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Alejandro Saporiti · Fernando Tohmé

Single-crossing, strategic voting and the median choice rule

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Abstract This paper studies the strategic foundation of the Representative Voter Theorem (Rothstein in: Pub Choice 72:193–212, 1991), also called the "second version" of the Median Voter Theorem. As a by-product, it also considers the existence of strategy-proof social choice functions over the domain of single-crossing preferences. The main result shows that single-crossing constitutes a domain restriction over the real line that allows not only majority voting equilibria, but also non-manipulable choice rules. In particular, this is true for the median rule, which is found to be group strategic-proof over the full set of alternatives and over every nonempty subset. In addition, the paper also examines the relation between single-crossing and order-restriction. And it uses this relation together with the strategy-proofness of the median rule to prove that the outcome predicted by the Representative Voter Theorem can be implemented in dominant strategies through a simple mechanism. This mechanism is a two-stage voting procedure in which, first, individuals select a representative among themselves, and then the winner chooses a policy to be implemented by the planner.

1 Introduction

In the last 25 years, *single-crossing* has become a "popular" feature of preferences within the field of Political Economy.[1] From the seminal works of Roberts (1977)

A. Saporiti (✉)
School of Economics, University of Manchester, Oxford Road,
Manchester M13 9PL, UK
E-mail: alejandro.saporiti@manchester.ac.uk

F. Tohmé
CONICET and Departamento de Economía, Universidad Nacional del Sur,
12 de Octubre y San Juan, 8000 Bahía Blanca, Argentina
E-mail: ftohme@criba.edu.ar

[1] See, for example, Persson and Tabellini (2000).

and Grandmont (1978) and, more recently, due to the theoretical contributions of Rothstein (1990, 1991), Gans and Smart (1996), and Austen-Smith and Banks (1999), it is now well known that this domain restriction is sufficient to guarantee the existence of equilibria in one-dimensional models of majority voting, especially in situations where single-peakedness does not hold.

This restriction is not only technically convenient, but it also makes sense in many political settings. In few words, the single-crossing property used in the context of voting, which is similar to that used in the principal-agent literature, says that, given any two policies, one of them more to the right than the other, the more rightist an individual is (with respect to another individual) the more he will "tend to prefer" the right-wing policy over the left-wing one.[2]

Thus, unlike single-peakedness, single-crossing is a restriction *across* individual preferences, on the character of the voters' heterogeneity, rather than on the shape of individual preferences. The main idea behind it is that, in many circumstances, *ordering* people according to a single parameter (like income, productivity, intertemporal preferences, ideological position, etc.) may be more natural than ordering alternatives. This condition projects the conflict of interests among individuals over a one-dimensional space, and then the *type* of each agent is located along this left–right scale in such a way that, for any pair of alternatives, the set of types preferring one of the alternatives all lie to one side of those who prefer the other.

It turns out that single-crossing not only guarantees the existence of majority voting equilibria but also provides a simple characterization of the core of the majority rule. In effect, under this condition the core is simply the ideal point of the median type agent, where the latter is defined over the ordering of individual types for which the preference profile is single-crossing.[3] This result is sometimes referred to in the literature as the *Representative Voter Theorem* (Rothstein 1991) (henceforth RVT) or, alternatively, as "the second version" of the Median Voter Theorem (Myerson 1996; Gans and Smart 1996).

The main problem is that, unlike the *original* Median Voter Theorem over single-peaked preferences, whose noncooperative foundation was provided by Black (1948), first, and then by Moulin (1980), the RVT is based on the assumption that individuals honestly reveal their preferences. That is, it is derived assuming *sincere voting*. Clearly, this assumption is difficult to maintain in applications that focus on policy choices made in strategic frameworks. Hence, a natural question arises with respect to its applicability in those models.

This paper studies the strategic foundation of the RVT. As a by-product, it also considers the existence of nontrivial strategy-proof social choice functions on the domain of single-crossing preferences. There are several reasons that justify carrying out this analysis. But the first and more important one is that, even though single-crossing is now largely used in models of collective decision-making, nothing has been said in the literature about the possibility of manipulation over this domain. In particular, the "single-crossing version" of the Median Voter Theorem is used without caring much about its strategic foundation. So, one of the main purposes here is filling out this gap.

[2] The formal definition is given in Sect 2, Definition 2.

[3] In contrast, under single-peaked preferences, the core of the majority rule is given by the median ideal point over the ordering of alternatives with respect to which the profile is single peaked.

In addition, the study is also motivated by a more technical fact, though not less important. Single-crossing assumes the existence of a specific kind of correlation or interdependence among individual preferences. As a result, the set of single-crossing preference profiles, i.e., the domain of the social choice function, cannot be expressed as a product set. However, this contrasts with much of the work on strategy-proofness, which usually focuses on social choice rules defined over Cartesian preference domains.[4]

The main result of the paper is that single-crossing constitutes a domain restriction over the real line that allows not only majority voting equilibria, but also non-manipulable choice rules. In particular, this is true for the median rule, which is found to be strategy-proof and group strategic-proof over the full set of alternatives and over every nonempty subset.

In addition, the paper also analyzes the relation between single-crossing and order-restriction (Rothstein 1990, 1991). And it extends the former, by introducing the domain of *broad single-crossing* preferences, to make both concepts equivalent. Finally, it uses this relation together with the strategy-proofness of the median choice rule to prove that the outcome predicted by the RVT can be implemented in dominant strategies. This is carried out through a simple mechanism, in which first individuals select a representative among themselves, and then the winner chooses a policy to be implemented by the planner.

The paper is organized as follows. Section 2 presents the model, the notation, and definitions. Section 3 exhibits the equivalence between single-crossing and order-restriction for preferences indexed by the types of the agents. Section 4 presents the nonstrategic version of the RVT. Results related to strategy-proofness are discussed in Sect 5. Section 6 provides the *indirect* implementation of the median rule and the game-theoretic counterpart of the RVT. Final remarks are made in Sect 7.

2 Preliminaries

Consider a society with a finite number of agents, represented by the elements of the set $I = \{1, \ldots, n\}$, where $|I| = n$ is odd and $n > 2$. These agents face a collective choice problem, which consists in choosing an alternative (for example, the level of a public good) from a finite subset of the real line. They make this choice by voting.

The set of all possible outcomes is $X = \{x_1, \ldots, x_l\}$, $|X| > 2$, where X is a finite subset of the non-negative real line $\Re_+$. The set of feasible alternatives may be either the entire X or just one of its nonempty subsets. We denote $\tilde{X}$ a generic subset of X and $A(X) = \{\tilde{X}\colon \tilde{X} \in 2^X \setminus \emptyset\}$. In words, X is the universal set of outcomes, whereas a particular situation, or *agenda*, involves a $\tilde{X} \in A(X)$. Following the standard notation, for a vector $(x_1, \ldots, x_n) \in \Re^n_+$, we let $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ and $(\hat{x}_i, x_{-i}) = (x_1, \ldots, x_{i-1}, \hat{x}_i, x_{i+1}, \ldots, x_n)$, where $\hat{x}_i \in \Re_+$. In addition, for any group of agents $S \subseteq I$, we denote $(x_S, x_{\bar{S}}) = ((x_i)_{i \in S}, (x_j)_{j \in \bar{S}})$, where $\bar{S} = I \setminus S$.

[4] An exception is Campbell and Kelly (2003a,b), who characterize the family of strategy-proof social choice rules on the domain where a Condorcet winner always exists.

Let $P(X)$ be the set of all complete, transitive, and antisymmetric binary orderings of X. We say $P(X)$ is the *universal domain* of individual preferences.[5] We assume agent i's preferences over X are completely characterized by a single parameter $\theta_i \in \Theta \subset \Re$. As usual, we interpret θ_i as being agent i's *type*. That is, we assume there exists a function $\Phi\colon \Theta \to P(X)$ that assigns a unique binary relation $\Phi(\theta) \in P(X)$ to each type $\theta \in \Theta$. Then, we say that $\succ_i$ represents the preferences of an agent i of type θ_i if,

$$\forall x,\ y \in X,\ x \succ_i y \Leftrightarrow x\ \Phi(\theta_i)\ y.$$

The following example, taken from Persson and Tabellini (2000), illustrates how these preferences can arise naturally in political-economic models:

Example 1 Consider the following simplified version of the redistributive distortionary taxation model of Roberts (1977). Suppose individual $i \in I$ has preferences $u(c_i, l_i) = c_i + v(l_i)$, where c_i denotes individual consumption, l_i leisure and $v_l > 0$ and $v_{ll} \leq 0$.[6] The individual's budget constraint is $c_i \leq (1-t)h_i + f$, where $t \in (0, 1)$ is an income tax rate, $f \in \Re_+$ a lump-sum transfer and h_i the individual labor supply. The real wage is exogenous and normalized at unity. Individuals are heterogenous in a productivity parameter $\theta_i \in \Theta \subset \Re$, which is distributed in the population with mean $\bar{\theta}$. Given these different productivities, each individual i faces an "effective" time constraint $1 - \theta_i \geq l_i + h_i$. Finally, the government runs a balanced budget, so that $nf \leq t \sum_{i\in I} h_i$.

Solving the model and substituting the solution into the individual utility function, the *induced* preferences of i over different tax rates can be expressed as $w_i(t) = h(t) + v[1 - h(t) - \bar{\theta}] - (1-t)(\theta_i - \bar{\theta})$, where $h(t) = 1 - \bar{\theta} - v_l^{-1}(1-t)$ is the average labor supply. Thus, for each individual $i \in I$, $w_i(t)$ is completely determined by θ_i.

