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Weighted a priori estimates with powers of the distance function for elliptic equations.

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Abstract. Let u be a weak solution of $(-\Delta)^m u = f$ with Dirichlet boundary conditions in a smooth bounded domain $\Omega \subset \mathbb{R}^n$. In this paper we obtain some estimates for the Green function associate to this problem. Moreover, under appropiate conditions on p, we prove some weighted Sobolev a priori estimates for the solution u, where the weight is a power of the distance function. This result extends a previous one joint R. Durán and M. Sanmartino (Indiana Univ. Math. J, 57(7):3463-3478, 2008 and Anal. Theory Appl. 26(4):339-349, 2010).

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1 Introduction

We will use the standard notation for Sobolev spaces and for derivatives, namely, if α is a multi-index, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{Z}_+^n$ we denote $|\alpha| = \sum_{j=1}^n \alpha_j,\ D^\alpha = \partial_{x_1}^{\alpha_1} ... \partial_{x_n}^{\alpha_n}$ and

$$W^{k,p}(\Omega) = \{v \in L^p(\Omega) \,:\, D^\alpha v \in L^p(\Omega) \quad \forall\, |\alpha| \le k\}.$$

For $u \in W^{k,p}(\Omega)$, its norm is given by

$$\|u\|_{W^{k,p}(\Omega)} = \sum_{|\alpha| \le k} \|D^\alpha u\|_{L^p(\Omega)}\,.$$

We consider the homogeneous problem

$$\begin{cases} (-\Delta)^m u = f & in \Omega \\ \left(\frac{\partial}{\partial \nu}\right)^j u = 0 & \text{on } \partial\Omega, \quad 0 \le j \le m-1, \end{cases} \tag{1.1}$$

where $\frac{\partial}{\partial \nu}$ is the normal derivative.

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For u solution of (1.1) on a smooth bounded domain Ω was proved in [2] for $m = 1$ and [3] for $m \geq 1$ that there exists a constant $C > 0$ such that

$$\|u\|_{W^{2m,p}_{d^\gamma}(\Omega)} \leq C \|f\|_{L^p_{d^\gamma}(\Omega)}, \tag{1.2}$$

where $d(x) := d(x, \partial\Omega)$ is the distance from x to de boundary of Ω and γ such that d^γ belongs to the Muckenhoupt class A_p, i.e. $-1 < \gamma < p - 1$. Since this kind of weights has a particular importance, it is natural to ask whether analogous estimates can be proved for differents values of p.

The goal of this paper is to arrive to some estimates for the Green function associate to the problem (1.1) and, consequently for the solution of this problem. More precisely we obtain that

$$\|u\|_{W^{2m-1,p}_{d^\gamma}(\Omega)} \leq C \|f\|_{L^p_{d^\gamma}(\Omega)},$$

for $\max\left\{1, \frac{\gamma}{m+\frac{1}{q^*}}\right\} < p < \infty$ and, if $|\alpha| = 2m$

$$\|D^\alpha u\|_{L^p_{d^\gamma}(\Omega)} \leq C\|f\|_{L^p_{d^{\gamma-p}}(\Omega)},$$

for $\gamma > m - 1$ and $\max\left\{1, \frac{\gamma}{m+\frac{1}{q^*}}\right\} < p < \infty$, where $q^* = \frac{q}{q-1}$ and $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$.

2 Definitions and main results

The theory developed by Muckenhoupt in [6] provides necessary and sufficient conditions on a weight ω in order to obtain weighted estimates for the usual Hardy-Littlewood maximal funtion of f

$$Mf(x) = \sup \frac{1}{|B|} \int_B |f(y)| dy,$$

where the supremum is taken over the family of balls B containing x. These functions ω are known as ***Muckenhoupt class A_p*** and they are defined for $1 < p < \infty$ as the set of all non-negative locally integrable function ω for which there exists a constant C such that the inequality

$$\left(\frac{1}{|Q|} \int_Q \omega(x)\, dx\right) \left(\frac{1}{|Q|} \int_Q \omega(x)^{-\frac{1}{p-1}}\, dx\right)^{p-1} \leq C$$

holds for every cube $Q \subset \mathbb{R}^n$.

