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Well-posedness and exponential stability of a thermoelastic Joint–Leg–Beam system with Robin boundary conditions

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ABSTRACT

An important class of proposed large space structures features a triangular truss backbone. In this paper we study thermomechanical behavior of a truss component; namely, a triangular frame consisting of two thin-walled circular beams connected through a joint. Transverse and axial mechanical motions of the beams are coupled though a mechanical joint. The nature of the external solar load suggests a decomposition of the temperature fields in the beams leading to two heat equations for each beam. One of these fields models the circumferential average temperature and is coupled to axial motions of the beam, while the second field accounts for a temperature gradient across the beam and is coupled to beam bending. The resulting system of partial and ordinary differential equations formally describes the coupled thermomechanical behavior of the joint–beam system. The main work is in developing an appropriate state-space form and then using semigroup theory to establish well-posedness and exponential stability.

1. Introduction

Rigidizable/Inflatable (RI) materials offer the possibility of deployable large space structures [5] and so are of interest in applications where large optical or RF apertures are needed. In this paper we study the dynamics of a basic truss component consisting of two RI beams connected through a joint (see Fig. 2.1). One of the more important characteristics of such space systems is their response to changing thermal loads, as they move in/out of the Earth's shadow. In this paper we study the thermoelastic behavior of a two-beam truss element subjected to solar heating. Additionally, the beams are fabricated as thin-walled circular cylinders.

2. Thermoelastic model

The equations of motion for the Joint–Leg–Beam system depicted in Fig. 2.1 has been derived in [1] as the following:

$$\rho_i A_i \frac{\partial^2 u^i(t,s_i)}{\partial t^2} = E_i A_i \frac{\partial^2 u^i(t,s_i)}{\partial s_i^2}, \tag{2.1}$$

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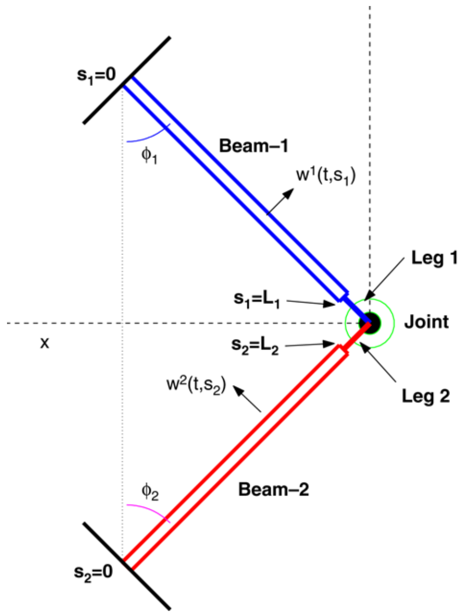

Fig. 2.1. Basic structure of the Joint–Leg–Beam system.

$$\rho_i A_i \frac{\partial^2 w^i(t,s_i)}{\partial t^2} = -E_i I_i \frac{\partial^4 w^i(t,s_i)}{\partial s_i^4}, \tag{2.2}$$

$$\mathbf{M}\frac{\mathrm{d}^2}{\mathrm{d}t}\begin{bmatrix} x(t) \\ y(t) \\ \theta_1(t) \\ \theta_2(t) \end{bmatrix} = \mathbf{C}\begin{bmatrix} M_1(t) \\ N_1(t) \\ M_2(t) \\ N_2(t) \\ F_1(t) \\ F_2(t) \end{bmatrix} \tag{2.3}$$

for time $t > 0$ and spatial variable $s_i \in [0, L_i]$, where

$$\mathbf{M} = \begin{bmatrix} m & 0 & -m_1 d_1 \cos\varphi_1 & m_2 d_2 \cos\varphi_2 \\ 0 & m & +m_1 d_1 \sin\varphi_1 & m_2 d_2 \sin\varphi_2 \\ -m_1 d_1 \cos\varphi_1 & m_1 d_1 \sin\varphi_1 & I_{1\ell} + m_1 d_1^2 & 0 \\ m_2 d_2 \cos\varphi_2 & m_2 d_2 \sin\varphi_2 & 0 & I_{2\ell} + m_2 d_2^2 \end{bmatrix}, \tag{2.4}$$

$$\mathbf{C} = \begin{bmatrix} 0 & -\cos\varphi_1 & 0 & \cos\varphi_2 & \sin\varphi_1 & \sin\varphi_2 \\ 0 & \sin\varphi_1 & 0 & \sin\varphi_2 & \cos\varphi_1 & -\cos\varphi_2 \\ 1 & \ell_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \ell_2 & 0 & 0 \end{bmatrix}, \tag{2.5}$$

and the other functions and parameters are as described below:

- $u^i(t,s_i)$, $w^i(t,s_i)$ — longitudinal and transversal displacements of the beam i, $i = 1, 2$;
- $x(t)$, $y(t)$ — horizontal and vertical displacements of the joint's tip;
- $\theta_i(t)$ — rotation angle of the leg i, $i = 1, 2$;
- ρ_i, A_i, L_i, E_i, I_i — mass density, cross sectional area, length, Young's modulus, moment of inertia of the beam i, $i = 1, 2$;
- m_i, d_i, ℓ_i, I_ℓ^i — mass, center of mass, length, moment of inertia of leg i, $i = 1, 2$;
- m_p — mass of the joint, $m = m_1 + m_2 + m_p$;
- φ_1 — initial angle of leg 1 with positive y axis;
- φ_2 — initial angle of leg 2 with negative y axis;
- $F_i(t)$ — extensional force of beam i at the end $s_i = L_i$, $i = 1, 2$;
- $N_i(t)$ — shear force of beam i at the end $s_i = L_i$, $i = 1, 2$;
- $M_i(t)$ — bending moment of beam i at the end $s_i = L_i$, $i = 1, 2$.

Each beam is clamped at the end $s_i = 0$. Thus the boundary conditions at $s_i = 0$ are

$$u^i(t,0) = w^i(t,0) = \frac{\partial w^i}{\partial s_i}(t,0) = 0, \quad i = 1, 2. \tag{2.6}$$

At the other end of each beam, we have geometric compatibility conditions that can be written in the form:

$$\begin{bmatrix} -\dfrac{\partial}{\partial s_1} w^1(t, L_1) \\ w^1(t, L_1) \\ -\dfrac{\partial}{\partial s_2} w^2(t, L_2) \\ w^2(t, L_2) \\ -u^1(t, L_1) \\ -u^2(t, L_2) \end{bmatrix} = \begin{bmatrix} \theta_1(t) \\ -x(t)\cos\varphi_1 + y(t)\sin\varphi_1 + \ell_1\theta_1(t) \\ \theta_2(t) \\ x(t)\cos\varphi_2 + y(t)\sin\varphi_2 + \ell_2\theta_2(t) \\ x(t)\sin\varphi_1 + y(t)\cos\varphi_1 \\ x(t)\sin\varphi_2 - y(t)\cos\varphi_2 \end{bmatrix} = \mathbf{C}^{\mathrm{T}} \begin{bmatrix} x(t) \\ y(t) \\ \theta_1(t) \\ \theta_2(t) \end{bmatrix}. \tag{2.7}$$

