# Economic Theory and the Alternative Set Theory $\mathbf{AFA}^{-} + \mathbf{AD} + \mathbf{DC}$

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## Abstract

Many authors in the discipline as well as outsiders have claimed that the main results from Mathematical Economics are far removed from real world phenomena. A more precise version of this position is that one of the main reasons for this unrealistic stance is the use of the wrong formal tools. So, for example, it has been pointed out that the computability of choice functions as well as the existence of economic equilibria and of states of the world may not be ensured in general if the assumed set theory is **ZFC**. We will show that there exists a very natural set theory that overcomes some formal limitations of contemporary economic theory. A switch to an alternative set theory helps to obtain in a more natural way results widely accepted by mathematical economists. Moreover, alternative set-theoretical frameworks convey different intuitions about how agents behave when solving problems. We claim that $\mathbf{AFA}^{-}+\mathbf{AD}+\mathbf{DC}$ is the adequate alternative set-theoretical universe for economic theory.

## 1 Introduction

Economic theory is constituted by an ever-growing body of mathematical formalisms developed in order to explain both the behavior of economies and of individual agents in them. In most of those formalisms the key question is to find *optimal* results. This is a direct consequence of assuming that the behavior of agents is guided by the principle of *rationality*. Moreover, even in the modeling of uncertain situations, the solutions that provide the explanations sought by economic theorists are those in which the beliefs and expectations of the agents are consistent with rational behavior [Kreps 1990].

Even if most economic theorists agree with this point of view, a few economists have some concerns about the meaning of the theoretical constructions and particularly about the portion of reality that they represent.[1] The question they pose can be rephrased as: *Does the formal apparatus of economic theory impose extra requirements on the cognitive abilities of agents, other than their rationality?* If so, can we be sure that these extra requirements do not require abilities beyond those that can be deemed reasonable?

Our focus will be on the computational and set-theoretical requirements for the

[1] A particularly influential alternative point of view was forcefully advanced by Herbert Simon, who claimed that human beings are only *boundedly rational* and that they, instead of optimizing when deciding, just use *procedures* to find solutions that satisfy their needs [Simon 1982]. Even this position seems "mechanicist" for more radical critics [Mirowski 2002].

 

solutions to both individual and collective decision-making problems. It will be shown that some usual innocent looking claims follow only if we assume alternative set-theoretic frameworks (instead of the usual **ZFC** set theory).

Beyond the question of the problem-solving abilities required by the characterization of economic agents, we are also interested in the intrinsic mathematical problems that arise in the modeling of economic behavior. In that sense, to choose a set-theoretical framework just to solve an analytical problem may seem an instance of what Bertrand Russell called the "advantage of theft over honest toil". Our answer is that, although this is not entirely a false claim, in the case of the modeling of agents, the choice of set theory implies the assumption of a given problem-solving ability that may be absent in another framework. Since one of the main goals of economic theory is to exhibit the basis on which agents solve their individual and collective decision-making problems, we may resort to set-theoretic postulates just to represent some cognitive abilities.

The idea of changing the underlying set-theory just in order to make the mathematical objects and constructions correspond to real-world counterparts is not new. The entire program of research that Jon Barwise initiated on *situation theory* and related formalisms, based on his idea that logic is a branch of applied mathematics, is clearly an inspiration for the approach we follow here [Barwise-Etchemendy 1987], [Barwise-Moss 1996],[Barwise-Seligman 1997]. In computer science, the advantages of adopting set theories closer to common sense have been emphasized in [Pakkan-Akman 1994] [Akman-Pakkan 1996]. More recently, Jan Mycielski has presented a universal set theory in which the class of mathematical objects can be given physical interpretations [Mycielski 2006], [Mycielski 2007]. We intend to do something similar for Economic Theory.

Our goal is to present an approach to economic theory based on choosing as its foundation a set theory that supports all the results widely accepted. The idea is that economic entities with their intended properties should be represented in the underlying set theory. This has consequences beyond the mere rephrasing of previous results. New entities will have a right to exist in this framework, which lack legitimate counterparts in **ZFC**. As said, an important aspect of this is that the alternative set-theoretical foundation should, on one hand, not give way to undesirable properties or entities, while on the other it should keep all the desirable results. Even if they are not presented exactly in these terms, the introduction of *O-minimal* structures as the right models for certain results ([Zame-Blume 1992],[Richter-Wong 2000]) points out the need for new formal alternatives to the usual interpretation of theoretical results.

The approach followed here runs in the opposite direction of a wide ranging project pushed forward, in a series of papers of the 1980s and 1990s, by Alain Lewis [Lewis 1985],[Lewis 1990],[Lewis 1991],[Lewis 1992]. The main idea advanced there was that economic theories should be formulated in an *effective* framework. That is, that every entity or property defined in them should be computable. To make his case Lewis tried to show, in an impressive exhibition of scholarship, that the key notions in economic theory are not effective and therefore that they should be redefined. We take issue with this claim and with Lewis' program in general, and a good deal of this paper will be devoted to showing how to overcome the limitations denounced by Lewis.

The set-theoretical framework we think is a better fit for economics is $\mathbf{AFA}^- +$

$\mathbf{AD} + \mathbf{DC}$. That is, the theory of sets that can be derived from the axioms of Zermelo-Frenkel set theory (except the axiom of Regularity, which is replaced by the Anti-Foundation axiom) as well as from the axiom of Dependent Choices (weaker than the Axiom Choice) and the Axiom of Determinacy. We claim that a lot of insight is gained in this switch. We will profit in our discussion from the wealth of results concerning the mathematical universes built upon **AD** and **AFA**. We do not intend this paper to be seen as a contribution to set theory nor claim that the proposed mathematical framework is more "real" than more classical ones. We just want to show how a domain of empirical discussion may be simplified by the choice of an alternative foundation.

In this paper we will emphasize on several consequences of this change of underlying set theory:

- Some economical meaningful entities can be more easily shown to be effectively definable. In particular, individual demands and economic equilibria (Lemmas 3.3 and 4.2, respectively).
- Sequences of economies with shrinking cores are shown to converge to an economy in which competitive equilibria are socially optimal (Proposition 4.8).
- The distinction between "infinite" and "very large" societies is eliminated (Proposition 4.8 again and Proposition 4.9).
- No matter how beliefs are represented, they can be unfolded in $\omega$ steps (Lemma 5.2).

Notice that all these claims do either not follow in **ZFC** or, if they do, their proofs are much more complicated than in $\mathbf{AFA}^- + \mathbf{AD} + \mathbf{DC}$.

In section 2 we introduce a brief description of the problems and goals of economic theory as well as the main difficulties of a logical nature we can find there. One of the most discussed problems is that of the computability of choice procedures, which will be analyzed in section 3. In section 4 we consider the problem of existence of competitive equilibria. In section 5 we analyze the problem of finding a common prior in situations of asymmetric information. Finally, in section 6 we return to our initial discussion of the legitimacy of solving analytic problems in economics by means of a change of the underlying set theory.

## 2 The Main Problems in Economics

Although it is difficult to summarize the core of a discipline in a few words, the main questions in economics, leaving all the technicalities aside, may be the following:

- How do agents make choices?
- How does so much order arise from individual choices?

These two simple questions have been analyzed under the assumption that agents are *rational*. That is, that they have *preferences*, face *constraints* and choose options in such a form that their elections satisfy the constraints and are consistent with their preferences. In most models this means that agents *maximize* their preferences over their constrained sets of options.

This last line of reasoning serves as the basis for the answer of the first question. We conclude that the entire schema of choices of the agents may be derived from the maximization of preferences varying the constraints (which represent the environment in which they have to choose). On the other hand, the second question actually asks for a precise characterization of the environment in which all the agents interact, in order to make their choices mutually consistent.

As an example of how these general problems have been attacked, let us consider an economy $\epsilon$ in which a finite number of agents interact. Each agent $i$ $(i = 1 \ldots n)$ can be identified with the triple $\langle \mathbf{X}_i, \preceq_i, w_i \rangle$, where $\mathbf{X}_i$ is the set of possible consumptions of $i$,[2] $\preceq_i$ is a preorder on $\mathbf{X}_i$ (representing $i$'s preferences) and $w_i \in \mathbf{X}_i$ are the initial endowments of the agent [Debreu 1959],[Arrow-Hahn 1971],[MWG 1995].

In words: an agent orders the alternative vectors of consumption. The order is transitive and each pair of alternative vectors ("consumption baskets") is comparable. The goal is to choose the most preferred alternative among those that can be purchased in the market using the proceeds from the sale of an initial vector of resources.

An additional piece of information about this economy, in which the only allowed transaction is the exchange of goods, is that the variable representing the environment is the system of prices, $p$.[3] That is, the only information available to the agents about the entire economy is encoded in the prices. Moreover, no agent can modify them.

The decision of an agent $i$ is to choose an amount $x_i^* \in \mathbf{X}_i$ that maximizes $\preceq_i$ over $\mathbf{B}(p, w_i) = \{x \in \mathbf{X}_i\colon p \cdot x \leq p \cdot w_i\}$. That is, $i$ will choose the amount of goods $x_i^*$ that is among the most preferred of the amounts that have the same or less value than the initial endowments.

If we vary the environmental variable, $p$, we change $x_i^*$. We obtain then a **demand** function, $x_i^*(p)$.[4] On the other hand, a market consistency condition is that the actual prices $p^*$ should satisfy that $\sum_{i=1}^n x_i^*(p^*) = \sum_{i=1}^n w_i$, that is, that the economy is in **equilibrium**. In other words, at the given equilibrium price, the amounts demanded by all the agents should be equal to the amounts that are available (the endowments). Notice that in order to find $p^*$ we have to know each $x_i^*(\cdot)$, i.e. the demand function, and not only the amount demanded at a given price.

This simple model abstracts the idea that in a perfectly competitive economy individual agents are not able to modify prices. On the other hand, since we do not assume the presence of an authority that enforces equilibrium, when the number of agents is small this creates incentives for strategic behavior. That is, agents may declare their demands dishonestly, in order to achieve a certain system of prices that allows them to get to a more preferred amount. This raises the question of what conditions ensure that the equilibrium outcome cannot be improved by individual or group deviations.

Finally, one of the more basic considerations in the analysis of economic interactions is whether agents are fully informed about the characteristics of other agents. If not, the context is said to consist of *incomplete information*. The only possibility for agents to coordinate on an equilibrium arises (otherwise than by sheer luck) if they share a *common prior* [Binmore 1990], [Fudenberg-Tirole 1991], [Osborne-Rubinstein 1994]. That is, if all of them initially evaluate the possible situations with the same prob-

[2] In general $\mathbf{X}_i$ is assumed to be a subset of an Euclidean subspace $\Re^l_+$. We will follow this convention here.

[3] If $\mathbf{X}_i \subseteq \Re^l_+$ then $p \in \Re^l_+$, i.e. there are as many prices as goods in the economy.

[4] Notice that for a given price $p$ there might exist several maximal elements for $\preceq_i$ over $\mathbf{B}(p, w_i)$. We will not consider this possibility here and will assume that each $x_i^*(p)$ is a singleton.

ability distribution over situations. After that, they may update their evaluations in different ways, according to the information they obtain during their interaction.[5] To achieve this, they have to converge to a shared assessment of the environment, the behavior of other agents and their beliefs. The objective data plus the common assessment made by the agents is called a *state of the world*.

