&\leqslant \|x\|^2 \left(\sum_{i=N+1}^{\infty} \alpha_i^2 + 2 \sum_{i=N+1}^{\infty} \alpha_i \beta_i + \sum_{i=N+1}^{\infty} \beta_i^2 \right) \quad \left(\text{since } \xi_i^2 \leqslant \|x\|^2 \; \forall i \in \mathbb{N} \right) \\
&\leqslant \|x\|^2 \left[\sum_{i=N+1}^{\infty} \alpha_i^2 + 2 \left(\sum_{i=N+1}^{\infty} \alpha_i^2 \right)^{\frac{1}{2}} \left(\sum_{i=N+1}^{\infty} \beta_i^2 \right)^{\frac{1}{2}} + \sum_{i=N+1}^{\infty} \beta_i^2 \right]
\end{aligned}
$$

(by Cauchy–Schwartz).

Since the sequences $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ are both in ℓ^2, it then follows that $\|A - \tilde{A}_N\| \to 0$ as $N \to \infty$. Hence, A, being uniform limit of compact operators, is itself compact.

We will now check that the operator A is injective. Let $x = \sum_{i=1}^{\infty} \xi_i e_i \in X$ and suppose $Ax = 0$, then $\sum_{i=1}^{\infty} \alpha_i \xi_i e_i = -\sum_{i=1}^{\infty} \beta_i \xi_1 e_i$. Equating coefficients of e_1 we have that $\alpha_1 \xi_1 = -\beta_1 \xi_1$. Since $\beta_1 = 0$ and $\alpha_1 \neq 0$ it follows that $\xi_1 = 0$. Similarly for all $i \geqslant 2$, $\alpha_i \xi_i = -\beta_i \xi_1 = 0$ which implies $\xi_i = 0$ since $\xi_1 = 0$ and $\alpha_i \neq 0$. Hence, $x = 0$ and the operator A is injective.

Next, we will show that $\mathcal{R}(A)$ is dense in X. Let $x = \sum_{i=1}^{\infty} \xi_i e_i \in X$. For each $N \in \mathbb{N}$, there are scalars $c_1, c_2, \ldots, c_N$ such that $z_N \doteq \sum_{i=1}^{N} c_i e_i \in X_N$ satisfies

$$
A z_N = \sum_{i=1}^{N} (\alpha_i c_i + \beta_i c_1) e_i = \sum_{i=1}^{N} \xi_i e_i.
$$

In fact, equating the coefficients of e_i, $i = 1, \ldots, N$, in this expression and recalling that $\beta_1 = 0$ and $\alpha_i \neq 0$ for all $i \in \mathbb{N}$, we immediately obtain $c_1 = \frac{\xi_1}{\alpha_1}$ and $c_i = \alpha_i^{-1} \left(\xi_i - \frac{\beta_i \xi_1}{\alpha_1} \right)$ for $2 \leqslant i \leqslant N$. Moreover,

$$
\|x - A z_N\|^2 = \left\| \sum_{i=N+1}^{\infty} \xi_i e_i \right\|^2 = \sum_{i=N+1}^{\infty} \xi_i^2,
$$

and since $\{\xi_i\}_{i=1}^{\infty}$ is a sequence in ℓ^2 we conclude that $\|x - A z_N\| \to 0$ as $N \to \infty$. Hence, $\mathcal{R}(A)$ is dense in X. This finishes the proof of the lemma. $\qquad \square$

In the following result, we show that for any arbitrary infinite-dimensional separable Hilbert space X and for any non-zero element $b \in X$, there exists a linear operator A such that the least-squares solutions of the problem $Ax = b$ in X_N diverge from the exact solution with arbitrarily large speed. More precisely, we have the following theorem.

Theorem 3.4. *Let X be an infinite-dimensional separable Hilbert space and $s : \mathbb{N} \to \mathbb{R}^+$ an arbitrary nonnegative increasing function. Then for each $b \in X$, $b \neq 0$, there exist an increasing sequence of subspaces X_N whose union is dense in X and a linear, compact, injective operator $A = A(b, s) : X \to X$, whose range is dense in X, such that $b = Ax^*$ for some $x^* \in X$, and if x_N denotes the least-squares solution of $Ax = b$ in X_N, then $\|x_N - x^*\| \geqslant s(N)$ for every $N \in \mathbb{N}$.*

Proof. Let $b \in X$, $b \neq 0$. From lemma 3.2 there exists an orthonormal basis $B \doteq \{e_n\}_{n=1}^{\infty}$ of X such that b is strongly non-degenerate with respect to B. Define $X_N \doteq \mathrm{span}\{e_1, \ldots, e_N\}$, let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ be any two sequences in ℓ^2 satisfying (4) and let A be the linear operator defined in (5). By virtue of lemma 3.3 A is compact, injective and its range is dense in X.

Next, we will show that given any element $b \in X$, strongly non-degenerate with respect to B, i.e. such that $b_n \doteq \langle b, e_n \rangle \neq 0$ for every $n \in \mathbb{N}$, it is possible to choose the sequences $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty} \in \ell^2$ in such a way that, besides satisfying (4), one has $b \in \mathcal{R}(A)$, and if x_N denotes the least-squares solution of $Ax = b$ in X_N, then $\|x_N - A^{-1}b\| \geqslant s(N)$ for every $N \in \mathbb{N}$.

The sequences $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ shall be constructed recursively as follows:

Step 1. Choose α_1 arbitrary, $\alpha_1 \neq 0$ and $\beta_1 \doteq 0$.

Step 2. $\alpha_2 \doteq \frac{1}{2}$,

$$\beta_2 \doteq \begin{cases} \frac{\alpha_1 b_2}{2 b_1}, & \text{if } b_2 = \frac{\alpha_2}{2^{\frac{2}{2}+1}} = \frac{1}{8}, \\ \frac{\alpha_1}{b_1}\left(b_2 - \frac{\alpha_2}{2^{\frac{2}{2}+1}}\right) = \frac{\alpha_1}{b_1}\left(b_2 - \frac{1}{8}\right), & \text{in other case.} \end{cases}$$

It suffices to choose $\beta_2 \neq 0$ such that $\frac{b_1 \beta_2}{\alpha_1} \in \left(b_2 - \frac{\alpha_2}{2^{\frac{2}{2}}}, b_2\right)$.