3. Thermal dynamics

Following Thornton [8], for each beam, the external heat flux in the space normal to the beam's surface is given by

$$S_i \doteq S_0 \cos\left(\xi_i - \frac{\partial w^i}{\partial s_i}\right), \tag{3.1}$$

where S_0 is the solar flux and ξ_i is the angle of orientation of the solar vector with respect to beam i. Since $\frac{\partial w^i}{\partial s_i}$ is small, it is negligible. We denote by $T^i(t, s_i, \phi_i)$ the deviation of the temperature of (thin-walled circular) beam i with respect to a reference temperature T_0^i at time t and at the point on the beam corresponding to axial coordinate s_i and circumferential coordinate ϕ_i. Then, conservation of energy for a small segment of circular cylinder including longitudinal and circumferential conduction in the cylinder wall and radiation from the cylinder's surface yields:

$$\rho_i c_i \frac{\partial T^i}{\partial t} - \frac{k_c^i}{R_i^2}\frac{\partial^2 T^i}{\partial \phi_i^2} - k_a^i \frac{\partial^2 T^i}{\partial s_i^2} + \frac{\sigma\epsilon_i}{h_i}(T_0^i + T^i)^4 = \frac{\alpha_s^i}{h_i} S_i \cos(\phi_i)\delta(\phi_i) \tag{3.2}$$

where k_a^i and k_c^i are the axial and circumferential thermal conductivity coefficients of the ith beam, respectively, c_i is the specific heat, R_i is the cylinder radius, h_i is the wall thickness, ϵ_i is the surface emissivity and α_s^i is the surface absorptivity, all of beam i, σ is the Stefan–Boltzmann constant, $\delta(\phi_i) = 1$ for $\phi_i \in (-\frac{\pi}{2}, \frac{\pi}{2})$, and $\delta(\phi_i) = 0$ for $\phi_i \in [-\pi, -\frac{\pi}{2}] \cup [\frac{\pi}{2}, \pi]$. The heat flux distribution on the RHS of Eq. (3.2) can be written as

$$S_i \cos(\phi_i)\delta(\phi_i) = S_i\left(\frac{1}{\pi} + g(\phi_i)\right) = \frac{S_i}{\pi} + S_i g(\phi_i) \tag{3.3}$$

where

$$g(\phi_i) = \begin{cases} \cos(\phi_i) - \dfrac{1}{\pi}, & \text{for } \phi_i \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] \\ -\dfrac{1}{\pi}, & \text{for } \phi_i \in \left[-\pi, -\dfrac{\pi}{2}\right) \cup \left(\dfrac{\pi}{2}, \pi\right]. \end{cases}$$

Clearly $g(\phi_i)$ is continuous and it has zero average in $[-\pi, \pi]$.

For each beam, the temperature distribution is separated into two parts, namely:

$$T^i(t, s_i, \phi_i) = T^i(t, s_i) + T^{m,i}(t, s_i)g(\phi_i), \tag{3.4}$$

where $T^i(t, s_i)$ is independent of ϕ_i and corresponds to the uniform part of the flux, $\frac{S_i}{\pi}$, in (3.3), and $T^{m,i}(t, s_i)g(\phi_i)$ accounts for the circumferential variation of the flux in (3.3). Note that for every $s_i \in [0, L_i]$, $t \geq 0$; $T^{m,i}(t, s_i) = T^i(t, s_i, 0) - T^i(t, s_i, \pi) = T^i(t, s_i, 0) - T^i(t, s_i, \phi)$ for any $\phi \in [-\pi, -\frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$. Hence, $T^{m,i}(t, s_i)$ can be thought of as the thermal gradient between the top and the bottom of the beam at the axial location s_i.

Also, we approximate the thermal radiation term $(T_0^i + T^i(t, s_i, \phi_i))^4$ in (3.2) by linearizing $T(t, s_i, \phi_i)$ around $T(t, s_i, \phi_i) = T_s^i$ (where T_s^i, to be determined later, is the steady-state constant temperature increment produced on the undeformed beam i by the solar flux S_i), i.e., we approximate $(T_0^i + T^i(t, s_i, \phi_i))^4$ by $(T_0^i + T_s^i)^4 + 4(T_0^i + T_s^i)^3 (T^i(t, s_i) - T_s^i + T^{m,i}(t, s_i)g(\phi_i))$. Hence Eq. (3.2) is replaced by

$$\begin{aligned} &\rho_i c_i \frac{\partial T^i(t, s_i)}{\partial t} + \rho_i c_i \frac{\partial T^{m,i}(t, s_i)}{\partial t} g(\phi_i) - \frac{k_c^i}{R_i^2} T^{m,i}(t, s_i) g''(\phi_i) - k_a^i \frac{\partial^2 T^i(t, s_i)}{\partial s_i^2} - k_a^i \frac{\partial^2 T^{m,i}(t, s_i)}{\partial s_i^2} g(\phi_i) \\ &\quad + \frac{\sigma\epsilon_i}{h_i}\left[(T_0^i + T_s^i)^4 + 4(T_0^i + T_s^i)^3\left(T^i(t, s_i) - T_s^i + T^{m,i}(t, s_i)g(\phi_i)\right)\right] = \frac{\alpha_s^i S_i}{h_i}\left[\frac{1}{\pi} + g(\phi_i)\right]. \end{aligned} \tag{3.5}$$

Integration of Eq. (3.5) over the cylinder's cross sectional area yields

$$\rho_i c_i \frac{\partial T^i(t, s_i)}{\partial t} - k_a^i \frac{\partial^2 T^i(t, s_i)}{\partial s_i^2} + \frac{4\sigma\epsilon_i (T_0^i + T_s^i)^3}{h_i}[T^i(t, s_i) - T_s^i] = \left[\frac{\alpha_s^i S_i}{\pi h_i} - \frac{\sigma\epsilon_i (T_0^i + T_s^i)^4}{h_i}\right] \doteq f_i. \tag{3.6}$$

Since $g'(\phi_i)$ is discontinuous at $\phi_i = \pm\frac{\pi}{2}$ the integration of $g''(\phi_i)$ above must be performed in the distributional sense. The value of T_s^i is now determined by setting the RHS, f_i, equal to zero. By doing so we obtain

$$T_s^i = \left(\frac{\alpha_s^i S_i}{\pi\sigma\epsilon_i}\right)^{\frac{1}{4}} - T_0^i. \tag{3.7}$$

Note that this value of T_s^i corresponds to the steady state $T^i(t, s_i) = T_s^i$ for the case of homogeneous Newman boundary conditions and, since usually $T^{m,i}(t, s_i)$ is small compared to T_0^i, the linearization of the thermal radiation term performed above, is justified near the steady-state solution.

Now multiplying Eq. (3.5) by $g(\phi_i)$ and integrating over the cylinder's cross sectional area, we obtain for $T^{m,i}$ the following equation:

$$\rho_i c_i \|g\|^2 \frac{\partial T^{m,i}(t, s_i)}{\partial t} - k_a^i \frac{\partial^2 T^{m,i}(t, s_i)}{\partial s_i^2}\|g\|^2 - \frac{k_c^i}{R_i^2} T^{m,i}(t, s_i)\int_{-\pi}^{\pi} g''(\phi_i)g(\phi_i)\mathrm{d}\phi_i + \frac{4\sigma\epsilon_i(T_0^i + T_s^i)^3}{h_i}\|g\|^2 T^{m,i}(t, s_i) = \frac{\alpha_s^i S_i}{h_i}\|g\|^2.$$

Since $\|g\|^2 = \int_{-\pi}^{\pi} g(\phi_i)^2\mathrm{d}\phi_i = \frac{\pi^2-4}{2\pi}$ and $\int_{-\pi}^{\pi} g''(\phi_i)g(\phi_i)\mathrm{d}\phi_i = -\frac{\pi}{2}$, the equation above reads:

$$\rho_i c_i \frac{\partial T^{m,i}(t, s_i)}{\partial t} - k_a^i \frac{\partial^2 T^{m,i}(t, s_i)}{\partial s_i^2} + \left(\frac{k_c^i \pi^2}{R_i^2(\pi^2-4)} + \frac{4\sigma\epsilon_i(T_0^i+T_s^i)^3}{h_i}\right) T^{m,i}(t, s_i) = \frac{\alpha_s^i S_i}{h_i}. \tag{3.8}$$

