A CATEGORICAL EQUIVALENCE BETWEEN SEMI-HEYTING ALGEBRAS AND CENTERED SEMI-NELSON ALGEBRAS

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Abstract. Inspired by an old construction due to R. Cignoli that relates Heyting algebras and centered Nelson algebras, in this paper we prove that there exists an equivalence between the category of semi-Heyting algebras and the category of centered semi-Nelson algebras.

1. Introduction

Inspired by results from [17] due to J. Kalman relating to lattices, R. Cignoli proved in [11, Theorem 2.4] that there exists an equivalence between the category of bounded distributive lattices and a particular full subcategory of centered Kleene algebras. Moreover, he also proved that there exists an equivalence between the category of Heyting algebras and the category of centered Nelson algebras [11, Theorem 3.14] (see also [10, 16]). In this paper we extend the previous result in order to show that there is an equivalence between the category of semi-Heyting algebras [20] and the category of semi-Nelson algebras [12] which have a center.

In the process of our research on the topic of the present paper, we have found useful to place our problems in the following general context.

We assume the reader is familiar with bounded distributive lattices and Heyting algebras [5]. A *De Morgan* algebra is an algebra $(H, \wedge, \vee, \sim, 0, 1)$ of type $(2, 2, 1, 0, 0, 0)$ such that $(H, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and $\sim$ fulfills the equations $\sim\sim x = x$ and $\sim(x \vee y) = \sim x \wedge \sim y$. An operation $\sim$ which satisfies the previous two equations is called *involution.* A *Kleene algebra* is a De Morgan algebra in which the inequality $x \wedge \sim x \leq y \vee \sim y$ holds. We say that an algebra $(H, \wedge, \vee, \sim, c, 0, 1)$ of type $(2, 2, 1, 0, 0)$ is a *centered Kleene algebra* if $(H, \wedge, \vee, \sim, 0, 1)$ is a Kleene algebra and c is such that $\mathrm{c} = \sim\mathrm{c}$ (this element is called *center*). It is immediate to see that c is necessarily unique. We write BDL for the category of bounded distributive lattices and $\mathsf{KA_c}$ for the category of centered Kleene algebras. In both cases the morphisms are the corresponding algebra homomorphisms. It is interesting to note that if T and U are centered Kleene algebras and $f : T \to U$ is a morphism of Kleene algebras, then f preserves necessarily the center, i.e., $f(\mathrm{c}) = \mathrm{c}$.

For an object $H \in \mathsf{BDL}$ we define

$$\mathrm{K}(H) := \{(a, b) \in H \times H : a \wedge b = 0\}.$$

This set could be endowed with the operations and the distinguished elements defined by:

$$\begin{array}{rcl}
(a, b) \vee (d, e) & := & (a \vee d, b \wedge e) \\
(a, b) \wedge (d, e) & := & (a \wedge d, b \vee e) \\
\sim(a, b) & := & (b, a) \\
0 & := & (0, 1) \\
1 & := & (1, 0) \\
\mathrm{c} & := & (0, 0)
\end{array}$$

In particular, $(\mathrm{K}(H), \wedge, \vee, \sim, \mathrm{c}, 0, 1) \in \mathsf{KA_c}$. For a morphism $f : H \to G \in \mathsf{BDL}$, the map $\mathrm{K}(f) : \mathrm{K}(H) \to \mathrm{K}(G)$, defined by $\mathrm{K}(f)(a, b) = (f(a), f(b))$, is a morphism in $\mathsf{KA_c}$. Moreover, K is a functor from BDL to $\mathsf{KA_c}$.

Let $(T, \wedge, \vee, \sim, \mathrm{c}, 0, 1) \in \mathsf{KA_c}$. The set

$$\mathrm{C}(T) := \{x \in T : x \geq \mathrm{c}\}$$

is the universe of a subalgebra of $(T, \wedge, \vee, \mathrm{c}, 1)$ and $(\mathrm{C}(T), \wedge, \vee, \mathrm{c}, 1) \in \mathsf{BDL}$. For a morphism $g : T \to U \in \mathsf{KA_c}$, the map $\mathrm{C}(g) : \mathrm{C}(T) \to \mathrm{C}(U)$, given by $\mathrm{C}(g)(x) = g(x)$, is a morphism in BDL. Moreover, C is a functor from $\mathsf{KA_c}$ to BDL.

Let $H \in \mathsf{BDL}$. The map $\alpha_H : H \to \mathrm{C}(\mathrm{K}(H))$ given by $\alpha_H(a) = (a, 0)$ is an isomorphism in BDL. If $T \in \mathsf{KA_c}$, the map $\beta_T : T \to \mathrm{K}(\mathrm{C}(T))$ given by $\beta_T(x) = (x \vee \mathrm{c}, \sim x \vee \mathrm{c})$ is injective and it is a morphism in $\mathsf{KA_c}$, but it is not necessarily surjective (see [16]).

Let $T \in \mathsf{KA_c}$. Consider the following algebraic condition:

$$\text{(CK)} \qquad (\forall x, y \geq c)(x \wedge y = \mathrm{c} \longrightarrow (\exists z)(z \vee \mathrm{c} = x \ \& \ \sim z \vee \mathrm{c} = y)).$$

This condition characterizes the surjectivity of β_T, that is, for every $T \in \mathsf{KA_c}$ we have that T satisfies (CK) if and only if β_T is a surjective map. The condition (CK) is not necessarily verified in every centered Kleene algebra, see for instance [10, Figure 1].

We write $\mathsf{KA_c^{CK}}$ for the full subcategory of $\mathsf{KA_c}$ whose objects satisfy (CK). The functor K can then be seen as a functor from BDL to $\mathsf{KA_c^{CK}}$.

The following result is [10, Theorem 2.7] (see also [11, Theorem 2.4]).

Theorem 1. *The functors* K *and* C *establish a categorical equivalence between* BDL *and* $\mathsf{KA_c^{CK}}$ *with natural isomorphisms* α *and* β.

Let $H \in \mathsf{BDL}$ and $a, b \in H$. If the relative pseudocomplement of a with respect to b exists, then we denote it by $a \to_{\mathsf{HA}} b$. Recall that a *Nelson algebra* [11, 23] is a Kleene algebra such that for each pair x, y there exists the binary operation $\Rightarrow$ given by $x \Rightarrow y := x \to_{\mathsf{HA}} (\sim x \vee y)$ and for every x, y, z it holds that $(x \wedge y) \Rightarrow z = x \Rightarrow (y \Rightarrow z)$. The binary operation $\Rightarrow$ so defined is called the weak implication. Nelson algebras can be seen as algebras $(H, \wedge, \vee, \Rightarrow, \sim, 0, 1)$ of type $(2, 2, 2, 1, 0, 0)$. The class of Nelson algebras is a variety [6, 7, 18].

We say that an algebra $(T, \wedge, \vee, \Rightarrow, \sim, \mathrm{c}, 0, 1)$ is a *centered Nelson algebra* if the reduct $(T, \wedge, \vee, \Rightarrow, \sim, 0, 1)$ is a Nelson algebra and c satisfies $\sim\mathrm{c} = \mathrm{c}$. It is a known fact that centered Nelson algebras satisfy the condition (CK) (see [10]). M. Fidel [14] and D. Vakarelov [22] proved independently that if $(H, \wedge, \vee, \to, 0, 1)$ is a Heyting algebra then $(\mathrm{K}(H), \wedge, \vee, \Rightarrow, \sim, \mathrm{c}, 0, 1)$ is a centered Nelson algebra, where $\Rightarrow$ is defined as follows, for pairs (a, b) and (d, e) in $\mathrm{K}(H)$:

$$(1) \qquad (a, b) \Rightarrow (d, e) := (a \to d, a \wedge e).$$

Let HA be the category of Heyting algebras and $\mathsf{NA_c}$ the category of centered Nelson algebras. The following result is a reformulation of [11, Theorem 3.14] (see also [9]).

Theorem 2. *The functors* K *and* C *establish a categorical equivalence between* HA *and* $\mathsf{NA_c}$ *with natural isomorphisms* α *and* β.

In what follows we recall the definition of a semi-Heyting algebra, which were introduced by H.P. Sankappanavar in [20] as an abstraction of Heyting algebras. Semi-Heyting algebras share with Heyting algebras the following properties: they are pseudocomplemented, distributive lattices and their congruences are determined by the lattice filters. The relationship between the variety of semi-Heyting algebras and the varieties of Heyting algebras (and its expansions) have been studied lately in [1, 2, 3, 4, 19].