Given a preference $\succ_i$, we define agent i's preferences over the agenda $\tilde{X} \in A(X)$, noted $\tilde{\succ}_i$, as follows: $\forall x,\ y \in \tilde{X}, x \tilde{\succ}_i y$ if and only if $x \succ_i y$. The maximal set associated with the pair $(\tilde{X}, \tilde{\succ}_i)$ is $M(\tilde{X}, \tilde{\succ}_i) = \{x \in \tilde{X}\colon \forall y \in \tilde{X}\backslash\{x\},\ x \tilde{\succ}_i y\}$. Notice that since preferences are strict, maximal sets are singletons. That is, $M(\tilde{X}, \tilde{\succ}_i) = \{\tau(\tilde{\succ}_i)\}$, where $\tau(\tilde{\succ}_i)$ is agent i's most preferred alternative in $\tilde{X}$ according to his preference relation $\tilde{\succ}_i$.

A preference profile on $\tilde{X} \in A(X)$, associated to a profile of types $\theta = (\theta_1, \ldots, \theta_n) \in \Theta^n$, is an n-tuple $(\tilde{\succ}_1, \ldots, \tilde{\succ}_n) = (\tilde{\Phi}(\theta_1), \ldots, \tilde{\Phi}(\theta_n))$ in $P(\tilde{X})^n$, where $\tilde{\Phi}$ represents the restriction of Φ over $\tilde{X}$. We assume each agent observes θ, so there is complete information among agents about their preferences over $\tilde{X}$. Extending our earlier conventions to preference profiles, we have that $\tilde{\succ}_{-i} = (\tilde{\succ}_1, \ldots, \tilde{\succ}_{i-1}, \tilde{\succ}_{i+1}, \ldots, \tilde{\succ}_n)$. Similarly, the profile obtained by changing agent i's preferences for $\hat{\succ}_i \in P(\tilde{X})$ is $(\hat{\succ}_i, \tilde{\succ}_{-i}) = (\tilde{\succ}_1, \ldots, \tilde{\succ}_{i-1}, \hat{\succ}_i, \tilde{\succ}_{i+1}, \ldots, \tilde{\succ}_n)$. And, for any group of agents $S \subseteq I$, $(\tilde{\succ}_S, \tilde{\succ}_{\bar{S}}) = ((\tilde{\succ}_i)_{i\in S}, (\tilde{\succ}_j)_{j\in\bar{S}})$. Finally,

[5] Indifference between alternatives is not allowed. This is a quite common assumption when the set of alternatives is finite. In this paper, it is also adopted to simplify the proofs of our main results.

[6] As usual, v_l and v_{ll} denote, respectively, the first and the second derivate of the function $v(l_i)$.

given a profile $\tilde{\succsim} = (\tilde{\succsim}_1, \ldots, \tilde{\succsim}_n) \in P(\tilde{X})^n$, we denote $\Theta_I(\tilde{\succsim}) = \{\theta \in \Theta\colon \exists\, i \in I$ such that $\tilde{\succsim}_i = \tilde{\Phi}(\theta)\}$ the set of *actual* types.

These preferences can be aggregated. The input for this aggregation process is the set of individuals' *declarations*. These declarations are intended to provide information about their true types, although their sincerity may not be ensured.

The aggregation process is represented by a social choice function. For any $\tilde{X} \in A(X)$, a *social choice function* f on $P(\tilde{X})^n$ is a single-valued mapping $f\colon P(\tilde{X})^n \to \tilde{X}$ that associates to each preference profile $\tilde{\succsim} = (\tilde{\succsim}_1, \ldots, \tilde{\succsim}_n) \in P(\tilde{X})^n$ a unique outcome $f(\tilde{\succsim}) \in \tilde{X}$.

We are primarily interested in aggregation procedures conducted by pairwise majority voting. Then, we focus the analysis on a particular social choice function: the *median choice rule*. Let $m : \Re^n_+ \to \Re_+$ be the *median function* on $\Re^n_+$. Then, for all $x \in \Re^n_+$, $m(x)$ is said the *median* of $x = (x_1, \ldots, x_n)$ if and only if $|\{x_i \in \Re_+\colon x_i \le m(x)\}| \ge (n+1)/2$ and $|\{x_j \in \Re_+ : m(x) \le x_j\}| \ge (n+1)/2$. Since n is odd, this function is always well defined.

Definition 1 *The social choice function f^m on $P(\tilde{X})^n$ is called the median choice rule if for each $(\tilde{\succsim}_1, \ldots, \tilde{\succsim}_n) \in P(\tilde{X})^n$, $f^m(\tilde{\succsim}_1, \ldots, \tilde{\succsim}_n) = m(\tau(\tilde{\succsim}_1), \ldots, \tau(\tilde{\succsim}_n))$.*

In the following sections, we study the incentive compatibility properties of the median choice rule over the domain of single-crossing preference profiles. So let us introduce now the formal definition of this condition.

Definition 2 *A preference profile $(\Phi(\theta_1), \ldots, \Phi(\theta_n))$ is single-crossing on X if for all $\{x, y\} \subset X$, $i, j \in I$,*

$$[y > x,\ \theta_j > \theta_i,\ \text{and}\ y\,\Phi(\theta_i)\,x] \;\Rightarrow\; [y\,\Phi(\theta_j)\,x]. \tag{SC}$$

We denote SC(X) the set of all single-crossing preference profiles on X (with respect to the linear order $\ge$).[7] Notice that the property of being single-crossing is preserved in the induced preferences. That is, if $(\succ_1, \ldots, \succ_n) \in \mathrm{SC}(X)$, then $(\tilde{\succsim}_1, \ldots, \tilde{\succsim}_n) \in \mathrm{SC}(\tilde{X})$ for all $\tilde{X} \in A(X)$, where SC$(\tilde{X})$ is the set of all single-crossing preference profiles on $\tilde{X}$.

In the political arena, single-crossing makes sense if, for example, individual types are interpreted as being different ideological characters, arranged in a left–right scale, and alternatives are policies or candidates to be chosen by the society. Put in this way, it says that, given any two policies, one of them more to the right than the other, the more rightist a type the more will he tend to prefer the right-wing policy over the left-wing one.

The recent interest on this preference domain is due to the fact that, like single-peakedness, single-crossing has been shown to be sufficient to guarantee the existence of majority voting equilibria. However, apart from this fact, it should be clear that both conditions are independent, in the sense that neither property is logically implied by the other.

[7] Other restrictions related to single-crossing are *hierarchical adherence*, *intermediateness*, *order-restriction* and *unidimensional alignment*. For more on them, see Roberts (1977), Grandmont (1978), Rothstein (1990, 1991), Gans and Smart (1996), Myerson (1996), Austen-Smith and Banks (1999), and List (2001).

To see this more formally, let us define the single peaked condition in the following two alternative ways. For a set A, let $Q(A)$ denote the set of all linear orders on A.[8] For example, if $A \subseteq \Re$, then the usual order on the real line $\geq \in Q(A)$. For each ordering $q \in Q(A)$, define the antisymmetric part of q, noted $\sim q$, as follows: $a \sim q\, b$ if and only if $a\, q\, b$ and $\neg[b\, q\, a]$. For instance, the antisymmetric part of the "greater than or equal to" relation over the real numbers is the "strictly greater than" relationship $>$.

Definition 3 *A profile* $(\succ_1, \ldots, \succ_n) \in P(X)^n$ *is single-peaked on X if there exists a linear order* $q \in Q(X)$ *such that* $\forall i \in I$, $\{x, y, z\} \subseteq X$,

$$[x \succ_i y \text{ and } x \sim q\, y \sim q\, z] \Rightarrow [y \succ_i z]. \qquad (\mathrm{SP}_1)$$

We denote SP(X, q) the set of all single-peaked (with respect to q) preference profiles on X. Since q is common to all these profiles, we can omit it and simply write SP(X). Clearly, given that individual preferences are strict, $\succ \in$ SP(X) implies $\tilde{\succ} \in$ SP$(\tilde{X})$, for all $\tilde{X} \in A(X)$.

Alternatively, as long as we may be interested in neutral social choice rules (like f^m), we can allow q to change from one profile to another. This produces a second definition of single -peakedness:

Definition 4 *A profile* $\succ = (\succ_1, \ldots, \succ_n) \in P(X)^n$ *is single peaked on X if there exists a linear order* $q(\succ) \in Q(X)$*, associated to* $\succ$*, such that* $\forall i \in I$, $\{x, y, z\} \subseteq X$,

$$[x \succ_i y \text{ and } x \sim q(\succ)\, y \sim q(\succ)\, z] \Rightarrow [y \succ_i z]. \qquad (\mathrm{SP}_2)$$

We denote SP$(X, \{q(\succ)\})$ the set of all preference profiles $\succ$ on X for which there exists an ordering $q(\succ) \in Q(X)$ such that Definition 4 is satisfied. Example 2 below shows that SP$(X) \subset$ SP$(X, \{q(\succ)\})$.

Example 2 Consider the preference profiles of Tables 1 and 2 over the set of alternatives $X = \{x, y, z\}$.

It is easy to see that there is no $q \in Q(X)$ such that $(\succ_1, \succ_2, \succ_3)$ and $(\hat{\succ}_1, \hat{\succ}_2, \hat{\succ}_3)$ simultaneously belong to $SP(X, q)$. However, both profiles are in SP$(X, \{q(\succ)\})$, since $(\succ_1, \succ_2, \succ_3)$ is single peaked with respect to the linear order $y < z < x$ and $(\hat{\succ}_1, \hat{\succ}_2, \hat{\succ}_3)$ with respect to $z < y < x$.

In words, a single-peaked profile (in both versions) is one in which the set of alternatives can be linearly ordered in such a way that, each agent has a unique most preferred alternative (or *ideal point*) over this common ordering, and the individual's ranking of other alternatives falls as one moves away from his ideal point. Such profiles capture the fact that, for example, an individual may have a most preferred ideological position on some left–right political scale, and the more distant is a candidate's ideological position from his most preferred alternative the more the individual may dislike the candidate.

Going back again to the relation between single-crossing and single-peakedness, it is clear for instance in Example 1 that the profile of induced policy preferences $(w_1, \ldots, w_n)$ satisfies single-crossing on the interval $(0, 1)$. However, for

[8] A linear order over A is a complete, transitive, and antisymmetric binary relation between the elements of A.