An other relevant operator in the theory of real and harmonic analysis is the fractional integral

$$I_t f(x) = \int_\Omega \frac{f(y)}{|x-y|^{n-t}} dy,$$

for $0 < t < n$. In [7], the authors give necesary and sufficient condition for a function ω such that there exists a constant C such that $\|I_t f(x)\omega(x)\|_{L^p} \leq C\|f(x)\omega(x)\|_{L^q}$ hold for $\frac{1}{p} = \frac{1}{q} - \frac{t}{n}$, with $1 < q < \frac{n}{t}$. These class of weights is the ***Muckenhoupt and Wheeden class A_{pq}*** and it is defined as the set of all non-negative locally integrable function ω for which there exists a constant C such that the inequality

$$\left(\frac{1}{|Q|}\int_Q \omega(x)^p\right)^{1/p} \left(\frac{1}{|Q|}\int_Q \omega(x)^{-q^*}\right)^{1/q^*} \leq C$$

holds for every cube $Q \subset \mathbb{R}^n$, where $q^* = \frac{q}{q-1}$.

Remark 2.1. Observe that ω belongs to the Muckenhoupt and Wheeden class A_{pq} if and only if ω^p belongs to the Muckenhoupt class A_s, with $s = \frac{p}{q^*} + 1$. In the particular case of $\omega(x) = d(x)^\nu$, this is equivalent to $-1 < \nu p < s - 1$ (see [2]).

The solution of (1.1) is given by the representation formula

$$u(x) = \int_\Omega G_m(x, y)\, f(y)\, dy, \tag{2.1}$$

where $G_m(x, y)$ is the Green function of the operator $(-\Delta)^m$ in Ω.

In what follows we consider that Ω is a bounded domain with $\partial\Omega \in C^{6m+4}$ for $n = 2$ and $\partial\Omega \in C^{5m+2}$ for $n > 2$ (the regularity on the boundary is necessary in order to use the results of the Green function given in [1]).

We will say $f \preceq g$ in $\Omega \times \Omega$ if an only if there exists a constant C such that $f(x, y) \preceq g(x, y)$ for all $x, y \in \Omega$.

The key to estimate the derivatives of the solution of (1.1) is given by the following lemma wich can be relevant itself. Since the proof is very technical it will be given in Section 3.

Lemma 2.2. *Let u be the solution of* (1.1) *and G_m the Green function associate to this problem. Then we have for $|\alpha| \leq 2m - 1$*

$$|D_x^\alpha G_m(x, y)| \preceq d(x)^{m-|\alpha|-1} \frac{d(y)^m}{|x-y|^{n-1}},$$

and consequently

$$|D^\alpha u(x)| \preceq d(x)^{m-|\alpha|-1} I_1(f\chi_\Omega d^m)(x),$$

where χ_Ω *denotes the characteristic function of* Ω*.*

We can now state and prove our first result.

Theorem 2.3. *Let* u *be the solution of* (1.1) *and* $|\alpha| \leq 2m-1$*. Then we have*

$$\int_\Omega |D_x^\alpha u(x)|^p\, d(x)^\gamma\, dx \preceq \int_\Omega f(x)^p d(x)^\gamma\, dx, \tag{2.2}$$

for $\gamma > m-1$ *and* $\max\left\{1, \frac{\gamma}{\frac{1}{q^*}+m}\right\} < p < \infty$.

Proof. By the representation formula (2.1), Lemma 2.2 and that $d(x)^{m-|\alpha|-1} \leq d(x)^{-m}$ for $|\alpha| \leq 2m-1$, we obtain that

$$\begin{aligned}
\int_\Omega |D_x^\alpha u(x)|^p\, d(x)^\gamma\, dx &\preceq \int_\Omega I_1(f\chi_\Omega d^m)^p(x) d(x)^{(-m+\frac{\gamma}{p})p}\, dx \\
&\preceq \left(\int_\Omega (f d^m)^q(x) d(x)^{(-m+\frac{\gamma}{p})q}\, dx\right)^{p/q} \\
&= \left(\int_\Omega f(x)^q d(x)^{\frac{\gamma q}{p}}\, dx\right)^{p/q} \\
&\preceq \int_\Omega f(x)^p d(x)^\gamma\, dx,
\end{aligned}$$

provided that $d(x)^{-m+\frac{\gamma}{p}}$ is in the Muckenhoupt and Wheeden class A_{pq}. But we can see from Remark 2.1 that it is when $-1 < -mp+\gamma < \frac{p}{q^*}$ or more precisely, $\max\left\{1, \frac{\gamma}{m+\frac{1}{q^*}}\right\} < p < 1 + \frac{\gamma+1}{m}$.