Definition 3. An algebra $(H, \wedge, \vee, \to, 0, 1)$ of type $(2,2,2,0,0)$ is a *semi-Heyting algebra* if the following conditions hold for every a, b, d in H:

(SH1) $(H, \wedge, \vee, 0, 1)$ is a bounded lattice,
(SH2) $a \wedge (a \to b) = a \wedge b$,
(SH3) $a \wedge (b \to d) = a \wedge [(a \wedge b) \to (a \wedge d)]$,
(SH4) $a \to a = 1$.

We write SH for the category of semi-Heyting algebras. The underlying lattice of a semi-Heyting lattice is necessarily distributive, as it is shown in [20].

In what follows we recall some definition given in [12] we shall use later.

Definition 4. An algebra $(T, \wedge, \vee, \to, \sim, 1)$ of type $(2,2,2,1,0)$ is a pre-semi-Nelson algebra if for every $x, y, z \in H$ the following conditions are satisfied:

(SN1) $x \wedge (x \vee y) = x$,
(SN2) $x \wedge (y \vee z) = (z \wedge x) \vee (y \wedge x)$,
(SN3) $\sim\sim x = x$,
(SN4) $\sim (x \wedge y) = \sim x \vee \sim y$,
(SN5) $x \wedge \sim x = (x \wedge \sim x) \wedge (y \vee \sim y)$,
(SN6) $x \wedge (x \to_N y) = x \wedge (\sim x \vee y)$,
(SN7) $x \to_N (y \to_N z) = (x \wedge y) \to_N z$,
(SN8) $(x \to_N y) \to_N [(y \to_N x) \to_N [(x \to z) \to_N (y \to z)]] = 1$,
(SN9) $(x \to_N y) \to_N [(y \to_N x) \to_N [(z \to x) \to_N (z \to y)]] = 1$,

where $x \to_N y$ stands for the term $x \to (x \wedge y)$.

We write PSN for the category of pre-semi-Nelson algebras. Notice that a pre-semi-Nelson algebra is, in particular, a Kleene algebra.

Definition 5. A pre-semi-Nelson algebra $(T, \wedge, \vee, \to, \sim, 1)$ is a *s*emi-Nelson algebra if it also verifies the following conditions for every $x, y \in H$:

(SN10) $(\sim (x \to y)) \to_N (x \wedge \sim y) = 1$,
(SN11) $(x \wedge \sim y) \to_N (\sim (x \to y)) = 1$.

Semi-Nelson algebras were introduced by J.M. Cornejo and I. Viglizzo in [12] as a generalization of Nelson algebras. We write SN for the category of semi-Nelson algebras and $\mathsf{SN_c}$ for the category of centered semi-Nelson algebras whose objects are algebras $(T, \wedge, \vee, \to, \sim, \mathrm{c}, 0, 1)$ of type $(2,2,2,1,0,0,0)$, where $(T, \wedge, \vee, \to, \sim, 1)$ is a semi-Nelson algebra, $0 = \sim 1$ and c satisfies that $c = \sim c$. The morphisms of $\mathsf{SN_c}$ are the algebra homomorphisms.

The fact that Kalman's construction can be extended consistently to Heyting algebras led us to believe that some of the picture could be lifted to the varieties SH and $\mathsf{SN_c}$. More precisely, it arises the natural question if it is possible to prove that there exists an equivalence between SH and $\mathsf{SN_c}$, making the following diagram commute:

$$\begin{array}{ccc} \mathsf{SH} & \underset{\mathrm{C}}{\overset{\mathrm{K}}{\rightleftarrows}} & \mathsf{SN_c} \\ {\scriptstyle i_1}\uparrow & & \uparrow{\scriptstyle i_2} \\ \mathsf{HA} & \underset{\mathrm{C}}{\overset{\mathrm{K}}{\rightleftarrows}} & \mathsf{NA_c} \end{array}$$

where i_1 is the inclusion functor from HA to SH and i_2 is the inclusion functor from $\mathsf{NA_c}$ to $\mathsf{SN_c}$.

We give a table with the categories we consider in this paper:

Category	Objects
BDL	Bounded distributive lattices
$\mathsf{KA_c}$	Centered Kleene algebras
HA	Heyting algebras
NA	Nelson algebras
$\mathsf{NA_c}$	Centered Nelson algebras
$\mathsf{NL_c}$	Centered Nelson lattices
SH	Semi-Heyting algebras
PSN	Pre-semi-Nelson algebras
SN	Semi-Nelson algebras
$\mathsf{SN_c}$	Centered semi-Nelson algebras
KSH	Centered Kleene algebras endowed with a particular binary operation

The results studied in the present paper are motivated by ideas coming from different varieties of algebras, as Heyting algebras and Nelson algebras, and by the categorical equivalence between the category of Heyting algebras and the category of centered Nelson algebras (see Theorem 2). Our main goal is to extend the above mentioned equivalence by considering the category of semi-Heyting algebras and the category of centered semi-Nelson algebras. We think that the properties studied here can be of interest for future work about the understanding of the categories of semi-Heyting algebras and centered semi-Nelson algebras respectively.

The paper is organized as follows. In Section 2 we study some properties concerning centered semi-Nelson algebras. In Section 3 we generalize the equivalence between the categories HA and $\mathsf{NA_c}$ (see Theorem 2) in the framework of SH and $\mathsf{SN_c}$. Finally, in Section 4 we study the relationship between $\mathsf{SN_c}$ and the category KSH (see Definition 22) introduced in [16], which is also equivalent to the category SH.

2. Basic results

In this section we give some basic about (centered) semi-Nelson algebras we shall use in the next section in order to show that there exists an equivalence between SH and $\mathsf{SN_c}$. Since the results given here are very technical, we recommend to the reader don't read the proofs of them in a first lecture of the present paper.

Lemma 6. [12] *Let $T \in \mathsf{PSN}$ and $x, y, z \in T$. Then*

(a) $1 \to_N x = x$,
(b) $x \to_N x = 1$,
(c) *if* $x \leq y$ *then* $x \to_N y = 1$,
(d) $x \leq y$ *if and only if* $x \to_N y = 1$ *and* $\sim y \to_N \sim x = 1$,
(e) *If* $x \to_N y = y \to_N z = 1$ *then* $x \to_N z = 1$,
(f) $(x \wedge y) \to_N y = 1$,
(g) *if* $x \to_N y = 1$ *and* $y \to_N z = 1$ *then* $x \to_N z = 1$.

Lemma 7. [13] *Let $T \in \mathsf{SN}$ and $x, y, z \in T$. Then*

(a) $(x \wedge \sim x) \to_N y = 1$,
(b) $(x \to_N y) \to_N ((x \to_N z) \to_N (x \to_N (y \wedge z))) = 1$,
(c) $(x \to_N z) \to_N ((y \to_N z) \to_N ((x \vee y) \to_N z)) = 1$,
(d) $x \to_N y = x \to_N (x \wedge y)$.

We will use the previous lemmas in order to show the following result.

Lemma 8. *Let $T \in \mathsf{SN}$ and $x, y, z \in T$. Then*

(a) *If* $x \to_N y = 1$ *and* $x \to_N z = 1$ *then* $x \to_N (y \wedge z) = 1$.
(b) *If* $x \to_N z = 1$ *and* $y \to_N z = 1$ *then* $(x \vee y) \to_N z = 1$.
(c) *If* $x \to_N y = 1$ *then* $(x \vee z) \to_N (y \vee z) = 1$.

Proof. (a) By (b) of Lemma 7 we have that

$$(x \to_N y) \to_N ((x \to_N z) \to_N (x \to_N (y \wedge z))) = 1.$$

Hence, by (a) of Lemma 6 we obtain that $x \to_N (y \wedge z) = 1$.

(b) It follows from (c) of Lemma 7 that

$$(x \to_N z) \to_N ((y \to_N z) \to_N ((x \vee y) \to_N z)) = 1.$$

Thus, by (a) of Lemma 6 we deduce the equality $(x \vee y) \to_N z = 1$.

