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J. Math. Anal. Appl. 312 (2005) 105–120

Journal of
MATHEMATICAL
ANALYSIS AND
APPLICATIONS

www.elsevier.com/locate/jmaa

Comparison of Hardy–Littlewood and dyadic maximal functions on spaces of homogeneous type [☆]

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Received 18 August 2004

Available online 12 April 2005

Submitted by L. Chen

Abstract

We obtain a comparison of the level sets for two maximal functions on a space of homogeneous type: the Hardy–Littlewood maximal function of mean values over balls and the dyadic maximal function of mean values over the dyadic sets introduced by M. Christ in [M. Christ, A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral, *Colloq. Math.* 60/61 (1990) 601–628]. As applications to the theory of A_p weights on this setting, we compare the standard and the dyadic Muckenhoupt classes and we give an alternative proof of reverse Hölder type inequalities.

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Keywords: Maximal functions; Spaces of homogeneous type; Calderón–Zygmund

1. Introduction

The partition process of a cube in \mathbb{R}^n involved in the original Calderón–Zygmund decomposition of the domain of a given integrable function f , sometimes can be substituted

[☆] The authors were supported in part by CAI+D and Conicet.

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by a selection method generally provided by a covering lemma. Spaces of homogeneous type are natural settings in which covering lemmas are available.

In some analytical problems the Calderón–Zygmund method needs to be applied to a function defined in a given cube of \mathbb{R}^n or a given ball of an abstract metric measure space. Such is the case if we try to extend the proof given by Coifman and Fefferman [6] of reverse Hölder inequalities for Muckenhoupt weights.

The basic facts concerning Muckenhoupt A_p classes on the euclidean space for $1 < p < \infty$, are consequences of the implicit reverse Hölder inequality contained in the A_p condition. From the technical point of view, dealing with the boundedness of operators, the basic fact used is that if $w \in A_p$ then there exists a positive ϵ , such that $w \in A_{p-\epsilon}$. The proof of this fact is the key argument in [6] in order to show that $w \in A_p$ if and only if the Hardy–Littlewood maximal function is bounded as an operator on $L^p(w)$.

To prove “ $A_p \Rightarrow A_{p-\epsilon}$,” Calderón–Zygmund decomposition is the standard and powerful tool. The Calderón–Zygmund decomposition is associated to the weight w and to a special sequence of levels and has to be obtained on cubes or balls in an uniform way.

The first generalization of the Muckenhoupt theory to the setting of quasi-metric measure spaces with the additional assumption of continuity of the measure of balls as functions of the radius, was given by Calderón in [3].

As it was pointed out by Macías and Segovia in [11], balls of a space of homogeneous type need not be subspaces of homogeneous type with the inherited measure and metric structures. Examples of parabolic distances on \mathbb{R}^2 for which the family of all balls is not a uniform family of subspaces of homogeneous type are also given in [11]. Nevertheless they are able to construct on a general space of homogeneous type (X, d, μ) another quasi-distance δ equivalent to d ($c_1 d \leq \delta \leq c_2 d$, for some constants c_1 and c_2) in such a way that δ -balls are uniformly subspaces of homogeneous type. Therefore the Calderón–Zygmund decomposition technique can be applied to functions given on balls with respect to this new distance. Actually Macías and Segovia [11] use their above mentioned construction to give a proof of the reverse Hölder inequality in the setting of spaces of homogeneous type, extending the technique introduced by Coifman and Fefferman in [6].

A different proof, of the sufficiency of A_p for the $L^p(w)$ boundedness of the Hardy–Littlewood maximal operator, avoiding reverse Hölder type inequalities, given by Christ and Fefferman [5] in the euclidean case, can be rather easily adapted to the setting of space of homogeneous type (see [1]).

In this note we intend to get a Calderón–Zygmund decomposition that goes back to the original partitioning argument, even in metric measure spaces. This method is based in the construction of dyadic type families given by Christ in [4]. Our goal is to compare the level sets of the Hardy–Littlewood maximal function and the level sets of the dyadic maximal function, built on these dyadic families. As applications we shall compare the Muckenhoupt classes defined through the d -balls and through this dyadic sets and prove reverse Hölder inequalities for A_p weights on spaces of homogeneous type.

In Section 2 we give the construction, due to Christ [4], of the dyadic family \mathcal{D} in the general measureless setting of quasi-metric spaces with finite Assouad metric dimension. We also prove that for a doubling measure μ on (X, d) , Christ’s construction is providing a tiling sequence of the space with the special property that the family $\{(Q, d, \mu): Q \in \mathcal{D}\}$ is a uniform family of spaces of homogeneous type and we state the Calderón–Zygmund de-

composition of the domain of a real integrable function. Section 3 contains the elementary but central comparison of level sets for the dyadic maximal and for the Hardy–Littlewood maximal functions. In Section 4 we introduce standard and dyadic A_p -Muckenhoupt weights, and we prove their equivalence under the assumption of the doubling property. Section 5 is devoted to apply the result of Section 4 to prove reverse Hölder inequalities. Even when the reverse Hölder inequality in the dyadic setting can be obtained from the general results for martingales see [8], we give, for the sake of completeness, an elementary proof in the spirit of [6].

2. Dyadic type partitions on spaces of homogeneous type

Let X be a set. A quasi-distance on X is a non-negative symmetric function defined on $X \times X$ such that $d(x, y) = 0$ if and only if $x = y$ and there exists a constant K such that the inequality

$$d(x, y) \leq K[d(x, z) + d(z, y)],$$

holds for every $x, y, z \in X$.

A well-known result due to Macías and Segovia (see [10]) provides a distance ρ and a real number α , generally larger than one, such that d is equivalent to $\rho^\alpha =: d'$.

Since a quasi-distance d on X induces a topology through the neighborhood system $\{B(x, r) : r > 0\}$ of each point $x \in X$ (see [7]), we consider on X this topology. A basic corollary of the above mentioned theorem of Macías and Segovia is the fact that for any quasi-distance d on X it is always possible to construct an equivalent quasi-distance d' such that every d' -ball is an open set.