To other hand, we have (2.2) for $p > m+1$ (see [2]). Then, by an interpolation argument (see for example [8]), (2.2) hold for every p such that

$$\frac{1}{p} = \frac{1-\eta}{p_1} + \frac{\eta}{p_2}$$

with $0 \leq \eta \leq 1$, $p_1 > m+1$ and $\max\left\{1, \frac{\gamma}{m+\frac{1}{q^*}}\right\} < p_2 < 1 + \frac{\gamma+1}{m}$. Finally we have (2.2) for $\max\left\{1, \frac{\gamma}{m+\frac{1}{q^*}}\right\} < p < \infty$ as we desire. □

Remark 2.4. Observe that in the proof we used $d(x)^{m-|\alpha|-1} \leq d(x)^{-m}$ for $|\alpha| \leq 2m-1$. This estimate is weaker than the given in Lemma 2.2. However, if we used the optimal one, the range of p will be less than the obtained (empty in some cases).

To obtain some estimates for the derivatives of order $2m$ we will use a result given in [5], where the authors proved that

$$\sum_{|\alpha|\leq 2m} \|d^{|\alpha|} D^\alpha u\|_{L^p_\omega(\Omega)} \preceq \|u\|_{L^p_\omega(\Omega)} + \|d^{2m} f\|_{L^p_\omega(\Omega)}, \tag{2.3}$$

with $\omega = d^\beta$ for any exponent $\beta \in \mathbb{R}$. Then, taking $\beta = -2mp+\gamma$ we have for $|\alpha| = 2m$ that

$$\int_\Omega |D^\alpha u|^p \, d^\gamma dx \preceq \int_\Omega |u|^p \, d^\beta dx + \int_\Omega |f|^p d^\gamma dx. \tag{2.4}$$

In order to estimate the first term in (2.4), observe that from Lemma 2.2 we have

$$|G_m(x,y)| \preceq \; d(x)^{-m-1} \, \frac{d(y)^m}{|x-y|^{n-1}}.$$

Then, using the simetry of the Green function we obtain

$$|G_m(x,y)| \preceq \; d(y)^{-m-1} \, \frac{d(x)^m}{|x-y|^{n-1}},$$

and consequently

$$|u(x)| \preceq \; d(x)^m \, I_1(f\chi_\Omega d^{m-1})(x).$$

Finally, on the same way that Theorem 2.3 we can prove the following result.

Theorem 2.5. *Let u be the solution of* (1.1) *and* $|\alpha| = 2m$. *Then we have*

$$\int_\Omega |D^\alpha_x u(x)|^p \, d(x)^\gamma \, dx \preceq \int_\Omega f(x)^p d(x)^{\gamma-p} \, dx,$$

for $\gamma > m-1$ *and* $\max\left\{1, \frac{\gamma}{m+\frac{1}{q^*}}\right\} < p < \infty$.

Remark 2.6. Generalizing the classical imbedding Theorems of Sobolev spaces to weighted Sobolev spaces (as we have done in [2], Theorem 3.4) we have a consequence of the main result: Under the hypotheses of Theorem 2.5 with $\omega =$

d^γ, where $\gamma = k\beta$, $k \in \mathbb{N}$ and $0 \le \beta \le 1$. If $1/p - 1/q \le 2m/(n+k)$ (with $q < \infty$ when $2mp = n+k$), then

$$\|u\|_{L^q_{d^\gamma}(\Omega)} \preceq \|f\|_{L^p_{d^{\gamma-p}}(\Omega)}. \tag{2.5}$$

The principal interest of this estimate is given by the limit case $1/p - 1/q = 2m/(n+k)$. In fact, it has proved in [1] that for $1/p - 1/q < 2m/(n+k)$, the solution $u \in L^q_{d^\gamma}(\Omega)$ and $\|u\|_{L^q_{d^\gamma}(\Omega)} \preceq \|f\|_{L^p_{d^\gamma}(\Omega)}$. This result have been extend in [3] for the limit case and $p > m+1$.

Moreover, if $1/p - 1/q > 2m/(n+k)$ we have proved in [4] that there exists $f \in L^q_{d^\gamma}(\Omega)$ such that $u \notin L^p_{d^\gamma}(\Omega)$, with u solution of problem (1.1).

Finally let us mention that, to our knowledge, it is not known what happens in general in the limit case and $p < m+1$.

Our results shows that in the limit case $1/p - 1/q = 2m/(n+k)$ with $\gamma > m-1$ and $\max\left\{1, \frac{\gamma}{m+\frac{1}{q^*}}\right\} < p < \infty$ we have some estimate for the solution u in the weighted space $L^p_{d^\gamma}$ given by (2.5).

3 Proof of Lemma 2.2

Under our assumption on the boundary of the domain, we have the following known estimates for the Green function $G_m(x,y)$ given in [1].