We can summarize the previous discussion by saying that the main questions of economics may be reduced to solving the following formal problems:

1. To compute choice functions (like $x_i^*(p)$).
2. To determine the conditions in which an equilibrium–like $p^*$– exists.
3. To find conditions for the convergence of the beliefs of the agents to a common state of the world.

Each of the next three sections is devoted to one of these problems.

## 3 The Existence of Choice Functions

The hypothesis of rationality of agents allows us to assume that each choice made by an agent is optimal, according to her preferences. Moreover, we assume that this is true for any given choice situation. Otherwise we would not be able to solve a problem like the determination of a market equilibrium. That is, we ask the choice function to be **realizable**, i.e. *computable* [Campbell 1978].

Given the well-known Church's Thesis, we have that the realizability of a choice function amounts to the existence of a Turing machine that yields the choice as an output when the environmental conditions are given as inputs. To make this a bit more precise, let us introduce some definitions.

Given a set of options $\mathbf{X}$, and $\mathcal{F}$ a subset of $2^{\mathbf{X}}$, a choice function $\mathcal{C} : \mathcal{F} \to \mathbf{X}$ is such that for each $\mathbf{B} \in \mathcal{F}$, $\mathcal{C}(\mathbf{B}) \in \mathbf{B}$. In words: for each of its feasible subsets $\mathbf{B}$, the choice function yields only one element in $\mathbf{B}$.

We assume that there exists a preorder over $\mathbf{X}$, denoted $\preceq$, representing the preferences of the agent. Then, we say that $\mathcal{C}$ is realizable if there exists a recursive function[6] $f : \mathbf{X} \to N$, such that[7]

- if $x, y \in \mathbf{X}$, $x \preceq y$ if and only if $f(x) \leq f(y)$
- for all $\mathbf{B} \in \mathcal{F}$, $\mathcal{C}(\mathbf{B})= \{x \in \mathbf{B}\text{: } f(y) \leq f(x), \text{for all } y \in \mathbf{B}\}$.

$\mathcal{C}$ is said to be **recursively realizable** if given its *graph*, $\mathcal{G} = \{\langle \mathbf{B}, \mathcal{C}(\mathbf{B})\rangle\}_{\mathbf{B}\in\mathcal{F}}$ there exists a recursive function $\phi$ such that

$$\phi(\langle \mathbf{B}, \mathcal{C}(\mathbf{B})\rangle) = \begin{cases} 1 & if\ \langle \mathbf{B}, \mathcal{C}(\mathbf{B})\rangle \in \mathcal{G} \\ 0 & otherwise \end{cases}$$

The difference between the problem of the existence of $f$ and that of the existence of $\phi$ is crucial here. The existence of $f$ ensures that $\mathcal{C}$ is *recursively enumerable* (R.E.)[8]

[5] This *Bayesian* conception can be relaxed, just assuming that agents conceive the same set of possible situations and may update it afterwards.

[6] That, is, there exists a Turing machine, that given two natural numbers, $x$ and $y$, answers **yes** to the question *"is $f(x)$ equal to $y$?"* if $y = f(x)$, and **no** otherwise.

[7] $f$ is an equivalent representation of the preference ordering of an agent, while $\mathcal{C}$ yields the chosen alternative on any subset of consumption baskets.

[8] This means, that there exists a Turing machine that can generate each and all the elements of the image of $\mathcal{C}$.

while the existence of $\phi$ yields the recursivity of $\mathcal{C}$. That is, if $\mathcal{C}$ is R.E. we know that given a set **B** the chosen option may be found, while if it is recursive we know **which** option is chosen in each **B**.

In the context of our market example, if a demand function $x_i^*(\cdot)$ is R.E. we may determine, given a price $p$, the chosen consumption. If we know more, namely that $x_i^*(\cdot)$ is recursive, we are able to find the price system $p^*$ that yields the market equilibrium ($\sum_{i=1}^n x_i^*(p^*) = \sum_{i=1}^n w_i$).

The problem of whether $\mathcal{C}$ is recursive or not, leads to another, more general problem, which is to find to which class in the *Arithmetic Hierarchy* it corresponds. This hierarchy, which constitutes a form of classifying degrees of *uncomputability*, is defined as follows: given a set of natural numbers **A**, consider a first-order formula in the language of the theory of numbers $\psi(x)$ such that **A**$= \{x : \psi(x)\}$. Then [Putnam 1973]:

- **A** is in $\Sigma_0^0$ and in $\Pi_0^0$ if $\psi(\cdot)$ is a recursive predicate.
- **A** is in $\Sigma_n^0$ if $\psi(x) \equiv \exists y_1, \forall y_2 \exists y_3 \cdots \Psi(y_1, \ldots, y_n; x)$ where $\Psi(\cdots)$ is a recursive predicate.
- **A** is in $\Pi_n^0$ if $\psi(x) \equiv \forall y_1, \exists y_2 \forall y_3 \cdots \Psi(y_1, \ldots, y_n; x)$ where $\Psi(\cdots)$ is a recursive predicate.

A set **A** is said recursive if its decision problem ("does $x$ belong to **A**?") which has a **yes** answer if $x \in \mathbf{A}$ and a **no** answer if $x \notin \mathbf{A}$, has its answer provided by a Turing machine, which computes the recursive characteristic function of **A**. By definition, each recursive set is in $\Sigma_0^0$ and in $\Pi_0^0$.

**A** is R.E. if its characteristic function is R.E., i.e., its elements can be generated (enumerated) by a Turing machine. It follows that if **A** and its complement $\mathbf{A}^c$ are both R.E., **A** is recursive. A well-known result in Recursion Theory shows that any R.E. set **A** is in $\Sigma_1^0$, while $\mathbf{A}^c$ is in $\Pi_1^0$ [Ash-Knight 2000].

This correspondence between degrees of recursion and classes in the Arithmetic Hierarchy can be extended beyond $\Sigma_1^0$ and $\Pi_1^0$. A set **A** is said $n$-enumerable if it is in $\Sigma_n^0$. That is, it is such that its elements can be enumerated provided that the other elements in the corresponding $n$-ary relation that defines **A** have been enumerated. To see this, suppose **A** is a $\Sigma_n^0$ set. Then, $x \in \mathbf{A}$ if and only if $\exists y_1, \forall y_2 \exists y_3 \cdots \Psi(y_1, \ldots, y_n; x)$. A similar characterization indicates when **A** is in $\Pi_n^0$ (the complement of $\Sigma_n^0$) [Kleene 1943]. A straightforward property of this hierarchy is that if **A** is in $\Sigma_n^0$ ($\Pi_n^0$), it is also in $\Sigma_{n+1}^0$ ($\Pi_{n+1}^0$), for all $n$.

Given two sets **A** and **B** we say that **A** is *Turing reducible* to **B** if there exists a Turing machine that translates the problem of enumerating **A** into a decision problem for **B**. That is, if **B** were recursive, then **A** would be recursively enumerable. If **B** is in either $\Sigma_n^0$ or $\Pi_n^0$, **A** will be in $\Pi_{n+1}^0$ or in $\Sigma_{n+1}^0$, respectively. Then, if **A** is reducible to **B**, it is at least as complex as **B**.

With all these notions at hand, we can present the following negative result:[9]

[9] The tools of *Constructive Analysis* have been also applied to the analysis of the computability of choice functions and equilibria. This variant of Analysis has been characterized as the result of applying Intuitionistic Logic to classical mathematics [Bridges 1994]. In this sense, every entity to be defined must be *constructed* (not just shown to be contradiction-free). Using this approach, a negative result for economic theory has been found [Richter-Wong 1999]. On the other hand, alternative definitions of convexity allow to obtain positive results [Bridges 1992]. Also see [Velupillai 2004] for a relativization of the importance of this approach for Economic Theory.

THEOREM 3.1
[Lewis 1985] In the case that $X$ is the recursive representation of a compact and convex subset of $\Re^l_+$ (see [Moschovakis 1964]), the graph of $\mathcal{C}$ ($\mathcal{G}$) is not a recursive set.

**Sketch of Lewis' argument**: *If we assume that the graph of $\mathcal{C}$ is recursive, its image, $Im(\mathcal{C})$ should also be recursive. On the other hand, $Im(\mathcal{C}) \subseteq [\alpha^-, \alpha^+]$, where $\alpha^-, \alpha^+$ are Gödel numbers corresponding to recursively defined real numbers. But $[\alpha^-, \alpha^+]$ can be Turing reduced to the decision problem of $\mathcal{ALG}([\alpha^-, \alpha^+])$, the set of Gödel codes of algebraic numbers corresponding to elements in $[\alpha^-, \alpha^+]$. If so, the complexity of $Im(\mathcal{C})$ must be at least the same as that of $\mathcal{ALG}([\alpha^-, \alpha^+])$. According to a result in [Shapiro 1956], $\mathcal{ALG}([\alpha^-, \alpha^+])$ is in $\Sigma^0_2$. Since $Im(\mathcal{C})$ is the projection of the graph $\mathcal{G}$ and is $\Sigma^0_2$, $\mathcal{G}$ must also be at least in $\Sigma^0_2$, and therefore it cannot be a recursive set.*

The key tool in the Lewis' "proof" is Shapiro's Theorem $II.15$, which states that the characteristic function of an interval of algebraic real numbers is not computable [Shapiro 1956]. He uses it after the quite suspicious claim that the computability of $\mathcal{C}$ is the same as that of the characteristic function of $[\alpha^-, \alpha^+]$. There is, to our knowledge, no justification for this claim, but if we accept it, Theorem 3.1 shows a serious limitation to the ideal of the realizability of economic theory. In fact, Lewis claims that Theorem 3.1 is similar to a celebrated result in game theory, namely that not every winning strategy is computable. Consider a two-person, zero-sum, perfect information game $\Gamma$, defined in terms of a total recursive function $h : \mathbf{N} \to \mathbf{N}$:

- Player $I$ chooses $i \in \mathbf{N}$.
- Player $II$, knowing $i$ chooses $j \in \mathbf{N}$.
- Player $I$, knowing $i$ and $j$, chooses $k \in \mathbf{N}$.

$\Gamma$ ends there. If $h(k) = i + j$, $I$ wins, otherwise, $II$ wins.
Now assume that $h$ enumerates a *simple* set $\mathbf{S} \subset N$, i.e.:

- $\mathbf{S}$ is infinite and R.E.,
- $\mathbf{N} - \mathbf{S}$ is infinite and there does not exist a R.E. infinite set $\mathbf{W} \subseteq \mathbf{N} - \mathbf{S}$.

Then we have:

THEOREM 3.2
[Rabin 1957] $\Gamma$ has no computable winning strategy.