Let us briefly introduce the Assouad dimension of a quasi-metric space. We shall say that a subset A of X is ϵ -disperse ($\epsilon > 0$) if $d(x, y) \geq \epsilon$ for every x and y in A with $x \neq y$. The Assouad dimension of X , $\dim_A X$, is the infimum of all those positive numbers s such that the inequality

$$\#(B(x, \lambda r) \cap A) \leq C\lambda^s$$

holds for some constant C , every $\lambda \geq 1$, every $x \in X$, every r -disperse subset A of X and every $r > 0$. It is not difficult to prove that $\dim_A X < \infty$ is equivalent to the fact that, for some $N > 0$, every r -disperse subset A of X has at most N points in each ball $B(x, 2r)$ for every $x \in X$ and every $r > 0$.

Let (X, d) be a quasi-metric space with finite Assouad dimension. Assume that the d -balls are open sets. Take $0 < \delta < 1$ and $j \in \mathbb{Z}$. We shall say that \mathcal{N}_j is a δ^j -net in X if \mathcal{N}_j is a maximal δ^j -disperse subset of X . Notice that for every $j \in \mathbb{Z}$ there exists a non-empty δ^j -net \mathcal{N}_j , since of course we are assuming $X \neq \emptyset$. We can write $\mathcal{N}_j = \{x_k^j : k \in \mathcal{K}(j)\}$, where $\mathcal{K}(j)$ is an initial interval of natural numbers that may coincide with all of \mathbb{N} . In fact $\mathcal{K}(j)$ is finite for some j if and only if it is finite for every j . Actually $\mathcal{K}(j)$ is finite for some j if and only if (X, d) is bounded.

The first step in the Christ's construction is to introduce a tree structure on the index set $\mathcal{A} = \bigcup_{j \in \mathbb{Z}} (\{j\} \times \mathcal{K}(j))$ that is closely related to the metric structure on X .

Lemma 2.1 [4, Lemma 13]. *There exists a partial order \preceq on \mathcal{A} satisfying the following tree properties:*

- (1) $(j_1, k_1) \preceq (j_2, k_2)$ implies $j_2 \leq j_1$;
- (2) for every $(j_1, k_1) \in \mathcal{A}$ and every $j_2 \leq j_1$, there exists a unique $k_2 \in \mathcal{K}(j_2)$ such that $(j_1, k_1) \preceq (j_2, k_2)$;
- (3) if $(j_1, k_1) \preceq (j_1 - 1, k_2)$, then $d(x_{k_1}^{j_1}, x_{k_2}^{j_1-1}) < \delta^{j_1-1}$;
- (4) if $d(x_{k_1}^{j_1}, x_{k_2}^{j_1-1}) < \frac{\delta^{j_1-1}}{2K}$, then $(j_1, k_1) \preceq (j_1 - 1, k_2)$.

For a helpful visualization we may think \mathcal{A} as a family tree in which $(j_1, k_1) \preceq (j_2, k_2)$ if and only if (j_2, k_2) is an ancestor of (j_1, k_1) .

Now we are in position to construct the building blocks of the partitions. Define for $(j, k) \in \mathcal{A}$, the set

$$Q_k^j = \bigcup_{(i,l) \preceq (j,k)} B(x_l^i, a\delta^i) \tag{2.1}$$

for a positive number a . Choosing a and δ appropriately, we get the desired dyadic properties for the family $\{Q_k^j : k \in \mathcal{K}(j), j \in \mathbb{Z}\}$.

Theorem 2.2. *Let (X, d) be a quasi-metric space with finite Assouad dimension such that the d -balls are open sets. Then there exist $a > 0$, $C > 0$, and $0 < \delta < 1$ such that the sets Q_k^j satisfy the following properties:*

- (D.1) Q_k^j is an open set for every $(j, k) \in \mathcal{A}$;
- (D.2) $B(x_k^j, a\delta^j) \subset Q_k^j$, for every $(j, k) \in \mathcal{A}$;
- (D.3) $Q_k^j \subset B(x_k^j, C\delta^j)$, for every $(j, k) \in \mathcal{A}$;
- (D.4) for every $(j, k) \in \mathcal{A}$ and every $i < j$ there exists a unique $\ell \in \mathcal{K}(i)$ such that $Q_k^j \subseteq Q_\ell^i$;
- (D.5) for $j \geq i$ then either $Q_k^j \subseteq Q_\ell^i$ or $Q_k^j \cap Q_\ell^i = \emptyset$, $k \in \mathcal{K}(j)$ and $\ell \in \mathcal{K}(i)$;
- (D.6) there exists a constant N such that $\#\{k \in \mathcal{K}(j) : Q_k^j \subset Q_\ell^{j-1}\} \leq N$ for every $\ell \in \mathcal{K}(j - 1)$ and every $j \in \mathbb{Z}$;
- (D.7) for every $j \in \mathbb{Z}$, $Q_k^j \cap Q_\ell^j = \emptyset$ for $k \neq \ell$ both in $\mathcal{K}(j)$ and the set $\bigcup_{k \in \mathcal{K}(j)} Q_k^j$ is dense in X ;
- (D.8) X is bounded if and only if there exists $(j, k) \in \mathcal{A}$ such that $X = Q_k^j$.

Proof. Notice that properties (D.2) to (D.5) can be proved as in [4] since there only the finiteness of the Assouad dimension is actually used. Property (D.1) follow from the fact that the d -balls are open sets. From (D.2) and (D.3) it follows that the sequence of points x_k^j such that $Q_k^j \subset Q_\ell^{j-1}$ is an $a\delta^j$ -disperse subset of $B(x_\ell^{j-1}, C\delta^{j-1})$. Since (X, d) has finite Assouad dimension, we get (D.6). The first statement in (D.7) follows from the definition of Q_k^j and (2) in Lemma 2.1. The second follows from the fact that being \mathcal{N}_i is a maximal δ^i -dense subset of X for every $i \in \mathbb{Z}$, and $\bigcup_{k \in \mathcal{K}(j)} Q_k^j \supseteq \bigcup_{i \geq j} \mathcal{N}_i$. If $X = B(x_0, R)$, it is possible to find j , negative enough, such that $2KR < a\delta^j$. For this j and every $k \in \mathcal{K}(j)$ we have $B(x_k^j, a\delta^j) \supset B(x_0, R)$. Since, from (D.2), Q_k^j contains the ball $B(x_k^j, a\delta^j)$, (D.8) follows. \square

Let us denote by \mathcal{D} the class of all *dyadic* sets defined by (2.1). With $\mathcal{D}_j = \{Q_k^j : k \in \mathcal{K}(j)\}$, we have that $\mathcal{D} = \bigcup_{j \in \mathbb{Z}} \mathcal{D}_j$. Given Q and Q' two elements in \mathcal{D} we say that they are of the same level if $Q = Q_k^j$, $Q' = Q_\ell^j$ for some $j \in \mathbb{Z}$ and some $k, \ell \in \mathcal{K}(j)$.