1. For $|\alpha| \ge m$:

(a) if $|\alpha| > 2m - n$, then

$$|D^\alpha_x G_m(x,y)| \preceq |x-y|^{2m-n-|\alpha|} \min\left\{1, \frac{d(y)}{|x-y|}\right\}^m,$$

(b) if $|\alpha| = 2m - n$, then

$$|D^\alpha_x G_m(x,y)| \preceq \log\left(2 + \frac{d(y)}{|x-y|}\right) \min\left\{1, \frac{d(y)}{|x-y|}\right\}^m,$$

(c) if $|\alpha| < 2m - n$, then

$$|D^\alpha_x G_m(x,y)| \preceq d(y)^{2m-n-|\alpha|} \min\left\{1, \frac{d(y)}{|x-y|}\right\}^{n+|\alpha|-m},$$

2. For $|\alpha| < m$:

(a) if $|\alpha| > 2m - n$, then

$$|D_x^\alpha G_m(x,y)| \preceq |x-y|^{2m-n-|\alpha|} \min\left\{1, \frac{d(x)}{|x-y|}\right\}^{m-|\alpha|} \min\left\{1, \frac{d(y)}{|x-y|}\right\}^{m},$$

(b) if $|\alpha| = 2m - n$, then

$$|D_x^\alpha G_m(x,y)| \preceq \log\left(2 + \frac{d(y)}{|x-y|}\right) \min\left\{1, \frac{d(y)}{|x-y|}\right\}^{m} \min\left\{1, \frac{d(x)}{|x-y|}\right\}^{m-|\alpha|},$$

(c) if $|\alpha| < 2m - n$, and moreover

i. $|\alpha| \geq m - \frac{1}{2}n$, then

$$|D_x^\alpha G_m(x,y)| \preceq d(y)^{2m-n-|\alpha|} \min\left\{1, \frac{d(x)}{|x-y|}\right\}^{m-|\alpha|} \min\left\{1, \frac{d(y)}{|x-y|}\right\}^{n-m+|\alpha|},$$

ii. $|\alpha| < m - \frac{1}{2}n$, then

$$|D_x^\alpha G_m(x,y)| \preceq d(y)^{m-\frac{n}{2}} d(x)^{m-\frac{n}{2}-|\alpha|} \min\left\{1, \frac{d(x)d(y)}{|x-y|^2}\right\}^{\frac{n}{2}}.$$

We are now in conditions to prove the result.

of Lemma 2.2. For the representation formula (2.1) we have that

$$|D_x^\alpha u(x)| \leq \int_\Omega |D_x^\alpha G_m(x,y)|\, f(y)\, dy$$

for every multindex α. Then, it is enought to prove that

$$|D_x^\alpha G_m(x,y)| \preceq d(x)^{m-|\alpha|-1} \frac{d(y)^m}{|x-y|^{n-1}}.$$

To do that we will analize each case given above.

1. For $|\alpha| \geq m$:

(a) if $|\alpha| > 2m - n$

$$|D_x^\alpha G_m(x,y)| \preceq |x-y|^{2m-n-|\alpha|}\,\frac{d(y)^m}{|x-y|^m}\,\chi_{\Omega_1}(y) + |x-y|^{2m-n-|\alpha|}\chi_{\Omega_2}(y) =: A_1 + A_2,$$

where $\Omega_1 = \{y \in \Omega : |x-y| > \frac{d(x)}{2}\}$ and $\Omega_2 = \{y \in \Omega : |x-y| \le \frac{d(x)}{2}\}$. Using that $m \le |\alpha| \le 2m-1$ we have

$$A_1 = |x-y|^{m-|\alpha|-1}\,\frac{d(y)^m}{|x-y|^{n-1}}\,\chi_{\Omega_1}(y) \preceq d(x)^{m-|\alpha|-1}\,\frac{d(y)^m}{|s-y|^{n-1}}\,\chi_{\Omega_1}(y)$$

and, since $\frac{1}{2}d(x) \le d(y) \le \frac{3}{2}d(x)$ for $y \in \Omega_2$

$$\begin{aligned} A_2 &= |x-y|^{2m-|\alpha|-1}\,\frac{1}{|x-y|^{n-1}}\,\chi_{\Omega_2}(y) \\ &\preceq d(x)^{-m}\,|x-y|^{2m-|\alpha|-1}\,\frac{d(y)^m}{|x-y|^{n-1}}\,\chi_{\Omega_2}(y) \\ &\preceq d(x)^{m-|\alpha|-1}\,\frac{d(y)^m}{|x-y|^{n-1}}\,\chi_{\Omega_2}(y) \end{aligned}$$

as wanted to proved.