Table 1

$\succ_1$	$\succ_2$	$\succ_3$
x	z	y
z	y	z
y	x	x

Table 2

$\hat{\succ}_1$	$\hat{\succ}_2$	$\hat{\succ}_3$
x	z	y
y	y	z
z	x	x

$h(t)$ sufficiently convex, it could violate single-peakedness. Examples 3 and 4 below also illustrate this point.

Example 3 Suppose three agents, with types $\theta_1 < \theta_2 < \theta_3$. Let $X = \{x, y, z\} \subset \Re_+$, where $x < y < z$. Assume preferences are as in Table 3. It is easy to see that this profile is single-crossing on $\{x, y, z\}$. However, for any ordering of the alternatives, the profile violates single-peakedness.[9]

Example 4 Suppose three individuals, 1, 2, and 3, who choose an alternative from $\{a, b, c, d\} \subset \Re_+$. Assume their preferences are as in Table 4. Then, the profile $(\succ_1, \succ_2, \succ_3)$ is single peaked with respect to the linear order $c < a < b < d$. However, if each preference ordering is associated with a different type and each agent is identified with its corresponding type, then for every ordering of the types the profile of Table 4 violates single-crossing.[10]

From the perspective of the analysis of strategy-proofness, there is also a substantial difference between these two preference domains. Considering

Table 3 Single-crossing

$\Phi(\theta_1)$	$\Phi(\theta_2)$	$\Phi(\theta_3)$
x	x	z
y	z	y
z	y	x

Table 4 Single-peakedness

$\succ_1$	$\succ_2$	$\succ_3$
a	d	b
b	b	a
d	a	c
c	c	d

[9] Notice that every alternative appears in the bottom row of Table 3.

[10] Notice that it violates single-crossing not only for the ordering of alternatives $c < a < b < d$, but also for every ordering of them.

Definition 3, which is the standard definition of single-peakedness used in the literature of strategy-proofness,[11] it is clear that $SP(\tilde{X})$ is a Cartesian subset of $P(\tilde{X})^n$.[12] Instead, the set of all single-crossing profiles over $\tilde{X}$, $SC(\tilde{X})$, is not a Cartesian preference domain. The reason is each individual ordering (or type) in a profile $(\tilde{\succsim}_1, \ldots, \tilde{\succsim}_n) \in SC(\tilde{X})$ may depend on other orderings, in the way specified in Definition 2.

This has two important implications. First of all, strategy-proofness (yet to be defined) becomes a *conditional* property of a social choice function (see Definitions 5 and 6 below). Second, the revelation principle does not apply on this domain. That is, even if a social choice function were found to be (conditional) strategy proof on $SC(\tilde{X})$, the mechanism implementing it cannot be a *direct* one.[13] We will return to this point in Sect. 6.

Now we define the two incentive compatibility properties we seek in a social choice function on $SC(\tilde{X})$. These properties are called *conditional strategy-proofness* and *conditional group strategy-proofness*. They guarantee that agents, acting individually or in groups, never have incentives to misrepresent their preferences, at least if they have the opportunity to tell the true.

Definition 5 *A social choice function f is conditional strategy-proof on $SC(\tilde{X})$ if for all $i \in I$, all $(\tilde{\succsim}_i, \tilde{\succsim}_{-i}) \in SC(\tilde{X})$, and any deviation $\hat{\succsim}_i \in P(\tilde{X})$, either $f(\tilde{\succsim}_i, \tilde{\succsim}_{-i}) \tilde{\succ}_i f(\hat{\succsim}_i, \tilde{\succsim}_{-i})$ or $f(\tilde{\succsim}_i, \tilde{\succsim}_{-i}) = f(\hat{\succsim}_i, \tilde{\succsim}_{-i})$.*

If a social choice function f is not conditional strategy-proof on $SC(\tilde{X})$, then there exist $i \in I$ and $\hat{\succsim}_i \in P(\tilde{X})$, such that for some $\tilde{\succsim}_{-i} \in P(\tilde{X})^{n-1}$ and i's true preferences $\tilde{\succsim}_i$, $f(\hat{\succsim}_i, \tilde{\succsim}_{-i}) \tilde{\succ}_i f(\tilde{\succsim}_i, \tilde{\succsim}_{-i})$, where $(\tilde{\succsim}_i, \tilde{\succsim}_{-i}) \in SC(\tilde{X})$. Then, f is said *manipulable* at $(\tilde{\succsim}_i, \tilde{\succsim}_{-i})$, by i, via $\hat{\succsim}_i$.

As usual, Definition 5 says that a social choice function f is strategy-proof on $SC(\tilde{X})$ if for *any* preference declaration $\tilde{\succsim}_{-i}$ the rest of the agents could make, each individual $i \in I$ considers the outcome generated by declaring his true preferences $\tilde{\succsim}_i$, $f(\tilde{\succsim}_i, \tilde{\succsim}_{-i})$, at least as good as $f(\hat{\succsim}_i, \tilde{\succsim}_{-i})$, where $f(\hat{\succsim}_i, \tilde{\succsim}_{-i})$ is generated by i's deviation to any other ordering $\hat{\succsim}_i \in P(\tilde{X})$. However, since $SC(\tilde{X})$ is a non-Cartesian domain, the declarations of the other agents $\tilde{\succsim}_{-i}$ that can be considered are only those that are compatible with the fact that i reports his true binary relation $\tilde{\succsim}_i$ and the profile remains in the admissible domain.

That is, the conditional extension of strategy-proofness we introduce for non-Cartesian preference domains requires that agents have incentives to report their true preferences only in those cases (profiles) where they have the opportunity to do so. But it puts no restrictions on the individual's declaration in the other profiles. Proceeding in a similar way, we can also define conditional group strategy-proofness.

Definition 6 *A social choice function f is conditional group strategy-proof on $SC(\tilde{X})$ if for every coalition $S \subseteq I$, and all $(\tilde{\succsim}_S, \tilde{\succsim}_{\bar{S}}) \in SC(\tilde{X})$, there does not*

[11] See, for example, Moulin (1980, 1988), Barberà and Jackson (1994), Sprumont (1995), Ching (1997), Berga and Serizawa (2000), and Barberà (2001), among others.

[12] Of course, the second definition of single-peakedness, i.e., Definition 4, does not generate a product set.

[13] Remember that a direct mechanism is a game form in which the strategy space of each agent is the set of all possible individual characteristics.

exist a joint deviation $\hat{\succ}_S \in P(\tilde{X})^S$ *such that* $f(\hat{\succ}_S, \tilde{\succ}_{\bar{S}}) \tilde{\succ}_i f(\tilde{\succ}_S, \tilde{\succ}_{\bar{S}})$ *for all* $i \in S$.

In the following sections, we study how well the median choice rule performs, according to these manipulation criteria, on the domain of single-crossing preferences. But, since the main motivation to do this is to study the strategic foundation of the RVT, let us discuss first the connection between single-crossing and order-restriction, which is the original domain where this Theorem was formulated.

3 Single-crossing and order-restriction

Order-restriction, first formally introduced by Rothstein (1990, 1991), is a preference restriction that has been shown to be closely related to single-crossing (Gans and Smart 1996). Next we provide its definition and an *equivalence* theorem (up to renaming of types and alternatives) that parallels that result, but that emphasizes at the same time the differences between these two domains.[14]

For any two sets A and B of real numbers, let $A \gg B$, read "A is higher than B", if for every $a \in A$ and $b \in B$, $a > b$, where $>$ is the antisymmetric part of the "greater than or equal to" relation over the real line.

Definition 7 *A preference profile* $\Phi(\theta) = (\Phi(\theta_1), \ldots, \Phi(\theta_n)) \in P(X)^n$ *is order-restricted on* X *if there exists a permutation* $\gamma_\theta \colon \Theta \to \Theta$ *such that,* $\forall \{x, y\} \subset X$, $x \neq y$, *either*

$$\{\gamma_\theta(\theta_i) \in \Theta_I(\theta) \colon x\, \Phi(\theta_i)\, y\} \gg \{\gamma_\theta(\theta_j) \in \Theta_I(\theta) : y\, \Phi(\theta_j)\, x\}, \qquad \text{OR-1}$$

or

$$\{\gamma_\theta(\theta_j) \in \Theta_I(\theta) \colon y\, \Phi(\theta_j)\, x\} \gg \{\gamma_\theta(\theta_i) \in \Theta_I(\theta) : x\, \Phi(\theta_i)\, y\}. \qquad \text{OR-2}$$

We denote OR$(X, \{\gamma_\theta\})$ the set of all profiles $\Phi(\theta) = (\Phi(\theta_1), \ldots, \Phi(\theta_n))$ on X for which there exists a permutation $\gamma_\theta : \Theta \to \Theta$ such that Definition 7 is satisfied. In words, a preference profile is order-restricted on X if we can order the types of the agents in such a way that for any pair of alternatives the set of types preferring one of the alternatives all lie to one side of those who prefer the other. It is important to emphasize that the ordering of types is not conditional on the pair of alternatives under consideration, while the "cut-off" types may depend on the pair. Example 5, taken from Austen-Smith and Banks (1999), illustrates the concept.

Example 5 Consider the preferences displayed in Table 5, over the set $X = \{x, y, z\} \subset \Re_+$. Assume $x < y < z$ and $\theta_1 < \theta_2 < \theta_3$. Then, the profile is order-restricted over X, since there exists a permutation γ, defined by $\gamma(\theta_1) = \theta_2$, $\gamma(\theta_2) = \theta_1$ and $\gamma(\theta_3) = \theta_3$, such that under this renaming of types:

- $\{\gamma(\theta_i) : \ y\, \Phi(\theta_i)\, x\} = \{\theta_3\} \gg \{\theta_1, \theta_2\} = \{\gamma(\theta_j) : \ x\, \Phi(\theta_j)\, y\}$;
- $\{\gamma(\theta_i) : \ z\, \Phi(\theta_i)\, x\} = \{\theta_3\} \gg \{\theta_1, \theta_2\} = \{\gamma(\theta_j) : \ x\, \Phi(\theta_j)\, z\}$;
- $\{\gamma(\theta_i) : \ z\, \Phi(\theta_i)\, y\} = \{\theta_2, \theta_3\} \gg \{\theta_1\} = \{\gamma(\theta_j) : \ y\, \Phi(\theta_j)\, z\}$.