Step 3. $\alpha_4 \doteq \frac{1}{4}$,

$$\beta_4 \doteq \begin{cases} \frac{\alpha_1 b_4}{2 b_1}, & \text{if } b_4 = \frac{\alpha_4}{2^{\frac{4}{2}+1}} = \frac{1}{32}, \\ \frac{\alpha_1}{b_1}\left(b_4 - \frac{\alpha_4}{2^{\frac{4}{2}+1}}\right) = \frac{\alpha_1}{b_1}\left(b_4 - \frac{1}{32}\right), & \text{in other case.} \end{cases}$$

Also here, it suffices to choose $\beta_4 \neq 0$ such that $\frac{b_1 \beta_4}{\alpha_1} \in \left(b_4 - \frac{\alpha_4}{2^{\frac{4}{2}}}, b_4\right)$.

Step 4. $\alpha_6 \doteq \frac{1}{6}$,

$$\beta_6 \doteq \begin{cases} \frac{\alpha_1 b_6}{2 b_1}, & \text{if } b_6 = \frac{\alpha_6}{2^{\frac{6}{2}+1}} = \frac{1}{96}, \\ \frac{\alpha_1}{b_1}\left(b_6 - \frac{\alpha_6}{2^{\frac{6}{2}+1}}\right) = \frac{\alpha_1}{b_1}\left(b_6 - \frac{1}{96}\right), & \text{in other case.} \end{cases}$$

As before, it suffices to choose $\beta_6 \neq 0$ such that $\frac{b_1 \beta_6}{\alpha_1} \in \left(b_6 - \frac{\alpha_6}{2^{\frac{6}{2}}}, b_6\right)$. For simplicity, let us denote $K \doteq \alpha_1^2\left[1 + b_1^{-2}\left(\frac{5}{4}\|b\|^2 + \sqrt{2}\|b\| + 2\right)\right]$.

Step 5. α_3 is chosen according to the sign of b_3 as follows:

- If $b_3 < 0$, then

$$\alpha_3 \doteq \min\left\{\frac{1}{3}; \left|\frac{\alpha_1 b_3 \beta_6}{b_1}\right|\left(b_6 - \frac{\beta_6 b_1}{\alpha_1}\right)[Ks(4)]^{-1}\right\} > 0. \tag{6}$$

  Note that $\alpha_3 > 0$ since $K > 0$, and by the selection of β_6 made in step 4 we have that $b_6 - \frac{\beta_6 b_1}{\alpha_1} > 0$.
- If $b_3 > 0$, then we define

$$\alpha_3 \doteq \min\left\{\frac{1}{3}; 2^{\frac{3}{2}} b_3; b_3\left[\frac{Ks(4)}{\left|\frac{\alpha_1 \beta_6}{b_1}\right|\left(b_6 - \frac{\beta_6 b_1}{\alpha_1}\right)} + 2^{-\frac{3}{2}}\right]^{-1}\right\} > 0. \tag{7}$$

In (6) and (7), it actually suffices to pick α_3 as any positive number less than or equal to the respective minimum.

Step 6.

$$\beta_3 \doteq \begin{cases} \frac{\alpha_1 b_3}{2b_1}, & \text{if } b_3 = \frac{\alpha_3}{2^{\frac{3}{2}+1}}, \\ \frac{\alpha_1}{b_1}\left(b_3 - \frac{\alpha_3}{2^{\frac{3}{2}+1}}\right), & \text{in other case.} \end{cases}$$

Here, again it suffices to choose $\beta_3 \neq 0$ such that $\frac{b_1\beta_3}{\alpha_1} \in \left(b_3 - \frac{\alpha_3}{2^{\frac{3}{2}}}, b_3\right)$.

Steps similar to 4, 5 and 6 are followed to construct α_{2j}, β_{2j} and $\alpha_{2j-3}, \beta_{2j-3}$ for $j \geqslant 4$. More precisely:

Step 7. For all $j \geqslant 4$, define

$$\alpha_{2j} \doteq \frac{1}{2j}, \qquad \beta_{2j} \doteq \begin{cases} \frac{\alpha_1 b_{2j}}{2b_1}, & \text{if } b_{2j} = \frac{\alpha_{2j}}{2^{j+1}}, \\ \frac{\alpha_1}{b_1}\left(b_{2j} - \frac{\alpha_{2j}}{2^{j+1}}\right), & \text{in other case.} \end{cases}$$

Here, it suffices to choose $\beta_{2j} \neq 0$ such that $\frac{b_1\beta_{2j}}{\alpha_1} \in \left(b_{2j} - \frac{\alpha_{2j}}{2^j}, b_{2j}\right)$.

Step 8. For all $j \geqslant 4$, α_{2j-3} is chosen according to the sign of b_{2j-3} as follows:

- If $b_{2j-3} < 0$, define

$$\alpha_{2j-3} \doteq \min\left\{\frac{1}{2j-3}; \left|\frac{\alpha_1 b_{2j-3}\beta_{2j}}{b_1}\right| \left(b_{2j} - \frac{\beta_{2j}b_1}{\alpha_1}\right)[Ks(2j-2)]^{-1}\right\} > 0. \tag{8}$$

 Note that $\alpha_{2j-3} > 0$ since, by the choice of β_{2j} in step 7, we have that $b_{2j} - \frac{\beta_{2j}b_1}{\alpha_1} > 0$.

- If $b_{2j-3} > 0$, define

$$\alpha_{2j-3} \doteq \min\left\{\frac{1}{2j-3}; 2^{\frac{2j-3}{2}} b_{2j-3}; b_{2j-3}\left[\frac{Ks(2j-2)}{\left|\frac{\alpha_1\beta_{2j}}{b_1}\right|\left(b_{2j}-\frac{\beta_{2j}b_1}{\alpha_1}\right)} + 2^{-\frac{2j-3}{2}}\right]^{-1}\right\} > 0. \tag{9}$$

In (8) and (9), it suffices to choose α_{2j-3} to be any positive number less than or equal to the respective minimum.

Finally, define

$$\beta_{2j-3} \doteq \begin{cases} \frac{\alpha_1 b_{2j-3}}{2b_1}, & \text{if } b_{2j-3} = \frac{\alpha_{2j-3}}{2^{\frac{2j-3}{2}+1}}, \\ \frac{\alpha_1}{b_1}\left(b_{2j-3} - \frac{\alpha_{2j-3}}{2^{\frac{2j-3}{2}+1}}\right), & \text{in other case.} \end{cases} \tag{10}$$

It suffices to choose $\beta_{2j-3} \neq 0$ such that $\frac{b_1\beta_{2j-3}}{\alpha_1} \in \left(b_{2j-3} - \frac{\alpha_{2j-3}}{2^{\frac{2j-3}{2}}}, b_{2j-3}\right)$.