**Proof**: *Assume that $II$ has a computable winning strategy. Given $i$, $II$ chooses $j$ such that $i + j \notin \mathbf{S}$. That is, $II$'s strategy can be described by a function $\tau : \mathbf{N} \to \mathbf{N}$ such that $i + \tau(i) \notin \mathbf{S}$, for all $i \in \mathbf{N}$. It follows that for every choice $k$ of $I$, since $h(k) \in \mathbf{S}$ and $i + j = i + \tau(i) \notin \mathbf{S}$, $II$ wins. But then, since $w(i) = i + \tau(i)$ is R.E., we have that its image, $\mathbf{W}$, is a R.E. set such that $\mathbf{W} \subseteq \mathbf{N} - \mathbf{S}$. Absurd, since $\mathbf{S}$ is simple.*

This result indicates that even if it is possible to describe the game for each possible sequence of plays and determine its winner (by means of $h$), there is no recursive function that yields 1 if the choice of $II$ leads her to win and 0 otherwise.

The point here is that the failure of recursivity depends critically on the characterization of $\Gamma$. In Lewis' claim, the problem is due to the representation of $\mathcal{G}$. The difference between both cases is critical, despite Lewis' claim that they are analogous. In fact, in the latter case, the choice function $\mathcal{C}$ is represented by means of natural numbers (the codomain of $f$). Then, its recursivity is an **absolute** property, i.e. true in every extension of **ZF**.[10] So, if its recursivity can be proven by other means, Lewis' Theorem 3.1 is false. The rest of this section will be devoted to present an alternative set-theoretical framework in which the recursivity of $\mathcal{C}$ becomes easy to prove.

As a first step, let us consider, as in Theorem 3.2, a *Gale-Stewart* game, i.e. a zero-sum, perfect information game in which two players choose natural numbers and one wins if she can lead the sequence to be in a certain set and looses otherwise. To define this game, $\Gamma_{\mathcal{C}}$, we will consider the same prerequisites as those for Theorem 3.1: we have the recursive presentation of $\mathcal{C}$ over $X$, $f$ and a recursive presentation of the domain of $\mathcal{C}$, $\mathcal{F}$, i.e. a recursive function $F$ such that $F(\mathbf{B})=1$ if $\mathbf{B} \in \mathcal{F}$.

In $\Gamma_{\mathcal{C}}$, Agent $I$ (the *spoiler*) chooses a subset $\mathbf{B} \in \mathcal{F}$. $II$ replies with an element $x_0^{II} \in \mathbf{B}$, and player $I$ chooses $x_0^I \in \mathbf{B}$ such that $f(x_0^I) > f(x_0^{II})$. $II$ selects $x_1^{II} \in \mathbf{B}$ such that $f(x_1^{II}) > f(x_0^I)$. In turn $I$ will try to find $x_1^I \in \mathbf{B}$ such that $f(x_1^I) > f(x_1^{II})$, etc. The game finishes at any stage when either one of the players fails to find the response to the previous move. If $II$ has a winning strategy it must consist in choosing $x^* = \mathcal{C}(\mathbf{B})$, for any $\mathbf{B} \in \mathcal{F}$. Furthermore, if a winning strategy for $II$ exists and is recursively defined, $\mathcal{C}$ will be computable.

Therefore, the first step towards ensuring the computability of $\mathcal{C}(\cdot)$ is to force the existence of a recursively defined winning strategy for $II$. One method to achieve this is by including in our set theory the **Axiom of Determinacy**(**AD**). **AD** states that each Gale-Stewart game is **determined**, i.e. there exists a winning strategy for it.[11] On the other hand, the addition of **AD** to our underlying set theory forces us to drop the Axiom of Choice, although in **ZF** it is consistent with the weaker **DC** (Axiom of Dependent Choices).[12] That is, we can assume that for any binary relation $R \subseteq \mathbf{X} \times \mathbf{X}$ (for any $\mathbf{X} \neq \emptyset$), if for any $x \in \mathbf{X}$ there exists $y \in \mathbf{X}$ such that $\langle x, y\rangle \in R$, then there exists a countable sequence $(x_n)_{n \in N} \subseteq \mathbf{X}$ such that $\langle x_n, x_{n+1}\rangle \in R$ for every $n$. **DC** is, in **ZF**, equivalent to the *Principle of Recursive Constructions*, i.e., given $\mathbf{X} \neq \emptyset$ and the set of all finite sequences in $\mathbf{X}$, denoted ${}^{<\omega}\mathbf{X}$, if we have a function $G : {}^{<\omega}\mathbf{X} \to 2^{\mathbf{X}} \setminus \{\emptyset\}$, where $2^{\mathbf{X}}$ is the class of subsets of $\mathbf{X}$, then there exists $f_G : \mathbf{N} \to \mathbf{X}$ such that $f_G(0) \in G(\emptyset)$ and $f_G(n) \in G(f_G(0), \ldots, f_G(n-1))$, for all $n > 0$ [Just-Weese 1996].

Consider now the properties of *ultrafilters* over the set of natural numbers, $\omega$. A family of sets $\mathcal{U} \subseteq 2^{\mathbf{S}}$ is said an *ultrafilter* over a set $\mathbf{S}$ if: $(i)$ $\mathbf{S} \in \mathcal{U}$; $(ii)$ $\mathbf{A} \cap \mathbf{B} \in \mathcal{U}$, if $\mathbf{A}, \mathbf{B} \in \mathcal{U}$; $(iii)$ if $\mathbf{A} \in \mathcal{U}$, and $\mathbf{A} \subseteq \mathbf{B}$, then $\mathbf{B} \in \mathcal{U}$; $(iv)$ if each $\mathbf{A} \subseteq \mathbf{S}$ satisfies either $\mathbf{A} \in \mathcal{U}$ or $\mathbf{A} \in 2^{\mathbf{S}} \setminus \mathcal{U}$. A ultrafilter $\mathcal{U}$ is said to be *free* if $\cap_{\mathbf{A} \in \mathcal{F}} \mathbf{A} = \emptyset$. Otherwise it is called *principal*: there exists a singleton $\{a\}$ such that $\mathcal{U} = \{\mathbf{A} : \{a\} \subseteq \mathbf{A}\}$. To ensure the computability of $\mathcal{C}$ we can profit from the fact that in $\mathbf{ZF} + \mathbf{AD}$ every ultrafilter over $\omega$ is principal [Just-Weese 1996].

We have that:[13]

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[10] Jan Mycielski, personal communication.

[11] **AD** was introduced in [Mycielski-Steinhaus 1962]. There exists an already huge and growing literature on this axiom. Very readable presentations, which show how the topic evolved in time, can be found in [Fenstad 1971],[Jech 1973],[Mycielski 1992], [Marek-Mycielski 2001].

[12] The incorporation of **AD** and the loss of **AC** has important consequences, one of them being that every set of real numbers becomes Lebesgue-measurable [Jech 1973].

[13] Related arguments, in rather different contexts, lead to similar conclusions [Canning 1992], [Mihara 1997].

Lemma 3.3
In $\mathbf{ZF} + \mathbf{DC} + \mathbf{AD}$, $\mathcal{C}$ is recursive.

**Proof:** *If there exists a winning strategy $\sigma$ for $II$ in $\Gamma_{\mathcal{C}}$, we can ensure that for each $\mathbf{B} \in \mathcal{F}$, $\mathcal{C}(\mathbf{B}) \neq \emptyset$. We expect $\sigma(\mathbf{B})$ (the consequence of playing $\sigma$ from the initial choice $\mathbf{B}$ made by $I$) to yield the element $x^* = \mathcal{C}(\mathbf{B})$. If $\sigma$ can be recursively given, then, a Turing machine can be defined such that for every pair $\langle \mathbf{B}, \mathcal{C}(\mathbf{B}) \rangle$ yields a $YES$ answer if $\sigma(\mathbf{B}) = \mathcal{C}(\mathbf{B})$ and a $NO$ otherwise. Let us show how to construct such $\sigma$. Consider a family of sets drawn from $\mathbf{Im}(f(\mathbf{B}))$, the range of $f$ restricted to $\mathbf{B}$. This class $\mathcal{O} \subseteq 2^{\mathbf{Im}(f(\mathbf{B}))}$ satisfies that $\mathbf{A}_x \in \mathcal{O}$ iff $\mathbf{A}_x = \{f(y) :$ there exists $x \in \mathbf{B}$, $f(x) \leq f(y)$, and $f(y) \in \mathbf{Im}(f(\mathbf{B}))\}$. Consider now the class $\mathcal{O}_{\mathcal{C}}$, consisting of sets of the form $\mathbf{A}_n = \{f(x_n), f(x_{n+1}), \ldots\}$ where $x_n$ is a choice that can be made at the n-th stage of $\Gamma_{\mathcal{C}}$ by either $I$ or $II$ and $x_{n+1}, \ldots$ is a possible ensuing sequence of choices in the play. We have that*

- *$f(\mathbf{B}) \in \mathcal{O}_{\mathcal{C}}$, since all the elements in $\mathbf{B}$ can be played out in a play of $\Gamma_{\mathcal{C}}$.*
- *$\mathbf{A} \cap \mathbf{A}' \in \mathcal{O}_{\mathcal{C}}$, if $\mathbf{A}, \mathbf{A}' \in \mathcal{O}_{\mathcal{C}}$, since the common elements in two different sequences are ordered (since they arise in plays of $\Gamma_{\mathcal{C}}$) and therefore constitute a sequence in itself*
- *if $\mathbf{A} \in \mathcal{O}_{\mathcal{C}}$, and $\mathbf{A} \subseteq \mathbf{A}'$ ($\mathbf{A}' \in \mathcal{O}$), then $\mathbf{A}' \in \mathcal{O}_{\mathcal{C}}$, since $\mathbf{A}' = \mathbf{A}_x$ for a given $x$, and therefore can be seen as arising in the play of $\Gamma_{\mathcal{C}}$.*
- *each $\mathbf{A} \in \mathcal{O}$ verifies that $\mathbf{A} \in \mathcal{O}_{\mathcal{C}}$.*

*It follows that $\mathcal{O}_{\mathcal{C}}$ is trivially an ultrafilter over $f(\mathbf{B}) \subseteq \omega$ since each element in $f(\mathbf{B})$ is represented by a natural number. If the cardinality of $\mathbf{Im}(f(\mathbf{B}))$ is finite, $\mathcal{O}_{\mathcal{C}}$ is principal. Otherwise, since we are assuming that the underlying set theory is* $\mathbf{ZF} + \mathbf{AD}$, *$\mathcal{O}_{\mathcal{C}}$ cannot be free. That is, there exists a singleton $\{f(x^*)\}$ such that $\mathcal{O}_{\mathcal{C}} = \{\mathbf{A} : \{f(x^*)\} \subseteq \mathbf{A}\}$. This means that $f(x^*) \geq f(y)$ for every $y \in \mathbf{B}$. That is, $x^* = \mathcal{C}(\mathbf{B})$. Since $f(x^*)$ belongs to all sequences played out in $\Gamma_{\mathcal{C}}$, $II$ can always win. Let us see how to define the corresponding $\sigma(\mathbf{B})$. First of all, notice that for any sequence of values in $f(\mathbf{B})$ that may arise from a play of $\Gamma_{\mathcal{C}}$, we have a set $\mathbf{A}_n \subseteq f(\mathbf{B})$ that includes the sequence. Then, there exists, according to the* Principle of Recursive Constructions*(equivalent to* $\mathbf{DC}$ *in* $\mathbf{ZF}$*), a function $h_{\mathbf{B}} : \mathbf{N} \rightarrow f(\mathbf{B})$ such that $h_{\mathbf{B}}(0) \in f(\mathbf{B})$, $h_{\mathbf{B}}(1) \in \mathbf{A}_0$, ..., $h_{\mathbf{B}}(n) \in \mathbf{A}_{n-1}$, for all $n > 0$, where $\mathbf{A}_n = \{f(x) : h_{\mathbf{B}}(n) \leq f(x)\} \subseteq f(\mathbf{B})$. Since this is a primitive recursive function, there exists a Turing machine that, given as an input $\mathbf{B}$, it produces the course of values of $h_{\mathbf{B}}$. The machine can be so defined as to have on its initial tape the code number of $\mathbf{B}$ and by a process of writing and erasing, at the end of each stage $n$ to have on the tape the number corresponding to $h_{\mathbf{B}}(n)$. The machine will always stop with the value of $f(x^*)$ written on the tape. Since $f$ is recursive, another Turing machine can read $f(x^*)$ and output $x^*$. The combined action of these two Turing machines is independent of the particular $h_{\mathbf{B}}$. So, we can define $\sigma(\mathbf{B})$ as $x^*$. It follows that $\mathcal{C}$ is recursive.*