To consider the thermally induced vibration in the system, we use Hooke's law for the stress–strain relation in the form of

$$\epsilon_{11}^i = \frac{1}{E_i}\sigma_{11}^i + \alpha_i T^i,$$

where α_i is the thermal expansion coefficient, and T^i is, as before, the deviation from the reference temperature T_0^i. Note that at $T^i = 0$ thermal strain vanishes, so that T_0^i is interpreted as the (uniform) temperature of beam i in the unstressed, rest state. By the standard derivation of the Euler–Bernoulli beam equation, we modify the Joint–Leg–Beam system (2.1) and (2.2) as follows:

$$\rho_i A_i \frac{\partial^2 u^i(t, s_i)}{\partial t^2} = E_i A_i \frac{\partial}{\partial s_i}\left(\frac{\partial u^i(t, s_i)}{\partial s_i} - \alpha_i T^i(t, s_i)\right), \tag{3.9}$$

$$\rho_i A_i \frac{\partial^2 w^i(t, s_i)}{\partial t^2} = -E_i I_i \frac{\partial^2}{\partial s_i^2}\left(\frac{\partial^2 w^i(t, s_i)}{\partial s_i^2} + \frac{\alpha_i}{2R_i} T^{m,i}(t, s_i)\right). \tag{3.10}$$

The above beam equations are coupled to the heat equations modified from Eqs. (3.6) and (3.8) and with T_s^i chosen as in Eq. (3.7) (so that $f_i = 0$ in (3.6)), that is:

$$\rho_i c_i \frac{\partial T^i(t, s_i)}{\partial t} = k_a^i \frac{\partial^2 T^i(t, s_i)}{\partial s_i^2} - \frac{4\sigma\epsilon_i(T_0^i + T_s^i)^3}{h_i}\left(T^i(t, s_i) - T_s^i\right) - \alpha_i E_i T_0^i \frac{\partial^2}{\partial s_i \partial t} u^i(t, s_i), \tag{3.11}$$

and

$$\rho_i c_i \frac{\partial T^{m,i}(t, s_i)}{\partial t} = k_a^i \frac{\partial^2 T^{m,i}(t, s_i)}{\partial s_i^2} - \left[\frac{k_c^i \pi^2}{R_i^2(\pi^2 - 4)} + \frac{4\sigma\epsilon_i(T_0^i + T_s^i)^3}{h_i}\right] T^{m,i}(t, s_i) + \frac{\alpha_i E_i I_i T_0^i}{2R_i A_i}\frac{\partial^3}{\partial s_i^2 \partial t} w^i(t, s_i) + \frac{\alpha_s^i S_i}{h_i}. \tag{3.12}$$

We impose Robin type boundary conditions for the temperature at both ends of each beam, i.e.

$$\frac{\partial}{\partial s_i} T^i(t, s_i, \phi_i)\,|_{s_i=L_i} = \lambda_R^i\left(T^* - T_0^i - T^i(t, L_i, \phi_i)\right),$$
$$\frac{\partial}{\partial s_i} T^i(t, s_i, \phi_i)\,|_{s_i=0} = \lambda_L^i\left(T_0^i + T^i(t, 0, \phi_i) - T^*\right),$$

$\forall t \geq 0$, $\phi_i \in [-\pi, \pi]$, $i = 1, 2$, where T^* is the temperature of the surrounding medium and λ_L^i, λ_R^i, $i = 1, 2$, are nonnegative constants. By writing $T^i(t, s_i, \phi_i)$ in terms of the decomposition given in (3.4) these equations take the form:

$$\frac{\partial}{\partial s_i} T^i(t, L_i) + \frac{\partial}{\partial s_i} T^{m,i}(t, L_i)g(\phi_i) = \lambda_R^i\left(T^* - T_0^i - T^i(t, L_i) - T^{m,i}(t, L_i)g(\phi_i)\right),$$
$$\frac{\partial}{\partial s_i} T^i(t, 0) + \frac{\partial}{\partial s_i} T^{m,i}(t, 0)g(\phi_i) = \lambda_L^i\left(T_0^i + T^i(t, 0) + T^{m,i}(t, 0)g(\phi_i) - T^*\right).$$

Since these equations must hold for all $\phi_i \in [-\pi, \pi]$ it follows that

$$\frac{\partial}{\partial s_i} T^i(t, L_i) = \lambda_R^i\left(T^* - T_0^i - T^i(t, L_i)\right),$$

$$\frac{\partial}{\partial s_i} T^i(t, 0) = \lambda_L^i\left(T_0^i + T^i(t, 0) - T^*\right) \tag{3.13}$$

and

$$\frac{\partial}{\partial s_i}T^{m,i}(t,L_i) = -\lambda_R^i T^{m,i}(t,L_i),$$
$$\frac{\partial}{\partial s_i}T^{m,i}(t,0) = \lambda_L^i T^{m,i}(t,0), \tag{3.14}$$

for all $t \geq 0$, $i = 1, 2$. So, just like the dynamics for the temperature distribution (3.5) decoupling into Eqs. (3.11) and (3.12) for T^i and $T^{m,i}$, respectively, we observe that the boundary conditions also decouple. Note however in Eq. (3.12) that the boundary conditions for the axial component of the temperature, $T^i(t,s_i)$, are nonhomogeneous. By defining

$$\tilde{T}^i(t,s_i) \doteq T^i(t,s_i) - (T^* - T_0^i). \tag{3.15}$$

Eq. (3.11) can be written in the form

$$\rho_i c_i \frac{\partial \tilde{T}^i(t,s_i)}{\partial t} = k_a^i \frac{\partial^2 \tilde{T}^i(t,s_i)}{\partial s_i^2} - \frac{4\sigma\epsilon_i (T_0^i + T_s^i)^3}{h_i}\left(\tilde{T}^i(t,s_i) + T^* - T_0^i - T_s^i\right) - \alpha_i E_i T_0^i \frac{\partial^2}{\partial s_i \partial t}u^i(t,s_i), \tag{3.16}$$

while the boundary conditions (3.12) now take the form

$$\frac{\partial}{\partial s_i}\tilde{T}^i(t,L_i) = -\lambda_R^i \tilde{T}^i(t,L_i),$$
$$\frac{\partial}{\partial s_i}\tilde{T}^i(t,0) = \lambda_L^i \tilde{T}^i(t,0). \tag{3.17}$$

Observe now that these boundary conditions are exactly the same as those in (3.14) for the circumferential component of the temperature. Finally, note also that in Eq. (3.9), $T^i(t,s_i)$ can be replaced by $\tilde{T}^i(t,s_i)$ without any changes.

System (3.9)–(3.12) (or equivalently (3.9), (3.10), (3.12) and (3.16)), together with the joint–leg dynamics described by Eq. (2.3) constitute the thermoelastic Joint–Leg–Beam equations with the external solar heat source. The extensional forces, shear forces and bending moments of the beams at $s_i = L_i$ are now given by:

$$F_i(t) = E_i A_i \left(\frac{\partial u^i}{\partial s_i}(t,s_i) - \alpha_i T^i(t,s_i)\right)\Bigg|_{s_i=L_i} \tag{3.18}$$

$$N_i(t) = E_i I_i \frac{\partial}{\partial s_i}\left(\frac{\partial^2 w^i}{\partial s_i^2}(t,s_i) + \frac{\alpha_i}{2R_i}T^{m,i}(t,s_i)\right)\Bigg|_{s_i=L_i}, \tag{3.19}$$

$$M_i(t) = E_i I_i \left(\frac{\partial^2 w^i}{\partial s_i^2}(t,s_i) + \frac{\alpha_i}{2R_i}T^{m,i}(t,s_i)\right)\Bigg|_{s_i=L_i}. \tag{3.20}$$

4. Well-posedness

In this section, we consider the well-posedness of the Joint–Leg–Beam system with solar heat flux, i.e., Eqs. (3.9), (3.10), (3.12) and (3.16) subject to the geometric beam–leg interface compatibility conditions (2.7), the dynamic boundary conditions (3.18)–(3.20) and the boundary conditions (2.6), (3.14) and (3.17). We first rewrite the system as a first-order evolution equation in an appropriate Hilbert space. Well-posedness is then obtained by using semigroup theory. Since the corresponding system without thermal effects has been studied in [1], we will follow the notation used there as much as possible for consistency. Numerical results for that case have been reported in [2].