Figure 1 depicts the recursion order used for the construction of the sequences $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$. It is easy to check that these sequences satisfy conditions (4) and are both in ℓ^2. In fact, $\sum_{n=1}^{\infty}\alpha_n^2 \leqslant \sum_{n=1}^{\infty}\frac{1}{n^2} < \infty$, since $0 < \alpha_n \leqslant \frac{1}{n}$ for every $n \in \mathbb{N}$ due to the choice of α_n. On the other hand, by our choice of the sequence $\{\beta_n\}_{n=1}^{\infty}$, it follows that

$$\sum_{n=1}^{\infty}\beta_n^2 \leqslant \sum_{n=1}^{\infty}\left(\frac{\alpha_1 b_n}{2b_1}\right)^2 + \sum_{n=1}^{\infty}\left(\frac{\alpha_1}{b_1}\right)^2\left(b_n - \frac{\alpha_n}{2^{\frac{n}{2}+1}}\right)^2$$

$$= \frac{\alpha_1^2}{b_1^2}\left[\frac{1}{4}\sum_{n=1}^{\infty}b_n^2 + \sum_{n=1}^{\infty}\left(b_n^2 - \frac{b_n\alpha_n}{2^{\frac{n}{2}}} + \frac{\alpha_n^2}{2^{n+2}}\right)\right]$$

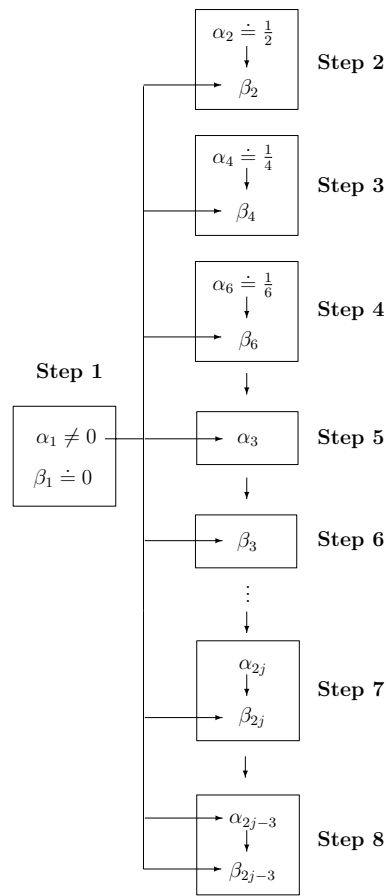**Figure 1.** Recursion order used for the construction of the sequences $\{\alpha_n\}$ and $\{\beta_n\}$.

$$\leqslant \frac{\alpha_1^2}{b_1^2} \left[\frac{5}{4} \|b\|^2 + \left(\sum_{n=1}^{\infty} b_n^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} \frac{\alpha_n^2}{2^n} \right)^{\frac{1}{2}} + \sum_{n=1}^{\infty} \alpha_n^2 \right]$$

(by Cauchy–Schwartz and since $2^{n+1} > 1 \ \forall n \in \mathbb{N}$)

$$\leqslant \frac{\alpha_1^2}{b_1^2} \left[\frac{5}{4} \|b\|^2 + \|b\| \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{\frac{1}{2}} + \sum_{n=1}^{\infty} \frac{1}{n^2} \right] \qquad \left(\text{since } \alpha_n \leqslant \frac{1}{n} \text{ and } 2^n > 1 \ \forall n \in \mathbb{N} \right)$$

$$\leqslant \frac{\alpha_1^2}{b_1^2} \left(\frac{5}{4} \|b\|^2 + \sqrt{2} \|b\| + 2 \right) \qquad \left(\text{since } \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < 2 \right)$$

$$= K - \alpha_1^2 < \infty. \tag{11}$$

We will check now that $b \in \mathcal{R}(A)$, i.e., that there exists $x^* = \sum_{n=1}^{\infty} \xi_n^* e_n \in X$ such that $Ax^* = b$. In order to do that, we will show that the system of equations derived from the equality $\sum_{n=1}^{\infty} (\alpha_n \xi_n^* + \beta_n \xi_1^*) e_n = \sum_{n=1}^{\infty} b_n e_n$ has a unique solution $\{\xi_n^*\} \in \ell^2$. Equating first the coefficients of e_1 we have $\alpha_1 \xi_1^* + \beta_1 \xi_1^* = b_1$. Since $\beta_1 = 0$, it follows that $\xi_1^* = \frac{b_1}{\alpha_1}$. Similarly, equating the coefficients of e_n for $n \geqslant 2$ it follows that $\alpha_n \xi_n^* + \beta_n \xi_1^* = b_n$, which

implies

$$\xi_n^* = \left(b_n - \frac{b_1 \beta_n}{\alpha_1}\right)\alpha_n^{-1}. \tag{12}$$

Hence,

$$
\begin{aligned}
\sum_{n=1}^{\infty} \xi_n^{*2} &= \left(\frac{b_1}{\alpha_1}\right)^2 + \sum_{n=2}^{\infty}\left(b_n - \frac{b_1 \beta_n}{\alpha_1}\right)^2 \alpha_n^{-2} \\
&= \left(\frac{b_1}{\alpha_1}\right)^2 + \sum_{\substack{n=2 \\ b_n = \frac{\alpha_n}{2^{\frac{n}{2}+1}}}}^{\infty}\left(b_n - \frac{b_1 \beta_n}{\alpha_1}\right)^2 \alpha_n^{-2} + \sum_{\substack{n=2 \\ b_n \neq \frac{\alpha_n}{2^{\frac{n}{2}+1}}}}^{\infty}\left(b_n - \frac{b_1 \beta_n}{\alpha_1}\right)^2 \alpha_n^{-2} \\
&= \left(\frac{b_1}{\alpha_1}\right)^2 + \sum_{\substack{n=2 \\ b_n = \frac{\alpha_n}{2^{\frac{n}{2}+1}}}}^{\infty}\left(b_n - \frac{b_n}{2}\right)^2 \alpha_n^{-2} + \sum_{\substack{n=2 \\ b_n \neq \frac{\alpha_n}{2^{\frac{n}{2}+1}}}}^{\infty}\left(\frac{\alpha_n}{2^{\frac{n}{2}+1}}\right)^2 \alpha_n^{-2} \qquad \text{(by our choice of } \beta_n) \\
&\leqslant \left(\frac{b_1}{\alpha_1}\right)^2 + \sum_{n=2}^{\infty}\frac{1}{2^{n+4}} + \sum_{n=2}^{\infty}\frac{1}{2^{n+2}} < \infty.
\end{aligned}
$$

Thus, $\{\xi_n^*\}_{n=1}^{\infty} \in \ell^2$, $x^* = \sum_{n=1}^{\infty}\xi_n^* e_n \in X$ and $b = Ax^* \in \mathcal{R}(A)$.