For a given positive number R , we shall say that two dyadic sets Q_k^j and Q_ℓ^j of the same level $j \in \mathbb{Z}$ are R -neighbors if the inequality $d(x_k^j, x_\ell^j) \leq R\delta^j$ holds.

The next result is elementary but useful for the subsequent development. Let us write $\mathcal{N}_R(Q_k^j)$ to denote the set of all R -neighbors of Q_k^j .

Lemma 2.3. *For every $R > 0$ there exists a number $M = M(R)$ such that the number of elements of $\mathcal{N}_R(Q_k^j)$ is less than or equal to M for every $j \in \mathbb{Z}$ and every $k \in \mathcal{K}(j)$.*

Proof. Since the set $\{x_\ell^j : Q_\ell^j \in \mathcal{N}_R(Q_k^j)\}$ is δ^j -disperse and, from the definition of neighboring, is contained in a ball with radius $R\delta^j$, the finiteness of the Assouad dimension gives the desired estimate. Moreover we can take $M \leq N^{1+\log_2 R}$, where N is the constant associated to the finiteness of the metric dimension. \square

Let (X, d) be a quasi-metric space with finite Assouad dimension such that the d -balls are open sets. If the space (X, d) is complete, in the Cauchy sense, we can apply the results of Vol'berg and Konyagin [13], Wu [14] and Luukkainen and Saksman [9] to get a Borel measure μ on X satisfying the doubling condition

$$0 < \mu(B(x, 2r)) \leq A\mu(B(x, r)) < \infty \tag{2.2}$$

for some constant A , every $x \in X$ and every $r > 0$. As usual we shall say that (X, d, μ) is a space of homogeneous type if (X, d) is a quasi-metric space and μ is a measure defined on a σ -algebra Σ containing the d -balls that satisfies (2.2). We will refer to the triangle constant K and the doubling constant A as the *geometric constants* of the space.

Let us notice that if (X, d, μ) is a space of homogeneous type then the fact that (X, d) has finite Assouad dimension is proved in [7]. So that in a space of homogeneous type the above construction of dyadic sets is available. But as Christ shows in [4], a result stronger than (D7) holds in this case: the measure of the boundaries of dyadic sets vanishes.

Theorem 2.4 [4]. *Let (X, d, μ) be a space of homogeneous type such that d -balls are open sets and continuous functions are dense in $L^1(X)$, then*

$$\mu(\partial Q_k^j) = 0 \quad \text{for every } (j, k) \in \mathcal{A},$$

where ∂Q_k^j is the boundary of Q_k^j .

We would like to observe that under metric completeness the density of Lipschitz β functions with compact support, for some $\beta > 0$, can actually be obtained (see [2] and [12]).

In the next lemma we sketch the proof of an interesting additional feature of these dyadic families: $\{(Q, d, \mu) : Q \in \mathcal{D}\}$ is a uniform family of spaces of homogeneous type for any doubling Borel measure μ on (X, d) .

Lemma 2.5. *Let (X, d, μ) be a space of homogeneous type with geometric constants K and A such that d -balls are open sets. Let \mathcal{D} be a dyadic family with constants C, a and δ . Then there exists a constant \tilde{A} (depending only on K, A, C, a and δ), such that for every $(j, k) \in \mathcal{A}$, (Q_k^j, d, μ) is a space of homogeneous type with geometric constants K and \tilde{A} .*

Proof. Notice first that if a Borel measure ν is given on a quasi-metric space (Y, ρ) with open balls and it satisfies the doubling property, with a doubling constant A_0 , for the balls centered on a dense subset of Y , then ν also satisfies the doubling property for every ball with a constant \tilde{A}_0 that depends only on A_0 and the triangle constant for ρ . With this fact in mind, take $x \in Q \setminus \bigcup_{Q' \in \mathcal{D}} \partial Q'$ for a fixed but general $Q \in \mathcal{D}$. Let r be a positive given number. Assume that Q belongs to the level $j_0 \in \mathbb{Z}$, that is $Q = Q_{k_0}^{j_0} \in \mathcal{D}_{j_0}$ for some $k_0 \in \mathcal{K}(j_0)$. With $B_Q(x, r)$ we shall denote the d -balls of the space (Q, d, μ) . Observe that if $r \geq 2KC\delta^{j_0}$, we have that $B_Q(x, r) = B_Q(x, 2r) = Q$, so that the doubling property trivially holds with constant equal to one. Let us, then assume that $0 < r < 2KC\delta^{j_0}$. Pick $j_1 \geq j_0$ such that $2KC\delta^{j_1+1} \leq r < 2KC\delta^{j_1}$. Let $k_1 \in \mathcal{K}(j_1+1)$ be such that $x \in Q_{k_1}^{j_1+1} \subset Q$. Then

$$B(x_{k_1}^{j_1+1}, a\delta^{j_1+1}) \subset Q_{k_1}^{j_1+1} \subset B_Q(x, r).$$

On the other hand,

$$B_Q(x, 2r) \subset B\left(x_{k_1}^{j_1+1}, CK\left(\frac{4K}{\delta} + 1\right)\delta^{j_1+1}\right).$$

Thus

$$\begin{aligned} \mu(B_Q(x, 2r)) &\leq \mu\left(B\left(x_{k_1}^{j_1+1}, CK\left(\frac{4K}{\delta} + 1\right)\delta^{j_1+1}\right)\right) \\ &\leq \tilde{A}\mu(B(x_{k_1}^{j_1+1}, a\delta^{j_1+1})) \\ &\leq \tilde{A}\mu(B_Q(x, r)), \end{aligned}$$

with \tilde{A} depending only on K, A, C, a and δ , not on Q . Let us finally observe that since $B_Q(x, 2r)$ is an open set, we have that $\mu(B_Q(x, 2r)) > 0$ and since $B_Q(x, r)$ is bounded, $\mu(B_Q(x, r))$ is finite. \square

Let us finish this section by proving a dyadic version of Calderón–Zygmund decomposition. We shall use the standard notation for mean values: $m_Q(f) = \frac{1}{\mu(Q)} \int_Q f \, d\mu$, for $Q \in \mathcal{D}$ and $m_X(f) = \frac{1}{\mu(X)} \int_X f \, d\mu$ if $\mu(X) < \infty$ and $m_X(f) = 0$ if $\mu(X) = +\infty$.