[14] In this section, we make definitions and proofs over X, but everything is equally valid for any $\tilde{X} \in A(X)$.

Table 5 Order restriction

$\Phi(\theta_1)$	$\Phi(\theta_2)$	$\Phi(\theta_3)$
x	x	z
z	y	y
y	z	x

The following results exhibit the closed relationship between order-restriction and single-crossing:

Lemma 1 *If a preference profile* $(\Phi(\theta_1), \ldots, \Phi(\theta_n))$*, is single-crossing on* X*, then it satisfies order-restriction on* X*. That is,* $SC(X) \subseteq OR(X, \{\gamma_\theta\})$*.*

Proof Consider a profile $\succ = (\succ_1, \ldots, \succ_n) \in \mathrm{SC}(X)$, derived from $\Phi\colon \Theta \to P(X)$. Take any pair of distinct alternatives $\{x, y\} \subset X$ and, without loss of generality, assume $y > x$. Let $\bar{\theta}$ be the smallest type in the set $\Theta_I(\succ)$, with respect to the linear order $>$, such that $y\,\Phi(\bar{\theta})\,x$. If such type does not exist, then $x\,\Phi(\theta_i)\,y$ for all $\theta_i \in \Theta_I$ and order-restriction follows immediately. Otherwise, define the sets $\underline{\Theta}^{x,y} = \{\theta_j \in \Theta_I(\succ) : \bar{\theta} > \theta_j\}$ and $\overline{\Theta}^{x,y} = \{\theta_i \in \Theta_I(\succ)\colon \theta_i \geq \bar{\theta}\}$. Clearly, by definition, $\overline{\Theta}^{x,y} \gg \underline{\Theta}^{x,y}$. On the other hand, by SC, $\overline{\Theta}^{x,y} = \{\theta_i \in \Theta_I(\succ) : y\,\Phi(\theta_i)\,x\}$ and $\underline{\Theta}^{x,y} = \{\theta_j \in \Theta_I(\succ)\colon x\,\Phi(\theta_j)\,y\}$, being the last equality a consequence of the completeness of the orders $\Phi(\cdot)$. Therefore, taken the invariant permutation γ_θ, (such that $\gamma_\theta(\theta_i) = \theta_i$ for each $\theta_i \in \Theta$), it follows that $\{\gamma_\theta(\theta_i) \in \Theta_I(\succ) : y\,\Phi(\theta_i)\,x\} \gg \{\gamma_\theta(\theta_j) \in \Theta_I(\succ)\colon x\,\Phi(\theta_j)\,y\}$. But, since the pair x and y was arbitrary chosen, by OR $-$ 2, we have that $(\succ_1, \ldots, \succ_n) \in \mathrm{OR}(X, \{\gamma_\theta\})$. □

It is easy to see that the converse of Lemma 1 is not true. That is, $\mathrm{OR}(X, \{\gamma_\theta\}) \nsubseteq \mathrm{SC}(X)$. In effect, consider for instance the original profile in Example 5. As we showed, it is in $\mathrm{OR}(X, \{\gamma_\theta\})$. However, it is not in $\mathrm{SC}(X)$ as, for example, $z\,\Phi(\theta_1)\,y$ while $y\,\Phi(\theta_2)\,z$, being $z > y$ and $\theta_2 > \theta_1$. Nevertheless, the equivalence between single-crossing and order-restriction is obtained if we enlarge the domain of single-crossing preferences in a way analogous to Definition 4, by allowing the order over X and Θ to change from one profile to another.

Definition 8 *A profile* $\Phi(\theta) = (\Phi(\theta_1), \ldots, \Phi(\theta_n))$ *is broad single-crossing over* X *if there exist* $q(\theta) \in Q(X)$ *and* $p(\theta) \in Q(\Theta)$ *such that* $\forall\{x, y\} \subset X, i, j \in I$*,*

$$[y \sim q(\theta)\,x,\ \theta_j \sim p(\theta)\,\theta_i,\ \ \textit{and}\ \ y\,\Phi(\theta_i)\,x] \ \Rightarrow\ [y\,\Phi(\theta_j)\,x]. \qquad \text{(BSC)}$$

We denote $\mathrm{BSC}(X, \{q(\theta),\ p(\theta)\})$ the set of all preference profiles $\Phi(\theta) = (\Phi(\theta_1), \ldots, \Phi(\theta_n))$ on X for which there exist $q(\theta) \in Q(X)$ and $p(\theta) \in Q(\Theta)$ such that Definition 8 is satisfied. Example 6 below shows that $\mathrm{SC}(\tilde{X}) \subset \mathrm{BSC}(\tilde{X}, \{q(\theta),\ p(\theta)\})$, meaning that broad single-crossing is a larger preference domain.

Example 6 Consider three individuals and the profiles of Tables 6 and 7, over the set of alternatives $X = \{x, y, z\} \subset \Re_+$. Then, $(\Phi(\theta_1), \Phi(\theta_2), \Phi(\theta_3)) \in \mathrm{BSC}(X, \{q(\theta),\ p(\theta)\})$ for $x < z < y$ and $\theta_1 < \theta_3 < \theta_2$. On the other hand, $(\Phi(\hat{\theta}_1), \Phi(\hat{\theta}_2), \Phi(\hat{\theta}_3)) \in \mathrm{BSC}(X, \{q(\theta),\ p(\theta)\})$ for $x < y < z$ and $\hat{\theta}_1 < \hat{\theta}_2 < \hat{\theta}_3$. However, it is clear that $(\Phi(\theta_1), \Phi(\theta_2), \Phi(\theta_3))$ and $(\Phi(\hat{\theta}_1), \Phi(\hat{\theta}_2), \Phi(\hat{\theta}_3))$ cannot simultaneously belong to $\mathrm{SC}(X)$.

Table 6

$\Phi(\theta_1)$	$\Phi(\theta_2)$	$\Phi(\theta_3)$
x	y	z
z	z	y
y	x	x

Table 7

$\Phi(\hat{\theta}_1)$	$\Phi(\hat{\theta}_2)$	$\Phi(\hat{\theta}_3)$
x	x	z
y	z	y
z	y	x

Lemma 2 *For any profile $\succ^* \in OR(X, \{\gamma_\theta\})$ there exists a profile $\succ \in BSC(X, \{q(\theta), p(\theta)\})$ such that $\succ^*$ and $\succ$ are equivalent up to two permutations $g^* : \Theta \to \Theta$ and $h^* : X \to X$. Conversely, for each profile $\succ \in BSC(X, \{q(\theta), p(\theta)\})$ there exists a profile $\succ^* \in OR(X, \{\gamma_\theta\})$ such that $\succ$ and $\succ^*$ are equivalent up to two permutations g: $\Theta \to \Theta$ and h: $X \to X$.*

Proof (1) Suppose $\succ \in$ BSC$(X, \{q(\theta), p(\theta)\})$. Then, $(X, q(\theta))$ is isomorphic to X under the natural order $\geq$. That is, there exists $h : X \to X$ such that, if $x\, q(\theta)\, y$, then $h(x) \geq h(y)$, for any $x, y \in X$. On the other hand, $(\Theta, p(\theta))$ is isomorphic to Θ under $\geq$ i.e., there exists $g : \Theta \to \Theta$ such that, if $\theta^{'}\, p(\theta)\, \theta^{''}$, then $g(\theta^{'}) \geq g(\theta^{''})$, for any pair $\theta^{'}, \theta^{''}$ in Θ. Therefore, since $\succ$ verifies BSC we have that, for all $x, y \in X, i, j \in I$,

$$[h(y) > h(x), g(\theta_j) > g(\theta_i), \text{ and } h(y)\Phi(g(\theta_i))h(x)] \Rightarrow [h(y)\, \Phi(g(\theta_j))\, h(x)].$$

That is, $(\Phi(g(\theta_1)), \ldots, \Phi(g(\theta_n)))$ is single-crossing over $h(X) = X$ under $\geq$, for the family of types $g(\Theta) = \Theta$ with the corresponding natural order. Then, if we denote $g(\theta_i) \equiv \theta_i^*$ and $\Phi(\theta_i^*) \equiv \succ_i^*$, we have, by Lemma 1, that $\succ^* = (\succ_1^*, \ldots, \succ_n^*) \in$ OR$(X, \{\gamma_\theta\})$.

(2) Consider a profile $\succ^* = (\succ_1^*, \ldots, \succ_n^*) \in$ OR$(X, \{\gamma_\theta\})$, associated to a profile of types $\theta^* = (\theta_1^*, \ldots, \theta_n^*)$. By definition, there exists $\gamma_{\theta^*} : \Theta \to \Theta$ such that, for any two distinct alternatives $x, y \in X$, either OR $-$ 1 or OR $-$ 2 holds. Define a binary relation q^* on X in the following way. For any pair $x, y \in X$, $x \neq y$, set $x\, q^*\, y$ and $\neg[y\, q^*\, x]$ if and only if OR$-$1 holds for $\{x, y\}$. Otherwise, fix $y\, q^*\, x$ and $\neg[x\, q^*\, y]$. On the other hand, for any pair $x, y \in X$ such that $x = y$, just set $x\, q^*\, y$ and $y\, q^*\, x$.