First, we define the following Hilbert spaces with their corresponding inner products:

$$\begin{cases} \mathcal{H}_z = L^2(0,L_1)\times L^2(0,L_2)\times L^2(0,L_1)\times L^2(0,L_2), \\ \langle z_1, z_2\rangle_{\mathcal{H}_z} \doteq \displaystyle\sum_{i=1}^{2} \rho_i A_i \left[\langle w_1^i, w_2^i\rangle + \langle u_1^i, u_2^i\rangle\right]; \end{cases}$$

$$\begin{cases} \mathcal{H}_b = [\ker(C)]^\perp = \mathrm{range}(C^{\mathrm{T}}), \\ \langle b_1, b_2\rangle_{\mathcal{H}_b} = \langle b_1, (C^{\mathrm{T}}M^{-1}C)^\dagger b_2\rangle_{\mathbb{R}^6}; \end{cases}$$

$$\begin{cases} \mathcal{H}_\zeta = L^2(0,L_1)\times L^2(0,L_2)\times L^2(0,L_1)\times L^2(0,L_2), \\ \langle \zeta_1, \zeta_2\rangle_{\mathcal{H}_\zeta} \doteq \displaystyle\sum_{i=1}^{2} \frac{\rho_i c_i A_i}{T_0^i}\left[\langle T_1^{m,i}, T_2^{m,i}\rangle + \langle \tilde{T}_1^i, \tilde{T}_2^i\rangle\right]; \end{cases}$$

where $z_j \doteq \left(w_j^1, w_j^2, u_j^1, u_j^2\right)^{\mathrm{T}}$, $\zeta_j \doteq \left(T_j^{m,1}, T_j^{m,2}, \tilde{T}_j^1, \tilde{T}_j^2\right)^{\mathrm{T}}$, and $(C^{\mathrm{T}}M^{-1}C)^\dagger$ denotes the Moore–Penrose generalized inverse of $C^{\mathrm{T}}M^{-1}C$.

We also define the operators $\mathcal{A}_z : \mathcal{H}_z \to \mathcal{H}_z$ and $\mathcal{B}_z : \mathcal{H}_\zeta \to \mathcal{H}_z$ by

$$\mathrm{dom}(\mathcal{A}_z) \doteq H^2_\ell \cap H^4(0, L_1) \times H^2_\ell \cap H^4(0, L_2) \times H^1_\ell \cap H^2(0, L_1) \times H^1_\ell \cap H^2(0, L_2),$$

$$\mathcal{A}_z \doteq \begin{pmatrix} \frac{E_1 I_1}{\rho_1 A_1} D^4 & 0 & 0 & 0 \\ 0 & \frac{E_2 I_2}{\rho_2 A_2} D^4 & 0 & 0 \\ 0 & 0 & -\frac{E_1}{\rho_1} D^2 & 0 \\ 0 & 0 & 0 & -\frac{E_2}{\rho_2} D^2 \end{pmatrix},$$

$$\mathrm{dom}(\mathcal{B}_z) \doteq H^2(0, L_1) \times H^2(0, L_2) \times H^1(0, L_1) \times H^1(0, L_2),$$

$$\mathcal{B}_z \doteq \begin{pmatrix} -\frac{\alpha_1 E_1 I_1}{2R_1 \rho_1 A_1} D^2 & 0 & 0 & 0 \\ 0 & -\frac{\alpha_2 E_2 I_2}{2R_2 \rho_2 A_2} D^2 & 0 & 0 \\ 0 & 0 & -\frac{\alpha_1 E_1}{\rho_1} D & 0 \\ 0 & 0 & 0 & -\frac{\alpha_2 E_2}{\rho_2} D \end{pmatrix}$$

where $D^n \doteq \frac{\mathrm{d}^n}{\mathrm{d}s_i^n}$ and for $n \in \mathbb{N}$, $H^n_\ell(0, L)$ denotes the space of functions in $H^n(0, L)$ that vanish, together with all derivatives up to the order $n-1$, at the left boundary. With this notation, Eqs. (3.9) and (3.10) can now be written as the following abstract second-order ODE in $\mathcal{H}_z$:

$$\ddot{z}(t) + \mathcal{A}_z z(t) - \mathcal{B}_z \zeta(t) = 0. \tag{4.1}$$

Next we define the operators $\mathcal{A}_\zeta : \mathcal{H}_\zeta \to \mathcal{H}_\zeta$ and $\mathcal{B}_\zeta : \mathcal{H}_z \to \mathcal{H}_\zeta$ by

$$\mathrm{dom}(\mathcal{A}_\zeta) \doteq H^2_{rb}(0, L_1) \times H^2_{rb}(0, L_2) \times H^2_{rb}(0, L_1) \times H^2_{rb}(0, L_2),$$

$$\mathcal{A}_\zeta \zeta = \mathcal{A}_\zeta \begin{pmatrix} T^{m,1} \\ T^{m,2} \\ \tilde{T}^1 \\ \tilde{T}^2 \end{pmatrix} \doteq \begin{pmatrix} -\frac{k_a^1}{\rho_1 c_1} D^2 T^{m,1} + \left[\frac{k_c^1 \pi^2}{\rho_1 c_1 R_1^2 (\pi^2 - 4)} + \frac{4\sigma\epsilon_1 (T_0^1 + T_s^1)^3}{\rho_1 c_1 h_1} \right] T^{m,1} \\ -\frac{k_a^2}{\rho_2 c_2} D^2 T^{m,2} + \left[\frac{k_c^2 \pi^2}{\rho_2 c_2 R_2^2 (\pi^2 - 4)} + \frac{4\sigma\epsilon_2 (T_0^2 + T_s^2)^3}{\rho_2 c_2 h_2} \right] T^{m,2} \\ -\frac{k_a^1}{\rho_1 c_1} D^2 \tilde{T}^1 + \frac{4\sigma\epsilon_1 (T_0^1 + T_s^1)^3}{\rho_1 c_1 h_1} \tilde{T}^1 \\ -\frac{k_a^2}{\rho_2 c_2} D^2 \tilde{T}^2 + \frac{4\sigma\epsilon_2 (T_0^2 + T_s^2)^3}{\rho_2 c_2 h_2} \tilde{T}^2 \end{pmatrix},$$

$$\mathrm{dom}(\mathcal{B}_\zeta) \doteq H^2(0, L_1) \times H^2(0, L_2) \times H^1(0, L_1) \times H^1(0, L_2),$$

$$\mathcal{B}_\zeta z \doteq \begin{pmatrix} \frac{\alpha_1 E_1 I_1 T_0^1}{2R_1 \rho_1 c_1 A_1} D^2 & 0 & 0 & 0 \\ 0 & \frac{\alpha_2 E_2 I_2 T_0^2}{2R_2 \rho_2 c_2 A_2} D^2 & 0 & 0 \\ 0 & 0 & -\frac{\alpha_1 E_1 T_0^1}{\rho_1 c_1} D & 0 \\ 0 & 0 & 0 & -\frac{\alpha_2 E_2 T_0^2}{\rho_2 c_2} D \end{pmatrix}$$

where $H^2_{rb}(0, L)$ denotes the space of functions in $H^2(0, L)$ satisfying the Robin boundary conditions (3.14) or equivalently (3.17). With this notation, Eqs. (3.12) and (3.16), can now be written as the following abstract first-order ODE in $\mathcal{H}_\zeta$:

$$\dot{\zeta}(t) - \mathcal{B}_\zeta \dot{z}(t) + \mathcal{A}_\zeta \zeta(t) = S \tag{4.2}$$

where

$$S \doteq \left(\frac{\alpha_s^1}{\rho_1 c_1 h_1} S_1, \quad \frac{\alpha_s^2}{\rho_2 c_2 h_2} S_2, \quad \frac{4\sigma\epsilon_1 (T_0^1 + T_s^1)^3}{\rho_1 c_1 h_1} (T_s^1 + T_0^1 - T^*), \quad \frac{4\sigma\epsilon_2 (T_0^2 + T_s^2)^3}{\rho_2 c_2 h_2} (T_s^2 + T_0^2 - T^*) \right)^{\mathrm{T}}.$$