Lemma 3.3 reveals that Lewis' Theorem 3.1 is mistaken. In fact, the absoluteness of the recursivity of $\mathcal{C}$ means that, in $\mathbf{ZF}$, given any $\mathbf{B}$ we will always obtain $x* = \mathcal{C}(\mathbf{B})$ by using the same Turing machines described in the proof of Lemma 3.3. Besides this important consequence, the construction used in the proof reveals how powerful becomes the adoption of **AD** as an axiom of the alternative set theory. The principal ultrafilter result, one of its main derivations, is the basis of the simple construction that yields a mechanical procedure of economic choice with ensured termination. Furthermore, this argument indicates that the change from **ZFC** to $\mathbf{ZF} + \mathbf{DC} + \mathbf{AD}$ provides a formal playground in which the main results of Economic Theory may be recasted in simpler terms.

Of course, the whole exercise would be trivial if $\mathbf{ZF}+\mathbf{AD}+\mathbf{DC}$ were not consistent. But consistency proofs have not been found even for **ZF** within itself. In fact, set theorists use normally large cardinals just to build *inner models* of **ZF** as well as of some other set theories, as a mechanism to evaluate the consistency of the theory up to a very large cardinal. It has been shown that $\mathbf{ZF}+\mathbf{AD}+\mathbf{DC}$ is satisfiable by the inner model $L[\mathbf{R}]$, given certain additional axioms of *large cardinals* ([Martin-Steel 1989]) and, moreover, it has been shown by Woodin that it is equiconsistent with **ZF** + "There are infinite Woodin cardinals" [Jech 2003].[14] For our purposes it suffices to know that these results led set-theoreticians to believe that $\mathbf{ZF} + \mathbf{AD} + \mathbf{DC}$ is not inconsistent.

To switch to this set theory could mean, in the context of choice functions in economics, that we assume that the internal deliberation process that leads to a choice reaches a result, and moreover, that under the same conditions the result must also be the same. A trivial form in which this could happen is if agents compute their choices with a certain bound in precision. Once an outcome has that degree of precision (say in solving a maximization problem using an approximation method), it is considered the chosen option.[15]

We cannot leave the topic of the computability of choice functions without a brief analysis of another kind of complexity. On one hand, in $\mathbf{ZF}+\mathbf{AD}+\mathbf{DC}$ the complexity of $\mathcal{C}$ equals that of a Turing machine. Another kind of complexity is defined in terms of the number of steps needed to get to a result. Given a finite choice set **B**, if the number of bits in its description is $m$, we say that it is *polynomial* if the number of steps needed to calculate its Gödel number is $\mathcal{O}(t^m)$, i.e. a polynomial of the type $\alpha_m t^m + \alpha_{m-1}t^{m-1} + \cdots + \alpha_1 t + \alpha_0$. It would be highly desirable that if $\mathcal{C}(\mathbf{B})$ can be described with, say, $r$ bits, the number of steps to compute its Gödel number were also polynomial, i.e. its complexity were $\mathcal{O}(t^r)$. The condition that allows this is characterized by:

Theorem 3.4
[Friedman 1984] The outcome of an optimization problem over polynomial inputs is polynomial if and only if $\mathbf{P} = \mathbf{NP}$.

That is, the encoding of the outcome of a finite optimization over a polynomial input is polynomial only in case that all the problems that have solutions that can be checked out in a polynomial number of steps can also be solved in a polynomial number of steps.[16] As it is well known, whether **P** is equal or not to **NP** is an

[14] The "large cardinals" axioms postulate the existence of cardinals that cannot be proven to exist in **ZF**. Each of such axioms implies that for every definable property of sets, there must exist very large sets satisfying them. [Marek-Mycielski 2001]

[15] This may be a formal form to present Norbert Wiener's quip that arithmetic in economics does not require more than two decimal digits [Wiener 1964]. Of course, in this case the range of values is discrete, requiring therefore only a rather simple set theory.

[16] The intuition is that, while the actual solution may be easily describable, its computation–as the result of the process of maximization–may require a high number of steps.

open problem, but there is a consensus among experts that the answer is for the negative. If so, according to Friedman's result, we should expect that the solution to an optimization problem over polynomial inputs will not be polynomial. In other words, the optimization may be *intractable*, being practically feasible only for small inputs. If we recall that $\mathcal{C}(\mathbf{B})$ is the solution to a maximization problem we can see the relevance of this problem for economic theory. But the change of set theory to $\mathbf{ZF} + \mathbf{AD} + \mathbf{DC}$ does not facilitate the solution of the $\mathbf{P} \neq \mathbf{NP}$ question.[17]

## 4 Existence of Equilibria

Market equilibria are fundamental entities in economic theory. If an economy is not in equilibrium, there are few characteristics of that economy that can be ascertained. On the other hand, the mere existence of an equilibrium does not ensure that the economy will end up there. But, in any case, to prove their existence is the first step towards the development of a more general model of economic interactions.

Kenneth Arrow and Gerard Debreu, applying the notion of *Nash equilibrium* found the first general proof of existence of equilibria. Further work by those authors led to a refinement of the proof. They got rid of the need of specifying a fictitious game (in order to find a Nash equilibrium) by using the main formal tool behind Nash's Theorem. That is, they used Kakutani's fixed-point theorem to determine the conditions that ensure the existence of a fixed point. The key to the proof is to define an *excess demand* function. In our leading example, $z(p) = \sum_{i=1}^n x_i^*(p) - \sum_{i=1}^n w_i$. Since $z(p) \in \Re^l_+$, if the price of good $j$ is such that its corresponding entry in the excess demand vector is $z_j(p) > 0$, it is increased from $p_j$ to $p_j^{'} > p_j$. If instead $z_j(p) < 0$ then $p_j^{'} < p_j$, while if $z_j(p) = 0$, $p_j$ remains unchanged. Therefore, we have a correspondence $F$ that takes $p$ as an argument and yields $p^{'}$ (given $p$ it determines $z(p)$ and then changes the prices according to the signs of its coordinates). An equilibrium is just a $p^*$ such that $p^* \in F(p^*)$. Standard properties of preferences, which translate into the functional form of $F(\cdot)$, ensure the existence of such $p^*$. Variants of this argument, involving all the available fixed-point theorems, have been applied to prove the existence of equilibria in different market structures.[18]

If we consider $\mathbf{E}$, the class of economies with a finite number of agents and endowments in a finite-dimensional Euclidean space, we can distinguish the corresponding space of prices $\Delta_{\mathbf{E}}$. Then, the existence of an equilibrium amounts to the claim that there exists a well defined correspondence $\mathcal{EQ} : \mathbf{E} \to \Delta_{\mathbf{E}}$, such that for $\epsilon = \{\langle X_i, \preceq_i, w_i, \rangle\}_{i=1}^n$, if $p^* \in \mathcal{EQ}(\epsilon)$, then $\sum_{i=1}^n x_i^*(p^*) - \sum_{i=1}^n w_i = 0$. Nice as this sounds, the following result casts doubt on its meaningfulness:

PROPOSITION 4.1
[Lewis 1992] The graph of $\mathcal{EQ}$ is not recursive.

**Sketch of Lewis' argument**:*Since each equilibrium is found by considering the individual demand functions, $x_i^*(\cdot)$, $\mathcal{EQ}$ is Turing-reducible to their sum. But each demand function is a choice function. Then, the complexity of $\mathcal{EQ}$ is at least that of*

[17] Unless $\mathbf{P} = \mathbf{NP}$ is shown undecidable [Tsuji-Da Costa-Doria 1998]. If so, there might be models of our set-theory in which it is true and others in which not. Extra axioms may make the claim decidable. But this is not an alternative that theoretical computer scientists consider seriously [Pudlák 1996].

[18] Even the topology-free Fixed-Point Theorem of Tarski has been shown useful [Tohmé 2003].

*a $\mathcal{C}$ (as discussed in section 3). According to Theorem 3.1, a $\mathcal{C}$ is in the class $\Sigma^0_2$. Therefore, $\mathcal{EQ}$ cannot be either in $\Sigma^0_0$ or $\Pi^0_0$, i.e. recursive.*

This result, obviously true if we accept Theorem 3.1, can be disposed again by means of **AD**:

Lemma 4.2
In $\mathbf{ZF} + \mathbf{DC} + \mathbf{AD}$, $\mathcal{EQ}$ is recursive.

**Proof** *Consider a game between "the economy" and a fictitious player, usually known in the literature as the "Walrasian auctioneer". For each economy $\epsilon$, in which the commodity space is the recursive representation of $\Re^l_+$, denoted $\widehat{\Re^l_+}$, the auctioneer begins by announcing a price system $p \in \widehat{\Delta_{\mathbf{E}}} \subset \widehat{\Re^l_+}$. $p$ determines the corresponding choice sets of each agent $i$: $\mathbf{B}_i(p) = \{x : p \cdot x \leq p \cdot \omega_i\} \subset \widehat{\Re^l_+}$. Each agent responds with the choice (recursively defined, according to Lemma 3.3) $x^*_i(p)$. That is, the economy responds with the vector of demands $(x^*_1(p), \ldots, x^*_n(p))$. If $z(p) = \sum_{i=1}^n x^*_i(p^*) - \sum_{i=1}^n w_i = 0$ the auctioneer wins, otherwise he proposes another system of prices $p^{'} \in \widehat{\Re^l_+}$ and the economy responds with the corresponding demands, etc. The game ends if a $p^*$ is reached such that $z(p^*) = 0$. Otherwise, the game goes on forever. To show that the auctioneer has a winning strategy, just notice that in $Z(p) = \{z(p) : p \in \widehat{\Delta_{\mathbf{E}}}\}$ we can distinguish a family of sets $\mathbf{A}_p = \{z(p^{'}) :$ there exists $p^{'} \in \widehat{\Delta_{\mathbf{E}}}, |z(p)| \geq |z(p^{'})|\}$. Over this class we can define trivially, as in the proof of Lemma 3.3, an ultrafilter based on the sets $\mathbf{A}_{p^W}$, where $p^W$ is any price that can be announced by the (Walrasian) auctioneer in any play of her game with the economy. In $\mathbf{ZF} + \mathbf{DC} + \mathbf{AD}$ this ultrafilter is principal, and there exists $p^*$ such that $\{z(p^*)\} \subseteq \mathbf{A}_p$ for any $p \in \widehat{\Delta_{\mathbf{E}}}$. By definition, $z(p^*) = 0$. That is, there exists a winning strategy for the auctioneer and it can be recursively defined as in the proof of Lemma 3.3.*

Another line of attack on the problem of existence of equilibria has also a long tradition in economics. Francis Y. Edgeworth initiated this approach to the analysis of exchange economies by introducing the notion of the **core** of an economy, that is, the set of resource allocations that cannot be improved upon for at least one agent without impoverishing the others. Edgeworth advanced the idea, later known as the *Core Equivalence Conjecture*, that the core of a competitive economy with a large number of traders is identical to the set of equilibrium allocations. The various proofs of this conjecture have been frequently cast as limit results. The key idea in those proofs is to make the relative importance of each agent decrease when the number of agents increases. One of the approaches consisted in assuming that in the limit there exists a continuum of agents, for example indexing them with the real numbers in the closed interval $[0, 1]$, a setting in which measure theory can be applied [Aumann 1966].