Theorem 2.6. *Let (X, d, μ) be a space of homogeneous type such that d -balls are open sets. Let $f \geq 0$ be a μ -integrable function defined on X and λ a positive number with $\lambda \geq m_X(f)$. Then there exists a family $\mathcal{F} \subset \mathcal{D}$ such that*

$$\text{if } Q \text{ and } Q' \text{ are distinct elements of } \mathcal{F}, \text{ then } Q \cap Q' = \emptyset; \tag{2.3a}$$

$$m_Q(f) > \lambda \text{ for every } Q \in \mathcal{F}; \tag{2.3b}$$

$$m_{\tilde{Q}}(f) \leq \lambda \text{ for every } \tilde{Q} \in \mathcal{D} \text{ such that } \tilde{Q} \not\supseteq Q \text{ for some } Q \in \mathcal{F}; \tag{2.3c}$$

$$m_{Q'}(f) \leq \lambda \text{ for every } Q' \in \mathcal{D} \text{ such that } Q' \cap \left(\bigcup_{Q \in \mathcal{F}} Q \right) = \emptyset. \tag{2.3d}$$

Proof. Let \mathcal{H} be the family of all dyadic sets $Q \in \mathcal{D}$ for which $m_Q(f) > \lambda$. If $\mathcal{H} = \emptyset$, taking $\mathcal{F} = \emptyset$ we trivially have that (2.3a–d) hold true for every $Q' \in \mathcal{D} = \mathcal{F}^c$. Let us then assume that $\mathcal{H} \neq \emptyset$. For each $Q \in \mathcal{H}$, the class of all dyadic sets \bar{Q} in \mathcal{H} such that $\bar{Q} \supset Q$ is bounded above. Of course this is true if (X, d) is bounded. For the unbounded case, as $f \in L^1(X, \mu)$, $m_{Q'}(f) \leq \frac{1}{\mu(Q')} \|f\|_1$ tends to zero if the diameter the dyadic set Q' grows to infinity and $Q' \supset Q$. So that for each $Q \in \mathcal{H}$ there is a unique cube $\bar{Q} \in \mathcal{D}$ which is maximal with the properties $\bar{Q} \in \mathcal{H}$ and $\bar{Q} \supset Q$. Let \mathcal{F} be the class of those \bar{Q} . In other words,

$$\mathcal{F} = \{ \bar{Q} : \bar{Q} \text{ is maximal with the property } m_{\bar{Q}}(f) > \lambda \}.$$

Properties (2.3a–d) for this class \mathcal{F} follow directly from its definition. \square

3. Comparison of the level sets of the dyadic and the standard maximal functions

Let (X, d, μ) be a space of homogeneous type. The non-centered Hardy–Littlewood maximal function is defined by

$$Mf(x) = \sup_{x \in B} \frac{1}{\mu(B)} \int_B |f| d\mu,$$

for a given locally integrable function f .

Taking d' a quasi-distance on X equivalent to d such that the d' -balls are open sets, we have a dyadic family \mathcal{D} satisfying the results of the previous section. For a locally integrable function f we define its dyadic maximal function by

$$M^{dy} f(x) = \sup_{x \in Q \in \mathcal{D}} \frac{1}{\mu(Q)} \int_Q |f(y)| d\mu(y),$$

for $x \in \bigcup_{Q \in \mathcal{D}} Q$ and $M^{dy} f(x) = 0$ otherwise.

The basic facts concerning boundedness of the dyadic maximal operator are contained in the next result.

Theorem 3.1. *With the notation introduced above we have*

(a) *For every integrable function f and every positive real number λ there exists a disjoint family $\mathcal{F} \subset \mathcal{D}$ such that*

$$\{x \in X : M^{dy} f(x) > \lambda\} = \bigcup_{Q \in \mathcal{F}} Q.$$

(b) *The weak type (1, 1) inequality*

$$\mu(\{x \in X : M^{dy} f(x) > \lambda\}) \leq \frac{1}{\lambda} \int_X |f| d\mu,$$

holds for every locally integrable function f and every $\lambda > 0$.

(c) If $1 < p \leq \infty$, there exists a constant $C_p > 0$ such that the inequality

$$\|M^{dy} f\|_p \leq C_p \|f\|_p,$$

holds for every locally integrable f .

Proof. (a) Let $\lambda > 0$ and f an integrable function. If $\lambda \geq m_X(|f|)$ we can apply Theorem 2.6 to $|f|$ and λ in order to obtain a family \mathcal{F} of disjoint dyadic cubes satisfying (2.3a–d). If, on the other hand, $\lambda < m_X(|f|)$, this is only because $\mu(X) < \infty$. Since μ is doubling we have that X is bounded and, from (D.8) $X = Q_k^j$ for some $(j, k) \in \mathcal{A}$. In this case we take as \mathcal{F} the family which contains only the element Q_k^j . Notice now that $\{M^{dy} f > \lambda\} = \bigcup_{Q \in \mathcal{F}} Q$. In fact, \mathcal{F} is empty if and only if $M^{dy} f \leq \lambda$. If \mathcal{F} has the only element $Q_k^j = X$, then both sets are the whole X . For the generic case, notice that if $x \in Q \in \mathcal{F}$, $M^{dy} f(x) \geq m_Q(|f|) > \lambda$. Given now a point x such that $M^{dy} f(x) > \lambda$, we have $m_Q(|f|) > \lambda$ for some $Q \in \mathcal{D}$ with $x \in Q$. From the construction of \mathcal{F} there exists a cube $Q' \supset Q$, $Q' \in \mathcal{F}$, hence x is an element of the set $\bigcup_{Q \in \mathcal{F}} Q$.