Let now x_N be the least-squares solution of $Ax = b$ in X_N and write $x_N = \sum_{n=1}^{N}\xi_n e_n$. We will show that $\|x_N - x^*\| \geqslant s(N)$ for every $N \in \mathbb{N}$. Defining the functional $\mathcal{J}(x_N) \doteq \frac{1}{2}\|Ax_N - Ax^*\|^2$ and equating to zero each one of its partial derivatives with respect to $\xi_i, i = 1, \ldots, N$ (for the sake of brevity we skip the details here), it follows immediately that

$$\xi_1 = \xi_1^* + \frac{\sum_{n=N+1}^{\infty}\alpha_n \beta_n \xi_n^*}{\alpha_1^2 + \sum_{n=N+1}^{\infty}\beta_n^2},$$

$$\xi_n = \xi_n^* - \frac{\beta_n}{\alpha_n}(\xi_1 - \xi_1^*) \qquad \text{for} \quad 2 \leqslant n \leqslant N.$$

Therefore,

$$
\begin{aligned}
\|x_N - x^*\|^2 &= \frac{\left(\sum_{n=N+1}^{\infty}\alpha_n \beta_n \xi_n^*\right)^2}{\left(\alpha_1^2 + \sum_{n=N+1}^{\infty}\beta_n^2\right)^2}\left(1 + \sum_{n=2}^{N}\frac{\beta_n^2}{\alpha_n^2}\right) + \sum_{n=N+1}^{\infty}\xi_n^{*2} \\
&\geqslant \frac{\left(\sum_{n=N+1}^{\infty}\alpha_n \beta_n \xi_n^*\right)^2}{\left(\alpha_1^2 + \sum_{n=N+1}^{\infty}\beta_n^2\right)^2}\sum_{n=2}^{N}\frac{\beta_n^2}{\alpha_n^2}. \tag{13}
\end{aligned}
$$

Hence,

(a) For N even,

$$
\begin{aligned}
\|x_N - x^*\|^2 &\geqslant \frac{\left(\alpha_{N+2}\beta_{N+2}\xi_{N+2}^*\right)^2}{\left(\alpha_1^2 + \sum_{n=N+1}^{\infty}\beta_n^2\right)^2}\frac{\beta_{N-1}^2}{\alpha_{N-1}^2} \\
&= \frac{\beta_{N+2}^2\left(b_{N+2} - \frac{b_1 \beta_{N+2}}{\alpha_1}\right)^2}{\left(\alpha_1^2 + \sum_{n=N+1}^{\infty}\beta_n^2\right)^2}\frac{\beta_{N-1}^2}{\alpha_{N-1}^2} \qquad \text{(by replacing } \xi_{N+2}^* \text{ by (12)).} \tag{14}
\end{aligned}
$$

(a.1) If $b_{N-1} < 0$, then by the selection of β_n we have $\frac{b_1 \beta_{N-1}}{\alpha_1} < b_{N-1} < 0$ and therefore $\beta_{N-1}^2 > \left(\frac{\alpha_1 b_{N-1}}{b_1}\right)^2$, which, together with our choice of α_{N-1} (see (8) in step 8), implies that

$$\begin{aligned}
\|x_N - x^*\|^2 &\geqslant \frac{[s(N)]^2 K^2}{\left(\alpha_1^2 + \sum_{n=N+1}^{\infty} \beta_n^2\right)^2} \\
&\geqslant \frac{[s(N)]^2 \left(\alpha_1^2 + \sum_{n=1}^{\infty} \beta_n^2\right)^2}{\left(\alpha_1^2 + \sum_{n=N+1}^{\infty} \beta_n^2\right)^2} \qquad \text{(from (11))} \\
&\geqslant [s(N)]^2.
\end{aligned}$$

(a.2) If $b_{N-1} > 0$, since α_{N-1} was chosen so that $\alpha_{N-1} \leqslant 2^{\frac{N-1}{2}} b_{N-1}$ (see (9) in step 8), and recalling our choice of β_{N-1} (see (10)), it follows that $0 \leqslant b_{N-1} - \frac{\alpha_{N-1}}{2^{\frac{N-1}{2}}} < \frac{b_1 \beta_{N-1}}{\alpha_1}$. Then, $\beta_{N-1}^2 > \frac{\alpha_1^2}{b_1^2}\left(b_{N-1} - \frac{\alpha_{N-1}}{2^{\frac{N-1}{2}}}\right)^2$. Substituting with this into (14), we finally obtain that

$$\begin{aligned}
\|x_N - x^*\|^2 &\geqslant \frac{\left[\beta_{N+2}\left(b_{N+2} - \frac{b_1 \beta_{N+2}}{\alpha_1}\right)\right]^2}{\left(\alpha_1^2 + \sum_{n=N+1}^{\infty} \beta_n^2\right)^2} \frac{\alpha_1^2}{b_1^2}\left(b_{N-1} - \frac{\alpha_{N-1}}{2^{\frac{N-1}{2}}}\right)^2 \frac{1}{\alpha_{N-1}^2} \\
&= \frac{\left[\beta_{N+2}\left(b_{N+2} - \frac{b_1 \beta_{N+2}}{\alpha_1}\right)\right]^2}{\left(\alpha_1^2 + \sum_{n=N+1}^{\infty} \beta_n^2\right)^2} \frac{\alpha_1^2}{b_1^2}\left(\frac{b_{N-1}}{\alpha_{N-1}} - \frac{1}{2^{\frac{N-1}{2}}}\right)^2 \\
&\geqslant \frac{[s(N)]^2 K^2}{\left(\alpha_1^2 + \sum_{n=N+1}^{\infty} \beta_n^2\right)^2} \qquad \text{(from (9))} \\
&\geqslant \frac{[s(N)]^2}{\left(\alpha_1^2 + \sum_{n=N+1}^{\infty} \beta_n^2\right)^2}\left(\alpha_1^2 + \sum_{n=1}^{\infty} \beta_n^2\right)^2 \qquad \text{(from (11))} \\
&\geqslant [s(N)]^2.
\end{aligned}$$

(b) For N odd, it follows from (13) that

$$\|x_N - x^*\|^2 \geqslant \frac{\left(\alpha_{N+3} \beta_{N+3} \xi_{N+3}^*\right)^2}{\left(\alpha_1^2 + \sum_{n=N+1}^{\infty} \beta_n^2\right)^2} \frac{\beta_N^2}{\alpha_N^2}.$$

Following similar steps as in the previous case (N even), it follows easily that now $\|x_N - x^*\|^2 \geqslant [s(N+1)]^2 \geqslant [s(N)]^2$, where the last inequality follows from the fact that s is an increasing function. This completes the proof of the theorem. $\qquad \square$

Later on, in corollary 3.7, we will see that under certain general additional assumptions on b, the operator A in the previous theorem can be constructed to further be self-adjoint.