In the case in which the number of agents is uncountable, the economy is seen as the limit of sequences of economies in which the measure of non-competitiveness converges to zero. In terms of the cores of those economies it means that they become "smaller" until, in the limit economy, the core has measure zero and coincides with the set of equilibria in the economy [Hildenbrand 1974]. A closely related approach consisted in assuming that in the limit economy each agent is indexed by an infinitesimal number.

In that setting the tools of non-standard analysis were applied to find an analogous result [Brown-Robinson 1975].

Besides measure theory and non-standard analysis other mathematical tools have been applied to proving versions of the Core Equivalence Theorem. This includes proofs using only elementary arguments. They begin by defining an economy like the one we presented in section 2. That is, a $n$-agent economy $\epsilon_n$ is:

$$\epsilon_n \ = \ \{\langle \mathbf{X}_i \ , \preceq_i, w_i\rangle\}_{i=1}^n$$

An *allocation* is defined as $x \ \in \ \mathbf{X}$, where $\mathbf{X} \ = \ X_1 \times \ldots \times X_n$. An allocation $x$ is said *feasible* if $\sum_{i=1}^n x_i \ = \ \sum_{i=1}^n w_i$. Then, the core of $\epsilon_n$ is defined as:

$$\begin{aligned} \mathcal{COR}(\epsilon_n) \ = \ \{\bar{x} \in X : & \text{ there is no } S \subseteq \{1, \ldots, n\} \text{ and } x \in X \text{ such that} \\ & \textstyle\sum_{i \in S} x_i = \sum_{i \in S} w_i \text{ and } \bar{x}_i \preceq_i x_i \text{ for all } i \in S\}, \end{aligned}$$

that is, $\bar{x}$ is in the core only if there is no other allocation, feasible for at least one group of agents, that makes them better off.

As the number of agents in $\epsilon_n$ is finite, we do not expect the core to coincide with an equilibrium allocation. But, if we consider a sequence of economies that converges in the sense that their cores shrink[19], then the result is that in the limit, the core has a single allocation $\bar{x}$, which coincides with the equilibrium allocation, $x^*(p^*)$.

The interest in elementary proofs resides in that they can be developed, in principle at least, without the aid of advanced mathematical tools. Even so, they have been subjected to a certain amount of criticism. As briefly noted in the Introduction, Alain Lewis pushed for the exclusive use of *effective* methods in economic theory. A first step in that direction was for him to restrict the mathematical arguments to be part of only *ordinary mathematics*, meaning the fragment of **absolute** first-order logic formulas in the language of set theory. That is, formulas of the form $\psi(x_1, \ldots, x_n)$, with free variables $x_1, \ldots, x_m$ such that for any model of **ZF**, $\mathcal{M}$, for all constants $a_1, \ldots, a_m \in \mathcal{M}$, $\mathcal{M} \models \psi(a_1, \ldots, a_m)$ if and only $\mathcal{M} \models \psi_{|\mathcal{M}}(a_1, \ldots, a_m)$, where $\psi_{|\mathcal{M}}$ is $\psi$ with its bounded (i.e. in the scope of a quantifier) variables also restricted to be in $\mathcal{M}$ [Just-Weese 1996].[20]

Lewis' concern was that the proofs and arguments should be effective, that is, that they must only involve recursively defined steps. Since ordinary mathematics does not use the non-effective **AC** he found that **ZF** was the right setting for his program. In the cases in which a direct proof of non-recursiveness was not available he tried to show that there exists a model of **ZF** in which the involved notions are not true (indicating that they are not part of ordinary mathematics) and therefore that they could not be effective.

Using this kind of argument, Lewis made the claim that even the elementary core equivalence results were not effective [Lewis 1991]. Let us analyze this claim. If $\mathbf{M}_{\epsilon_n} \ = \ \max\limits_{S \subseteq \{1, \ldots, n\}} \max\limits_{j=1, \ldots, l} \{\sum_{i \in \mathbf{S}} w_i^j\}$ we have that:

LEMMA 4.3
[Anderson 1978] If $\bar{x} \in \mathcal{COR}(\epsilon_n)$ there exists a system of prices $p^*$ such that:

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[19] In the sense that the difference between the class of equilibria (which remains fixed in all the sequence) and that of allocations in the core becomes progressively reduced [Hildenbrand-Kirman 1988].

[20] A whole research program, known as *Reverse Mathematics* consists in searching for a set-theoretical foundation, less general than **ZF**, for ordinary mathematics [Friedman 1981].

- $\sum_{i=1}^n |p^* \cdot (\bar{x}_i - w_i)| \ \leq 2\mathbf{M}_{\epsilon_n}$.
- $\sum_{i=1}^n |\inf\{p^* \cdot (x_i - w_i) : \ \bar{x}_i \preceq_i x_i\}| \ \leq 2\mathbf{M}_{\epsilon_n}$.

Of course, if $\bar{x} \in \mathcal{COR}(\epsilon)$ already coincides with an equilibrium allocation $\langle x_1^*(p^*), \ldots, x_n^*(p^*)\rangle$ corresponding to the system of prices $p^*$ this lemma follows immediately. More generally, if we have a sequence of economies $\{\epsilon_n\}_{n>1}$ such that:

- $\frac{\mathbf{M}_{\epsilon_n}}{n} \to 0$.
- $\sup_n \frac{\max_{k=1\cdots l}\{\sum_{i=1}^n w_i^k\}}{n} < \infty$.

with a few additional technical conditions, including Lemma 4.3, we have the following result:

Theorem 4.4

[Anderson 1981] There exists a sequence of prices $\{p_n^*\}_{n>1}$ such that for all $\mathbf{S} \subseteq \{1, \ldots, n\}$ we have that:

$$\frac{\max_{k=1,\ldots,l}\{|\sum_{i\in\mathbf{S}} \bar{x}_i^j - x_i^j(p_n^*)|\}}{n} \to 0.$$

That is, when $n$ grows, the number of goods for which the core allocation differs from the equilibrium allocation decreases. In the limit the core coincides with the equilibrium allocation. But we have that:

Proposition 4.5

[Lewis 1992] There exists a model of **ZF**, $\mathcal{M}$, in which, given the conditions of Theorem 4.4, every sequence $\{\epsilon_n\}_{n>1}$ is such that for each $n$, $\mathcal{COR}(\epsilon_n)$ is either $\emptyset$ or includes at most one allocation.

The argument given by Lewis is that if at least for one $n$, $|\mathcal{COR}(\epsilon_n)| > 1$, then there must exist a choice function $g : \omega \to \cup_n \mathcal{COR}(\epsilon_n)$, such that $g(n) \in \mathcal{COR}(\epsilon_n)$. Since this, according to Lewis, requires the application of the full Axiom of Choice (**AC**), and this axiom is independent of **ZF**, there exists a model of **ZF** in which $g$ cannot be defined.

This argument can be criticized on several grounds, but let us concentrate on Lewis' ultimate goal of showing that the Core Equivalence cannot be proven effectively. In this sense, if a non-effective axiom is required then we cannot expect the outcome to be effective. But, in fact, the existence of $g$ *does not* require the full **AC** but only a much weaker form, called Axiom of Countable Choice (**CC**) that allows to select an element from a each set in a countable class.

**CC** can be derived from **DC** [Just-Weese 1996]. Therefore, if we assume **DC** it is enough to ensure the existence of $g$. Of course **DC** is independent of **ZF** and Proposition 4.5 will still stand. But as remarked above, **DC** is equivalent in **ZF** to the Principle of Recursive Constructions. That is, it is effective. And therefore, non-trivial sequences of prices $\{p_n^*\}_{n>1}$ fulfilling the conditions of Theorem 4.4 can be effectively defined in $\mathbf{ZF} + \mathbf{DC}$.

Interestingly enough, Lewis overlooks a more interesting source of possible problems for the validity of Theorem 4.4. The proof of Lemma 4.3 uses a variant of the Hahn-Banach theorem, Minkowski's theorem, that ensures that two disjoint convex sets can be separated by a hyperplane. Only one variant of the Hahn-Banach theorem can be proved in **ZF**, the so-called Finite Extension Lemma (FEL) [Schechter 1997].[21] But FEL is too weak to imply Minkowski's Theorem ($\Re^l$ is not a convex hull of the union of any of its hyperspaces and a finite number of points). Therefore Lemma 4.3 requires more than **ZF** to be true. But then, it is known that **DC** implies the Hahn-Banach theorem in separable spaces [Bridges 1994]. Since Euclidean spaces like $\Re^l$ are separable, in $\mathbf{ZF} + \mathbf{DC}$ Lemma 4.3 can be effectively established.

Another form of the Core Equivalence result can be obtained considering **market games**. This approach permits us to translate the structure of an economy into a coalitional game form. Let us consider an economy

$$\epsilon_n \ = \ \{\langle \mathbf{X}_i, u_i, w_i\rangle\}_{i=1}^n$$

where each $u_i : \mathbf{X}_i \to \Re$ is a representation of the preferences $\preceq_i$, i.e., for $x, y \in \mathbf{X}_i$, $u_i(x) \leq u_i(y)$ if and only if $x \preceq_i y$.

Then, a market game is a cooperative (or coalitional) game $\Gamma = \langle \mathbf{I}, A(\mathbf{I}), \mu\rangle$, where $\mathbf{I}= \{1, \ldots, n\}$ is the set of agents, $A(\mathbf{I}) \subseteq 2^{\mathbf{I}}$ the class of coalitions of agents and $\mu$: $A(\mathbf{I}) \to \Re$ is the payoff function. Given $\epsilon_n$, a game $\Gamma$ can be defined such that for each coalition $\mathbf{S} \subseteq \mathbf{I}$, $\mu(\mathbf{S})= \max_{x^{\mathbf{S}}} \sum_{i\in\mathbf{S}} u_i(x_i^{\mathbf{S}})$, where $x^{\mathbf{S}}= \sum_{i\in\mathbf{S}} x_i^{\mathbf{S}}$ such that $\sum_{i\in\mathbf{S}} x_i^{\mathbf{S}}= \sum_{i\in\mathbf{S}} w_i$. In words, the payoff to a coalition $\mathbf{S}$ is the best sum of utilities that the members of the coalition can achieve in a redistribution of their endowments.