(c) By hypothesis we have that $x \to_N y = 1$. It follows from (c) of Lemma 6 the equality $y \to_N (y \vee z) = 1$. Then, using (g) of Lemma 6 we obtain that $x \to_N (y \vee z) = 1$. Observe that $z \to_N (y \vee z) = 1$, which follows again from (c) of Lemma 6. In view of (c) of Lemma 7 it holds that

$$(x \to_N (y \vee z)) \to_N ((z \to_N (y \vee z)) \to_N ((x \vee z) \to_N (y \vee z))) = 1.$$

Therefore, by (a) of Lemma 6 we conclude that $(x \vee z) \to_N (y \vee z) = 1$.

□

The following lemma presents some useful properties of centered semi-Nelson algebras.

Lemma 9. *Let $T \in \mathsf{SN_c}$ and $x, y, z \in T$. Then*

(a) $\mathrm{c} \to_N x = 1$,
(b) $(x \wedge \mathrm{c}) \to_N y = 1$,
(c) $(\sim (x \to y) \vee \mathrm{c}) \to_N ((x \vee \mathrm{c}) \wedge (\sim y \vee \mathrm{c})) = 1$,
(d) $((x \vee \mathrm{c}) \wedge (\sim y \vee \mathrm{c})) \to_N (\sim (x \to y) \vee \mathrm{c}) = 1$,
(e) $[\sim ((x \vee \mathrm{c}) \wedge (\sim y \vee \mathrm{c}))] \to_N [\sim (\sim (x \to y) \vee \mathrm{c})] = 1$,
(f) $[\sim (\sim (x \to y) \vee \mathrm{c})] \to_N [\sim ((x \vee \mathrm{c}) \wedge (\sim y \vee \mathrm{c}))] = 1$,
(g) $\sim (x \to y) \vee \mathrm{c} = (x \vee \mathrm{c}) \wedge (\sim y \vee \mathrm{c})$,
(h) $[\sim ((x \to y) \vee \mathrm{c})] \to_N [\sim ((x \vee \mathrm{c}) \to (y \vee \mathrm{c}))] = 1$,
(i) $[\sim ((x \vee \mathrm{c}) \to (y \vee \mathrm{c}))] \to_N [\sim ((x \to y) \vee \mathrm{c})] = 1$,
(j) $(x \to y) \to_N ((x \vee \mathrm{c}) \to (y \vee \mathrm{c})) = 1$,
(k) $((x \vee \mathrm{c}) \to (y \vee \mathrm{c})) \to_N (x \to y) = 1$,
(l) $(x \vee \mathrm{c}) \to (y \vee \mathrm{c}) = (x \to y) \vee \mathrm{c}$.

Proof. Let $x, y \in T$.

(a) Note that $\mathrm{c} \to_N x = (\mathrm{c} \wedge \mathrm{c}) \to_N x = (\mathrm{c} \wedge \sim \mathrm{c}) \to_N x = 1$, which follows from (a) of Lemma 7.

(b) By item (a) we have that $\mathrm{c} \to_N y = 1$ and it follows from (f) of Lemma 6 that $(x \wedge \mathrm{c}) \to_N \mathrm{c} = 1$. Thus, it follows from (g) of Lemma 6 that $(x \wedge \mathrm{c}) \to_N y = 1$.

(c) The fact that $\sim (x \to y) \to_N (x \wedge \sim y) = 1$ follows from (SN10). By (f) of Lemma 6 we obtain that $(x \wedge \sim y) \to_N x = 1$. Then, applying (e) of Lemma 6 we obtain that $\sim (x \to y) \to_N x = 1$. Thus, taking into account (c) of Lemma 8 we have that

(2) $$(\sim (x \to y) \vee \mathrm{c}) \to_N (x \vee \mathrm{c}) = 1.$$

Similarly we can show that

(3) $$(\sim (x \to y) \vee \mathrm{c}) \to_N (\sim y \vee \mathrm{c}) = 1.$$

By (2), (3) and (a) of Lemma 8 it is possible to verify that

$$(\sim (x \to y) \vee \mathrm{c}) \to_N ((x \vee \mathrm{c}) \wedge (\sim y \vee \mathrm{c})).$$

(d) By (SN11) we have that $(x \wedge \sim y) \to_N (\sim (x \to y)) = 1$. Thus, by (c) of Lemma 8 we deduce the equality

$$((x \wedge \sim y) \vee \mathrm{c}) \to_N ((\sim (x \to y)) \vee \mathrm{c}) = 1.$$

Consequently,

$$((x \vee \mathrm{c}) \wedge (\sim y \vee \mathrm{c})) \to_N (\sim (x \to y) \vee \mathrm{c}) = ((x \wedge \sim y) \vee \mathrm{c}) \to_N (\sim (x \to y) \vee \mathrm{c}) = 1.$$

(e) Let us notice that (f) of Lemma 6 implies the equality

$$(\sim x \wedge \sim \mathrm{c}) \to_N (\sim \mathrm{c}) = 1. \tag{4}$$

Hence, using (a) we have that

$$\mathrm{c} \to_N (\sim \mathrm{c}) = 1 \tag{5}$$

and

$$\mathrm{c} \to_N (x \to y) = 1. \tag{6}$$

Hence, (5), (6) and (a) of Lemma 8 can be used to verify that

$$\mathrm{c} \to_N ((\sim \mathrm{c}) \wedge (x \to y)) = 1.$$

Since $\mathrm{c} = \sim \mathrm{c}$ then

$$(\sim \mathrm{c}) \to_N ((\sim \mathrm{c}) \wedge (x \to y)) = 1. \tag{7}$$

It follows from (4), (7) and (g) of Lemma 6 that

$$(\sim x \wedge \sim \mathrm{c}) \to_N ((\sim \mathrm{c}) \wedge (x \to y)) = 1. \tag{8}$$

Similarly we can show the equality

$$(y \wedge \sim \mathrm{c}) \to_N ((\sim \mathrm{c}) \wedge (x \to y)) = 1. \tag{9}$$

By (b) of Lemma 8 combined with (8) and (9) we deduce that

$$((\sim x \wedge \sim \mathrm{c}) \vee (y \wedge \sim \mathrm{c})) \to_N ((\sim \mathrm{c}) \wedge (x \to y)) = 1. \tag{10}$$

In consequence, it follows from (10) that

$$\begin{aligned}
& [\sim ((x \vee \mathrm{c}) \wedge (\sim y \vee \mathrm{c}))] \to_N [\sim (\sim (x \to y) \vee \mathrm{c})] \\
= \; & [[\sim ((x \vee \mathrm{c})] \vee [\sim (\sim y \vee \mathrm{c})]] \to_N [\sim (\sim (x \to y) \vee \mathrm{c})] \\
= \; & ((\sim x \wedge \sim \mathrm{c}) \vee (\sim\sim y \wedge \sim \mathrm{c})) \to_N [\sim (\sim (x \to y) \vee \mathrm{c})] \\
= \; & ((\sim x \wedge \sim \mathrm{c}) \vee (y \wedge \sim \mathrm{c})) \to_N [\sim (\sim (x \to y) \vee \mathrm{c})] \\
= \; & ((\sim x \wedge \sim \mathrm{c}) \vee (y \wedge \sim \mathrm{c})) \to_N [(\sim\sim (x \to y) \wedge (\sim \mathrm{c}))] \\
= \; & ((\sim x \wedge \sim \mathrm{c}) \vee (y \wedge \sim \mathrm{c})) \to_N [((x \to y) \wedge (\sim \mathrm{c}))] \\
= \; & 1.
\end{aligned}$$

(f) Item (b) implies the equality

$$((x \to y) \wedge \mathrm{c}) \to_N [\sim ((x \vee \mathrm{c}) \wedge (\sim y \vee \mathrm{c}))] = 1. \tag{11}$$

Hence, it follows from (11) that

$$\begin{aligned}
& [\sim (\sim (x \to y) \vee \mathrm{c})] \to_N [\sim ((x \vee \mathrm{c}) \wedge (\sim y \vee \mathrm{c}))] \\
= \; & [(\sim\sim (x \to y)) \wedge (\sim \mathrm{c})] \to_N [\sim ((x \vee \mathrm{c}) \wedge (\sim y \vee \mathrm{c}))] \\
= \; & [((x \to y)) \wedge (\sim \mathrm{c})] \to_N [\sim ((x \vee \mathrm{c}) \wedge (\sim y \vee \mathrm{c}))] \\
= \; & [((x \to y)) \wedge \mathrm{c}] \to_N [\sim ((x \vee \mathrm{c}) \wedge (\sim y \vee \mathrm{c}))] \\
= \; & 1.
\end{aligned}$$

(g) Using items (c), (d), (e), (f) and (d) of Lemma 6, the desired identity holds.