For our next corollary, the following lemma will be needed.

Lemma 3.5. *Let X be a Hilbert space, H a closed subspace of X, $A : X \to X$ a bounded linear operator, M a closed subspace of X such that M and $M^\perp$ are invariant under A and $b \in M$. If x^* is the least-squares solution of $Ax = b$ in H, then the least-squares solution of $Ax = b$ in $H \cap M$ is $P_M x^*$, where P_M is the orthogonal projection of X onto M.*

Proof. Let us suppose that $P_M x^*$ is not the least-squares solution of $Ax = b$ in $H \cap M$. Then, there exists $z \in H \cap M$, $z \neq P_M x^*$, such that

$$\|Az - b\| < \|A P_M x^* - b\|. \tag{15}$$

Let $\widetilde{x^*} \doteq z + P_{M^\perp}x^* \in H$. Since $z \neq P_M x^*$, it turns out that $\widetilde{x^*} \neq x^*$. Then, since $b \in M, z \in M$ and $M, M^\perp$ are both invariant under A, we have that

$$
\begin{aligned}
\|A\widetilde{x^*} - b\|^2 &= \|Az + AP_{M^\perp}x^* - b\|^2 \\
&= \|Az - b\|^2 + \|AP_{M^\perp}x^*\|^2 \\
&< \|AP_M x^* - b\|^2 + \|AP_{M^\perp}x^*\|^2 \quad \text{(from (15))} \\
&= \|Ax^* - b\|^2,
\end{aligned}
$$

which is a contradiction since $\widetilde{x^*} \in H$ and x^* is the least-squares solution of $Ax = b$ in H. Hence, $P_M x^*$ is the least-squares solution of $Ax = b$ in $H \cap M$. $\qquad\square$

In actual applications of the least-squares method, the basis $B = \{e_n\}_{n=1}^\infty$ of X is usually given and X_N is chosen as $\mathrm{span}\{e_1, \ldots, e_N\}$. In the next corollary, we shown that in this case, for any $b \in X$ weakly non-degenerate with respect to that basis and for any function $s : \mathbb{N} \to \mathbb{R}^+$ as before, it is also possible to construct an operator $A = A(b, s)$ such that the least-squares solutions in X_N of $Ax = b$ diverge from the exact solution with speed greater than or equal to s.

Corollary 3.6. *Let X be an infinite-dimensional separable Hilbert space, $s : \mathbb{N} \to \mathbb{R}^+$ an arbitrary nonnegative increasing function, $B \doteq \{e_n\}_{n=1}^\infty$ an orthonormal basis of X, $b \in X$ weakly non-degenerate with respect to B and $X_N \doteq \mathrm{span}\{e_1, \ldots, e_N\}$. Then there exists a linear, compact, injective operator $A = A(b, s) : X \to X$, whose range is dense in X, such that $b = A\hat{x}$ for some $\hat{x} \in X$, and if x_N is the least-squares solution of $Ax = b$ in X_N, then $\|x_N - \hat{x}\| \geqslant s(N)$ for every $N \in \mathbb{N}$.*

Proof. Let us define

$$
\begin{aligned}
\Delta &\doteq \{n \in \mathbb{N} : \langle b, e_n \rangle \neq 0\}, & \Gamma &\doteq \mathbb{N} \setminus \Delta, \\
B^\Delta &\doteq \{e_{n_j} : n_j \in \Delta\}, & B^\Gamma &\doteq \{e_{m_j} : m_j \in \Gamma\},
\end{aligned}
\tag{16}
$$

and let $X_b \doteq \overline{\mathrm{span}\, B^\Delta}$. Clearly,

$$
\overline{\mathrm{span}\, B^\Gamma} = X_b^\perp
\tag{17}
$$

and b is strongly non-degenerate with respect to the basis B^Δ of X_b. Since X_b is an infinite-dimensional separable Hilbert space, theorem 3.4 implies that there exists a compact, injective, linear operator $A^\Delta : X_b \to X_b$ whose range is dense in X_b such that $b = A^\Delta \hat{x}$ for some $\hat{x} \in X_b$, and if x_M^Δ is the least-squares solution of $A^\Delta x = b$ in

$$
X_M^\Delta \doteq \mathrm{span}\{e_{n_j}\}_{j=1}^M \subset X_b,
\tag{18}
$$

then

$$
\|x_M^\Delta - \hat{x}\| \geqslant \tilde{s}(M) \,\forall M \in \mathbb{N},
\tag{19}
$$

where $\tilde{s} : \mathbb{N} \to \mathbb{R}^+$ is defined by $\tilde{s}(j) \doteq s(n_{j+1})$.

Next, the operator A will be constructed by appropriately extending A^Δ to the whole space X. The way in which this extension is made depends upon the cardinality of the set Γ defined in (16).

Case I. The set Γ is not finite. In this case, $X_b^\perp$ is an infinite-dimensional separable Hilbert space. Let $\{\gamma_n\}_{n=1}^\infty$ be any sequence in ℓ^2 such that $\gamma_n \neq 0 \,\forall n \in \mathbb{N}$ and define $\tilde{b} \doteq \sum_{n \in \Gamma} \gamma_n e_n$. Clearly, $\tilde{b}$ is strongly non-degenerate with respect to the basis B^Γ of $X_b^\perp$. By virtue of theorem 3.4, there exists a compact, injective linear operator $A^\Gamma : X_b^\perp \to X_b^\perp$ whose range is dense in $X_b^\perp$ such that $\tilde{b} = A^\Gamma \tilde{x}$ for some $\tilde{x} \in X_b^\perp$. We then define A as follows:

$$
Ax \doteq A^\Delta x_1 + A^\Gamma x_2,
\tag{20}
$$

where $x = x_1 + x_2$ with $x_1 \in X_b$ and $x_2 \in X_b^\perp$.