The core of the market game $\Gamma$ is

$$\mathcal{COR}(\Gamma) = \{(\bar{\mu_1}, \ldots, \bar{\mu_n}) : \text{for all } \mathbf{S} \in A(\mathbf{I}), \sum_{i\in\mathbf{S}} \bar{\mu_i} \geq \mu(\mathbf{S})\}$$

That is, the core consists of the vector of individual payoffs that cannot be improved upon by any coalition. Given the endowments associated to $\Gamma$, $\{w_i\}_{i\in\mathbf{I}}$, the core corresponds to their redistribution among the agents in $\mathbf{I}$ such that no individual nor coalition could get a higher payoff in another distribution without reducing the payoffs corresponding to other agents. In **ZFC** we have that:

Theorem 4.6
[Shapley-Shubik 1969] Every market game has a non-empty core.

A proof of this result is based on the simple fact that if $x^*$ is an equilibrium allocation in the economy $\epsilon_n$, then if for each $i$, $\bar{\mu_i}= u_i(x_i^*)$, then $(\bar{\mu_1}, \ldots, \bar{\mu_n}) \in \mathcal{COR}(\Gamma_n)$. And this can be proven entirely in **ZF**. Of course, this last proof assumes that a market game is associated with an economy. When this connection is not assumed a proof can be given using again a form of Minkowski's Theorem. We already discussed that this obtains in $\mathbf{ZF} + \mathbf{DC}$.

A Core Equivalence result requires to consider a countable sequence of market games, $\{\Gamma_k\}_{k<\omega}$, in which each $\Gamma_k$ is $\langle \mathbf{I}_k, A(\mathbf{I}_k), \mu^k\rangle$, where $\mathbf{I}_k \subseteq \mathbf{I}$, $A(\mathbf{I}_k) \subseteq 2^{\mathbf{I}_k}$ the class of coalitions of agents and $\mu^k$: $A(\mathbf{I}_k) \to \Re$ is the payoff function. Furthermore the cores shrink with $k$. The limit of this sequence must be a game $\Gamma^*$ such that $\mathcal{COR}(\Gamma^*)$ is a singleton (corresponding to the equilibrium allocation). In **ZF** this sequence may not exist:

[21] The version of FEL that is of interest for us is as follows: *Suppose that over a linear subspace* $\mathbf{H} \subseteq \Re^l$*, a linear map* $f^0 : \mathbf{H} \to \Re^l$ *is such that* $f^0 \leq p$*, where* $p$ *is a convex map* $p : \Re^l \to \Re^l$*. Suppose furthermore that there exists* $\mathbf{S}= \{s_1, \ldots, s_m\} \subset \Re^l$ *such that* $\Re^l$ *is the convex hull of* $\mathbf{H} \cup \mathbf{S}$*. Then,* $f^0$ *can be extended to* $f : \Re^l \to \Re^l$*, such that* $f \leq p$ *over* $\Re^l$*.*

PROPOSITION 4.7
[Lewis 1990] There exists a model $\mathcal{M}$ of **ZF** in which there exists an infinite class of market games $\{\Gamma_k\}_{k<\omega}$ all of which satisfy the property that $\mathcal{C}OR(\Gamma_k) = \emptyset$.

**Proof** *(Sketch): Consider a model of* **ZF**, $\mathcal{M}$ *such that a measure* $\mathbf{\Upsilon}$ *is defined satisfying that* $\mathbf{\Upsilon}(\mathbf{S})$ *is either* 1 *or* 0*, for each* $\mathbf{S} \subseteq \omega$*. Furthermore, we can assume* $\mathbf{\Upsilon}(\mathbf{S}) = 0$ *for every finite set* $\mathbf{S}$*. Then, for each* $\mathbf{I}_k$ *define a game* $\langle \mathbf{I}_k, A(\mathbf{I}_k), \mu^k\rangle$*, where* $\mu^k$ *is the restriction of* $\mathbf{\Upsilon}$ *on* $2^{\mathbf{I}_k}$*. In each of these games all the coalitions can be improved upon. That is, the core is empty, because all redistributions are feasible.*

A model of **ZF** with this property was originally obtained by Paul Cohen, using his method of **forcing**. Cohen produced a model in which every Boolean Algebra $2^\omega$ admits a real measure $\mathbf{\Upsilon} : 2^\omega \rightarrow \Re$ which is either 0 or 1, that vanishes over all the finite subsets of $\omega$ [Jech 1973].[22]

In turn, the existence of a bi-valued measure $\mathbf{\Upsilon}$ over the Boolean Algebra $2^\omega$ that vanishes over all the finite sets implies in **ZF** that there exists a non-measurable subset of $\omega$ [Sikorski 1969]. On the other hand, in $\mathbf{ZF} + \mathbf{AD} + \mathbf{DC}$ every set of real numbers is measurable. [23] In fact, we have that

PROPOSITION 4.8
In $\mathbf{ZF} + \mathbf{AD} + \mathbf{DC}$, given a sequence of economies $\{\epsilon_n\}_{n\in\omega}$, with each $\epsilon_n = \{\langle \mathbf{X}_i, u_i, w_i\rangle\}_{i=1}^n$ such that:

- $\sum_{i=1}^n w_i \rightarrow_{n\rightarrow\infty} \mathbf{w} < \infty$ (i.e. the available endowments are bounded for every economy in the sequence).
- For each pair of economies $\epsilon_n$ and $\epsilon_{n+1}$ and for each $i \leq n$, the corresponding endowments $w_i^n$ and $w_i^{n+1}$ satisfy that $w_i^{n+1} \not> w_i^n$ (with more agents, no one gets a larger individual endowment).
- $\mathbf{\Upsilon}(\mathcal{C}OR(\epsilon_n)) \rightarrow_n 0$ in any suitable notion of measure $\mathbf{\Upsilon}$ over Euclidean spaces.

Then, to each $\epsilon_n$ there corresponds a market game $\Gamma_n$, with a non-empty core, such that the sequence of market games $\langle\Gamma_n\rangle_{n<\omega}$, converges to a limit game $\Gamma^* = \langle\omega, A(\omega), \mu\rangle$ in which $\mathcal{C}OR(\Gamma^*)$ is a singleton $\{(\bar{\mu}_i)_{i\in\omega}\}$ and there exists an allocation $x^*$ that satisfies for each $i$, $u_i(x_i^*) = \bar{\mu}_i$, which is an equilibrium allocation in a competitive economy $\epsilon^*$, the limit of $\{\epsilon_n\}_{n\in\omega}$.

**Proof**: *Let us consider a sequence of market games* $\langle\Gamma_n\rangle_{n<\omega}$*, where each* $\Gamma_n$ *obtains from the economy* $\epsilon_n$*. Notice that each* $\Gamma_n$ *has an associated payoff function* $\mu^n$ *that satisfies that for every pair of finite sets* $\mathbf{S}, \mathbf{T} \in 2^\omega$ *such that* $\mathbf{S} \subseteq \mathbf{T}$, $\mu^n(\mathbf{S}) \leq \mu^n(\mathbf{T})$*. This follows from the definition:*

$$\mu^n(\mathbf{T}) = \max_{x^{\mathbf{T}}} \sum_{i\in\mathbf{T}} u_i(x_i^{\mathbf{T}}) \textit{ such that } \sum_{i\in\mathbf{T}} x_i^{\mathbf{T}} = \sum_{i\in\mathbf{T}} w_i.$$

[22] The *forcing conditions* that yield $\mathcal{M}$ are the finite functions $f : \omega \times \omega \rightarrow \{0, 1\}$. Then, real numbers are defined as $\alpha_n = \{m \in \omega$: there exists $f, f(n, m) = 1\}$. $\mathcal{M}$ is the submodel of Cohen's that contains all the sets $\alpha_n$, but not the collection of all of them $\bar{\mathbf{A}} = \{\alpha_n : n \in \omega\}$ [Pincus 1973],[Jech 1973].

[23] Alternatively, it has been shown that **ZF** + "existence of large cardinals", also supports the conclusion that every *reasonably definable* set of real numbers is measurable [Shelah-Woodin 1990]. But each finite coalition in our market games– in which each agent is identified by a natural (therefore real) number–constitutes a "reasonably definable" set of real numbers (it can be defined by simple enumeration).

*But then we have that* $\sum_{i\in\mathbf{T}} u_i(x_i^{\mathbf{T}}) = \sum_{i\in\mathbf{S}} u_i(x_i^{\mathbf{S}}) + \sum_{i\in\mathbf{T}\setminus\mathbf{S}} u_i(x_i^{\mathbf{T}\setminus\mathbf{S}})$ *while* $\sum_{i\in\mathbf{S}} x_i^{\mathbf{S}} + \sum_{i\in\mathbf{T}\setminus\mathbf{S}} x_i^{\mathbf{T}\setminus\mathbf{S}} = \sum_{i\in\mathbf{S}} w_i + \sum_{i\in\mathbf{T}\setminus\mathbf{S}} w_i$. *It follows that* $\mu^n(\mathbf{S}) + \mu^n(\mathbf{T}\setminus\mathbf{S}) \leq \mu^n(\mathbf{T})$. *On the other hand, consider any finite coalition* $\mathbf{S}$ *which is in both* $\Gamma_n$ *and* $\Gamma_{n+1}$. *Then* $\mu^{n+1}(\mathbf{S}) \leq \mu^n(\mathbf{S})$. *We know that*

$$\mu^{n+1}(\mathbf{S}) = \max_{^{(n+1)}x^{\mathbf{S}}} \sum_{i\in\mathbf{S}} u_i(^{(n+1)}x_i^{\mathbf{S}})$$

*such that* $\sum_{i\in\mathbf{S}} {}^{(n+1)}x_i^{\mathbf{S}} = \sum_{i\in\mathbf{S}} w_i^{n+1}$, *where each* $w_i^{n+1}$ *is the initial endowment of agent* $i$ *in the economy* $\epsilon_{n+1}$ *while* $^{(n+1)}x_i$ *is a possible allocation to* $i$ *in that economy. But since total endowments remain bounded from above we have that* $\sum_{i\in\mathbf{S}} w_i^{n+1} \not> \sum_{i\in\mathbf{S}} w_i^n$. *Otherwise, there would be a redistribution of resources in the economy. It follows then that*

$$\mu^{n+1}(\mathbf{S}) = \max_{^{n+1}x^{\mathbf{S}}} \sum_{i\in\mathbf{S}} u_i(^{n+1}x_i^{\mathbf{S}}) \leq \max_{^{n}x^{\mathbf{S}}} \sum_{i\in\mathbf{S}} u_i(^{n}x_i^{\mathbf{S}}) = \mu^n(\mathbf{S}).$$