(b) if $|\alpha| = 2m - n$

$$\begin{aligned} |D_x^\alpha G_m(x,y)| &\preceq \log\left(2 + \frac{d(y)}{|x-y|}\right)\,\frac{d(y)^m}{|x-y|^m}\,\chi_{\Omega_1}(y) \\ &\quad + \log\left(2 + \frac{d(y)}{|x-y|}\right)\,\chi_{\Omega_2}(y) =: B_1 + B_2. \end{aligned}$$

Taking $D_1 = \{y \in \Omega : d(y) < 2|x-y|\}$ and $D_2 = \{y \in \Omega : d(y) \ge 2|x-y|\}$ we have $\log(2 + \frac{d(y)}{|x-y|}) \le \log(4)$ for $y \in D_1$ and $\log(2 + \frac{d(y)}{|x-y|}) \preceq \frac{d(y)}{|x-y|}$ for $y \in D_2$. Then

$$\begin{aligned} B_1 &\preceq \frac{d(y)^m}{|x-y|^m}\,\chi_{\Omega_1 \cap D_1}(y) + \frac{d(y)^{m+1}}{|x-y|^{m+1}}\,\chi_{\Omega_1 \cap D_2}(y) \\ &\preceq \frac{d(y)^m}{|x-y|^{n-1}}\,|x-y|^{n-1-m}\,\chi_{\Omega_1 \cap D_1}(y) \\ &\quad + \frac{d(y)^{m+1}}{|x-y|^{n-1}}\,|x-y|^{n-2-m}\,\chi_{\Omega_1 \cap D_2}(y), \end{aligned}$$

and, since $m \le |\alpha| = 2m - n$ and $2|x-y| \le d(y) \le 3|x-y|$ for $y \in \Omega_1 \cap D_2$ we have

$$B_1 \preceq d(x)^{m-|\alpha|-1} \frac{d(y)^m}{|x-y|^{n-1}}.$$

To other hand, using that $d(y)$ is equivalent to $d(x)$ for $y \in \Omega_2$

$$B_2 \preceq \chi_{\Omega_2 \cap D_1}(y) + \frac{d(y)}{|x-y|} \chi_{\Omega_2 \cap D_2}(y) \preceq d(x)^{m-|\alpha|-1} \frac{d(y)^m}{|x-y|^{n-1}}.$$

(c) if $|\alpha| < 2m - n$

$$\begin{aligned} |D_x^\alpha G_m(x,y)| &\preceq |x-y|^{m-|\alpha|-1} \frac{d(y)^m}{|x-y|^{n-1}} \chi_{\Omega_1}(y) + d(y)^{2m-n-|\alpha|} \chi_{\Omega_2}(y) \\ &\preceq d(x)^{m-|\alpha|-1} \frac{d(y)^m}{|x-y|^{n-1}}. \end{aligned}$$

2. For $|\alpha| < m$:
(a) if $|\alpha| > 2m - n$

$$\begin{aligned} |D_x^\alpha G_m(x,y)| &\preceq d(x)^{m-|\alpha|} \frac{d(y)^m}{|x-y|^n} \chi_{\Omega_1}(y) + |x-y|^{m-n-|\alpha|} d(y)^m \chi_{\Omega_2}(y) \\ &\preceq d(x)^{m-|\alpha|-1} \frac{d(y)^m}{|x-y|^{n-1}}. \end{aligned}$$

(b) if $2m - n - |\alpha| = 0$, the estimate is obtained on the same way of $1.(b)$
(c) if $|\alpha| < 2m - n$
i. $|\alpha| \ge m - \frac{1}{2}n$. Since $|\alpha| < m$,

$$\begin{aligned} |D_x^\alpha G_m(x,y)| \preceq\ & d(x)^{m-|\alpha|} \frac{d(y)^m}{|x-y|^n} \chi_{\Omega_1}(y) \\ & + |x-y|^{m-|\alpha|-1} \frac{d(y)^m}{|x-y|^{n-1}} \chi_{\Omega_2}(y) \\ & \preceq d(x)^{m-|\alpha|-1} \frac{d(y)^m}{|x-y|^{n-1}}. \end{aligned}$$

ii. $|\alpha| < m - \frac{1}{2}n$.

$$\begin{aligned} |D_x^\alpha G_m(x,y)| &\preceq d(x)^{m-|\alpha|} \frac{d(y)^m}{|x-y|^n} \chi_{\Omega_1}(y) + d(y)^{m-\frac{n}{2}} d(x)^{m-\frac{n}{2}-|\alpha|} \chi_{\Omega_2}(y) \\ &\preceq d(x)^{m-|\alpha|-1} \frac{d(y)^m}{|x-y|^{n-1}}. \end{aligned}$$

□

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