Case II. The set Γ is finite. In this case, we simply define A as

$$Ax \doteq A^\Delta x_1 + x_2, \tag{21}$$

where as before $x = x_1 + x_2$, with $x_1 \in X_b$ and $x_2 \in X_b^\perp$.

It is clear that in both cases the operator A so defined is linear. Let us now prove that this operator is also injective, compact and that its range is dense in X.

Let us first consider case I (see (20)). If $Ax = 0$, then $A^\Delta x_1 = -A^\Gamma x_2$. Since $A^\Delta x_1 \in X_b$ and $-A^\Gamma x_2 \in X_b^\perp$ we have that $A^\Delta x_1 = -A^\Gamma x_2 = 0$ and therefore $x_1 = x_2 = 0$ since A^Δ and A^Γ are both injective operators. Thus, $x = x_1 + x_2 = 0$ and therefore A is injective.

Let us now show that $\mathcal{R}(A)$ is dense in X. By definition of A and due to the fact that $\mathcal{R}(A^\Delta) \subset X_b$ and $\mathcal{R}(A^\Gamma) \subset X_b^\perp$, it follows that $\mathcal{R}(A^\Delta)$ and $\mathcal{R}(A^\Gamma)$ are subspaces of X which are orthogonal to each other. Therefore, $\mathcal{R}(A) = \mathcal{R}(A^\Delta) \oplus \mathcal{R}(A^\Gamma)$. Then,

$$
\begin{aligned}
\overline{\mathcal{R}(A)} &= \overline{\mathcal{R}(A^\Delta)} \oplus \overline{\mathcal{R}(A^\Gamma)} \\
&= (X_b \cap \overline{\mathcal{R}(A^\Delta)}) \oplus \left(X_b^\perp \cap \overline{\mathcal{R}(A^\Gamma)} \right) \quad \left(\text{since } \mathcal{R}(A^\Delta) \subset X_b \text{ and } \mathcal{R}(A^\Gamma) \subset X_b^\perp \right) \\
&= \overline{\mathcal{R}(A^\Delta)}^{X_b} \oplus \overline{\mathcal{R}(A^\Gamma)}^{X_b^\perp} \\
&= X_b \oplus X_b^\perp \quad \left(\text{since } \mathcal{R}(A^\Delta) \text{ and } \mathcal{R}(A^\Gamma) \text{ are dense in } X_b \text{ and } X_b^\perp, \text{ respectively} \right) \\
&= X.
\end{aligned}
$$

Therefore, $\mathcal{R}(A)$ is dense in X.

To show that A is compact, we note that A can be written in the form

$$A = A^\Delta P_{X_b} + A^\Gamma P_{X_b^\perp},$$

where P_{X_b} and $P_{X_b^\perp}$ are the orthogonal projections of X onto the subspaces X_b and $X_b^\perp$, respectively. Since A^Δ and A^Γ are both compact and P_{X_b} and $P_{X_b^\perp}$ are both bounded, it follows that A, being the sum of two compact operators, is compact.

Consider now case II, that is, when A is defined by (21). The operator A is injective. In fact, if $Ax = 0$ then $A^\Delta x_1 = -x_2$ and since $A^\Delta x_1 \in X_b$ and $-x_2 \in X_b^\perp$ we have that $A^\Delta x_1 = -x_2 = 0$. Due to the fact that A^Δ is injective it follows that $x_1 = 0$, and therefore $x = x_1 + x_2 = 0$.

Let us now check that $\mathcal{R}(A)$ is dense in X. By definition of A, and since $\mathcal{R}(A^\Delta) \subset X_b$, if follows that $\mathcal{R}(A^\Delta)$ and $X_b^\perp$ are subspaces of X orthogonal to each other. Then,

$$
\begin{aligned}
\overline{\mathcal{R}(A)} &= \overline{\mathcal{R}(A^\Delta)} \oplus \overline{X_b^\perp} \\
&= (X_b \cap \overline{\mathcal{R}(A^\Delta)}) \oplus X_b^\perp \quad \left(\text{since } \mathcal{R}(A^\Delta) \subset X_b \text{ and } X_b^\perp \text{ is closed} \right) \\
&= \overline{\mathcal{R}(A^\Delta)}^{X_b} \oplus X_b^\perp \\
&= X_b \oplus X_b^\perp \quad \left(\text{since } \mathcal{R}(A^\Delta) \text{ is dense in } X_b \right) \\
&= X.
\end{aligned}
$$

We will now show that in this case also the operator A is compact. For that we write $A = A^\Delta P_{X_b} + P_{X_b^\perp}$. Since in this case Γ is finite, if follows from (16) and (17) that $P_{X_b^\perp}$ is a finite rank operator and, therefore, compact. Then, A is compact since it is the sum of two compact operators.

Let us now check that in both cases (I and II) $b \in \mathcal{R}(A)$. Since $\hat{x} \in X_b$ and $A^\Delta \hat{x} = b$, it follows from (20) and (21) that $A\hat{x} = A^\Delta \hat{x} + A^\Gamma 0 = b$ in case I and $A\hat{x} = A^\Delta \hat{x} + 0 = b$ in case II.

Finally, let x_N denote the least-squares solution of $Ax = b$ in $X_N \doteq \text{span}\{e_1, \ldots, e_N\}$, $x_N = \sum_{j=1}^{N} \xi_j e_j$. We will show that $\|x_N - \hat{x}\| \geqslant s(N)$ for every $N \in \mathbb{N}$:

$$
\begin{aligned}
\|x_N - \hat{x}\|^2 &= \left\| \sum_{j=1}^{N} \xi_j e_j - \hat{x} \right\|^2 \\
&= \left\| \sum_{j=1, j \in \Delta}^{N} \xi_j e_j + \sum_{j=1, j \in \Gamma}^{N} \xi_j e_j - \hat{x} \right\|^2 \\
&= \left\| \sum_{j=1, j \in \Delta}^{N} \xi_j e_j - \hat{x} \right\|^2 + \left\| \sum_{j=1, j \in \Gamma}^{N} \xi_j e_j \right\|^2 \\
&\quad \left(\text{since } \sum_{j=1, j \in \Delta}^{N} \xi_j e_j - \hat{x} \in X_b \text{ and } \sum_{j=1, j \in \Gamma}^{N} \xi_j e_j \in X_b^{\perp} \right) \\
&\geqslant \left\| \sum_{j=1, j \in \Delta}^{N} \xi_j e_j - \hat{x} \right\|^2 .
\end{aligned}
\tag{22}
$$