(h) It is consequence from (b) as follows:

$$\begin{aligned}
& [\sim ((x \to y) \vee \mathrm{c})] \to_N [\sim ((x \vee \mathrm{c}) \to (y \vee \mathrm{c}))] \\
= \; & [(\sim (x \to y)) \wedge (\sim \mathrm{c})] \to_N [\sim ((x \vee \mathrm{c}) \to (y \vee \mathrm{c}))] \\
= \; & [(\sim (x \to y)) \wedge \mathrm{c}] \to_N [\sim ((x \vee \mathrm{c}) \to (y \vee \mathrm{c}))] \\
= \; & 1.
\end{aligned}$$

(i) By (SN10) we obtain that

$$[\sim((x \vee \mathrm{c}) \to (y \vee \mathrm{c}))] \to_N [(x \vee \mathrm{c}) \wedge (\sim(y \vee \mathrm{c}))] = 1. \tag{12}$$

Taking into account (b) we have that

$$\begin{aligned} & [(x \vee \mathrm{c}) \wedge (\sim(y \vee \mathrm{c}))] \to_N [\sim((x \to y) \vee \mathrm{c})] \\ = \; & [(x \vee \mathrm{c}) \wedge (\sim y) \wedge (\sim \mathrm{c})] \to_N [\sim((x \to y) \vee \mathrm{c})] \\ = \; & [(x \vee \mathrm{c}) \wedge (\sim y) \wedge \mathrm{c}] \to_N [\sim((x \to y) \vee \mathrm{c})] \\ = \; & 1. \end{aligned}$$

Then,

$$[(x \vee \mathrm{c}) \wedge (\sim(y \vee \mathrm{c}))] \to_N [\sim((x \to y) \vee \mathrm{c})] = 1. \tag{13}$$

To finish off the proof of this item we can apply (g) of Lemma 6 to equations (12) and (13).

(j) Using item (a) and (b) of Lemma 6 we can verify that $\mathrm{c} \to_N x = 1$ and $x \to_N x = 1$ respectively. By (b) of Lemma 8 we obtain that

$$(x \vee \mathrm{c}) \to_N x = 1. \tag{14}$$

Also, it follows from (c) of Lemma 6 that

$$x \to_N (x \vee \mathrm{c}) = 1. \tag{15}$$

Similarly we can show that

$$(y \vee \mathrm{c}) \to_N y = 1 \quad \text{and} \quad y \to_N (y \vee \mathrm{c}) = 1. \tag{16}$$

Then,

$$\begin{aligned} & (x \to y) \to ((x \vee \mathrm{c}) \to y) \\ = \; & 1 \to_N [(x \to y) \to ((x \vee \mathrm{c}) \to y)] \\ & \text{by (a) of Lemma 6} \\ = \; & ((x \vee \mathrm{c}) \to_N x) \to_N [(x \to y) \to ((x \vee \mathrm{c}) \to y)] && \text{by (14)} \\ = \; & 1 \to_N [((x \vee \mathrm{c}) \to_N x) \to_N [(x \to y) \to ((x \vee \mathrm{c}) \to y)]] \\ & \text{by (a) of Lemma 6} \\ = \; & (x \to_N (x \vee \mathrm{c})) \to_N [((x \vee \mathrm{c}) \to_N x) \to_N [(x \to y) \to ((x \vee \mathrm{c}) \to y)]] && \text{by (15)} \\ = \; & 1 && \text{by (SN8).} \end{aligned}$$

Then,

$$(x \to y) \to ((x \vee \mathrm{c}) \to y) = 1. \tag{17}$$

On the other hand,

$$\begin{aligned} & ((x \vee \mathrm{c}) \to y) \to ((x \vee \mathrm{c}) \to (y \vee \mathrm{c})) \\ = \; & 1 \to_N [((x \vee \mathrm{c}) \to y) \to ((x \vee \mathrm{c}) \to (y \vee \mathrm{c}))] \\ & \text{by (a) of Lemma 6} \\ = \; & ((y \vee \mathrm{c}) \to_N y) \to_N [((x \vee \mathrm{c}) \to y) \to ((x \vee \mathrm{c}) \to (y \vee \mathrm{c}))] && \text{by (16)} \\ = \; & 1 \to_N [((y \vee \mathrm{c}) \to_N y) \to_N [((x \vee \mathrm{c}) \to y) \to ((x \vee \mathrm{c}) \to (y \vee \mathrm{c}))]] \\ & \text{by (a) of Lemma 6} \\ = \; & (y \to_N (y \vee \mathrm{c})) \to_N [((y \vee \mathrm{c}) \to_N y) \to_N [((x \vee \mathrm{c}) \to y) \to ((x \vee \mathrm{c}) \to (y \vee \mathrm{c}))]] && \text{by (16)} \\ = \; & 1 && \text{by (SN9).} \end{aligned}$$

Thus, we have the equality

$$((x \vee \mathrm{c}) \to y) \to ((x \vee \mathrm{c}) \to (y \vee \mathrm{c})) = 1. \tag{18}$$

Finally, taking into account (g) of Lemma 6 in conditions (17) and (18) we conclude that $(x \to y) \to ((x \vee \mathrm{c}) \to (y \vee \mathrm{c})) = 1$.

(k) The proof of this item is similar to the one used in (j).

(l) It follows from item (j) that $(x \to y) \to_N ((x \vee \mathrm{c}) \to (y \vee \mathrm{c})) = 1$. Also, $\mathrm{c} \to_N ((x \vee \mathrm{c}) \to (y \vee \mathrm{c})) = 1$ follows from (a). Hence, by (b) of Lemma 8 we deduce that

$$((x \to y) \vee \mathrm{c}) \to_N ((x \vee \mathrm{c}) \to (y \vee \mathrm{c})) = 1. \tag{19}$$

Besides, from item (k) we know that $((x \vee \mathrm{c}) \to (y \vee \mathrm{c})) \to_N (x \to y) = 1$. Also, by (c) of Lemma 6 we have that $(x \to y) \to_N ((x \to y) \vee \mathrm{c})$. Thus, by (e) of Lemma 6 the equality

$$((x \vee \mathrm{c}) \to (y \vee \mathrm{c})) \to_N ((x \to y) \vee \mathrm{c}) = 1 \tag{20}$$

is satisfied. Therefore, applying (d) of Lemma 6 in (19),(20), (h) and (i) we can verify that $(x \vee \mathrm{c}) \to (y \vee \mathrm{c}) = (x \to y) \vee \mathrm{c}$.

□

In what follows we will use previous results in order to prove the following lemma.

Lemma 10. *Let $T \in \mathsf{SN_c}$ and $x, y \in T$. Then $\mathrm{c} \leq x \to (y \vee \mathrm{c})$.*

Proof. By (b) of Lemma 9 we have that

$$\begin{aligned} (x \wedge \sim (y \vee \mathrm{c})) \to_N (\sim \mathrm{c}) &= (x \wedge (\sim y) \wedge (\sim \mathrm{c})) \to_N (\sim \mathrm{c}) \\ &= (x \wedge (\sim y) \wedge \mathrm{c}) \to_N (\sim \mathrm{c}) \\ &= 1. \end{aligned}$$

Hence, $(x \wedge \sim (y \vee \mathrm{c})) \to_N (\sim \mathrm{c}) = 1$. Besides, it follows from (SN10) that $[\sim (x \to (y \vee \mathrm{c}))] \to_N (x \wedge \sim (y \vee \mathrm{c})) = 1$. Then, using (g) of Lemma 6 we obtain that

$$[\sim (x \to (y \vee \mathrm{c}))] \to_N (\sim \mathrm{c}) = 1. \tag{21}$$

Notice that

$$\mathrm{c} \to_N (x \to (y \vee \mathrm{c})) = 1 \tag{22}$$

in view of (a) of Lemma 9. Therefore, taking into account (21), (22) and using (d) of Lemma 6 we conclude that $\mathrm{c} \leq x \to (y \vee \mathrm{c})$. □

3. Categorical equivalence between SH and $\mathsf{SN_c}$

In this section we will prove that there is a categorical equivalence between SH and $\mathsf{SN_c}$. We will use the same notation and constructions given in Section 1 about the functors $\mathrm{K} : \mathsf{HA} \to \mathsf{NA_c}$, $\mathrm{C} : \mathsf{NA_c} \to \mathsf{HA}$ and the isomorphisms α_H (for $H \in \mathsf{HA}$) and β_T (for $T \in \mathsf{NA_c}$).