(b) From (a) we easily obtain

$$\mu(\{x \in X: M^{dy} f(x) > \lambda\}) = \sum_{Q \in \mathcal{F}} \mu(Q) \leq \frac{1}{\lambda} \int_{\bigcup_{Q \in \mathcal{F}} Q} |f| d\mu \leq \frac{1}{\lambda} \int_X |f| d\mu.$$

(c) From Marcinkiewicz interpolation, we get the L^p boundedness of M^{dy} for $1 < p \leq \infty$. \square

Of course, inequalities of type (b) and (c) follow also from the inequality $M^{dy} f \leq C Mf$ which follows from (D2), (D3) and the doubling property for μ . Observe that the pointwise inequality in the opposite sense is not possible in general, but as the next theorems show a control of the level sets of Mf in terms of those of $M^{dy} f$ is still possible.

Theorem 3.2. *Let (X, d, μ) be a space of homogeneous type and Mf the non-centered Hardy–Littlewood maximal function. Let d' be any quasi-distance on X equivalent to d for which the balls are open sets. Let \mathcal{D} be any dyadic family on (X, d', μ) as in Section 2. Then there exist $R_0 > 0$ and $L > 0$ such that for every locally integrable function f and every positive real number λ , we have that*

$$\{x: Mf(x) > L\lambda\} \subseteq \bigcup_{Q \in \mathcal{F}} \left(\bigcup_{Q' \in \mathcal{N}_{R_0}(Q)} Q' \right) \cup \mathcal{Z}, \tag{3.1}$$

where \mathcal{F} is the family associated to f and λ given in Theorem 3.1, $\mathcal{N}_{R_0}(Q)$ denotes the R_0 -neighbors of Q defined in Section 2 and $\mathcal{Z} = \bigcup_{Q \in \mathcal{D}} \partial Q$ is a set of zero μ -measure.

Proof. Let us first notice that by taking L large enough it suffices to prove (3.1) for the centered maximal function

$$M^c f(x) = \sup_{r>0} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f| d\mu$$

instead of $Mf(x)$ and for the case in which actually the d -balls are open sets. The constants R_0 and L will become explicit at the end of our estimates. Let us take a point x which does not belong to the set $\bigcup_{Q \in \mathcal{F}} (\bigcup_{Q' \in \mathcal{N}_{R_0}(Q)} Q') \cup \mathcal{Z}$. In order to obtain an upper estimate for the $M^c f(x)$, let us pick a ball $B(x, r)$ centered at x with positive radius r and let us estimate $m_{B(x,r)}(|f|)$. Take $j \in \mathbb{Z}$ such that $\delta^{j+1} \leq r < \delta^j$. Define the subclass of dyadic sets

$$\mathcal{G}(x, r) = \{ \tilde{Q} \in \mathcal{D}_j : \tilde{Q} \cap B(x, r) \neq \emptyset \}.$$

Claim. No $\tilde{Q} \in \mathcal{G}(x, r)$ is contained in a $Q \in \mathcal{F}$.

Let us assume that the claim is proved. The family \mathcal{D} can be partitioned in two disjoint subfamilies $D^1 = \{Q \in \mathcal{D} : m_Q(|f|) > \lambda\}$ and $D^2 = \{Q \in \mathcal{D} : m_Q(|f|) \leq \lambda\}$. From the claim and Theorem 2.6 we see that $\mathcal{G}(x, r) \subset D^2$. Notice also that the number of elements of the class $\mathcal{G}(x, r)$ is bounded by a constant M_1 which does not depend on x or $r > 0$. Hence

$$\begin{aligned} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f| d\mu &= \frac{1}{\mu(B(x, r))} \sum_{\tilde{Q} \in \mathcal{G}(x, r)} \int_{B(x, r) \cap \tilde{Q}} |f| d\mu \\ &\leq \sum_{\tilde{Q} \in \mathcal{G}(x, r)} \frac{\mu(\tilde{Q})}{\mu(B(x, r))} m_{\tilde{Q}}(|f|) \\ &\leq M_1 A_2 \lambda, \end{aligned}$$

where A_2 satisfies $\mu(\tilde{Q}) \leq A_2 \mu(B(x, r))$, which follows from the fact that \tilde{Q} is contained in the ball centered at x with radius Cr for some fixed C . So, we have that $M^c f(x) \leq L\lambda$ with $L = M_1 A_2$. Let us, finally, prove the claim. Let us assume there exist $\tilde{Q} \in \mathcal{G}(x, r)$ and $Q \in \mathcal{F}$ such that $\tilde{Q} \subset Q$. Since Q is also a dyadic set, we have that $Q = Q_l^i$ for some $i \leq j$ and some $l \in \mathcal{K}(i)$. Let us show that for an appropriate choice of the constant R_0 , we have the contradiction: $x \in \bigcup_{Q' \in \mathcal{N}_{R_0}(Q)} Q'$. Take a point $y \in \tilde{Q} \cap B(x, r)$, then $d(x, x_l^i) \leq K[d(x, y) + d(y, x_l^i)] < K(r + C\delta^i) < K(\delta^j + C\delta^i) \leq K(1 + C)\delta^i$. In other words x is a point in the ball $B(x_l^i, K(1 + C)\delta^i)$ which does not belong to the residual boundaries \mathcal{Z} . Hence $x \in Q_m^i$ for some $m \in \mathcal{K}(i)$. So that from (D.3) $d(x, x_m^i) < C\delta^i$ then,

$$\begin{aligned} d(x_m^i, x_l^i) &\leq K(d(x_m^i, x) + d(x, x_l^i)) < K(C\delta^i + K(1 + C)\delta^i) \\ &= K(C + K(1 + C))\delta^i. \end{aligned}$$

In other words $x \in Q_m^i$ and Q_m^i and $Q = Q_l^i$ are R_0 -neighbors with $R_0 = K(C + K(1 + C))$. \square

Theorem 3.2 allows us to obtain distribution function estimates for the Hardy–Littlewood maximal function in terms of the dyadic maximal function for a given doubling measure ν on X which is absolutely continuous with respect to μ .

Theorem 3.3. Let (X, d, μ) , Mf , d' and \mathcal{D} as in Theorem 3.2. Let $\nu \ll \mu$ be a doubling measure. Then there exist two positive and finite constants L and C such that the inequality

$$\nu^* (\{Mf > L\lambda\}) \leq C \nu (\{M^{d_y} f > \lambda\}),$$

holds for every locally integrable f and every $\lambda > 0$, where $\nu^*(E) = \inf \nu(F)$ with $F \supset E$ and $F \in \Sigma$.