For every $N \in \mathbb{N}$, define $k_N \doteq \max\{j : n_j \leqslant N\}$ or, equivalently, $k_N \doteq \#\{e_j : j \leqslant N, e_j \in B^{\Delta}\} = \#(B_N \cap B^{\Delta})$, where $B_N \doteq \{e_1, \ldots, e_N\}$. It follows from (18) that $X_{k_N}^{\Delta} = \text{span} \{e_{n_j}\}_{j=1}^{k_N}$. Since X_b is a closed subspace of X such that X_b and $X_b^{\perp}$ are invariant under A and $b \in X_b$, it follows from lemma 3.5 that the least-squares solution of $Ax = b$ in $X_b \cap X_N = X_{k_N}^{\Delta}$ is

$$
P_{X_b} x_N = P_{X_{k_N}^{\Delta}} x_N = \sum_{j=1, j \in \Delta}^{N} \xi_j e_j,
$$

where $x_N = \sum_{j=1}^{N} \xi_j e_j$ is the least-squares solution of $Ax = b$ in X_N.

Now, since $A|_{X_b} = A^{\Delta}$ and $X_{k_N}^{\Delta} \subset X_b$, it follows that $A\eta = A^{\Delta}\eta \; \forall \eta \in X_{k_N}^{\Delta}$ and therefore $\sum_{\substack{j=1 \\ j \in \Delta}}^{N} \xi_j e_j$ is also the least-squares solution of $A^{\Delta} x = b$ in $X_{k_N}^{\Delta}$, that is, $x_{k_N}^{\Delta} = \sum_{\substack{j=1 \\ j \in \Delta}}^{N} \xi_j e_j$ (see (18)). Then, substituting with this into (22) and using (19) it follows that

$$
\|x_N - \hat{x}\|^2 \geqslant \left\| x_{k_N}^{\Delta} - \hat{x} \right\|^2 \geqslant [\tilde{s}(k_N)]^2 = \left[s(n_{k_N+1}) \right]^2 .
\tag{23}
$$

Now, by definition of k_N, if $j_0 \in \mathbb{N}$ and $n_{j_0} \leqslant N$ then $k_N \geqslant j_0$. Therefore, the inequality $n_{k_N+1} \leqslant N$ would imply that $k_N \geqslant k_N + 1$, which is a contradiction. Hence, $n_{k_N+1} > N$ or, equivalently, $n_{k_N+1} \geqslant N + 1$.

Since s is increasing and nonnegative, it follows that

$$
\left[s(n_{k_N+1}) \right]^2 \geqslant [s(N+1)]^2 \geqslant [s(N)]^2 .
\tag{24}
$$

From (23) and (24), we finally conclude that $\|x_N - \hat{x}\| \geqslant s(N) \; \forall N \in \mathbb{N}$. $\qquad \square$

Next, we show that the operator A both in theorem 3.4 and corollary 3.6 can be chosen to further be self-adjoint.

Corollary 3.7. *Let X be an infinite-dimensional separable Hilbert space, $B \doteq \{e_n\}_{n=1}^{\infty}$ an orthonormal basis of X, $b \in X$ non-degenerate with respect to B, $X_N \doteq \text{span}\{e_1, \ldots, e_N\}$ and $s : \mathbb{N} \to \mathbb{R}^+$ an arbitrary nonnegative increasing function. Then, there exist a compact, self-adjoint, injective, linear operator $A = A(b, s) : X \to X$, whose range is dense in X, and a unitary operator $V : X \to X$ such that $b^* \doteq Vb \in \mathcal{R}(A)$, and if x_N is the least-squares solution of $Ax = b^*$ in X_N, then $\|x_N - A^{-1}b^*\| \geqslant s(N)$ for every $N \in \mathbb{N}$.*

Proof. Since $b \in X$ is non-degenerate with respect to B, it follows from corollary 3.6 that there exists a compact, injective, linear operator $\tilde{A} = \tilde{A}(b, s)$ whose range is dense in X such that $b \in \mathcal{R}(\tilde{A})$, and if x_N denotes the least-squares solution of $\tilde{A}x = b$ in X_N, then $\|x_N - \tilde{A}^{-1}b\| \geqslant s(N)$ for every $N \in \mathbb{N}$. Let $\{(\sigma_n; v_n, u_n)\}$ be the singular system associated with the compact operator $\tilde{A}$ and $V : X \to X$ the linear unitary operator defined by $Vu_n = v_n$. Then, it can be easily proved that the linear operator $A \doteq V\tilde{A}$, besides being compact, injective and having range dense in X, is self-adjoint. This follows immediately from the fact that $V\tilde{A}v_n = \tilde{A}^* V^* v_n$ for all $n \in \mathbb{N}$.

Since $b \in \mathcal{R}(\tilde{A})$, we have that $b^* \doteq Vb \in \mathcal{R}(A)$. Now, since $\|Ax - b^*\| = \|\tilde{A}x - b\|$, it follows that x_N is also the least-squares solution of $Ax = b^*$ in X_N. Therefore, $s(N) \leqslant \|x_N - \tilde{A}^{-1}b\| = \|x_N - A^{-1}b^*\|$ for every $\in \mathbb{N}$. $\qquad\square$

One may think that the result of corollary 3.6 has little relevance from a practical point of view since it could happen that, in a neighbourhood of b, the only element $\eta \in \mathcal{R}(A)$ for which the least-squares solution in X_N of $Ax = \eta$ satisfying $\|x_N - A^{-1}\eta\| \geqslant s(N)$ for every $N \in \mathbb{N}$ is precisely $\eta = b$. However, this is not the case. In fact, it can be easily proved that in every reduced neighbourhood of b of radius δ there exists a sequence $\{b_k^\delta\}_{k=1}^\infty \subset \mathcal{R}(A)$ such that $b_k^\delta \to b$ as $k \to \infty$ and $\|x_N^k - A^{-1}b_k^\delta\| \geqslant s(N)$ for every $N \in \mathbb{N}$, where x_N^k is the least-squares solution of $Ax = b_k^\delta$ in X_N. That is, for each element of the sequence $\{b_k^\delta\}_{k=1}^\infty$ (i.e., for each fixed k), the approximating least-squares solutions $\{x_N^k\}_{N=1}^\infty$ also diverge from the exact solution with speed arbitrarily large, in the same way that the least-squares solutions obtained with data equal b do.