It is immediate to note that q^* is complete and antisymmetric. It is also easy to prove that q^* is transitive. By contradiction, assume there exists $\{x, y, z\} \subseteq X$ such that $x\, q^*\, y$ and $y\, q^*\, z$, but $z\, q^*\, x$ and $\neg[x\, q^*\, z]$. Then, $x \neq z$. Let $\bar{\theta} = \max \{\gamma_{\theta^*}(\theta_i^*) \in \Theta_{\succ^*} : z\, \Phi(\theta_i^*)\, x\}$. Notice that, by hypothesis, this set is nonempty.[15]. Therefore, $\bar{\theta}$ is well defined. Furthermore, according to OR $-$ 1, $\bar{\theta} = \max\ \gamma_{\theta^*}(\Theta_{\succ^*})$. Then, we have three cases to consider:

[15] If it were empty, then $x\, \Phi(\theta_i^*)\, z$ for all θ_i^*. Therefore, it would follow $x\, q^*\, z$

1. If $x = y$ and $y \neq z$, then $x\, q^*\, y$ and $y\, q^*\, x$, while $\neg[z\, q^*\, y]$. Again, $\{\gamma_{\theta^*}(\theta_i^*) : y\, \Phi(\theta_i^*)\, z\} \neq \emptyset$. Hence, $y\, \Phi(\bar{\theta})\, z$; and, by the transitivity of $\Phi(\cdot)$, $y\, \Phi(\bar{\theta})\, x$. Contradiction.
2. If $y = z$, then $y\, \Phi(\bar{\theta})\, x$. Thus, by definition of $\bar{\theta}$, $y\, q^*\, x$ and $\neg[x\, q^*\, y]$. Contradiction.
3. Finally, if $x \neq y$ and $y \neq z$, then $x\, \Phi(\bar{\theta})\, y$ and $y\, \Phi(\bar{\theta})\, z$. Thus, by the transitivity of $\Phi(\cdot)$, it follows that $x\, \Phi(\bar{\theta})\, z$. Contradiction.

Therefore, by (1)–(3), $q^* \in Q(X)$. Now consider the linear order p^* over Θ induced by γ_{θ^*}. That is, for any two types $\theta', \theta'' \in \Theta$, set $\theta'\, p^*\, \theta''$ and $\neg[\theta''\, p^*\, \theta']$ if $\gamma_{\theta^*}(\theta') > \gamma_{\theta^*}(\theta'')$; and $\theta'\, p^*\, \theta''$ and $\theta''\, p^*\, \theta'$ if $\gamma_{\theta^*}(\theta') = \gamma_{\theta^*}(\theta'')$. Then, for *any* pair of distinct alternatives $x, y \in X$, if $y \sim q^*\, x$ and $y \succ_i^* x$ for some $i \in I$, it follows that $y \succ_j^* x$ for all $j \in I$ such that $\theta_j \sim p^*\, \theta_i$. Hence, the profile $(\succ_1^*, \ldots, \succ_n^*)$ is broad single-crossing over X with respect to q^* and p^*. □

4 Representative Voter Theorem

The domain of order-restricted (broad single-crossing) preferences has two properties that are extremely important for collective decision-making analysis. First, as it was already mentioned in other parts of the paper, it guarantees the existence of majority voting equilibria. Second, it offers a simple characterization of the core of the majority rule. In effect, when preferences are order-restricted, the *median type* agent over the order of Θ, which is unique in our framework, is *decisive* in all pairwise majority contests between alternatives in $\tilde{X}$, for all $\tilde{X} \in A(X)$. This result is sometimes referred to as the RVT or, alternatively, as the "second version" of the Median Voter Theorem.

In this section we formally derive the RVT, leaving for Sect. 6 its game-theoretic counterpart.[16] In order to do that, let $f^m : \mathrm{OR}(\tilde{X}, \{\gamma_\theta\}) \to \tilde{X}$ be the median choice rule on the domain of order-restricted preferences. The non-strategic version of the Theorem is as follows:

Theorem 1 *For every agenda* $\tilde{X} \in A(X)$ *and each preference profile* $\tilde{\Phi}(\theta) = (\tilde{\Phi}(\theta_1), \ldots, \tilde{\Phi}(\theta_n)) \in OR(\tilde{X}, \{\gamma_\theta\})$, $f^m(\tilde{\Phi}(\theta_1), \ldots, \tilde{\Phi}(\theta_n)) = M\left(\tilde{X}, \tilde{\Phi}(\gamma_\theta(\theta_r))\right)$, *where* $\gamma_\theta(\theta_r) = m\,(\gamma_\theta(\theta_1), \gamma_\theta(\theta_2), \ldots, \gamma_\theta(\theta_n))$.

Proof Consider a profile $\succ \in \mathrm{OR}(X, \{\gamma_\theta\})$. By Lemma 2, there exists a pair of permutations $\gamma : \Theta \to \Theta$ and $\rho : X \to X$ such that $\succ^\gamma \in \mathrm{SC}(\rho(X))$, where $\succ^\gamma = (\succ_1^\gamma, \ldots, \succ_n^\gamma)$ is generated by $\Phi : \gamma(\Theta) \to P(\rho(X))$. Take the agenda $\rho(\tilde{X})$, corresponding to $\tilde{X} \in A(X)$, and the restriction of $\succ^\gamma$ to $\rho(\tilde{X})$, noted $\tilde{\succ}^\gamma$. Let $T\left(\rho(\tilde{X}), \tilde{\succ}^\gamma\right) = \{\tau(\tilde{\succ}_1^\gamma), \ldots, \tau(\tilde{\succ}_i^\gamma), \ldots, \tau(\tilde{\succ}_n^\gamma)\}$ be the set of individuals' top-ranked alternatives in $\rho(\tilde{X})$ according to $\tilde{\succ}^\gamma$. Then, for all $i, j \in I, \theta_i^\gamma < \theta_j^\gamma$ implies $\tau(\tilde{\succ}_i^\gamma) \leq \tau(\tilde{\succ}_j^\gamma)$, where $\theta_i^\gamma, \theta_j^\gamma \in \gamma(\Theta)$. Suppose not. That is, assume there exist $i, j \in I$ such that $\theta_i^\gamma < \theta_j^\gamma$ while $\tau(\tilde{\succ}_i^\gamma) > \tau(\tilde{\succ}_j^\gamma)$. Since

[16] Notice that we present a simplified version of the RVT, since neither individual indifference nor an even number of voters is considered. For a complementary analysis, see Rothstein (1991), Myerson (1996), Gans and Smart (1996), and Austen-Smith and Banks (1999).

$\tau(\tilde{\succ}_i^\gamma) \tilde{\succ}_i^\gamma \tau(\tilde{\succ}_j^\gamma)$ and $\theta_i^\gamma < \theta_j^\gamma$, by single-crossing, we have that $\tau(\tilde{\succ}_i^\gamma) \tilde{\succ}_j^\gamma \tau(\tilde{\succ}_j^\gamma)$. Contradiction. Therefore, $f^m(\tilde{\succ}_1^\gamma, \ldots, \tilde{\succ}_n^\gamma) = \tau(\tilde{\succ}_r^\gamma) = M(\rho(\tilde{X}), \tilde{\Phi}(\theta_r^\gamma))$, where $\theta_r^\gamma = m\left(\theta_1^\gamma, \theta_2^\gamma, \ldots, \theta_n^\gamma\right)$. Moreover, since $\tilde{\succ}^\gamma$ is equivalent to $\tilde{\succ}$ under γ and ρ, it follows that $f^m(\tilde{\succ}_1, \ldots, \tilde{\succ}_n) = M(\rho(\tilde{X}), \tilde{\Phi}(\theta_r^\gamma))$. But, according to part (2) in the Proof of Lemma 2, the order over Θ induced by γ coincides with the order induced by γ_θ.[17] Thus, $\theta_r^\gamma = m(\theta_1^\gamma, \ldots, \theta_n^\gamma) = m(\gamma_\theta(\theta_1), \ldots, \gamma_\theta(\theta_n)) = \gamma_\theta(\theta_r)$, implying that $M(\rho(\tilde{X}), \tilde{\Phi}(\theta_r^\gamma)) = M(\rho(\tilde{X}), \tilde{\Phi}(\gamma_\theta(\theta_r)))$. Finally, since $\rho(\tilde{X}) = \tilde{X}$, we have $M(\rho(\tilde{X}), \tilde{\Phi}(\gamma_\theta(\theta_r))) = M(\tilde{X}, \tilde{\Phi}(\gamma_\theta(\theta_r)))$. □

In words, Theorem 1 says that, given any subset of policies $\tilde{X} \in A(X)$, the alternative chosen by a society with order-restricted preferences coincides with the most preferred alternative of the median type agent.[18] This result also holds under single-peakedness if individual preferences are symmetric, but not in other cases. Lemma 3 and Fig. 1 below illustrate this point.

Lemma 3 *If $\succ_i$ is single peaked on X and symmetric around $\tau(\succ_i)$ for all $i \in I$, then $(\tilde{\succ}_1, \ldots, \tilde{\succ}_n) \in OR(\tilde{X}, \{\gamma_\theta\})$, for all $\tilde{X} \in A(X)$.*