We start with some preliminary definitions and properties.

Let $H \in \mathsf{SH}$. In [12, Theorem 4.1] it was proved that $(\mathrm{K}(H), \wedge, \vee, \Rightarrow, 1) \in \mathsf{SN}$. Thus, if $H \in \mathsf{SH}$ then $\mathrm{K}(H) \in \mathsf{SN_c}$. It is immediate that if f is a morphism in SH, then $\mathrm{K}(f)$ is a morphism in $\mathsf{SN_c}$. Hence, we obtain the following result.

Proposition 11. *There exists a functor* K *from* SH *to* $\mathsf{SN_c}$.

Let $T \in \mathsf{SN}$ (or $T \in \mathsf{SN_c}$). The binary relation $\equiv$ on T given by

$$x \equiv y \text{ if and only if } x \to y = 1 \text{ and } y \to x = 1$$

is an equivalence relation on T compatible with the operations $\wedge$, $\vee$ and $\to$, as it is shown in [12, Lemma 3.1]. If $x \in T$ we write $[\![x]\!]$ for the equivalence class associated to $\equiv$. As usual, we write $T/_{\equiv}$ for the set of equivalence classes.

We denote by $\mathbf{sH}(T)$ the algebra $(T/_{\equiv}, \cap, \cup, \rightsquigarrow, [\![0]\!], [\![1]\!])$, where the operations are defined by:

- $[\![x]\!] \cap [\![y]\!] = [\![x \wedge y]\!]$,
- $[\![x]\!] \cup [\![y]\!] = [\![x \vee y]\!]$,
- $[\![x]\!] \rightsquigarrow [\![y]\!] = [\![x \to y]\!]$.

It follows from [12, Theorem 3.4] that $\mathbf{sH}(T) \in \mathsf{SH}$.

Proposition 12. *Let* $T \in \mathsf{SN_c}$. *Then* $\mathrm{C}(T) \in \mathsf{SH}$.

Proof. Let $T \in \mathsf{SN_c}$. It follows from Lemma 10 that $x \to y \in \mathrm{C}(T)$ whenever $x, y \in \mathrm{C}(T)$.

Consider the map $h : T \to \mathrm{K}(\mathbf{sH}(T))$ given by $h(x) = ([\![x]\!], [\![\sim x]\!])$. We will write $\to$ for the implication of T and $\Rightarrow$ for the implication in $\mathrm{K}(\mathbf{sH}(T))$. The function h is an injective morphism in SN, see [12, Corollary 5.2]. Straightforward computations show that h preserves the bottom and the center, so $T \cong h(T)$ in $\mathsf{SN_c}$. Thus, by definition of C and SH we have that $\mathrm{C}(T) \in \mathsf{SH}$ if and only if $\mathrm{C}(h(T)) \in \mathsf{SH}$. Notice that $h(x) \in \mathrm{C}(h(T))$ if and only if $[\![\mathrm{c}]\!] \leq [\![x]\!]$ and $[\![\sim x]\!] \leq [\![\mathrm{c}]\!]$.

In what follows we will prove that $\mathrm{C}(h(T)) \in \mathsf{SH}$. We will use the fact that $\mathbf{sH}(T) \in \mathsf{SH}$. Let $x, y, z \in T$ such that $h(x), h(y), h(z) \in \mathrm{C}(h(T))$. Then $[\![\mathrm{c}]\!] \leq [\![x]\!]$, $[\![\mathrm{c}]\!] \leq [\![y]\!]$, $[\![\mathrm{c}]\!] \leq [\![z]\!]$, $[\![\sim x]\!] \leq [\![\mathrm{c}]\!]$, $[\![\sim y]\!] \leq [\![\mathrm{c}]\!]$ and $[\![\sim z]\!] \leq [\![\mathrm{c}]\!]$.

Taking into account that $[\![\sim y]\!] \leq [\![\mathrm{c}]\!] \leq [\![x]\!]$ and $[\![x \wedge (x \to y)]\!] = [\![x \wedge y]\!]$, the condition (SH2) in $\mathbf{sH}(T)$, we obtain that

$$\begin{array}{rcl} h(x) \cap (h(x) \Rightarrow h(y)) & = & ([\![x]\!], [\![\sim x]\!]) \cap (([\![x]\!], [\![\sim x]\!]) \Rightarrow ([\![y]\!], [\![\sim y]\!])) \\ & = & ([\![x]\!], [\![\sim x]\!]) \cap ([\![x \to y]\!], [\![x]\!] \cap [\![\sim y]\!]) \\ & = & ([\![x]\!], [\![\sim x]\!]) \cap ([\![x \to y]\!], [\![\sim y]\!]) \\ & = & ([\![x \wedge (x \to y)]\!], [\![\sim x \vee \sim y]\!]) \\ & = & ([\![x \wedge y]\!], [\![\sim x \vee \sim y]\!]) \\ & = & ([\![x]\!] \cap [\![y]\!], [\![\sim x]\!] \cup [\![\sim y]\!]) \\ & = & h(x) \cap h(y). \end{array}$$

Then $\mathrm{C}(h(T))$ satisfies (SH2).

In order to prove (SH3), first note that since $[\![\sim z]\!] \leq [\![\mathrm{c}]\!] \leq [\![y]\!]$ then

$$\begin{array}{rcl} h(x) \cap (h(y) \Rightarrow h(z)) & = & ([\![x]\!], [\![\sim x]\!]) \cap (([\![y]\!], [\![\sim y]\!]) \Rightarrow ([\![z]\!], [\![\sim z]\!])) \\ & = & ([\![x]\!], [\![\sim x]\!]) \cap ([\![y \to z]\!], [\![y]\!] \cap [\![\sim z]\!]) \\ & = & ([\![x]\!], [\![\sim x]\!]) \cap ([\![y \to z]\!], [\![\sim z]\!]) \\ & = & ([\![x \wedge (y \to z)]\!], [\![\sim x \vee \sim z]\!]). \end{array}$$

Then

$$h(x) \cap (h(y) \Rightarrow h(z)) = ([\![x \wedge (y \to z)]\!], [\![\sim x \vee \sim z]\!]). \tag{23}$$

Besides, straightforward computations based in the distributivity of the underlying lattice of T shows that

$$h(x) \cap ((h(x) \cap h(y)) \Rightarrow (h(x) \cap h(z)) = ([\![x \wedge ((x \wedge y) \to (x \wedge z))]\!], [\![\sim x \vee \sim z]\!]). \tag{24}$$

Using (SH3) in $\mathbf{sH}(T)$ we have that $[\![x \wedge (y \to z)]\!] = [\![x \wedge ((x \wedge y) \to (x \wedge z))]\!]$. Thus, it follows from (23) and (24) that

$$h(x) \cap (h(y) \Rightarrow h(z)) = h(x) \cap ((h(x) \cap h(y)) \Rightarrow (h(x) \cap h(z)),$$

which is the condition (SH3) in $\mathrm{C}(h(T))$.

Finally,

$$\begin{array}{rcl} h(x) \Rightarrow h(x) & = & ([\![x \to x]\!], [\![x \wedge \sim x]\!]) \\ & = & ([\![1]\!], [\![0]\!]), \end{array}$$

i.e., the condition (SH4) is also satisfied in $\mathrm{C}(h(T))$. Therefore, $\mathrm{C}(T) \in \mathsf{SH}$. $\square$

It is immediate that if f is a morphism in SH, then $\mathrm{C}(f)$ is a morphism in $\mathsf{SN_c}$. Therefore, we conclude the following result.

Proposition 13. *There exists a functor* C *from* $\mathsf{SN_c}$ *to* SH.

It follows from (SH3) and (SH4) that for every $H \in \mathsf{SH}$, $a \leq b \to (a \wedge b)$ holds for every $a, b \in H$. Thus, the next lemma follows from Proposition 12.

Lemma 14. *Let* $T \in \mathsf{SN_c}$ *and* $x, y \in T$. *Then* $x \vee \mathrm{c} \leq (y \vee c) \to ((x \vee c) \wedge (y \vee c))$.