*Consider now the limit game* $\Gamma^*$ *with payoff* $\mu$. *This payoff function must satisfy that for each finite* $\mathbf{S}$, $\mu^n(\mathbf{S}) \to \mu(\mathbf{S})$. *It could be that* $\mu(\mathbf{S}) = 0$, *but then, this is not possible, because it leads to contradiction in the presence of* $\mathbf{AD}$.[24] *Therefore, there must exist a finite* $\mathbf{S}$ *such that* $\mu(\mathbf{S}) \neq 0$. *In particular, it must exist an agent* $i$ *such that* $\mu(\{i\}) \neq 0$. *That means that* $i$*'s endowment in* $\epsilon^*$ *is* $w_i^* > 0$. *By the usual Archimedean property,*[25] *there must exist only a finite number of agents with this property, say* $n^*$. *That means that* $\mathcal{COR}(\Gamma^*)$ *must coincide with* $\mathcal{COR}(\Gamma_{n^*})$ *(up to a renaming of the agents). To see this suppose that* $\bar{(\mu_i)}_{i\in\omega} \in \mathcal{COR}(\Gamma^*)$ *is such that* $\bar{\mu_j} > 0$, *for* $j$ *with* $w_j^* = 0$. *This would mean that there is at least one agent* $k$ *with* $w_k^* > 0$ *receiving* $\mu_k$ *such that* $\mu(\{k\}) > \bar{\mu_k}$. *Contradiction.*

*So we have that the sequence of market games converges to a finite market game, which we know has a non-empty core. Now suppose that* $|\mathcal{COR}(\Gamma^*)| > 1$. *Say that* $\bar{\mu} = (\bar{\mu}_i)_{i\in\omega}$ *and* $\bar{\mu}' = (\bar{\mu}'_i)_{i\in\omega}$ *are both in* $\mathcal{COR}(\Gamma^*)$. *Since for each* $\mathbf{S}$, $\sum_{i\in\mathbf{S}} \bar{\mu}_i \geq \mu(\mathbf{S})$ *as well as* $\sum_{i\in\mathbf{S}} \bar{\mu}'_i \geq \mu(\mathbf{S})$, *we have that for any* $0 \leq \alpha \leq 1$, *there exists the convex combination* $\alpha\bar{\mu} + (1-\alpha)\bar{\mu}'$. *Then, there exists a homeomorphism between* $[0,1]$ *and* $\mathcal{COR}(\Gamma^*)$. *But, for every* $\bar{\mu} \in \mathcal{COR}(\Gamma^*)$ *there exists* $x \in \prod_{i\in\omega} X_i$ *such that* $\bar{\mu_i} = u_i(x_i)$ *if* $w_i^* > 0$ *and* $\bar{\mu_i} = 0$ *otherwise. But then* $x$ *is in* $\mathcal{COR}(\epsilon^*)$. *But this means that* $\mathbf{\Upsilon}(\mathcal{COR}(\epsilon_n)) > 0$. *Contradiction. Therefore, there is only one allocation* $\bar{\mu} \in \mathcal{COR}(\Gamma^*)$. *Finally, notice that for every* $\epsilon_n$, *its equilibrium allocation* $^n x^*$ *verifies that* $\bar{\mu}$ *such that* $\bar{\mu_i} = u_i(^n x_i^*)$ *is in* $\mathcal{COR}(\Gamma_n)$. *Therefore,* $\mathcal{COR}(\Gamma^*)$, *being equivalent to* $\mathcal{COR}(\Gamma_{n^*})$, *includes the allocation corresponding to the equilibrium allocation* $^{n^*}x^*$.

This result shows that in $\mathbf{ZF} + \mathbf{AD} + \mathbf{DC}$ every economy with an infinite number of agents is actually equivalent to a finite one. That is, only a finite number of agents determine the final outcome. It is interesting to note that in another collective-decision making context (the theory of *Social Choice*) we have a similar result:

[24] The existence of a measure that vanishes over all finite subsets of natural numbers implies that some subsets of real numbers do not verify the *Baire property* (which we do not define here). But $\mathbf{AD}$ implies that every set of real numbers verifies this property [Jech 2003].

[25] The construction of a non-Archimedean field (the *hyperreal* line) obtains as a non-standard model of the real numbers. But Robinson's construction requires the existence of a non-principal ultrafilter over $\omega$ [Goldblatt 1998]. As we already mentioned in Section 3, in $\mathbf{ZF} + \mathbf{AD}$ every ultrafilter over $\omega$ is principal.

Proposition 4.9
In $\mathbf{ZF} + \mathbf{AD} + \mathbf{DC}$, consider a society $\mathcal{S} = \langle \mathbf{S}, \mathbf{X}, \{\preceq_i\}_{i \in \mathbf{S}} \rangle$, where $\mathbf{S}$ is the set of agents, $\mathbf{X}$ the set of options and each $\preceq_i$ is a reflexive, antisymmetric, transitive and complete ordering for $\mathbf{X}$. Assume that $\mathbf{S}$ is countable. Then, let $\preceq_{\mathbf{S}}$ be a social preference that verifies the conditions of Arrow's Theorem (see [Arrow 1951]). Then, there exist an $i \in \mathbf{S}$ (the *dictator*) such that $\preceq_i \equiv \preceq_{\mathbf{S}}$.

A well-known result in social choice theory is that the set of *decisive* agents (the coalitions that can impose their preferences over the society) constitutes an ultrafilter [Kirman-Sondermann 1972]. If $\mathbf{S}$ is finite, every ultrafilter in $2^{\mathbf{S}}$ is principal. Therefore, since this is true for the class of decisive sets there exists an agent $i$ that is in each decisive set and moreover, is decisive by himself. This $i$ is the dictator. As said, in $\mathbf{ZF}+\mathbf{AD}$ every ultrafilter over $\omega$ is principal. Therefore, there exists also a dictator in the case that $\mathbf{S} = \omega$. That is, in the context of $\mathbf{AD}$, the society $\mathcal{S}$ behaves like a finite society.

## 5 Existence of States of the World

The dynamics of beliefs has been always part of the explanatory mechanisms of economists. No sound analysis of complex situations, both in macro and microeconomics, can disregard the importance of the beliefs held by agents, and moreover, how they evolve in time. Despite this fact, these considerations were always used as additions to the actual theoretical constructions, merely as parts of their intended interpretations. Even John von Neumann shied away from beliefs and deemed the games of incomplete information as ill-defined.

It was not until John Harsanyi postulated that agents interact in an implicitly agreed-on environment (a common prior), that beliefs became legitimate part of the modeling toolbox of economists [Harsanyi 1967]. In fact, he advanced the idea that agents in an interaction build their beliefs incorporating the possible beliefs of the *others*.

The entire description of the physical resources and the possible beliefs of the agents in an interaction is called the **state of the world**, $\alpha$. The description of the physical aspects of the context is summarized by the *state of the nature* of the situation, $s$, while the beliefs that are held in $\alpha$ can be represented as $\mathcal{B}(\alpha)$. Then, we have that a state of the world is a fixed point in a description operator:

$$\alpha \; = \; \langle s, \mathcal{B}(\alpha) \rangle$$

To see whether this $\alpha$ exists, the usual procedure is to unfold it, beginning with the state of nature and then adding the beliefs of the agents:

$$\mathcal{B}^0(\alpha) = s$$

$$\mathcal{B}^1(\alpha) = \mathcal{B}^0(\alpha) \times bel(\mathcal{B}^0(\alpha))$$

$$\mathcal{B}^2(\alpha) = \mathcal{B}^1(\alpha) \times bel(\mathcal{B}^1(\alpha))$$

$$\ldots$$

Then, the conjecture is that $\mathcal{B}(\alpha)$ is the **limit** of this sequence,[26] i.e.:

$$\mathcal{B}(\alpha) = \mathcal{B}^{\omega}(\alpha) = \lim_{n<\omega} \mathcal{B}^n(\alpha)$$

This construction has been shown to be sound, usually by means of topological assumptions about the space of states of nature [Mertens-Zamir 1984], [Brandenburger-Dekel 1993], [Dekel-Gul 1997]. But a critical assumption that is *always* included in the proofs of the conjecture, is that agents have *consistent* beliefs. That is, that the belief operator verifies that, if $\beta$ is a limit ordinal:

$$\lim_{\beta} \mathcal{B}^{\beta}(\alpha) = \mathcal{B}(\lim_{\beta} \alpha^{\beta})$$

This consistency requirement arises naturally in any Bayesian model. Such a probabilistic structure generates a potential tree, where the probability of an outcome is identified with the measure of the entire branch that ends in that outcome. This measure, in turn, allows to reconstruct, via *backtracking*, the probabilities at each intermediate node [Chuaqui 1991]. That is, the sequence of probabilities at nodes along a branch is consistent. Therefore, consistency constitutes a necessary condition of Bayesian belief updating.

While Bayesian reasoning, and therefore the consistency of beliefs in a sequence, can be considered a reasonable condition to ask of rational agents, it is easy to imagine situations where consistency leads to absurd results. In fact, a number of examples exhibit that agents may form their beliefs discontinuously, but even so end up converging to a single state of world. The most interesting for the foundations of economic theory is known as *Newcomb's paradox* [Dekel-Gul 1997]. Let us assume a human agent who has to play against a Genie who claims she can predict the human's choices. There are two boxes **A** and **B**, the first translucent, the second opaque. The Genie offers the human to take either both boxes or only box **B**. The agent can see that box **A** has \$1000 inside, but the Genie tells him that if he chooses **B** he will leave \$1000000 instead. Otherwise, if he predicts that the agent will grab both boxes, he will leave **B** empty. The final decision made by the agent depends on what he believes about the powers of the Genie. A possibility is to think that the Genie may be able to predict correctly the agent's choice. According to that, the agent should choose only box $B$ and, if so, the Genie should leave the million dollars inside the box. But then, at the moment of grabbing only box **B** the agent reflects that either his belief was right and therefore there is \$ 1000000 inside the box or he was wrong and therefore the box is empty. Then, he looses nothing grabbing *both* boxes. In other words, the belief that constitutes part of the state of the world (i.e. that both boxes have money inside) is inconsistent with the beliefs he held in the process.

While examples like this seem far from real-world decision-making situations, just consider that for many authors, the well-known Prisoner's Dilemma can be seen as a two-sided version of Newcomb's problem (i.e. each player conceives the other as making decisions based on a prediction about her own decision) [Sobel 1991]. In fact, many situations in which strategic uncertainty plays a role can be seen as variants of this same phenomenon.

[26] If $bel(\cdot)$ yields probability distributions, the limit may arise as described by Kolmogorov's Extension Theorem on sequences of distributions [Halmos 1974]. Otherwise, in the case that $bel(\cdot)$ yields families of propositional formulas, the limit obtains as the class of formulas representing all the possible beliefs of the agents [Fagin et al. 1999]. A similar limit construction on classes of formulas allows to rationalize decision procedures [Lipman 1991].

Notice that various forms of non-monotonic reasoning processes can be represented as a discontinuous sequences of intermediate beliefs, that is sequences in which an iterated limit is the initial state for a new round of iterations. Therefore, in the case that neither the number of possible intermediate beliefs nor that of steps is bounded, no proofs of termination of the reasoning process can be given, at least in **ZFC**. Even so, since obviously the agents reach some state of the world, we have to explain the convergence to it, without the assumption of consistency of $\mathcal{B}$. One way of handling the problem is by assuming that agents are able to handle non-well founded objects like $\alpha$ when $\mathcal{B}$ does not warrant the convergence of the iterative construction.