Proof Consider a preference profile $\succ = (\succ_1, \ldots, \succ_n) \in P(X)^n$ that satisfies the hypothesis of Lemma 3. Define a permutation $\gamma \colon \Theta \to \Theta$ in the following way. Consider the order (over X) of the sequence $\{\tau(\succ_i)\}_{i \in I}$. Suppose the j-th element in that sequence is $\tau(\succ_i)$, where $\Phi(\theta_i) = \succ_i$. Then, set $\gamma(\theta_i) = \theta_j$. Since $\tau(\succ_i)$ is unique for each $i \in I$, γ is always well defined. Take an agenda $\tilde{X} \in A(X)$ and the restriction of $\succ$ to $\tilde{X}$, noted $(\tilde{\succ}_1, \ldots, \tilde{\succ}_n)$. Without loss of generality, consider an arbitrary pair of distinct alternatives $\{x, y\} \subseteq \tilde{X}$, say $x < y$, and define the "cut point" $\alpha = (y - x)/2$ and the sets $\underline{T}^{x,y} = \{\tau(\succ_j) \in T(X, \succ) \colon \alpha > \tau(\succ_j)\}$ and $\overline{T}^{x,y} = \{\tau(\succ_i) \in T(X, \succ) \colon \tau(\succ_i) \geq \alpha\}$. Suppose $\overline{T}^{x,y}$ and $\underline{T}^{x,y}$ are non-empty. Then, by the construction of γ, it is clear that $\{\gamma(\theta_i) \in \Theta_I(\succ) \colon \tau(\succ_i) \in \overline{T}^{x,y}\} \gg \{\gamma(\theta_j) \in \Theta_I(\succ) : \tau(\succ_j) \in \underline{T}^{x,y}\}$. Furthermore, by symmetry and single-peakedness, $\{\gamma(\theta_i) \in \Theta_I(\succ) \colon \tau(\succ_i) \in \overline{T}^{x,y}\} = \{\gamma(\theta_i) \in \Theta_I(\succ) \colon y\, \Phi(\theta_i)\, x\}$ and $\{\gamma(\theta_j) \in \Theta_I(\succ) \colon \tau(\succ_j) \in \underline{T}^{x,y}\} = \{\gamma(\theta_j) \in \Theta_I(\succ) \colon x\, \Phi(\theta_j)\, y\}$. Therefore, $\{\gamma(\theta_i) \in \Theta_I(\tilde{\succ}) : y\, \tilde{\Phi}(\theta_i)\, x\} \gg \{\gamma(\theta_j) \in \Theta_I(\tilde{\succ}) \colon x\, \tilde{\Phi}(\theta_j)\, y\}$. That is, OR $-$ 2 holds. But, since $\{x, y\}$ was arbitrary chosen, it follows that $\tilde{\succ} \in \mathrm{OR}(\tilde{X}, \{\gamma_\theta\})$. On the other hand, if either $\overline{T}^{x,y}$ or $\underline{T}^{x,y}$ is empty, then order-restriction follows immediately, since either $x\, \tilde{\Phi}(\theta)\, y$ for all $\theta \in \Theta_I(\tilde{\succ})$ or $y\, \tilde{\Phi}(\theta)\, x$ for all $\theta \in \Theta_I(\tilde{\succ})$.

[17] Remember that γ_θ is the permutation that makes the original profile $\tilde{\succ}$ order-restricted on $\tilde{X}$.

[18] Rothstein (1991) has also shown that, when preferences are strict and the number of voters is odd, as in our case, the preference ordering generated by the majority rule coincides with the preference relation associated to the median type agent. This implies that the majority preference relation inherits all the properties of the median type agent's preference ordering, in particular, transitivity. Gans and Smart (1996) have proved a similar result for nonstrict preference orderings, but under *strict* single-crossing.

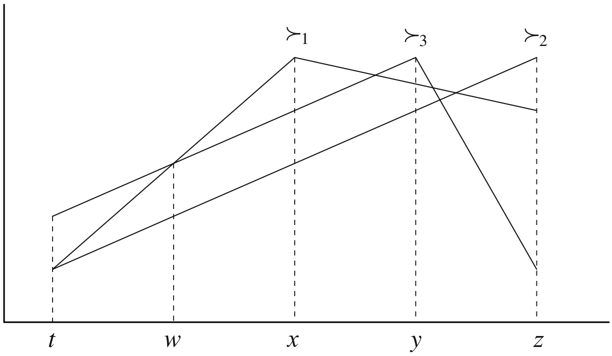

Fig. 1 Median voter identity

Figure 1 exhibits a case where individual preferences are single peaked, but not symmetric for all individuals. The picture shows that the decisive agent depends then on the particular agenda analyzed. In effect, consider first the set $X = \{t, w, x, y, z\}$. Since preferences are single peaked and $|I|$ is odd, the Median Voter Theorem says agent 3's unrestricted top, y, is the unique Condorcet winner in X. Now take the subset $\tilde{X} = \{t, w, z\}$. The induced profile is still single-peaked. However, agent 3's most preferred alternative in $\tilde{X}$, w, is defeated by z, which is the restricted top of agents 1 and 2, and the Condorcet winner in $\tilde{X}$.

As we have seen, this change in the identity of the median voter for different subsets of policies does not occur under order-restriction and broad single-crossing. In that case, Theorem 1 ensures that the individual who has the median type is decisive over any nonempty subset. However, is this result robust to individual and group manipulation? That is, can we expect that the society will end up choosing in the way predicted by the RVT when individuals behave strategically?

The RVT is a result derived under the assumption that individuals honestly reveal their preferences. This is obviously very restricted. In the next sections we show, however, that even if we allow strategic voting the RVT still holds. As we argue, the reason for this is that the median choice rule is conditional strategy-proof over single-crossing domains.

5 Conditional strategy-proofness

The manipulation of the median rule has been studied for a long time in the literature of social choice. The earliest reference goes back to the seminal paper of Black (1948). Since then, a lot of progress has been made towards the understanding of its properties. For instance, it is well known today that there exists a preference domain where this voting procedure performs quite well, in terms of its capacity to extract truthful information about the preferences of the agents. This domain is of course single-peakedness.

In this section, we analyze whether the median choice rule can be manipulated on a different preference domain, namely over single-crossing preferences. Even though this family of preferences is now employed in many models of collective decision-making, nothing has been said in the literature about the possibility of manipulation over this domain. The main purpose here is therefore to fill out this gap.

Our main result is the following:

Proposition 1 *The median choice rule f^m is conditional strategy-proof over $SC(\tilde{X})$, for any $\tilde{X} \in A(X)$.*

Proof Consider a profile $(\tilde{\succ}_i, \tilde{\succ}_{-i}) \in SC(\tilde{X})$, where agent i, of type θ_i, has preferences $\tilde{\succ}_i$. Suppose that there exists another type $\hat{\theta}_i$ such that $f^m(\hat{\succ}_i, \tilde{\succ}_{-i}) \tilde{\succ}_i f^m(\tilde{\succ}_i, \tilde{\succ}_{-i})$, where $\hat{\succ}_i = \tilde{\Phi}(\hat{\theta}_i)$. Furthermore, without loss of generality, assume $\tau(\tilde{\succ}_i) < f^m(\tilde{\succ}_i, \tilde{\succ}_{-i})$. We have two cases to consider:

1. $\tau(\hat{\succ}_i) \leq f^m(\tilde{\succ}_i, \tilde{\succ}_{-i})$. Then $f^m(\hat{\succ}_i, \tilde{\succ}_{-i}) = f^m(\tilde{\succ}_i, \tilde{\succ}_{-i})$. Contradiction;
2. $\tau(\hat{\succ}_i) > f^m(\tilde{\succ}_i, \tilde{\succ}_{-i})$. Then $f^m(\hat{\succ}_i, \tilde{\succ}_{-i}) > f^m(\tilde{\succ}_i, \tilde{\succ}_{-i})$. Denote $\tilde{\tau} = f^m(\tilde{\succ}_i, \tilde{\succ}_{-i})$ and $\hat{\tau} = f^m(\hat{\succ}_i, \tilde{\succ}_{-i})$. Since we assume $(\tilde{\succ}_i, \tilde{\succ}_{-i})$ verifies the single-crossing property, we have that $\hat{\tau}\, \tilde{\Phi}(\theta)\, \tilde{\tau}$ for all $\theta \geq \theta_i$. On the other hand, since $\tilde{\tau}$ is top-ranked in at least one ordering, say $\tilde{\succ}_j$, in the profile $(\tilde{\succ}_1, \ldots, \tilde{\succ}_n)$, it must be that the type corresponding to $\tilde{\succ}_j$, θ_j, is such that $\theta_j < \theta_i$. But then, since $\tau(\tilde{\succ}_i) < \tilde{\tau}$, by single-crossing we have that $\tilde{\tau}\, \tilde{\Phi}(\theta)\, \tau(\tilde{\succ}_i)$ for every $\theta \geq \theta_j$. In particular for θ_i. Contradiction. □

Proposition 1 proves that, in addition to single-peakedness, there exists another *meaningful* domain over the real line that ensures the existence of non manipulable choice rules. That is, it shows that single-crossing constitutes a restriction that guarantees not only majority voting equilibria, but also nontrivial (conditional) strategy-proof social choice functions. In particular, this is true for the median choice rule.[19]

Now we extend the result of Proposition 1 by showing that in fact these conditions, i.e., single-crossing and broad single-crossing, assure strategy proofness not only at the individual level but also at the group level.

Proposition 2 *The median choice rule f^m is conditional group strategy-proof over $SC(\tilde{X})$, for any $\tilde{X} \in A(X)$.*

Proof Consider a profile $(\tilde{\succ}_1, \ldots, \tilde{\succ}_n) \in \mathrm{SC}(\tilde{X})$, with associated types $(\theta_1, \ldots, \theta_n)$. Suppose there exists a coalition $S \subseteq I$ and a list of alternative types for members of S, $(\hat{\theta}_i)_{i \in S}$, such that $f^m(\hat{\succ}_S, \tilde{\succ}_{\bar{S}}) \tilde{\succ}_i f^m(\tilde{\succ}_S, \tilde{\succ}_{\bar{S}})$ for all $i \in S$, where $\hat{\succ}_S = (\tilde{\Phi}(\hat{\theta}_i))_{i \in S}$. For simplicity, denote $\tilde{\tau} = f^m(\tilde{\succ}_S, \tilde{\succ}_{\bar{S}})$ and $\hat{\tau} = f^m(\hat{\succ}_S, \tilde{\succ}_{\bar{S}})$. Notice that, by the definition of f^m, $\tilde{\tau}$ and $\hat{\tau}$ coincide with the tops corresponding to the orderings reported by some voters. Denote these agents j and j' and their

[19] Saporiti and Tohmé (2004) show that the whole family of anonymous, tops-only and dominant strategy implementable social choice functions over single-crossing preferences is given by a subclass of the *extended median rule*, obtained by distributing $n-1$ fixed parameters (also called *phantom voters*) at the extremes of the real line. This subclass, where each phantom is either a *leftist* or a *rightist*, is sometimes referred to as *positional dictator* choice rules (Moulin 1988). They select the kth ranked peak among the tops of reported orderings, for some $k = 1, \ldots, n$. For example, if $k = 1$, we have the *leftist rule*, which always chooses the smallest reported peak. Of course, the median choice rule is also a particular case.