Remark 15. Let $T \in \mathsf{SN_c}$ and $x, y \in T$. Throughout the rest of this section we will use the equalities $\sim (x \to y) \vee \mathrm{c} = (x \vee \mathrm{c}) \wedge (\sim y \vee \mathrm{c})$ and $(x \to y) \vee \mathrm{c} = (x \vee \mathrm{c}) \to (y \vee \mathrm{c})$, which appears in items (g) and (l) of Lemma 9, respectively.

The following lemma will be used latter.

Lemma 16. *Let* $T \in \mathsf{SN_c}$. *Then* T *satisfies* (CK).

Proof. In this proof we will use Lemma 14 and Remark 15.

Let $x, y \in T$ such that $x \geq \mathrm{c}$, $y \geq \mathrm{c}$ and $x \wedge y = \mathrm{c}$. Let $z = (y \to \sim y) \wedge x$. In particular, $\sim x \leq \mathrm{c}$ and $\sim y \leq \mathrm{c}$. We will prove that $x = z \vee \mathrm{c}$ and $y = \sim z \vee \mathrm{c}$.

The equality $z \vee \mathrm{c} = x$ can be proved as follows:

$$\begin{aligned} z \vee \mathrm{c} &= ((y \to \sim y) \wedge x) \vee (\mathrm{c} \wedge x) \\ &= x \wedge ((y \to \sim y) \vee \mathrm{c}) \\ &= x \wedge ((y \vee \mathrm{c}) \to (\sim y \vee \mathrm{c})) \\ &= (x \vee \mathrm{c}) \wedge ((y \vee \mathrm{c}) \to (x \wedge y)) \\ &= (x \vee \mathrm{c}) \wedge ((y \vee \mathrm{c}) \to ((x \vee \mathrm{c}) \wedge (y \vee \mathrm{c}))) \\ &= x \vee \mathrm{c} \\ &= x. \end{aligned}$$

Finally we have that

$$\begin{aligned} z \wedge \mathrm{c} &= ((y \to \sim y) \wedge \mathrm{c}) \wedge x \\ &= (\sim (\sim (y \to \sim y) \vee \mathrm{c})) \wedge x \\ &= (\sim (y \vee \mathrm{c})) \wedge x \\ &= \sim y \wedge \mathrm{c} \wedge x \\ &= \sim y \wedge \mathrm{c} \\ &= \sim y, \end{aligned}$$

so $\sim z \vee \mathrm{c} = y$. $\square$

Let $T \in \mathsf{SN_c}$. We will see that β_T is an isomorphism in $\mathsf{SN_c}$.

Proposition 17. *Let* $T \in \mathsf{SN_c}$. *Then* β_T *is an isomorphism in* $\mathsf{SN_c}$.

Proof. We know that β_T is an injective morphism in $\mathsf{KA_c}$. The fact that β_T preserves the implication is a direct consequence of Remark 15. Thus, β_T is an injective morphism in $\mathsf{SN_c}$. Finally, since the condition (CK) is equivalent to the surjectivity of β_T, then it follows from Lemma 16 that β_T is a surjective map. Therefore, β_T is an isomorphism in $\mathsf{SN_c}$. $\square$

Taking into account the previous results of this section, the fact that if $H \in \mathsf{SH}$ then α_H is an isomorphism in SH, and the categorical equivalence between $\mathsf{KA_c}$ and BDL, we obtain the main theorem of this paper.

Theorem 18. *The functors* K *and* C *establish a categorical equivalence between* SH *and* $\mathsf{SN_c}$ *with natural isomorphisms* α *and* β.

4. Connection with existing literature

In [16] it was proved that there exists a categorical equivalence between SH and an algebraic category denoted by KSH (see Definition 22). The original motivation to consider this algebraic category comes from a different definition of the binary operation given in (1) of Section 1 on $(\mathrm{K}(H), \wedge, \vee, \sim, \mathrm{c}, 0, 1)$, where $H \in \mathsf{SH}$. Combining the categorical equivalence between SH and KSH with Theorem 18, we obtain that there exists a categorical equivalence between $\mathsf{SN_c}$ and KSH. In this section we do a more detailed study about the connection between the categories $\mathsf{SN_c}$ and KSH.

We assume the reader is familiar with commutative residuated lattices [15]. An *involutive residuated lattice* is a bounded, integral and commutative residuated lattice $(T, \wedge, \vee, *, \rightarrow, 0, 1)$ such that for every $x \in T$ it holds that $\neg\neg x = x$, where $\neg x := x \rightarrow 0$ and 0 is the first element of T [8]. In an involutive residuated lattice it holds that $x * y = \neg(x \rightarrow \neg y)$ and $x \rightarrow y = \neg(x * \neg y)$. A *Nelson lattice* [8] is an involutive residuated lattice $(T, \wedge, \vee, *, \rightarrow, 0, 1)$ which satisfies the additional inequality $(x^2 \rightarrow y) \wedge ((\neg y)^2 \rightarrow \neg x) \leq x \rightarrow y$, where $x^2 := x * x$. See also [22].

Remark 19. a) Let $(T, \wedge, \vee, \Rightarrow, \sim, 0, 1)$ be a Nelson algebra. We define on T the binary operations $*$ and $\rightarrow$ by

$$x * y := \sim(x \Rightarrow \sim y) \vee \sim(y \Rightarrow \sim x),$$
$$x \rightarrow y := (x \Rightarrow y) \wedge (\sim y \Rightarrow \sim x).$$

Then, [8, Theorem 3.1] says that $(T, \wedge, \vee, \rightarrow, *, 0, 1)$ is a Nelson lattice. Moreover, $\sim x = \neg x = x \rightarrow 0$.

b) Let $(T, \wedge, \vee, *, \rightarrow, 0, 1)$ be a Nelson lattice. We define on T a binary operation $\Rightarrow$ and a unary operation $\sim$ by

$$x \Rightarrow y := x^2 \rightarrow y,$$
$$\sim x := \neg x,$$

where $x^2 := x * x$. In particular, $x \Rightarrow y = (\sim (x \rightarrow \sim x)) \rightarrow y$. Then, [8, Theorem] says that the algebra $(T, \wedge, \vee, \Rightarrow, \sim, 0, 1)$ is a Nelson algebra.

c) Notice that in [8, Theorem 3.11] it was also proved that the category of Nelson algebras and the category of Nelson lattices are isomorphic. Taking into account the construction of this isomorphism (see [8]), we obtain that the variety of Nelson algebras and the variety of Nelson lattices are term equivalent, and the term equivalence is given by the operations we have defined in items a) and b).

The results from [8] about the connections between Nelson algebras and Nelson lattices mentioned in Remark 19 are based on results from Spinks and Veroff [21]. More precisely, the term equivalence of the varieties of Nelson algebras and Nelson lattices was discovered by Spinks and Veroff in [21].

A *centered Nelson lattice* is an algebra $(T, \vee, \wedge, *, \rightarrow, \mathrm{c}, 0, 1)$, where the reduct $(T, \vee, \wedge, *, \rightarrow, 0, 1)$ is a Nelson lattice and c is an element of T such that $\neg \mathrm{c} = \mathrm{c}$. It follows from Remark 19 that the variety of centered Nelson algebras and the variety of centered Nelson lattices are term equivalent. We write $\mathsf{NL_c}$ for the category of centered Nelson lattices.

Remark 20. Let $(H, \wedge, \vee, \rightarrow, 0, 1) \in \mathsf{HA}$. We know that $(\mathrm{K}(H), \wedge, \vee, \Rightarrow, \sim, \mathrm{c}, 0, 1) \in \mathsf{NA_c}$, where $\Rightarrow$ is the operation given in (1). Hence, it follows from Remark 19 that $(\mathrm{K}(H), \wedge, \vee, *, \rightarrow, \mathrm{c}, 0, 1) \in \mathsf{NL_c}$, where for (a, b) and (d, e) in $\mathrm{K}(H)$ the operations $*$ and $\rightarrow$ take the form

$$(a, b) * (d, e) = (a \wedge d, (a \rightarrow e) \wedge (d \rightarrow b)), \tag{25}$$

$$(a, b) \rightarrow (d, e) = ((a \rightarrow d) \wedge (e \rightarrow b), a \wedge e). \tag{26}$$

We write $\to$ both for the implication in H as for the implication in $\mathrm{K}(H)$ as Nelson lattice.