This assumption amounts to dropping from **ZF** the Regularity Axiom,

$$\exists x F(x) \rightarrow \exists y \Big[ F(y) \wedge \forall z \neg \big( z \in y \wedge F(z) \big) \Big]$$

and replacing it with the so-called *Solution Lemma*, that states that every general system of equations $\varepsilon$ has a unique solution $\bar{sol}$:

- A *general system of equations* is a $\varepsilon = \langle \mathbf{X}, \mathbf{A}, \mathbf{e} \rangle$, where $\mathbf{X}$ is a set of indeterminates, $\mathbf{A}$ a set of "constants", $\mathbf{X} \cap \mathbf{A} = \emptyset$ and $\mathbf{e} : \mathbf{X} \rightarrow \mathcal{V}(\mathbf{X} \cup \mathbf{A})$, provides the equations (where $\mathcal{V}(\mathbf{X} \cap \mathbf{A})$ is the class of sets build up from elements in $\mathbf{A}$). Equations have the following form: $x = \mathbf{e}(x) \in \mathcal{V}(\mathbf{X} \cap \mathbf{A})$.
- A solution to $\varepsilon$ is a function $\bar{sol}$ on $\mathbf{X}$, such that $\bar{sol}(x) = \bar{sol}(\mathbf{e}(x))$. This $\bar{sol}$ is a *substitution* function, which assigns to each indeterminate a set in $\mathcal{V}(\mathbf{A})$ (i.e. the class of sets without indeterminates and constructed entirely of elements in $\mathbf{A}$).

The new set theory that obtains, **ZFC** $-$ Regularity $+$ Solution Lemma, is called **AFA** (for *antifoundation*)[Aczel 1988], [Devlin 1993] [Barwise-Moss 1996].

In the case of states of the world, we take $\alpha \in \mathbf{X}$ and $s \in \mathbf{A}$, i.e. our indeterminates are states of the world, and the constants are states of nature. Then we can prove the following result:

Theorem 5.1
[Tohmé 2005] In **AFA**, a state of the world $\alpha$, given its underlying state of nature $s$, obtains as a fixed point of the belief formation operator $\mathcal{B}(\cdot)$.

The key of the proof is just to show that $\alpha$ is either a fixed point or the process of belief formation is endless. Since every general system of equations has a solution (a bounded sequence), it cannot be endless. Therefore $\alpha$ has to be a fixed point.

This is different from the fixed point theorems we briefly mentioned in section 4, since in all those cases we had either continuity of the operators (like in Kakutani's theorem) or at least the property of being *increasing* in a structure in which each set has a unique maximal element (like in Tarski's fixed point theorem). Here we can ensure almost nothing about the state of the world, but at least we want to know whether it exists, which in this case means just that it can be described as a set.

Notice that the change from **ZFC** to **AFA** is not as radical as the one from **ZFC** to $\mathbf{ZF} + \mathbf{AD} + \mathbf{DC}$. In fact, every object that can be defined in **AFA** ends up being isomorphic to an element in the universe of **ZFC**. The only advantage in this change is that it simplifies the treatment of circularities, in particular those that arise in the use of information and knowledge [Akman-Pakkan 1996]. Using *coalgebraic* methods–i.e. characterizations of the "solutions of equations", particularly in the case of circular definitions–we could achieve similar results.[27] In any case, **AFA** seems to be the most intuitive form of ensuring the existence of states of the world.

For an alternative approach to the characterization of states of the world, consider the space of possible belief structures that can be constructed over a fixed space of states of nature $\mathbf{S}$. It is easy to conceive a game in which player $I$ chooses a state of nature $s \in \mathbf{S}$, $II$ replies with a belief $\phi_1 \in \mathcal{B}(s)$ to which $I$ responds with $\phi_2 \in \mathcal{B}(\langle s, \mathcal{B}(s)\rangle)$, to which $II$ answer is $\phi_3 \in \mathcal{B}(\langle s, \mathcal{B}^2(s)\rangle)$ etc. $II$ wins if it can lead the sequence to fixed-point, otherwise $I$ wins. Notice that this game is isomorphic to a Gale-Stewart game, except for the fact that since we do not assume continuity, the number of steps in the game could be transfinite, unless we can again apply the **AD**. But to do this, each belief structure must be codified (at least in principle) by a natural number. This depends, of course, on the language in which these beliefs are expressed.

Consider the class $\mathcal{B}^{\mathbf{ORD}}(\mathbf{S}) = \cup_{\beta \in \mathbf{ORD}} \mathcal{B}^\beta(\mathbf{S})$, where $\mathcal{B}^\beta(\mathbf{S})$ is the class of belief structures constructed over $\mathbf{S}$ for an ordinal level $\beta$. Let $\mathcal{L}_{\mathcal{B}^{\mathbf{ORD}}(\mathbf{S})}$ be a language such that for each $\phi \in \mathcal{B}^{\mathbf{ORD}}(\mathbf{S})$ there exists a formula $\mathbf{\Phi}$ expressed in $\mathcal{L}_{\mathcal{B}^{\mathbf{ORD}}(\mathbf{S})}$ such that $\lceil \mathbf{\Phi} \rceil = \phi$, that is, the interpretation of $\mathbf{\Phi}$ is the belief structure $\phi$.

It has been shown that not every $\phi \in \mathcal{B}^{\mathbf{ORD}}(\mathbf{S})$ can be defined in a language that does not permit inconsistencies [Brandenburger-Keisler 1999], [Tohmé 2005]. This indicates that completeness of states of the world is somehow hard to achieve, if we do not restrict the class of admissible beliefs. For our purposes it is enough for each belief structure $\phi$ to be such that the corresponding formula $\mathbf{\Phi}$ has a finite Gödel code. That is, the belief structure $\phi$ must be represented by a formula in $\mathcal{L}_{\mathcal{B}^{\mathbf{ORD}}(\mathbf{S})}$ that can be coded by a natural number. Let us call a belief structure $\phi \in \mathcal{B}^{\mathbf{ORD}}(\mathbf{S})$ *finitely representable* if it has this property. Then we have this trivial result:

Lemma 5.2
In $\mathbf{ZF} + \mathbf{AD} + \mathbf{DC}$, if a possible belief structure $\bar{\mathcal{B}}$ in an incomplete information context with underlying state of the world $s \in \mathbf{S}$ is finitely representable, there exists a state of the world $\alpha = \langle s, \mathcal{B}(\alpha)\rangle$ such that $\mathcal{B}(\alpha) = \bar{\mathcal{B}}$. Moreover, even if $\bar{\mathcal{B}}$ is not continuous, it can be unfolded in $\omega$ steps.

Admittedly, this sounds very restrictive, but on the other hand, there are no natural examples in economics in which beliefs cannot be represented by a finite expression. Since the usual form of representing beliefs is by means of probability distributions, if no finite expression were to exist for a given distribution or hierarchy of distributions, it would mean that it is not possible to compress its informational content in order to obtain a manageable form. While it is easy to infer that most possible probability structures cannot be represented by an expression shorter than its full extent (an almost immediate consequence of Chaitin's theorem [Chaitin 1974]), it is also true that it is quite unlikely that they could represent the beliefs held by any rational agent.

## 6 Discussion

In the previous sections we presented a variety of limitations to the theoretical answers that have been given to the main problems in economics. It is not clear that they

[27] But this requires to use category-theoretical methods instead of set-theoretical ones [Moss-Viglizzo 2004].

will (or should) affect the work of most economists. Although the existence of holes in the edifice of economic theory is somewhat worrisome, it is also true that none of the problems discussed here amounts to its demolition. In fact, most economists use the existence of choice functions, equilibria and states of the world as just metaphors and build models in which the difficulties discussed here are disregarded.[28]

On the other hand, even if the problems discussed here may have small impact in the practice of economics, they ask for more refined tools in the analysis of economic phenomena. In fact, Lewis, in his analysis of economic theory concludes, rather inconclusively, that the set-theoretical principles required by contemporary economic theory are far more demanding than what is needed in physics, both classic and quantum [Lewis 1990].[29] If this is true, this would mean that the search for the right set-theoretical foundations for economic theory should be pursued more actively. At least, it should be advisable for economists to know that the validity of their conclusions depends on very deep properties of the formal tools they use.

In this sense, we have shown that a change in the underlying set theory allows, at the very least, to simplify some proofs (in particular of recursivity) that are not easy to obtain in $ZFC$. In any case, if economic theory represents something, it must be the problem solving ability of decision-making agents. Our claim is that this ability may correspond to certain properties of the set-theoretic universe in which our models are represented. It is interesting, in this light, to discuss what abilities do our alternative set theories imply.

Let us consider $\mathbf{ZF} + \mathbf{DC} + \mathbf{AD}$. In this framework we can still use our finite mathematics and much of elementary calculus. What we add here is that agents can always find winning strategies in Gale-Stewart games. This means that we are assuming a very simplified universe of sets, that satisfies our intuition that economic theories describe “tame” environments, and therefore degenerate cases do not have place in them.

Moreover, in these mild environments we do not admit something like a finite coalition in a market game without a payoff. In fact, in $\mathbf{ZF} + \mathbf{DC} + \mathbf{AD}$ every set of reals is measurable. Therefore, if we increase the number of agents we will always have a sequence of games that ends up converging to a game $\Gamma$ such that the allocations supported in its core are the equilibria of the economy. In fact, every use we made of this particular system of axioms ends up showing some sort of “finitization” of the theory, making the problems more tractable, at least in principle.[30]

In the case in which problems arise with the information held by the agents, we saw that **AFA** allows us to ensure the convergence of beliefs, even if the process is discontinuous. An additional interest we may have in using **AFA** in economics is that it can be interpreted as indicating that agents are able to conceive states of the world as integral constructions instead of just processes that converge to them. This legitimates the methodology of applied economists, that just take states of the world (or equivalently, the **types** of agents in them) as data and not as objects that are to be derived.

In a few words, we think that $\mathbf{AFA}^- + \mathbf{DC} + \mathbf{AD}$ (where $\mathbf{AFA}^-$ is **AFA** without

[28] A clear exception is the complexity of computations (of the **NP** sort).

[29] According to Lewis, all the principles of physics are derivable in **ZF**.

[30] $\mathbf{ZF}$ + “large cardinals” is more difficult to interpret in terms of cognitive abilities, although it can be speculated that agents able to conceive inaccessible cardinals may also be able to handle discontinuous processes, which seems interesting for the problem of the existence of states of the world.

**AC**) is a intuitively good arena for economic theory. Furthermore, it is equiconsistent with $\mathbf{ZF} + \mathbf{DC} + \mathbf{AD}$.[31]

We conclude by noting that this procedure of choosing the right set theory for economic theory has still to be checked out, to see whether these ideas are sound enough. In any case, the study of this possibility as well as its extension to other economic problems (e.g. the treatment of economies with an infinite number of goods or with other types of dynamical structures) is matter of further work.

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