Proof. Let L and R_0 be the numbers given by Theorem 3.2. Let us first observe that, from Lemma 2.3, (D2) and (D3), there exists a constant $A(\nu, R_0)$ such that the inequality

$$\nu \left(\bigcup_{Q' \in \mathcal{N}_{R_0}(Q)} Q' \right) \leq A(\nu, R_0) \nu(Q)$$

holds for every $Q \in \mathcal{D}$. From this inequality and (3.1) we have that

$$\begin{aligned} \nu^* (\{Mf > L\lambda\}) &\leq \nu \left(\bigcup_{Q \in \mathcal{F}} \left(\bigcup_{Q' \in \mathcal{N}_{R_0}(Q)} Q' \right) \right) + \nu(\mathcal{Z}) \\ &\leq \sum_{Q \in \mathcal{F}} \nu \left(\bigcup_{Q' \in \mathcal{N}_{R_0}(Q)} Q' \right) \\ &\leq A(\nu, R_0) \sum_{Q \in \mathcal{F}} \nu(Q) \\ &= A(\nu, R_0) \nu \left(\bigcup_{Q \in \mathcal{F}} Q \right) \\ &= A(\nu, R_0) \nu (\{M^{d_y} f > \lambda\}), \end{aligned}$$

and the result holds with $C = A(\nu, R_0)$. \square

4. A_p and dyadic- A_p Muckenhoupt weights on space of homogeneous type

A non-negative, measurable and locally integrable function w defined on the space of homogeneous type (X, d, μ) , is said to be a Muckenhoupt weight of class $A_p(X, d, \mu)$ ($1 < p < \infty$) if the inequality

$$\left(\int_B w d\mu \right) \left(\int_B w^{-\frac{1}{p-1}} \right)^{p-1} \leq C \mu(B)^p, \tag{4.1}$$

holds for some constant C and every ball B in X . We say that $w \in A_1(X, d, \mu)$ if there exists a constant C such that the inequality

$$\frac{1}{\mu(B)} \int_B w d\mu \leq C \operatorname{ess\,inf}_B w, \tag{4.2}$$

holds for every ball B in X . Let us observe that the definitions of the A_p classes are invariant by change of equivalent quasi-distances.

Assuming that η is a quasi-distance on X such that η -balls are open sets and (X, η, μ) is a space of homogeneous type, we have dyadic families $\mathcal{D} = \mathcal{D}(\eta)$ given by the sets defined in (2.1). It is easy to see that if $w \in A_p(X, d, \mu)$, then the measure $w(x) d\mu(x)$ is doubling.

We say that a non-negative, measurable and locally integrable function w is a dyadic Muckenhoupt weight of class $A_p^{dy}(X, \eta, \mu)$ $1 < p < \infty$ (respectively $p = 1$) with respect to \mathcal{D} , if (4.1) (respectively (4.2)) holds with $Q \in \mathcal{D}$ instead of B .

Let us notice that $A_p(X, d, \mu)$ implies $A_p^{dy}(X, d', \mu)$ for $d' \simeq d$ with the d' -balls open sets. In fact, if $w \in A_p(X, d, \mu)$ and Q_k^j is any dyadic set in \mathcal{D} , we have

$$\begin{aligned} \left(\int_{Q_k^j} w d\mu \right) \left(\int_{Q_k^j} w^{-\frac{1}{p-1}} \right)^{p-1} &\leq \left(\int_{B(x_k^j, C\delta^j)} w d\mu \right) \left(\int_{B(x_k^j, C\delta^j)} w^{-\frac{1}{p-1}} \right)^{p-1} \\ &\leq C\mu(B(x_k^j, C\delta^j))^p \\ &\leq \tilde{C}\mu(B(x_k^j, a\delta^j))^p \\ &\leq \tilde{C}(\mu(Q_k^j))^p. \end{aligned}$$

The converse is generally false. For example the function defined on \mathbb{R} by $w(x) = 1$ if $x < 0$ and $w(x) = x^{1/2}$ if $x > 0$ belongs to A_2^{dy} but not to A_2 with respect to the usual dyadic intervals on \mathbb{R} .

The purpose of this section is to prove the next result.

Theorem 4.1. *Let (X, d, μ) be a space of homogeneous type. If $\nu \ll \mu$ is doubling measure on (X, d) such that for some $d' \simeq d$ with the d' -balls being open sets we have that $w = \frac{d\nu}{d\mu} \in A_p^{dy}(X, d', \mu)$, then $w \in A_p(X, d, \mu)$.*

Proof. Since $w \in A_p^{dy}(X, d', \mu)$, by Hölder inequality we get that, for every $Q \in \mathcal{D}$,

$$\begin{aligned} \frac{1}{\mu(Q)} \int_Q |f| d\mu &\leq \frac{1}{\mu(Q)} \left(\int_Q |f|^p w d\mu \right)^{1/p} \left(\int_Q w^{-\frac{1}{p-1}} d\mu \right)^{1/p'} \\ &\leq C \left(\frac{1}{w(Q)} \int_Q |f|^p w d\mu \right)^{1/p}, \end{aligned}$$

where $w(Q) = \int_Q w d\mu$. Then we get that

$$M^{dy} f(x) \leq C [M_w^{dy} (|f|^p)(x)]^{1/p}, \tag{4.3}$$

where $M_w^{dy} g(x) = \sup_{x \in Q, Q \in \mathcal{D}} \frac{1}{w(Q)} \int_Q |g| w d\mu$ for every $x \in \bigcup_{Q \in \mathcal{D}} Q$ and $M^{dy} g(x) = 0$ otherwise. Now, from Theorem 3.3, (4.3) and, since ν is doubling, applying Theorem 3.1(b) in the space of homogeneous type (X, d, ν) , we get

$$\begin{aligned}
 v^*({Mf > \lambda}) &\leq v\left(\left\{M^{dy} f > \frac{\lambda}{L}\right\}\right) \leq v\left(\left\{M_w^{dy}(|f|^p) > \left(\frac{\lambda}{CL}\right)^p\right\}\right) \\
 &\leq \frac{C^p L^p}{\lambda^p} \int_X |f|^p w \, d\mu
 \end{aligned}$$

for all locally integrable f and all $\lambda > 0$. Then, by standard arguments we obtain that $w \in A_p(X, d, \mu)$. In fact, let us consider a ball B and $f = w^{-\frac{1}{p-1}} \chi_B$. Then

$$B \subset \left\{Mf > \frac{1}{\mu(B)} \int_B w^{-\frac{1}{p-1}} \, d\mu - \varepsilon\right\},$$

for all $\varepsilon > 0$. So that

$$\begin{aligned}
 \int_B w \, d\mu = v(B) &\leq v^*\left(\left\{Mf > \frac{1}{\mu(B)} \int_B w^{-\frac{1}{p-1}} \, d\mu - \varepsilon\right\}\right) \\
 &\leq \frac{C}{\left(\frac{1}{\mu(B)} \int_B w^{-\frac{1}{p-1}} \, d\mu - \varepsilon\right)^p} \int_B w^{-\frac{1}{p-1}} \, d\mu,
 \end{aligned}$$

for all $\varepsilon > 0$. Then letting $\varepsilon \rightarrow 0$ we are done. \square