It was proved in [16] that K defines a functor from HA to $\mathsf{NL_c}$, where K is defined using the same construction given in Section 1 but changing the binary operation given in (1) of Section 1 by the binary operation given in (26) of Remark 20. Also it was proved in [16] that C is a functor from KSH to SH, where C is defined as in Section 1. Moreover, we have that the maps α_H for $H \in \mathsf{SH}$ and β_T for $T \in \mathsf{KSH}$ are isomorphisms. The following result is [16, Proposition 7].

Proposition 21. *The functors* K *and* C *establish a categorical equivalence between* HA *and* $\mathsf{NL_c}$ *with natural isomorphisms* α *and* β*.*

In what follows we recall the definition of the category KSH given in [16].

Definition 22. We write KSH for the algebraic category whose objects are algebras $(T, \wedge, \vee, \to, \sim, \mathrm{c}, 0, 1)$ of type $(2, 2, 2, 1, 0, 0, 0)$ such that $(T, \wedge, \vee, \sim, \mathrm{c}, 0, 1)$ is a centered Kleene algebra and $\to$ is a binary operation on T which satisfies the following conditions for every $x, y \in T$:

(1) $\mathrm{c} \leq x \to (y \vee \mathrm{c})$,
(2) $x \to x = 1$,
(3) $(x \to y) \wedge \mathrm{c} = (\sim x \wedge \mathrm{c}) \vee (y \wedge \mathrm{c})$,
(4) $(x \to \sim y) \vee \mathrm{c} = ((x \vee \mathrm{c}) \to (\sim y \vee \mathrm{c})) \wedge ((y \vee \mathrm{c}) \to (\sim x \vee \mathrm{c}))$,
(5) $x \wedge ((x \vee \mathrm{c}) \to (y \vee \mathrm{c})) = x \wedge (y \vee \mathrm{c})$,
(6) $x \wedge ((y \vee \mathrm{c}) \to (z \vee \mathrm{c})) = x \wedge (((x \vee \mathrm{c}) \wedge (y \vee \mathrm{c})) \to ((x \vee \mathrm{c}) \wedge (z \vee \mathrm{c})))$.

By considering the objects of KSH as algebras $(T, \wedge, \vee, \to, *, \sim, \mathrm{c}, 0, 1)$, where $*$ is defined as in (25) of Remark 20, we have that $\mathsf{NL_c}$ is a full subcategory of KSH [10, Proposition 4.4]. The construction from Proposition 21 can be extended to prove a categorical equivalence between SH and KSH, as it is shown in [16, Theorem 51].

Theorem 23. *The functors* K *and* C *establish a categorical equivalence between* SH *and* KSH *with natural isomorphisms* α *and* β*.*

The following result follows from theorems 18 and 23.

Theorem 24. *There exists a categorical equivalence between* $\mathsf{SN_c}$ *and* KSH*.*

Notice we are using the same notation K to refer us to a functor from SH to $\mathsf{SN_c}$ and also for a functor from SH to KSH (similarly with the notation C). We believe it is clear which is the corresponding functor considered in each case.

4.1. **The varieties $\mathsf{SN_c}$ and KSH are not term equivalent.** We know that the varieties $\mathsf{NA_c}$ and $\mathsf{NL_c}$ are term equivalent. We will prove that the varieties $\mathsf{SN_c}$ and KSH are not term equivalent by using the construction given in Remark 19.

Consider an algebra $(T, \wedge, \vee, \to, \sim, \mathrm{c}, 0, 1) \in \mathsf{SN_c}$ and define a binary operation $\to_{\mathsf{KSH}}$ by

$$x \to_{\mathsf{KSH}} y = (x \to y) \wedge (\sim y \to \sim x).$$

Remark 25. Let $H \in \mathsf{SH}$ and $\to$ the implication of H. Let $a, b, d, e \in H$ such that $(a, b), (d, e) \in \mathrm{K}(H)$. Then

$$\begin{aligned} (a, b) \to_{\mathsf{KSH}} (d, e) &= ((a, b) \Rightarrow (d, e)) \cap ((e, d) \Rightarrow (b, a)) \\ &= (a \to d, a \wedge e) \cap ((e \to b), a \wedge e) \\ &= ((a \to d) \wedge (e \to b), a \wedge e). \end{aligned}$$

Therefore, the implication of the algebra $\mathrm{K}(H) \in \mathsf{KSH}$ (see Remark 20) is exactly the binary operation $\to_{\mathsf{KSH}}$ defined in $\mathrm{K}(H)$.

In the following proposition we will see that if we consider an algebra in $\mathsf{SN_c}$ then we can define an algebra in KSH.

Proposition 26. *Let* $(T, \wedge, \vee, \to, \sim, \mathrm{c}, 0, 1) \in \mathsf{SN_c}$. *Then,* $(T, \wedge, \to_{\mathsf{KSH}}, \sim, \mathrm{c}, 0, 1) \in \mathsf{KSH}$.

Proof. We will write $\mathrm{K}^{\mathsf{SN_c}}$ for the functor K from SH to $\mathsf{SN_c}$ and $\mathrm{K}^{\mathsf{KSH}}$ for the functor K from SH to KSH. In this proof we will use theorems 18 and 23.

Let $(T, \wedge, \vee, \to, \sim, \mathrm{c}, 0, 1) \in \mathsf{SN_c}$. Hence, $(T, \wedge, \vee, \to, \sim, \mathrm{c}, 0, 1) \cong \mathrm{K}^{\mathsf{SN_c}}(\mathrm{C}(T))$, where the isomorphism from $(T, \wedge, \vee, \to, \sim, \mathrm{c}, 0, 1)$ to $\mathrm{K}^{\mathsf{SN_c}}(\mathrm{C}(T))$ is given by β_T. Since $\to_{\mathsf{KSH}}$ can be written in terms of $\to$, $\wedge$ and $\sim$, then β_T preserves the operation $\to_{\mathsf{KSH}}$, i.e., for every $x, y \in T$ it holds that $\beta_T(x \to_{\mathsf{KSH}} y) = \beta_T(x) \to_{\mathsf{KSH}} \beta_T(y)$. Thus, it follows from Remark 25 that $(T, \wedge, \vee, \to_{\mathsf{KSH}}, \sim, \mathrm{c}, 0, 1) \cong \mathrm{K}^{\mathsf{KSH}}(\mathrm{C}(T))$. However $\mathrm{K}^{\mathsf{KSH}}(\mathrm{C}(T)) \in \mathsf{KSH}$, so $(T, \wedge, \to_{\mathsf{KSH}}, \sim, \mathrm{c}, 0, 1) \in \mathsf{KSH}$ because KSH is a variety. □

Let $(T, \wedge, \vee, \to, \sim, \mathrm{c}, 0, 1) \in \mathsf{KSH}$. Define a binary operation $\hat{\to}$ by

$$x \hat{\to} y := (\sim (x \to \sim x)) \to y.$$

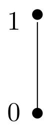
The definition of $\hat{\to}$ is motivated by item b) of Remark 19. Then, it naturally arises the following question: does it hold that $(T, \wedge, \hat{\to}, \sim, \mathrm{c}, 0, 1) \in \mathsf{SN_c}$? The answer is negative, as we show in what follows.

Let $(H, \wedge, \vee, \to, 0, 1) \in \mathsf{SH}$ and also write $\to$ for the implication in $\mathrm{K}(H) \in \mathsf{KSH}$. Let $(a, b), (d, e) \in \mathrm{K}(H)$. Then

$$\begin{aligned}(a,b) \hat{\to} (d,e) &= (\sim ((a,b) \to \sim (a,b))) \to (d,e) \\ &= (\sim (a \to b, a)) \to (d,e) \\ &= (a, a \to b) \to (d,e) \\ &= ((a \to d) \wedge (e \to (a \to b)), a \wedge e).\end{aligned}$$

Note that $(a,b) \hat{\to} (d,e) = (a \to d, a \wedge e)$ if and only if $a \to d \leq e \to (a \to b)$. In Heyting algebras the condition $a \to d \leq e \to (a \to b)$ is true whenever $a \wedge b = d \wedge e = 0$. However, in semi-Heyting algebras this condition is not necessarily true. For instance, consider the semi-Heyting algebra given by

$\bar{\mathbf{2}}$:



$\to$	0	1
0	1	0
1	0	1

with $a = d = 0$ and $b = e = 1$. In this case $a \wedge b = d \wedge e = 0$, $a \to d = 1$ and $e \to (a \to b) = 0$, so $a \to d \not\leq e \to (a \to b)$. Hence, we have that there exists $(T, \wedge, \vee, \to, \sim, \mathrm{c}, 0, 1) \in \mathsf{KSH}$ such that the algebra $(T, \wedge, \vee, \hat{\to}, \sim, \mathrm{c}, 0, 1) \notin \mathsf{SN_c}$. Therefore, the varieties $\mathsf{SN_c}$ and KSH are not term equivalent by using the construction given in Remark 19.