We also define three boundary projection operators P_1^B, P_2^B from $\mathcal{H}_z$ into $\mathbb{R}^6$ and P_3^B from $\mathcal{H}_\zeta$ into $\mathbb{R}^6$ by

$$\mathrm{dom}(P_1^B) \doteq H^2(0, L_1) \times H^2(0, L_2) \times H^1(0, L_1) \times H^1(0, L_2),$$

$$\mathrm{dom}(P_2^B) \doteq H^4(0,L_1)\times H^4(0,L_2)\times H^2(0,L_1)\times H^2(0,L_2),$$
$$\mathrm{dom}(P_3^B) \doteq H^2(0,L_1)\times H^2(0,L_2)\times H^1(0,L_1)\times H^1(0,L_2),$$

$$P_1^B\begin{pmatrix} w^1\\ w^2\\ u^1\\ u^2\end{pmatrix} \doteq \begin{pmatrix} -\frac{\partial}{\partial s_1}w^1(L_1)\\ w^1(L_1)\\ -\frac{\partial}{\partial s_2}w^2(L_2)\\ w_2(L_2)\\ -u^1(L_1)\\ -u^2(L_2)\end{pmatrix}, \qquad P_2^B\begin{pmatrix} w^1\\ w^2\\ u^1\\ u^2\end{pmatrix} \doteq \begin{pmatrix} \frac{\partial^2}{\partial s_1^2}w^1(L_1)\\ \frac{\partial^3}{\partial s_1^3}w^1(L_1)\\ \frac{\partial^2}{\partial s_2^2}w^2(L_2)\\ \frac{\partial^3}{\partial s_2^3}w^2(L_2)\\ \frac{\partial}{\partial s_1}u^1(L_1)\\ \frac{\partial}{\partial s_2}u_2(L_2)\end{pmatrix},$$

$$P_3^B\begin{pmatrix} T^{m,1}\\ T^{m,2}\\ \tilde{T}^1\\ \tilde{T}^2\end{pmatrix} \doteq \begin{pmatrix} T^{m,1}(L_1)\\ \frac{\partial}{\partial s_1}T^{m,1}(L_1)\\ T^{m,2}(L_2)\\ \frac{\partial}{\partial s_2}T^{m,2}(L_2)\\ \tilde{T}^1(L_1)\\ \tilde{T}^2(L_2)\end{pmatrix}.$$

Now, by using the geometric compatibility conditions (2.7) and the dynamic boundary conditions (3.18)–(3.20), the equation for the leg–joint dynamics (2.3) can be written as the following abstract second-order ODE in $\mathcal{H}_b$:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(P_1^B z(t)\right) - C^{\mathrm{T}}M^{-1}CE\left(P_2^B z(t) + \Lambda P_3^B\zeta(t)\right) = \tilde{R} \tag{4.3}$$

where

$$E \doteq \mathrm{diag}\,(E_1I_1, E_1I_1, E_2I_2, E_2I_2, E_1A_1, E_2A_2)\,,$$
$$\Lambda \doteq \mathrm{diag}\left(\frac{\alpha_1}{2R_1}, \frac{\alpha_1}{2R_1}, \frac{\alpha_2}{2R_2}, \frac{\alpha_2}{2R_2}, -\alpha_1, -\alpha_2\right),$$
$$\tilde{R} \doteq C^{\mathrm{T}}M^{-1}C\left(0,0,0,0,E_1A_1\alpha_1(T^*-T_0^1), E_2A_2\alpha_2(T^*-T_0^2)\right)^{\mathrm{T}}.$$

Next we define the Hilbert space $\mathcal{H}_{zb} \doteq \mathcal{H}_z \times \mathcal{H}_b$ with the usual inner product inherited from those in $\mathcal{H}_z$ and $\mathcal{H}_b$. In this Hilbert space we define the elastic operator $\mathcal{A}_{zb}$ by

$$\mathrm{dom}(\mathcal{A}_{zb}) \doteq \left\{\begin{pmatrix} z\\ b\end{pmatrix} \in \mathrm{dom}(\mathcal{A}_z)\times\mathcal{H}_b : P_1^B z = b\right\},$$

and

$$\mathcal{A}_{zb}\begin{pmatrix} z\\ b\end{pmatrix} \doteq \begin{pmatrix} \mathcal{A}_z z\\ -C^{\mathrm{T}}M^{-1}CEP_2^B z\end{pmatrix}.$$

Furthermore, we define $B_{zb} : \mathcal{H}_\zeta \to \mathcal{H}_{zb}$ by

$$\mathrm{dom}(\mathcal{B}_{zb}) \doteq H^2(0,L_1)\times H^2(0,L_2)\times H^1(0,L_1)\times H^1(0,L_2)$$

and

$$\mathcal{B}_{zb}\zeta \doteq \begin{pmatrix} \mathcal{B}_z\zeta\\ C^{\mathrm{T}}M^{-1}CE\Lambda P_3^B\zeta\end{pmatrix}.$$

Thus, Eqs. (4.1) and (4.3) can be combined as

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\begin{pmatrix} z(t)\\ b(t)\end{pmatrix} + \mathcal{A}_{zb}\begin{pmatrix} z(t)\\ b(t)\end{pmatrix} - \mathcal{B}_{zb}\zeta(t) = R \quad \text{on } \mathcal{H}_{zb}, \tag{4.4}$$

where $R \doteq (0,\tilde{R})^{\mathrm{T}}$. It has been proved in [1] that the operator $\mathcal{A}_{zb}$ is self-adjoint and strictly positive. Thus, we can define the state space

$$\mathcal{H} \doteq \mathrm{dom}(\mathcal{A}_{zb}^{1/2})\times\mathcal{H}_{zb}\times\mathcal{H}_\zeta$$

with the inner product

$$\left\langle \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}, \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \right\rangle_{\mathcal{H}} \doteq \langle \mathcal{A}_{zb}^{1/2} X_1, \mathcal{A}_{zb}^{1/2} Y_1 \rangle_{\mathcal{H}_{zb}} + \langle X_2, Y_2 \rangle_{\mathcal{H}_{zb}} + \langle X_3, Y_3 \rangle_{\mathcal{H}_\zeta}.$$

Finally, we define operator $\mathcal{A}$ on $\mathcal{H}$ by

$$\mathrm{dom}(\mathcal{A}) \doteq \left\{ \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \in \mathcal{H} | X_1 \in \mathrm{dom}(\mathcal{A}_{zb}), X_2 \in \mathrm{dom}(\mathcal{A}_{zb}^{1/2}), X_3 \in \mathrm{dom}(\mathcal{A}_\zeta) \right\},$$

$$\mathcal{A} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \doteq \begin{pmatrix} 0 & I & 0 \\ -\mathcal{A}_{zb} & 0 & \mathcal{B}_{zb} \\ 0 & (\mathcal{B}_\zeta, 0) & -\mathcal{A}_\zeta \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}. \tag{4.5}$$

Then, Eqs. (4.2) and (4.4) can be rewritten as a first-order nonhomogeneous evolution equation

$$\dot{X}(t) = \mathcal{A}X(t) + G \quad \text{on } \mathcal{H} \tag{4.6}$$

where

$$X \doteq \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}, \qquad X_1 \doteq \begin{pmatrix} z \\ b \end{pmatrix}, \qquad X_2 \doteq \dot{X}_1, \qquad X_3 \doteq \zeta \quad \text{and} \qquad G \doteq \begin{pmatrix} 0 \\ R \\ S \end{pmatrix}.$$