Notice that Theorem 4.1 proves that, under the hypothesis of doubling for a given weight w , the Muckenhoupt character of w can be described through its behavior on any dyadic system $\mathcal{D}(\eta)$ with $\eta \simeq d$. Since doubling condition on w , in Theorem 4.1, involves the family of all balls on (X, d) , one may think that the Muckenhoupt character of w is not completely described by a dyadic family of the type $\mathcal{D}(\eta)$. But if we look at the actual estimate in the proof of Theorem 3.3, we see that the doubling property used involves only dyadic sets. Moreover what matters is the boundedness of the measure of neighbors of a dyadic set Q in terms of the measure of Q itself. Nevertheless is not difficult to prove that this notion of doubling is equivalent to the standard one, so that all the information of the Muckenhoupt character of a weight w can be given in terms of its behavior on the dyadic sets. Let us state this remark in the next result.

Corollary 4.2. *Let (X, d, μ) be a space of homogeneous type such that d -balls are open sets and let $\mathcal{D} = \mathcal{D}(d)$. Let w be a non-negative locally integrable function defined on X . Then $w \in A_p(X, d, \mu)$ if and only if $w \in A_p^{dy}(X, d, \mu)$ and for each $R > 0$ there exists a constant $A(w, R)$ such that the inequality*

$$w(Q') \leq A(w, R)w(Q) \tag{4.4}$$

holds for every $Q' \in \mathcal{N}_R(Q)$ and for every $Q \in \mathcal{D}$.

Proof. Since it is clear from the definition of R -neighbors of a given $Q \in \mathcal{D}$, that the doubling condition implies (4.4), from Theorem 4.1, we only have to show the converse. Given $r > 0$, let $j \in \mathbb{Z}$ be such that $K(1 + C)\delta^j < r \leq K(1 + C)\delta^{j-1}$ where K is the triangle constant of d and C is the constant in Theorem 2.2. It is easy to see that there exists

$\ell \in \mathbb{Z}$ such that $Q_\ell^j \subset B(x, r)$ since \mathcal{N}_j is maximal with the property of δ^j -dispersion. We shall prove that there exists R_0 such that

$$B(x, 2r) \subset \left(\bigcup_{Q \in \mathcal{N}_{R_0}(Q_\ell^j)} Q \right) \cup \mathcal{Z},$$

where $\mathcal{Z} = X \setminus \bigcup_{k \in \mathcal{K}(j)} Q_k^j$. To prove this fact, take $y \in B(x, 2r) - \mathcal{Z}$. Then, there exists $m \in \mathcal{K}(j)$ such that $y \in Q_m^j$. Notice that

$$\begin{aligned} d(x_m^j, x_\ell^j) &\leq K [d(x_m^j, y) + d(y, x_\ell^j)] \\ &\leq K [C\delta^j + K [d(y, x) + d(x, x_\ell^j)]] \\ &\leq K (C\delta^j + 3Kr) \\ &\leq [KC + 3K^3(1 + C)]\delta^j. \end{aligned}$$

Then choosing $R_0 = KC + 3K^3(1 + C)/\delta$, we get that $Q_m^j \in \mathcal{N}_{R_0}(Q_\ell^j)$. Now, by Lemma 2.3,

$$v(B(x, 2r)) \leq v\left(\bigcup_{Q \in \mathcal{N}_{R_0}(Q_\ell^j)} Q\right) + v(\mathcal{Z}) \leq Cv(Q_\ell^j) \leq v(B(x, r)). \quad \square$$

Let us finally observe that (4.4) is not the “dyadic doubling” obtained naturally from A_p^{dy} relating the measure of a dyadic set to the measure of its father (first ancestor): if $Q = Q_k^j \in \mathcal{D}$ and $\tilde{Q} = Q_\ell^{j-1}$ are such that $\tilde{Q} \supseteq Q$, then

$$w(\tilde{Q}) \leq Cw(Q). \tag{4.5}$$

In fact, from the A_p^{dy} condition, the doubling property for μ and Hölder inequality, we get the following inequalities:

$$\begin{aligned} \left(\int_{\tilde{Q}} w d\mu\right) \left(\int_{\tilde{Q}} w^{-\frac{1}{p-1}} d\mu\right)^{p-1} &\leq C[\mu(\tilde{Q})]^p \leq \tilde{C}[\mu(Q)]^p \\ &\leq \tilde{C} \left(\int_Q w d\mu\right) \left(\int_Q w^{-\frac{1}{p-1}} d\mu\right)^{p-1} \\ &\leq \tilde{C} \left(\int_Q w d\mu\right) \left(\int_Q w^{-\frac{1}{p-1}} d\mu\right)^{p-1}, \end{aligned}$$

which give us (4.5).

5. Application to reverse Hölder inequalities

As it was mentioned in the introduction, we shall use the above results to give another proof of the next theorem.

Theorem 5.1. *Let (X, d, μ) be a space of homogeneous type such that continuous functions are dense in $L^1(X)$ and let $w \in A_p(X, d, \mu)$, then there exists a positive ϵ such that $w \in A_{p-\epsilon}(X, d, \mu)$.*

To prove the theorem let us take a quasi-distance d' equivalent to d such that d' -balls are open sets and let us construct a dyadic family $\mathcal{D} = \mathcal{D}(d')$ associated to this new quasi-distance. As we have observed in Section 4, since $w \in A_p(X, d, \mu)$, then $w \in A_p^{dy}(X, d', \mu)$. If we prove the desired result in the dyadic setting, i.e.: there exists a positive ϵ such that $w \in A_{p-\epsilon}^{dy}(X, d', \mu)$ we are done, since we can apply Theorem 4.1 because $w d\mu$ is a doubling measure. On the other hand, in order to prove $A_p^{dy} \Rightarrow A_{p-\epsilon}^{dy}$, it will suffice to obtain a reverse Hölder inequality in the dyadic context.