Up to now we have only considered problems with exact data, that is, equations of the form $Ax = b^*$, where $b^* \doteq Ax^*$ and x^* is the exact solution. As it is to be expected, the nonconvergence in corollary 3.6 can also occur with noisy data. Seidman [1] showed that if the operator $A : X \to X$ is compact (besides being linear, injective, positive, self-adjoint and with dense range), then for each increasing sequence of finite-dimensional subspaces $\{X_N\}$ whose union is dense in X and for each $b^* \in \mathcal{R}(A)$ there exists a sequence $b_N \to b^*$ for which the least-squares solution x_N of $Ax = b_N$ in X_N satisfy $\|x_N - x^*\| \to \infty$, where $Ax^* = b^*$. In a certain sense, the next corollary generalizes this result. More precisely, it is shown that also for the case of noisy data the careless application of the least-squares method can lead to arbitrary rates of divergence.

Corollary 3.8 (*arbitrary divergence rate with noisy data*)*. Let X be an infinite-dimensional separable Hilbert space, $B \doteq \{e_n\}_{n=1}^\infty$ an orthonormal basis of X, $b \in X$ non-degenerate with respect to B, $X_N \doteq \text{span}\{e_1, \ldots, e_N\}$, $s : \mathbb{N} \to \mathbb{R}^+$ an arbitrary nonnegative increasing function and $A = A(b, s)$ the operator whose existence was proved in corollary 3.6. Then $\forall \alpha \in (0, 1)$ there exists a sequence $\{b_N\}_{N=1}^\infty \subset \mathcal{R}(A)$ such that $b_N \to b$ as $N \to \infty$ and the least-squares solution x_N of $Ax = b_N$ in X_N satisfies $\|x_N - A^{-1}b\| \geqslant \alpha s(N)$ for every $N \in \mathbb{N}$.*

Proof. Let $\{\alpha_N\} \subset \mathbb{R}$ be a sequence such that $\alpha_N \neq 0$ and

$$|1 - \alpha_N| \leqslant \min\left\{ \frac{1}{\|x_N^*\|}, \frac{(1 - \alpha)s(N)}{\|x_N^*\|} \right\}, \tag{25}$$

where x_N^* is the least-squares solution of $Ax = b$ in X_N. By corollary 3.6,

$$\|x_N^* - A^{-1}b\| \geqslant s(N) \,\forall N \in \mathbb{N}, \tag{26}$$

then $\|x_N^*\| \to \infty$, which implies that $|1 - \alpha_N| \to 0$ and, therefore, $\alpha_N \to 1$ as $N \to \infty$. Let us define $b_N \doteq \alpha_N b$ for each $N \in \mathbb{N}$. Then, $b_N \to b$ and since $\mathcal{R}(A)$ is a subspace of

X, $\{b_N\}_{N=1}^{\infty} \subset \mathcal{R}(A)$. Since the least-squares solution x_N of $Ax = b_N = \alpha_N b$ in X_N is $\alpha_N x_N^*$, it follows that

$$
\begin{aligned}
\|x_N - A^{-1}b\| &\geqslant \|x_N^* - A^{-1}b\| - \|x_N^* - x_N\| \\
&= \|x_N^* - A^{-1}b\| - |1 - \alpha_N| \, \|x_N^*\| \\
&\geqslant s(N) - (1 - \alpha)s(N) \qquad \text{(from (25) and (26))} \\
&= \alpha s(N). \qquad\qquad\qquad\qquad\qquad\qquad\qquad \square
\end{aligned}
$$

Remark 3.9. During the proof above we made use of the following result: if x_N is the least-squares solution of $Ax = b$ in X_N, then σx_N, with $\sigma \neq 0$, is the least-squares solution of $Ax = \sigma b$ in X_N.

For simplicity, we have only considered the case in which X and Y are real Hilbert spaces. However, all results in this paper carry over to the case of complex Hilbert spaces with the obvious modifications. Also, a procedure similar to the one employed in corollary 3.6 can be used to show that the hypothesis of separability can be neglected.

4. Conclusions

In this paper, we have considered the application of the least-squares method to inverse ill-posed problems of the form $Ax = y$ where A is a compact linear invertible operator on a separable Hilbert space X. It was shown that under no additional assumptions on the finite-dimensional subspaces there is no guarantee that the corresponding sequence of approximating least-squares solutions will converge to the true solution. Moreover, it was shown that it is possible for the approximating solutions to diverge from the exact solution at any arbitrary speed, both for the case of exact and noisy data. In particular, theorem 3.4 and corollary 3.6 extend the original result of Seidman ([1], example 3.1). The main difficulty that arises in these extensions resides in the construction, starting from the element $b \in X$ and the function $s(\cdot)$, of the sequences $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ in such a way that they satisfy the constraints (4) and such that the operator A defined from them by (5) satisfies $b \in \mathcal{R}(A)$ and $\|x_N - A^{-1}b\| \geqslant s(N)$ for all $N \in \mathbb{N}$.

The use of general residual minimization procedures, such as the least-squares method, is very common in practice and it constitutes a widely used standard approach. Although the operators A constructed in theorem 3.4 and corollary 3.6 are somewhat artificial, they do serve to prove that in the case of ill-posed problems the convergence of the finite-dimensional approximating solutions obtained by direct application of the least-squares method is far from being automatically guaranteed and that, without any rigorous mathematical justification of this convergence, it is a completely unacceptable procedure. It is important to emphasize here that the lack of convergence of the sequence of least-squares solutions is not a consequence of the least-squares method, but rather the result of a very poor choice of the approximating spaces X_N, without taking into account the operator A which defines the ill-posed problem. Therefore, if the infinite-dimensional problem is ill-posed, extreme care must be taken during discretization and finite-dimensional approximation of the solutions in order to guarantee convergence.

If the operator $A : X \rightarrow X$ is given, as it happens most of the times in practice, then Seidman [1] showed that for almost any $b \in \mathcal{R}(A)$ there exists a sequence of subspaces X_N such that the corresponding sequence $\{x_N\}$ of least-squares solutions of $Ax = b$ is unbounded. Although we strongly believe that the results in this paper can also be extended to this case, i.e., that arbitrary rates of divergence can be obtained by the application of the

least-squares method when the operator A is given, we are not able to provide a rigorous proof of this conjecture at this time. Efforts in this direction are underway.

Acknowledgments

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