Theorem 4.1 (*Well-posedness*). *Let $\mathcal{A} : \mathcal{H} \to \mathcal{H}$ be as defined above. Then $\mathcal{A}$ is the infinitesimal generator of a strongly continuous semigroup of contractions $S(t)$ on $\mathcal{H}$ and hence, for any initial condition $X_0 = X(0) \in \mathrm{dom}(\mathcal{A})$, system (4.6) has a unique global solution $X(t)$ given by*

$$X(t) = S(t)X_0 + \int_0^t S(t-s)G\mathrm{d}s.$$

Proof. It follows from integration by parts that

$$\begin{aligned} \langle \mathcal{B}_{zb} X_3, X_2 \rangle_{\mathcal{H}_{zb}} &= \langle \mathcal{B}_z \zeta, \dot{z} \rangle_{\mathcal{H}_z} + \langle C^{\mathrm{T}} M^{-1} C E \Lambda P_3^B \zeta, \dot{b} \rangle_{\mathcal{H}_b} \\ &= -\langle \zeta, \mathcal{B}_\zeta \dot{z} \rangle_{\mathcal{H}_\zeta} - \langle E \Lambda P_3^B \zeta, P_1^B \dot{z} \rangle_{\mathbb{R}^6} + \langle C^{\mathrm{T}} M^{-1} C E \Lambda P_3^B \zeta, \dot{b} \rangle_{\mathcal{H}_b} \\ &= -\langle \zeta, \mathcal{B}_\zeta \dot{z} \rangle_{\mathcal{H}_\zeta} - \langle E \Lambda P_3^B \zeta, \dot{b} \rangle_{\mathbb{R}^6} + \langle C^{\mathrm{T}} M^{-1} C E \Lambda P_3^B \zeta, (C^{\mathrm{T}} M^{-1} C)^\dagger \dot{b} \rangle_{\mathbb{R}^6} \\ &= -\langle \zeta, \mathcal{B}_\zeta \dot{z} \rangle_{\mathcal{H}_\zeta}. \end{aligned}$$

Here we have used the facts $P_1^B z = b$ and $(C^{\mathrm{T}} M^{-1} C)(C^{\mathrm{T}} M^{-1} C)^\dagger$ is the orthogonal projection of $\mathbb{R}^6$ onto $\mathcal{H}_b$ and $C^{\mathrm{T}} M^{-1} C$ is symmetric. Therefore, we have

$$\begin{aligned} \langle \mathcal{A}X, X \rangle_{\mathcal{H}} &= \langle \mathcal{A}_{zb}^{1/2} X_2, \mathcal{A}_{zb}^{1/2} X_1 \rangle_{\mathcal{H}_{zb}} + \langle -\mathcal{A}_{zb} X_1 + \mathcal{B}_{zb} X_3, X_2 \rangle_{\mathcal{H}_{zb}} + \langle \mathcal{B}_\zeta \dot{z} - \mathcal{A}_\zeta X_3, X_3 \rangle_{\mathcal{H}_\zeta} \\ &= -\langle \mathcal{A}_\zeta X_3, X_3 \rangle_{\mathcal{H}_\zeta} \\ &= -\sum_{i=1}^{2} \frac{A_i}{T_0^i} \left[k_a^i \left(\left\| \frac{\partial T^{m,i}}{\partial s_i} \right\|^2 + \left\| \frac{\partial \tilde{T}^i}{\partial s_i} \right\|^2 + \lambda_L^i \left(T^{m,i}(0)^2 + \tilde{T}^i(0)^2 \right) + \lambda_R^i \left(T^{m,i}(L_i)^2 + \tilde{T}^i(L_i)^2 \right) \right) \right. \\ &\quad \left. + \frac{4\sigma\epsilon_i (T_0^i + T_s^i)^3}{h_i} \left(\|T^{m,i}\|^2 + \|\tilde{T}^i\|^2 \right) + \frac{k_c^i \pi^2}{R_i^2 (\pi^2 - 4)} \|T^{m,i}\|^2 \right] \\ &\le 0, \end{aligned} \tag{4.7}$$

which implies that $\mathcal{A}$ is dissipative.

We will prove now that $0 \in \rho(\mathcal{A})$. Consider the equation

$$\mathcal{A}X = f, \quad \forall f \in \mathcal{H} \tag{4.8}$$

where $f = (f_1, f_2, f_3)^{\mathrm{T}}$, i.e.,

$$X_2 = f_1 \in \mathrm{dom}(\mathcal{A}_{zb}^{1/2}), \tag{4.9}$$

$$-\mathcal{A}_{zb} X_1 + \mathcal{B}_{zb} X_3 = f_2 \in \mathcal{H}_{zb}, \tag{4.10}$$

$$(\mathcal{B}_\zeta, 0) X_2 - \mathcal{A}_\zeta X_3 = f_3 \in \mathcal{H}_\zeta. \tag{4.11}$$

Substituting (4.9) into (4.11), we get

$$\mathcal{A}_\zeta X_3 = -f_3 - (\mathcal{B}_\zeta, 0)f_1 \in \mathcal{H}_\zeta.$$

By the standard elliptic PDE theory, we have a unique solution

$$X_3 = -\mathcal{A}_\zeta^{-1}(f_3 + (\mathcal{B}_\zeta, 0)f_1) \in \mathrm{dom}(\mathcal{A}_\zeta). \tag{4.12}$$

Thus, Eq. (4.10) becomes

$$\mathcal{A}_{zb}X_1 = -f_2 + \mathcal{B}_{zb}(f_3 + (\mathcal{B}_\zeta, 0)f_1) \in \mathcal{H}_{zb}. \tag{4.13}$$

The fact that this equation does have a unique solution $X_1 \in \mathrm{dom}(\mathcal{A}_{zb})$ follows from the well-posedness of the operator $\begin{pmatrix} 0 & I \\ -\mathcal{A}_{zb} & 0 \end{pmatrix}$ on $\mathcal{H}_{zb}$, which has been proved in [1]. Thus, $0 \in \rho(\mathcal{A})$. Since $\mathrm{dom}(\mathcal{A})$ is dense in $\mathcal{H}$, it then follows from Theorem 1.2.4 in [6] that $\mathcal{A}$ generates a strongly continuous semigroup of contractions $S(t)$ on $\mathcal{H}$. The existence and uniqueness of solutions for system (4.6) for any initial condition $X_0 = X(0) \in \mathrm{dom}(\mathcal{A})$ finally follows from Corollary 2.10 in [7]. ∎

5. Exponential stability

We now turn our attention to the stability of system (4.6). It is well known that the semigroup associated with longitudinal and transversal motion of a thermoelastic Euler beam is exponentially stable [3,6]. System (4.6) consists of two thermoelastic beam equations plus the equations for the joint–leg dynamics. This type of system is often referred to as a "*hybrid system*". It is certainly an interesting problem to determine whether the thermal damping is strong enough by itself to induce exponential stability of this kind of system. We shall prove this in the affirmative.