types θ_j and $\theta_{j'}$, respectively. Since $\tilde{\tau} \neq \hat{\tau}$, assume $\tilde{\tau} < \hat{\tau}$. Then, for all $i \in S$, $\tau(\tilde{\succ}_i) > \tilde{\tau}$. Suppose not. That is, assume $\tau(\tilde{\succ}_i) \leq \tilde{\tau}$ for some agent $i \in S$. If $\tau(\tilde{\succ}_i) = \tilde{\tau}$, then $\tilde{\tau} \hat{\succ}_i \hat{\tau}$, which contradicts our initial hypothesis. Consider, instead, that $\tau(\tilde{\succ}_i) < \tilde{\tau}$. Since $\hat{\tau} \tilde{\succ}_i \tilde{\tau}$, by single-crossing we have that for all $\theta \geq \theta_i$, $\hat{\tau}\, \tilde{\Phi}(\theta)\, \tilde{\tau}$. Then, θ_j has to verify $\theta_j < \theta_i$; and, by single-crossing, $\tilde{\tau}\, \tilde{\Phi}(\theta_j)\, \tau(\tilde{\succ}_i)$ implies $\tilde{\tau}\, \tilde{\Phi}(\theta_i)\, \tau(\tilde{\succ}_i)$. Contradiction. Then, $\tau(\tilde{\succ}_i) > \tilde{\tau}$ for all $i \in S$. The rest of the proof is as follows. By definition,

$$f^m(\tilde{\succ}_S, \tilde{\succ}_{\bar{S}}) = m(\tau(\tilde{\succ}_1), \ldots, \tau(\tilde{\succ}_n)) = \tilde{\tau},$$

while

$$f^m(\hat{\succ}_S, \tilde{\succ}_{\bar{S}}) = m(\{\tau(\hat{\succ}_i)\}_{i \in S}, \{\tau(\tilde{\succ}_j)\}_{j \in \bar{S}}) = \hat{\tau}.$$

Two cases are possible:

1. For each $i \in S$, $\tau(\hat{\succ}_i) > \tilde{\tau}$. Then $\hat{\tau} = \tilde{\tau}$. Contradiction.
2. For some $i \in S$, $\tau(\hat{\succ}_i) \leq \tilde{\tau}$. Then, by rewritten $(\{\tau(\hat{\succ}_i)\}_{i \in S}, \{\tau(\tilde{\succ}_j)\}_{j \in \bar{S}})$ as $(y_1, \ldots, y_n)$, we have that

$$\left| \left\{ j \in \{1, \ldots, n\}\colon y_j \leq \tilde{\tau} \right\} \right| \geq \frac{(n+1)}{2}.$$

   But this implies that $m(y_1, \ldots, y_n) \leq \tilde{\tau}$. That is, $f^m(\hat{\succ}_S, \tilde{\succ}_{\bar{S}}) \leq f^m(\tilde{\succ}_S, \tilde{\succ}_{\bar{S}})$. Contradiction. □

Corollary 1 *The median choice rule f^m is conditional group strategy-proof over $BSC(\tilde{X}, \{q(\theta),\ p(\theta)\})$, for any $\tilde{X} \in A(X)$.*[20]

Proof Assume, by contradiction, that f^m is manipulable at some profile $(\tilde{\succ}_1, \ldots, \tilde{\succ}_n) \in \mathrm{BSC}(\tilde{X}, \{q(\theta),\ p(\theta)\})$. By the same argument applied in part (1) of the Proof of Lemma 2, the profile $(\tilde{\succ}_1, \ldots, \tilde{\succ}_n)$ is equivalent, up to permutations $h\colon \tilde{X} \to \tilde{X}$ and $g\colon \Theta \to \Theta$, to a profile $(\tilde{\succ}_1^{'}, \ldots, \tilde{\succ}_n^{'})$ which is also single-crossing over $h(\tilde{X}) = \tilde{X}$ under $\geq$, for the family of types $g(\Theta) = \Theta$ with the corresponding natural order. Then,

$$f^m(\tilde{\succ}^{'}) = m_{\geq}(\tau(\tilde{\succ}_1^{'}), \ldots, \tau(\tilde{\succ}_n^{'})) = m_{\geq}(h \circ \tau(\tilde{\succ}_{g(1)}), \ldots, h \circ \tau(\tilde{\succ}_{g(n)})),$$

where $m_{\geq}(\cdot)$ is the median under the order $\geq$. Notice that by the properties of both the median and the permutation g, we have that $m_{\geq}(h \circ \tau(\tilde{\succ}_{g(1)}), \ldots, h \circ \tau(\tilde{\succ}_{g(n)})) = m_{\geq}(h \circ \tau(\tilde{\succ}_1), \ldots, h \circ \tau(\tilde{\succ}_n))$. Then, we can apply the proof of Proposition 2, to show that no group $S \subseteq I$ can manipulate $m_{\geq}(h \circ \tau(\tilde{\succ}_1), \ldots, h \circ \tau(\tilde{\succ}_n))$ over $(\tilde{X}, \geq)$. Contradiction. □

Corollary 2 *The median choice rule f^m is conditional group strategy-proof over $OR(\tilde{X}, \{\gamma_\theta\})$, for any $\tilde{X} \in A(X)$.*

[20] The definition of conditional group strategy-proofness over $\mathrm{BSC}(\tilde{X}, \{q(\theta),\ p(\theta)\})$ follows from Definition 6, by changing in the appropriate way the domain of the choice rule. The same applies in Corollary 2 for $\mathrm{OR}(\tilde{X}, \{\gamma_\theta\})$.

Proof Immediate from Lemma 2 and Corollary 1. □

Summarizing, the results of this section show that the median rule is non manipulable (both at the individual and group level) neither over single-crossing nor order-restricted preferences. However, does it imply that it can be implemented in dominant strategies? According to the revelation principle, strategy-proofness is a *necessary* condition for truthful or *direct* implementation. However, it is not *sufficient*. It is sufficient when the preference domain is a product set. Otherwise, the direct mechanism is not well defined, in the sense that the set of strategies of each agent, i.e., the set of admissible individual preferences, depends on the strategies used by the others. In the next section, we analyze this problem.

6 Implementation

In this section, we propose an extensive game form that can be used to *indirectly* implement f^m in dominant strategies. We show that this game form is equivalent to a mechanism in normal form, and we prove that the latter succeeds in implementing the median rule. We also briefly discuss why the extensive game form and its associated reduced form work, but not a direct mechanism in which each individual simply declares his top. Finally, we derive the game-theoretic equivalent of Theorem 1.

Suppose individuals have preferences $(\succ_1, \ldots, \succ_n) \in \mathrm{SC}(X)$. Assume the election of an outcome in $\tilde{X} \in A(X)$, which is the planner's problem, is indirectly performed by the following two-stage voting procedure. In the first stage, individuals select by pairwise majority voting a *representative* individual from the set I. Then, in the second stage, the winner chooses an alternative in $\tilde{X}$, which is then the policy implemented by the planner.

Clearly, in the last stage each individual has a dominant strategy, which is simply to choose his most preferred alternative in $\tilde{X}$. Therefore, it is immediate to see that the extensive game form is equivalent to a strategic game form in which individuals choose by pairwise majority comparisons an alternative from the set of actual ideal points $T(\tilde{X}, \tilde{\succ}) = \{\tau(\tilde{\succ}_1), \ldots, \tau(\tilde{\succ}_n)\}$. Next we prove this mechanism can be used to implement f^m in dominant strategies.

Definition 9 *A mechanism Γ with consequences in $\tilde{X}$ is a strategic game form $\langle I, (S_i), \phi\rangle$, where S_i is the set of actions for each agent $i \in I$ and ϕ: $\prod_{i \in I} S_i \to \tilde{X}$ an outcome function that associates an alternative with every action profile.*

We say that Γ implements a social choice function f: $P(\tilde{X})^n \to \tilde{X}$ in dominant strategies if there exists a dominant strategy equilibrium for the mechanism, yielding the same outcome as f for each possible preference profile.

Definition 10 *The mechanism $\Gamma = \langle I, (S_i), \phi\rangle$ implements the social choice function f: $P(\tilde{X})^n \to \tilde{X}$ in dominant strategies if there exists a dominant strategy equilibrium of Γ, noted $s^*(\cdot) = (s_1^*(\cdot), \ldots, s_n^*(\cdot))$, such that $\phi(s^*(\tilde{\succ})) = f(\tilde{\succ})$ for all $\tilde{\succ} \in P(\tilde{X})^n$.*

Proposition 3 *For any $\tilde{X} \in A(X)$, there exists a mechanism that implements $f^m : SC(\tilde{X}) \to \tilde{X}$ in dominant strategies over $\tilde{X}$.*

Proof Consider a preference profile $\tilde{\succ} \in \mathrm{SC}(\tilde{X})$ and the mechanism $\Gamma = \langle I, (S_i), \phi \rangle$, where an action for agent $i \in I$ is simply to choose an element in $S_i = T(\tilde{X}, \tilde{\succ})$, and the outcome function $\phi(s_1, \ldots, s_n) = m(s_1, \ldots, s_n)$. We show that the action profile $(\tau(\tilde{\succ}_1), \ldots, \tau(\tilde{\succ}_n))$ constitutes a dominant strategy equilibrium of the game induced by Γ. That is,

$$\phi(s_1, \ldots, \tau(\tilde{\succ}_i), \ldots, s_n) \tilde{\succ}_i \phi(s_1, \ldots, \hat{s}_i, \ldots, s_n)$$

for all $i \in I$, $\hat{s}_i \neq \tau(\tilde{\succ}_i)$, $s_{-i} \in \prod_{j \neq i} S_j$. Since, by definition, $\phi(\cdot) = m(\cdot)$, we can easily recast the proof of Proposition 1 to fit in this scheme. Suppose that there exists such $\hat{s}_i$. Call $\tilde{s} = \phi(\tau(\tilde{\succ}_i), s_{-i})$ and $\hat{s} = \phi(\hat{s}_i, s_{-i})$. Without loss of generality, assume $\tau(\tilde{\succ}_i) < \tilde{s}$. We have two cases to consider:

1. $\hat{s}_i \leq \tilde{s}$. Then $m(\tau(\tilde{\succ}_i), s_{-i}) = m(\hat{s}_i, s_{-i})$. Therefore, $\phi(\tau(\tilde{\succ}_i), s_{-i}) = \phi(\hat{s}_i, s_{-i})$. Contradiction.
2. $\hat{s}_i > \tilde{s}$. Then the new median $\hat{s}$ belongs to the interval $(\tilde{s}, \hat{s}_i]$. By hypothesis, $\hat{s} \tilde{\succ}_i \tilde{s}$. Furthermore, since the preferences are single-crossing on $T(\tilde{X}, \tilde{\succ})$ and $\hat{s} > \tilde{s}$, for every $\theta \geq \theta_i$ we have that $\hat{s} \tilde{\Phi}(\theta) \tilde{s}$. On the other hand, notice that, since each $S_j = T(\tilde{X}, \tilde{\succ})$, there must exist $\theta_j \in \Theta_I(\tilde{\succ})$ such that $\tilde{s} = \tau(\tilde{\Phi}(\theta_j))$. Moreover, θ_j must be such that $\theta_j < \theta_i$. But then, since $\tau(\tilde{\succ}_i) < \tilde{s}$ and $\tilde{s} \tilde{\Phi}(\theta_j) \tau(\tilde{\succ}_i)$, by single-crossing, we have that $\tilde{s} \tilde{\Phi}(\theta_i) \tau(\tilde{\succ}_i)$. Contradiction.

Therefore, $(\tau(\tilde{\succ}_1), \ldots, \tau(\tilde{\succ}_n))$ is a dominant strategy equilibrium. □

Interestingly the fact that the alternative declared by each agent is restricted to be in the set $T(\tilde{X}, \tilde{\succ})$ is crucial for the proof of Proposition 3. It is easy to see that a mechanism based on direct declarations of most preferred alternatives in $\tilde{X}$ cannot be used to implement f^m. For instance, in Example 3, if each agent has to announce his most preferred alternative in $\tilde{X} = \{x, y, z\}$ and the collective decision is taken by the median function $m(\cdot)$, then manipulation cannot be avoided: If agent 1 and agent 3 declare y and z, respectively, then player 2 would prefer to announce z instead of his true top x.[21]

Instead, our *indirect* mechanism works because the induced preferences over the set $T(\tilde{X}, \tilde{\succ})$ are single peaked:

Lemma 4 *If a profile $\tilde{\succ} = (\tilde{\succ}_1, \ldots, \tilde{\succ}_n)$ is single-crossing over $\tilde{X}$, then the restriction of $\tilde{\succ}$ over the set $T(\tilde{X}, \tilde{\succ})$ is single peaked.*

Proof Given a profile $(\tilde{\succ}_1, \ldots, \tilde{\succ}_n) \in \mathrm{SC}(\tilde{X})$ and the associated set $T(\tilde{X}, \tilde{\succ})$, consider the restriction of $\tilde{\succ}$ to $T(\tilde{X}, \tilde{\succ})$, denoted $\tilde{\succ}^T = (\tilde{\succ}_1^T, \ldots, \tilde{\succ}_n^T)$. By contradiction, suppose $\tilde{\succ}^T \notin \mathrm{SP}(T)$, where $\mathrm{SP}(T)$ is the set of all single-peaked preference profiles over $T(\tilde{X}, \tilde{\succ})$ (with respect to the linear order $\leq$). Then, there exist an individual $i \in I$, with type $\theta_i \in \Theta$, and $x, y, \tau(\tilde{\succ}_i) \in T(\tilde{X}, \tilde{\succ})$ such that $x < y \leq \tau(\tilde{\succ}_i)$, but $x \tilde{\succ}_i^T y$.[22]

[21] Remember that Proposition 1 shows that individual manipulation is ruled out when agents declare a complete preference ordering, and not just the top alternative.

[22] The same argument applies if $\tau(\tilde{\succ}_i) \leq y < x$ and $x \tilde{\succ}_i^T y$.

Table 8 Counterexample

$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$
w	x	y	z
x	y	x	y
y	z	w	x
z	w	z	w

Thus, $y \neq \tau(\tilde{\succ}_i)$. Moreover, since $\tilde{\succsim}^T \in \mathrm{SC}(T)$, $x \tilde{\succ}_j^T y$ for all $\theta_j \leq \theta_i$. This means $y \neq \tau(\tilde{\succ}_j)$ for all $\theta_j \leq \theta_i$. However, since we assume $y \in T(\tilde{X}, \tilde{\succsim})$, then $y = \tau(\tilde{\succ}_k)$ for some individual $k \in I$, with type $\theta_k > \theta_i$. Then, $y \tilde{\succ}_k \tau(\tilde{\succ}_i)$ implies $y \tilde{\succ}_j \tau(\tilde{\succ}_i)$ for all $\theta_j \leq \theta_k$. In particular, for θ_i. Contradiction. Hence, $\tilde{\succsim}^T \in \mathrm{SP}(T)$. □

It is easy to show that the converse of Lemma 4 does not hold. That is, preferences can be single peaked over $T(\tilde{X}, \tilde{\succsim})$, but not necessarily single-crossing. The profile of Table 8 illustrates this.

Finally, we derive the following corollaries from Proposition 3.

Corollary 3 *For any $\tilde{X} \in A(X)$, there exists a mechanism that implements f^m: $OR(\tilde{X}, \{\gamma_\theta\}) \to \tilde{X}$ in dominant strategies over $\tilde{X}$.*

Proof Consider any preference profile $\tilde{\succsim} \in OR(\tilde{X}, \{\gamma_\theta\})$. By Lemma 2, there exist $\gamma : \Theta \to \Theta$ and $\rho : X \to X$ such that $\tilde{\succsim}^\gamma \in SC(\rho(\tilde{X}))$, where $\succ^\gamma$ is generated by $\Phi : \gamma(\Theta) \to P(\rho(X))$. Hence, the mechanism defined in Proposition 3 yields, as the outcome of its dominant strategy equilibrium, the median value of the maximal alternatives over $\rho(\tilde{X})$; i.e., $\phi(\tilde{\succsim}^\gamma) = m(\tau(\tilde{\succsim}_1^\gamma), \ldots, \tau(\tilde{\succsim}_n^\gamma)) = \tau(\tilde{\Phi}(\theta_r^\gamma))$. But then, applying the reasoning of the Proof of Theorem 1, it follows that $f^m(\tilde{\succsim}) = \phi(\tilde{\succsim}^\gamma)$; and, by the equivalence between $\tilde{\succsim}^\gamma$ and $\tilde{\succsim}$, we have $f^m(\tilde{\succsim}) = \phi(\tilde{\succsim})$. □

Corollary 4 *For any $\tilde{X} \in A(X)$, there exists a mechanism that implements f^m : $BSC(\tilde{X}, \{q(\theta), p(\theta)\}) \to \tilde{X}$ in dominant strategies over $\tilde{X}$.*

Proof Immediate from Lemma 2 and Corollary 3. □

Corollary 3 provides the *strategic counterpart* of Theorem 1. That is, it shows that, when preferences are order restricted, the outcome predicted by the RVT, i.e., the most preferred alternative of the median type agent, can be attained by a mechanism in which each agent is allowed to choose an alternative among the top-ranked alternatives in the feasible set of policies. Or, equivalently, it can be achieved by following a two-stage voting procedure in which, first, individuals select a representative among themselves, and then the representative voter chooses a policy to be implemented by the planner.

7 Final remarks

In this paper we present several results. First of all, we prove that, in addition to single-peakedness, there exist other *meaningful* domains over the real line that ensures the existence of nonmanipulable social choice rules. These are the domain of single-crossing and broad single-crossing (order-restricted) preferences, over which the median choice rule is shown to be not only strategy proof, but also group strategy proof.

The main feature to remark of this result is that single-crossing does not necessarily satisfy single-peakedness, and vice versa. But, in one-dimensional models of voting, the latter is usually invoked to guarantee strategy-proofness. Thus, our result shows that, at least for the median rule, certain amount of correlation or interdependence between individuals' preferences is also sufficient to prevent individual and group manipulation.

In addition, the paper explores the relation between single-crossing and order-restriction. A previous work in the same direction is Gans and Smart (1996), in which these preference domains are shown to be essentially equivalent. Our results differ from theirs in two ways. First, ours seem to be more consistent with Rothstein's original characterization of order-restriction. Second, particular attention is devoted here to the fact that these conditions may not be *directly* equivalent. The crucial point to understand this difference is that, unlike single-crossing, order-restriction does not assume any ordering on the set of possible alternatives. Furthermore, it is precisely this feature that make order-restriction interesting for multi-dimensional analysis.

Finally, the paper shows that the RVT has a well-defined strategic foundation. That is, it proves that the outcome predicted by the RVT can be implemented in dominant strategies through a simple mechanism. This mechanism is a two-stage voting procedure in which, first, individuals select a representative among themselves, and then the representative voter chooses a policy to be implemented by the planner. Given that the structure of this mechanism presents some features that we observe frequently in "real" voting processes, the analysis carried out here may also provide insights for the rationale of these "real" voting situations.

At the same time, there are significant issues that this paper does not cover. The most important one is the characterization of the whole family of strategy proof social choice functions over single-crossing preferences. Of course, the set that also satisfies other requirements, like tops-onliness, Pareto efficiency or combinations of them, should also be determined.

Another relevant question that we do not address is how sensitive are our results to the assumption that individual preferences are antisymmetric. In our model, this simplification is partially justified by the fact that the set of alternatives is finite. However, it is clear that a complete analysis requires to study what happens when indifference is admitted.

Finally, another interesting avenue for further research is to explore how single-crossing and order-restriction can be extended to multidimensional settings; i.e., to models with multidimensional choice sets and with conflicts of interests that cannot be projected over a one-dimensional space.

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