4.2. **Congruences.** Finally we study the connection between the congruences of $\mathsf{SN_c}$ and KSH. For $T_{\mathsf{SN_c}} = (T, \wedge, \vee, \to, \sim, \mathrm{c}, 0, 1) \in \mathsf{SN_c}$ consider the algebra $T_{\mathsf{KSH}} = (T, \wedge, \vee, \to_{\mathsf{KSH}}, \sim, \mathrm{c}, 0, 1) \in \mathsf{KSH}$ (see Proposition 26).

We start with the following preliminary lemma.

Lemma 27. *If* $T_{\mathsf{SN_c}} \in \mathsf{SN_c}$ *then* $\mathrm{C}(T_{\mathsf{SN_c}}) = \mathrm{C}(T_{\mathsf{KSH}})$.

Proof. Let $x, y \geq \mathrm{c}$. We will prove that $x \to y = x \to_{\mathsf{KSH}} y$. Since $x \to_{\mathsf{KSH}} y = (x \to y) \wedge (\sim y \to \sim x)$, it is enough to prove that $\sim y \to \sim x = 1$. In order to show it we use Remark 15 as follows:

$$\begin{aligned}(\sim y \to\sim x) \vee \mathrm{c} &= (\sim y \vee \mathrm{c}) \to (\sim x \vee \mathrm{c})\\ &= c \to c\\ &= 1\\ &= 1 \vee \mathrm{c}\end{aligned}$$

and

$$\begin{aligned}(\sim y \to\sim x) \wedge \mathrm{c} &= \sim (\sim (\sim y \to\sim x) \vee \mathrm{c})\\ &= \sim ((\sim y \vee \mathrm{c}) \wedge (x \vee \mathrm{c}))\\ &= \sim \mathrm{c}\\ &= \mathrm{c}\\ &= 1 \wedge \mathrm{c}.\end{aligned}$$

Hence, taking into account the distributivity of the underlying lattice of $T_{\mathsf{SN_c}}$ we deduce the equality $\sim y \to\sim x = 1$, which was our aim. □

If T is an algebra we write $\mathrm{Con}(T)$ for the lattice of congruences of T.

Proposition 28. *Let* $T_{\mathsf{SN_c}} \in \mathsf{SN_c}$. *Then* $\mathrm{Con}(T_{\mathsf{SN_c}}) = \mathrm{Con}(T_{\mathsf{KSH}})$

Proof. It follows from Proposition 26 that $\mathrm{Con}(T_{\mathsf{SN_c}}) \subseteq \mathrm{Con}(T_{\mathsf{KSH}})$.

Conversely, let $\theta \in \mathrm{Con}(T_{\mathsf{KSH}})$ and $x, y, z, w \in T$ such that $(x, y) \in \theta$ and $(z, w) \in \theta$. First note that it follows from Remark 15 that

$$(x \vee \mathrm{c}) \to (z \vee \mathrm{c}) = (x \to z) \vee \mathrm{c}, \tag{27}$$

$$(y \vee \mathrm{c}) \to (w \vee \mathrm{c}) = (y \to w) \vee \mathrm{c}. \tag{28}$$

Besides $(x \vee \mathrm{c}, y \vee \mathrm{c}) \in \theta$ and $(z \vee \mathrm{c}, w \vee \mathrm{c}) \in \theta$, so

$$((x \vee \mathrm{c}) \to_{\mathsf{KSH}} (z \vee \mathrm{c}), (y \vee \mathrm{c}) \to_{\mathsf{KSH}} (w \vee \mathrm{c})) \in \theta.$$

It follows from Lemma 27 that $(x \vee \mathrm{c}) \to_{\mathsf{KSH}} (z \vee \mathrm{c}) = (x \vee \mathrm{c}) \to (z \vee \mathrm{c})$ and $(y \vee \mathrm{c}) \to_{\mathsf{KSH}} (w \vee \mathrm{c}) = (y \vee \mathrm{c}) \to (w \vee \mathrm{c})$. Hence, it follows from (27) and (28) that

$$((x \to z) \vee \mathrm{c}, (y \to w) \vee \mathrm{c}) \in \theta. \tag{29}$$

On the other hand, it follows from Remark 15 that

$$(x \to z) \wedge \mathrm{c} = (\sim x \wedge \mathrm{c}) \vee (z \wedge \mathrm{c}), \tag{30}$$

$$(y \to w) \wedge \mathrm{c} = (\sim y \wedge \mathrm{c}) \vee (w \wedge \mathrm{c}). \tag{31}$$

Besides, since $(x, y) \in \theta$ and $(z, w) \in \theta$ then $(\sim x \wedge \mathrm{c}, \sim y \wedge \mathrm{c}) \in \theta$ and $(z \wedge \mathrm{c}, w \wedge \mathrm{c}) \in \theta$, so $((\sim x \wedge \mathrm{c}) \vee (z \wedge \mathrm{c}), (\sim y \wedge \mathrm{c}) \vee (w \wedge \mathrm{c})) \in \theta$. Thus, taking into account (30) and (31) we conclude that

$$((x \to z) \wedge \mathrm{c}, (y \to w) \wedge \mathrm{c}) \in \theta. \tag{32}$$

Hence, it follows from (29), (32) and the distributivity of the underlying lattice of $T_{\mathsf{SN_c}}$ that $(x \to z, y \to w) \in \theta$, so $\theta \in \mathrm{Con}(T_{\mathsf{SN_c}})$. Then, $\mathrm{Con}(T_{\mathsf{SN_c}}) = \mathrm{Con}(T_{\mathsf{KSH}})$. □

Let L be a distributive lattice. Recall that a non empty subset F of L is said to be a filter the following two conditions are satisfied, for $a, b \in L$:
1) if $a \leq b$ and $a \in F$ then $b \in F$,
2) $a \wedge b \in F$ whenever $a, b \in F$.
We write $\mathrm{Fil}(L)$ by the set of filters of L.

Corollary 29. *Let* $T_{\mathsf{SN_c}} \in \mathsf{SN_c}$. *Then* $\mathrm{Con}(T_{\mathsf{SN_c}}) \cong \mathrm{Fil}(\mathrm{C}(T_{\mathsf{SN_c}}))$.

Proof. It follows from Proposition 28 that $\mathrm{Con}(T_{\mathsf{SN_c}}) = \mathrm{Con}(T_{\mathsf{KSH}})$. Besides, it follows from [16, Proposition 61] that $\mathrm{Con}(T_{\mathsf{KSH}}) \cong \mathrm{Con}(\mathrm{C}(T_{\mathsf{KSH}}))$. Also note that $\mathrm{Con}(\mathrm{C}(T_{\mathsf{KSH}})) \cong \mathrm{Fil}(\mathrm{C}(T_{\mathsf{KSH}}))$ because $\mathrm{C}(T_{\mathsf{KSH}}) \in \mathsf{SH}$. Finally, taking into account Lemma 27 we have that $\mathrm{Fil}(\mathrm{C}(T_{\mathsf{KSH}})) = \mathrm{Fil}(\mathrm{C}(T_{\mathsf{SN_c}}))$. Therefore, $\mathrm{Con}(T_{\mathsf{SN_c}}) \cong \mathrm{Fil}(\mathrm{C}(T_{\mathsf{SN_c}}))$. □

In what follows we will use the notation of the proof of Proposition 26. Straightforward computations based in Lemma 27 proves that if $H \in \mathsf{SH}$ then

$$\mathrm{Con}(\mathrm{K}^{\mathsf{KSH}}(H)) = \mathrm{Con}(\mathrm{K}^{\mathsf{SN_c}}(H)).$$

Proposition 30. *Let* $T \in \mathsf{KSH}$. *Then* $\mathrm{Con}(T) \cong \mathrm{Con}(\mathrm{K}^{\mathsf{SN_c}}(\mathrm{C}(T)))$.

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