The following result by Huang [4] will be used:

Theorem 5.1. *Let H be a Hilbert space, $A : H \to H$ a closed, densely defined linear operator. Assume that A generates a C_0-semigroup of contractions $T(t)$ on H. Then $T(t)$ is exponentially stable if and only if*

$$i\mathbb{R} \cap \sigma(A) = \emptyset, \tag{5.1}$$

$$\lim_{\beta\to\infty} \|(\boldsymbol{i}\beta - A)^{-1}\| < \infty. \tag{5.2}$$

Theorem 5.2. *The C_0-semigroup of contractions $S(t)$ generated by $\mathcal{A}$ (see Theorem 4.1) is exponentially stable.*

Proof. We will first verify the condition (5.2). Suppose that (5.2) is false. Then there exist a sequence $\{\beta_n\} \subset \mathbb{R}$ with $\beta_n \to \infty$ and a sequence $\{X_n\} \subset D(\mathcal{A})$ with $\|X_n\|_{\mathcal{H}} = 1\ \forall n$ such that

$$\lim_{n\to\infty} \|(\boldsymbol{i}\beta_n - \mathcal{A})X_n\|_{\mathcal{H}} = 0. \tag{5.3}$$

Our goal is to show that (5.3) will yield the contradiction $\|X_n\|_{\mathcal{H}} \to 0$ as $n \to \infty$. For simplicity of notation we shall hereafter omit the subscript "n". Since

$$|\mathrm{Re}\langle\mathcal{A}X, X\rangle_{\mathcal{H}}| = |\mathrm{Re}\langle(\boldsymbol{i}\beta - \mathcal{A})X, X\rangle_{\mathcal{H}}| \le \|(\boldsymbol{i}\beta - \mathcal{A})X\|_{\mathcal{H}},$$

the dissipativeness of $\mathcal{A}$ proved in Theorem 4.1 (namely Eq. (4.7)) and (5.3) lead to

$$\|T^{m,i}\|_{H^1(0,L_i)}, \|\tilde{T}^i\|_{H^1(0,L_i)}, |T^{m,i}(0)|, |\tilde{T}^i(0)|, |T^{m,i}(L_i)|, |\tilde{T}^i(L_i)| \to 0. \tag{5.4}$$

The components of (5.3) related to the thermoelastic beam equations yield

$$\boldsymbol{i}\beta u^i - v^i \to 0 \quad \text{in } H^1_\ell(0, L_i), \tag{5.5}$$

$$\boldsymbol{i}\beta v^i - \gamma_1^i D^2u^i + \gamma_2^i D\tilde{T}^i \to 0 \quad \text{in } L^2(0, L_i), \tag{5.6}$$

$$\boldsymbol{i}\beta\tilde{T}^i + \gamma_3^i Dv^i - \gamma_4^i D^2\tilde{T}^i + \gamma_5^i\tilde{T}^i \to 0 \quad \text{in } L^2(0, L_i), \tag{5.7}$$

$$\boldsymbol{i}\beta w^i - y^i \to 0 \quad \text{in } H^2_\ell(0, L_i), \tag{5.8}$$

$$\boldsymbol{i}\beta y^i + \gamma_6^i D^4w^i + \gamma_7^i D^2T^{m,i} \to 0 \quad \text{in } L^2(0, L_i), \tag{5.9}$$

$$\boldsymbol{i}\beta T^{m,i} - \gamma_8^i D^2y^i - \gamma_9^i D^2T^{m,i} + \gamma_{10}^i T^{m,i} \to 0 \quad \text{in } L^2(0, L_i), \tag{5.10}$$

for $i = 1, 2$, where $\gamma_j^i, j = 1, \ldots, 10$ are positive coefficients depending on the physical parameters, more precisely, for $i = 1, 2$, $\gamma_1^i = \frac{E_i}{\rho_i}$, $\gamma_2^i = \frac{\alpha_i E_i}{\rho_i}$, $\gamma_3^i = \frac{\alpha_i E_i T_0^i}{\rho_i c_i}$, $\gamma_4^i = \frac{k_a^i}{\rho_i c_i}$, $\gamma_5^i = \frac{4\sigma\epsilon_i(T_0^i + T_s^i)^3}{\rho_i c_i h_i}$, $\gamma_6^i = \frac{E_i I_i}{\rho_i A_i}$, $\gamma_7^i = \frac{\alpha_i E_i I_i}{2R_i\rho_i A_i}$, $\gamma_8^i = \frac{\alpha_i E_i I_i T_0^i}{2R_i\rho_i c_i A_i}$, $\gamma_9^i = \frac{k_a^i}{\rho_i c_i}$, $\gamma_{10}^i = \frac{k_c^i\pi^2}{\rho_i c_i R_i^2(\pi^2-4)} + \frac{4\sigma\epsilon_i(T_0^i + T_s^i)^3}{\rho_i c_i h_i}$, and v^i, y^i are used to denote $\frac{\partial}{\partial t}u^i$ and $\frac{\partial}{\partial t}w^i$, respectively.

Let us deal first with the longitudinal motions in equations (5.5)–(5.7). Due to (5.5) we can replace the term Dv^i in (5.7) by $\boldsymbol{i}\beta Du^i$ and further remove the term $\tilde{T}^i$ since by (5.4) it converges to zero in $L^2(0, L_i)$. Then, divide the resulting equation by β and again remove the term $\tilde{T}^i$. We obtain

$$\boldsymbol{i}\gamma_3^i Du^i - \frac{\gamma_4^i}{\beta} D^2\tilde{T}^i \to 0 \quad \text{in } L^2(0, L_i).$$

Taking now the inner product in $L^2(0, L_i)$ of this expression with Du^i and integrating by parts we obtain

$$\boldsymbol{i}\gamma_3^i \|Du^i\|^2_{L^2(0,L_i)} - \gamma_4^i D\tilde{T}^i(s_i) \frac{Du^i(s_i)}{\beta}\bigg|_{s_i=0}^{s_i=L^i} + \frac{\gamma_4^i}{\beta}\langle D\tilde{T}^i, D^2u^i\rangle \to 0. \tag{5.11}$$

Dividing (5.6) by β and recalling that by (5.4) $D\tilde{T}^i \to 0$ in $L^2(0, L_i)$ and that v^i and Du^i are both bounded in $L^2(0, L_i)$ (since they both appear in $\|X\|^2_{\mathcal{H}}$), it follows from (5.6) that $\|Du^i/\beta\|_{H^1(0,L_i)}$ is bounded. Hence, the third term in (5.11) converges to zero. It can be shown that the second term in (5.11) also converges to zero. In fact, from the inequality

$$\frac{|Du^i(s)|}{\beta} \le C\|Du^i\|^{\frac{1}{2}}_{L_2(0,L_i)} \frac{\|Du^i\|^{\frac{1}{2}}_{H^1(0,L_i)}}{\beta}, \quad \text{for } s = 0, L_i, \tag{5.12}$$

we get the boundedness of $|Du^i(s)|/\beta$ on both boundaries. Since $\tilde{T}^i$ satisfies the Robin boundary condition (3.17), Eq. (5.4) also implies that $D\tilde{T}^i$ converges to zero at both boundaries. Eq. (5.11) is then reduced to

$$\|Du^i\|_{L^2(0,L_i)} \to 0. \tag{5.13}$$

Going back to (5.12), Eq. (5.13) yields

$$\frac{|Du^i(0)|}{\beta}, \frac{|Du^i(L_i)|}{\beta} \to 0. \tag{5.14}$$

Now by (5.4), $D\tilde{T}^i \to 0$ in $L^2(0, L_i)$. This fact, together with Eq. (5.6) implies that

$$\boldsymbol{i}\beta v^i - \gamma_1^i D^2u^i \to 0 \quad \text{in } L^2(0, L_i).$$

Taking the inner product in $L^2(0, L_i)$ of the above expression with u^i, using (5.5) to replace $\boldsymbol{i}\beta u^i$ by v^i and integrating by parts we obtain

$$-\|v^i\|^2_{L^2(0,L_i)} + \gamma_1^i\|Du^i\|^2_{L^2(0,L_i)} - \gamma_1^i Du^i(L_i)\overline{u}^i(L_i) \to 0.$$

The second term above tends to zero by (5.13) and so also the third one by virtue of the inequality

$$|Du^i(L_i)u^i(L_i)| \le C\|Du^i\|^{\frac{1}{2}}_{L^2(0,L_i)} \left(\frac{\|Du^i\|_{H^1(0,L_i)}}{\beta}\right)^{\frac{1}{2}} \|Du^i\|^{\frac{1}{2}}_{L^2(0,L_i)}\|\beta u^i\|^{\frac{1}{2}}_{L^2(0,L_i)} \to 0.$$