Even when the remaining dyadic reverse Hölder inequality could be obtained from the martingale setting [8], we shall briefly sketch how the result follows from the analytical tools given in Theorem 2.6 (Calderón–Zygmund decomposition) and Lemma 2.5 applying *mutatis mutandi* the technique introduced by Coifman and Fefferman in [6].

Lemma 5.2 (Reverse Hölder inequality). *Assume that the space satisfies the hypotheses of Theorem 5.1 and that each ball is an open set. Given a weight $w \in A_p^{dy}$ with $1 \leq p < \infty$ there exist positive constants C and δ depending only on p , the A_p^{dy} constant for w and the geometric constants such that the inequality*

$$\left(\frac{1}{\mu(Q)} \int_Q [w(x)]^{1+\delta} d\mu(x) \right)^{\frac{1}{1+\delta}} \leq \frac{C}{\mu(Q)} \int_Q w(x) d\mu(x) \tag{5.1}$$

holds for every $Q \in \mathcal{D}$.

Proof. Let w be a weight in A_p^{dy} , let Q be a given dyadic set in \mathcal{D} and let $\{\lambda_m: m = 0, 1, \dots\}$ be an increasing sequence with $\lambda_0 = m_Q(w)$. Since (Q, d, μ) is a space of homogeneous type (Lemma 2.5) we can apply Theorem 2.6 with $X = Q$, $f = w$ and $\lambda = \lambda_m$ in order to obtain a family $\mathcal{F}_m \subset \mathcal{D}$ satisfying (2.3a) to (2.3d). Set $\Omega_m = \bigcup_{Q' \in \mathcal{F}_m} Q'$. Notice that $\Omega_{m+1} \subseteq \Omega_m$, for every $m = 0, 1, 2, \dots$. The desired inequality (5.1) will follow from the next statement.

Claim. For each $\alpha \in (0, 1)$, we can choose two numbers $M > 1$ and $\beta \in (0, 1)$ such that $\lambda_m = \lambda_0 M^m$ and both inequalities

$$\mu(\Omega_m) \leq \alpha^m \mu(\Omega_0) \quad \text{and} \tag{5.2a}$$

$$w(\Omega_m) \leq \beta^m w(\Omega_0) \tag{5.2b}$$

hold for every Q and every $m = 0, 1, 2, \dots$.

Assuming our claim, let us finish the proof of the lemma. Pick $\delta > 0$ such that $\beta M^\delta < 1$. Let us estimate the desired mean value of $w^{1+\delta}$ over the set Q ,

$$\int_Q w^{1+\delta} d\mu = \int_{Q-\Omega_0} w^{1+\delta} d\mu + \sum_{m=0}^\infty \int_{\Omega_m-\Omega_{m+1}} w^{1+\delta} d\mu + \int_{\bigcap_{m=1}^\infty \Omega_m} w^{1+\delta} d\mu.$$

From the first assertion in our claim, we see that the last term above vanishes. For the first and the second terms we use (2.3d), the Lebesgue Differentiation Theorem and the second assertion in the claim in order to get

$$\begin{aligned} \int_Q w^{1+\delta} d\mu &\leq \lambda_0^\delta w(Q) + \sum_{m=0}^\infty \lambda_{m+1}^\delta w(\Omega_m) \leq \left(\lambda_0^\delta + \sum_{m=0}^\infty \lambda_{m+1}^\delta \beta^m \right) w(Q) \\ &= \lambda_0^\delta \left(1 + \sum_{m=0}^\infty M^{\delta(m+1)} \beta^m \right) w(Q) = C \left(\frac{w(Q)}{\mu(Q)} \right)^{1+\delta} \mu(Q). \end{aligned}$$

Let us finally sketch the proof of the claim. Take $Q \in \mathcal{D}$ and $\alpha \in (0, 1)$. As in the euclidean case, it is enough to show that there exists a constant M such that with $\lambda_m = \lambda_0 M^m$ we have the inequalities

$$\mu(\Omega_{m+1} \cap Q') \leq \alpha \mu(Q') \tag{5.2c}$$

for every $Q' \in \mathcal{F}_m$ and every $m = 0, 1, 2, \dots$. Once (5.2c) is proved, from the standard A_∞^{dy} -type inequality, we also have

$$w(\Omega_{m+1} \cap Q') \leq \beta w(Q'), \tag{5.2d}$$

for some $\beta < 1$. Adding, over $Q' \in \mathcal{F}_m$, in the inequalities (5.2c) and (5.2d), and then iterating, we obtain (5.2a) and (5.2b).

Let us sketch the proof of (5.2c). Take $Q' \in \mathcal{F}_m$. Since we are dealing with the dyadic sets in \mathcal{D} , the intersection of Q' and Ω_{m+1} is the disjoint union of those dyadic sets $Q'' \in \mathcal{F}_{m+1}$, which are contained in Q' . From property (2.3b) of the Calderón–Zygmund decomposition at level λ_{m+1} , we have

$$\begin{aligned} \mu(\Omega_{m+1} \cap Q') &= \sum_{\{Q'' \in \mathcal{F}_{m+1}: Q'' \subseteq Q'\}} \mu(Q'') < \frac{1}{\lambda_{m+1}} \sum_{\{Q'' \in \mathcal{F}_{m+1}: Q'' \subseteq Q'\}} \int_{Q''} w d\mu \\ &\leq \frac{1}{\lambda_{m+1}} \int_{Q'} w d\mu. \end{aligned}$$

Let us now consider the first ancestor \tilde{Q} of Q' , applying (2.3c), and using the fact that \tilde{Q} and Q' have comparable μ -measures, we get

$$\mu(\Omega_{m+1} \cap Q') \leq C \frac{\lambda_m}{\lambda_{m+1}} \mu(Q'),$$

which becomes (5.2c) provided that $\lambda_m = \left(\frac{C}{\alpha}\right)^m \lambda_0$ or, in other words $M = \frac{C}{\alpha}$. \square

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