We therefore conclude that

$$\|v^i\|_{L^2(0,L_i)} \to 0. \tag{5.15}$$

The transversal motions in Eqs. (5.8)–(5.10) can be treated in a similar way. Using (5.8) to replace the term D^2y^i in (5.10) by $\boldsymbol{i}\beta D^2w^i$, dividing through by β and using the fact that by (5.4) $T^{m,i} \to 0$ in $L^2(0, L_i)$, we obtain

$$\boldsymbol{i}\gamma_8^i D^2w^i + \frac{\gamma_9^i}{\beta} D^2T^{m,i} \to 0 \quad \text{in } L^2(0, L_i). \tag{5.16}$$

Taking the inner product in $L^2(0, L_i)$ of the above expression with D^2w^i and integrating by parts we obtain

$$\boldsymbol{i}\gamma_8^i\|D^2w^i\|_{L^2(0,L_i)} + \gamma_9^i DT^{m,i}(s_i)\frac{D^2w^i(s_i)}{\beta}\bigg|_{s_i=0}^{s_i=L_i} - \frac{\gamma_9^i}{\beta}\langle DT^{m,i}, D^3w^i\rangle \to 0. \tag{5.17}$$

It can be seen from (5.9) that $\|D^4w^i/\beta\|_{L^2(0,L_i)}$ is bounded. By interpolation, $\frac{\|D^2w^i\|_{H^1(0,L_i)}}{\beta^{1/2}}$ is also bounded. Hence, the third term in the last equation converges to zero, since by (5.4) $DT^{m,i} \to 0$ in $L^2(0, L_i)$. Now from the inequality

$$\frac{|D^2w^i(s)|}{\beta} \le C\|D^2w^i\|^{\frac{1}{2}}_{L^2(0,L_i)} \frac{\|D^2w^i\|^{\frac{1}{2}}_{H^1(0,L_i)}}{\beta} \quad \text{for } s = 0, L_i, \tag{5.18}$$

it follows that $\frac{|D^2w^i(s)|}{\beta}$ is bounded at both boundaries. Also by (5.4), $T^{m,i}(s) \to 0$ at both boundaries and since $T^{m,i}$ satisfies the Robin boundary conditions (3.14), it follows that $DT^{m,i}(s)$ also converges to zero at both boundaries. Hence, also the second term in (5.17) converges to zero. Thus Eq. (5.17) is reduced to

$$\|D^2w^i\|_{L^2(0,L_i)} \to 0. \tag{5.19}$$

Going back now to (5.18) we get

$$\frac{|D^2w^i(0)|}{\beta}, \frac{|D^2w^i(L_i)|}{\beta} \to 0. \tag{5.20}$$

Next, we take the inner product of (5.9) with w^i in $L^2(0, L_i)$, use (5.8) to replace the term $\boldsymbol{i}\beta w^i$ by y^i and integrate by parts to obtain

$$\|y\|^2_{L^2(0,L_i)} + \gamma_6^i\|D^2w^i\|^2_{L^2(0,L_i)} + \gamma_6^i D^3w^i(L_i)\overline{w}^i(L_i) - \gamma_6^i D^2w^i(L_i)D\overline{w}^i(L_i) + \gamma_7^i\langle D^2T^{m,i}, w^i\rangle \to 0. \tag{5.21}$$

Now, the second term in (5.21) converges to zero by (5.19) and the same happens with the last term, namely, $\gamma_7^i\langle D^2T^{m,i}, w^i\rangle$, since $\|\beta w^i\|_{L^2(0,L_i)}$ is bounded and $\frac{\|D^2T^{m,i}\|_{L^2(0,L_i)}}{\beta}$ converges to zero by virtue of (5.16) and (5.19). We claim that also both boundary terms in (5.21) converge to zero. By interpolation,

$$\|\beta^{\frac{1}{2}}Dw^i\|_{L^2(0,L_i)} \le C\|D^2w^i\|^{\frac{1}{2}}_{L^2(0,L_i)}\|\beta w^i\|^{\frac{1}{2}}_{L^2(0,L_i)} \to 0.$$

Thus,

$$|D^3w^i(L_i)\overline{w}^i(L_i)| \le C\left(\frac{\|D^3w^i\|_{L^2(0,L_i)}}{\beta^{\frac{1}{2}}}\right)^{\frac{1}{2}}\left(\frac{\|D^3w^i\|_{H^1(0,L_i)}}{\beta}\right)^{\frac{1}{2}}\|\beta w^i\|^{\frac{1}{2}}_{L^2(0,L_i)}\|\beta^{\frac{1}{2}}Dw^i\|^{\frac{1}{2}}_{L^2(0,L_i)} \to 0,$$

and

$$|D^2w^i(L_i)Dw^i(L_i)| \le C\|D^2w^i\|^{\frac{1}{2}}\left(\frac{\|D^2w^i\|}{\beta^{\frac{1}{2}}}\right)^{\frac{1}{2}}\|\beta^{\frac{1}{2}}Dw^i\|^{\frac{1}{2}}\|Dw^i\|^{\frac{1}{2}}_{H^1} \to 0$$

we conclude that

$$\|y^i\|_{L^2(0,L_i)} \to 0. \tag{5.22}$$

Furthermore,

$$\begin{aligned}\frac{|D^3w^i(L_i)|}{\beta} &\le \frac{C}{\beta}\|D^3w^i\|^{\frac{1}{2}}_{L^2(0,L_i)}\|D^3w^i\|^{\frac{1}{2}}_{H^1(0,L_i)}\\ &\le C\|D^2w^i\|^{\frac{1}{4}}_{L^2(0,L_i)}\left(\frac{\|D^2w^i\|^{\frac{3}{4}}_{H^2(0,L_i)}}{\beta}\right) \to 0.\end{aligned} \tag{5.23}$$

Notice that (5.14), (5.20), (5.23) and (5.4) together with (3.14), imply that the forces and moments at the right boundary of the beams satisfy

$$\frac{1}{\beta}C^{\mathrm{T}}M^{-1}CE(P_2^Bz - \Lambda P_3^B\zeta) \to 0 \quad \text{in } \mathcal{H}_b. \tag{5.24}$$

The components of (5.3) related to the joint–leg equations have the following form

$$\boldsymbol{i}\beta p - C^{\mathrm{T}}M^{-1}CE(P_2^Bz - \Lambda P_3^B\zeta) \to 0 \quad \text{in } \mathcal{H}_b \tag{5.25}$$

where $p = \dot{b}$. Eqs. (5.24) and (5.25) imply that

$$p \to 0 \quad \text{in } \mathcal{H}_b. \tag{5.26}$$

In summary, we have reached the contradiction $\|X\|_{\mathcal{H}} \to 0$.

If the condition (5.1) is false, then there exist $\beta \in \mathbb{R}$ and a sequence $\{X_n\} \subset D(\mathcal{A})$ with $\|X_n\|_{\mathcal{H}} = 1\ \ \forall n$, such that

$$\lim_{n\to\infty}\|(\boldsymbol{i}\beta - \mathcal{A})X_n\|_{\mathcal{H}} = 0. \tag{5.27}$$

By repeating the same arguments as before (note that we have intentionally avoided using the fact that $\beta \to \infty$) we get the contradiction $\|X_n\|_{\mathcal{H}} \to 0$. Hence $\mathcal{A}$ satisfies conditions (5.1) and (5.2) and therefore, the C_0-semigroup of contractions $S(t)$ generated by $\mathcal{A}$ is exponentially stable. Thus, the proof is completed. ∎

6. Conclusions

In this article we considered a system of two thermoelastic Euler–Bernoulli beams coupled to a joint through two legs. By means of semigroup theory the well-posedness of the system was proved and its exponential stability was derived. It is certainly of much interest to develop numerical approximations for our state-space model (4.6). Such numerical schemes will be useful in simulation and identification studies to predict and better understand the structural and thermal responses of space-borne observation systems. Efforts in this direction are already under way.

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