Commutators of singular integrals with kernels satisfying generalized Hörmander conditions and extrapolation results to the variable exponent spaces

Luciana Melchiori* and Gladis Pradolini†

Facultad de Ingeniería Química, CONICET-UNL, Santa Fe, Argentina.

Abstract

We obtain boundedness results for the higher order commutators of singular integral operators between weighted Lebesgue spaces, including L^p-BMO and L^p-Lipschitz estimates. The kernels of such operators satisfy certain regularity condition, and the symbol of the commutator belongs to a Lipschitz class. We also deal with commutators of singular integral operators with less regular kernels satisfying a Hörmander's type inequality. Moreover, we give a characterization result involving symbols of the commutators and continuity results for extreme values of p. Finally, by extrapolation techniques, we derive different results in the variable exponent context.

1 Introduction

In [5], A. P. Calderón proved that, if T is certain pseudo-differential operator and b is a Lipschitz continuous function, then the first order commutator of T with symbol b, $[b, T]$, is bounded between Lebesgue spaces. Later, in [7] and [8] the authors proved the same estimate for the case that $T = T_\sigma$, where the function σ belongs to a certain Hörmander class. This result was obtained by proving that, for each Lipschitz function b, the operator $[b, T]$ is a Calderón-Zygmund singular integral operator whose kernel constant is controlled by the Lipschitz norm of b.

2010 *Mathematics Subject Classification*: 42B25

Keywords and phrases: Commutators, Variable Lebesgue spaces, Extrapolation

*lmelchiori@santafe-conicet.gov.ar

†gpradolini@santafe-conicet.gov.ar, gladis.pradolini@gmail.com

On the other hand, in [28] the authors considered the commutators of singular integral operators with Lipschitz symbols and proved the boundedness between Lebesgue spaces, including the boundedness from Lebesgue spaces into Lipschitz spaces on non-homogeneous spaces. (See also [35] in the context of variable Lebesgue spaces). Moreover, in [35] the authors give characterizations of Lipschitz symbols by mean of the boundedness of commutators of singular integral operators between variable Lebesgue spaces.

Nevertheless, there is not enough information about the behavior of the commutators acting between weighted Lebesgue spaces, even less for extreme values of p, that is, the weighted L^p-BMO or L^p-Lipschitz boundedness. Hence, one of our main aims is, precisely, to give sufficient conditions on the weights in order to obtain these continuity properties. Some previous results in this direction were given in [2] where the authors study the boundedness between Lebesgue spaces with variable exponent for commutators of singular integral operators with BMO symbols. So, in this paper, we shall be concerned with commutators of singular integral operators with Lipschitz symbols. Recall that the first order commutator of a Calderón-Zygmund operator T is formally defined, for $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, by

$$[b, T]f(x) = b(x)Tf(x) - T(bf)(x).$$

We prove weighted results of the type described above for higher order commutators of T, defined, for $m \in \mathbb{N}$, by $T^m_b = [b, T^{m-1}_b]$. Inspired in a result in [20], we particularly prove a characterization result involving symbols of the commutators and continuity results for extreme values of p. We shall begin with kernels satisfying a Lipschitz type regularity and then we consider kernels with less regularity properties, associated to a given Young function. These type of operators include a great variety of operators and were introduced in [27] and [26]. See section 2.2 for examples and more related facts. (For information about the behavior of the singular integral operators see for example [1], [6], [9], [10], [13], [19], [20], [27], [30], [32] and [37]. More recent results related with commutators of singular integral operators can be found in [25], [21] and [18].)

The results mentioned above allow us to obtain corresponding results in the variable exponent spaces, which can be derived by extrapolations techniques (See section 3).

The paper is organized as follows. In section §2 we give the preliminaries definitions in order to state the main results of the article, which are also included in this section, and in section §3 we give some applications to the variable exponent spaces context by mean of extrapolation techniques. Then, in §4 we give some auxiliary results which allow us to prove the main results in §5.

2 Preliminaries and main results

In this section we give the definitions of the operators we shall be dealing with and the functional classes of the symbols in order to define the commutators. We also give some preliminaries.

We shall consider singular integral operators of convolution type T with kernel K, that is T is bounded on $L^2(\mathbb{R}^n)$ and if $x \notin \operatorname{supp} f$

$$Tf(x) = \int_{\mathbb{R}^n} K(x-y)f(y)dy. \tag{2.1}$$

The kernel K is a measurable function defined away from 0, satisfying certain smoothness condition to be described later. We shall also want to point out that the typical size condition on the kernel K given by

$$|K(x-y)| \leq \frac{C}{|x-y|^n} \tag{2.2}$$

is not be assumed in the $L^p - L^q$ estimates (Theorems 2.1 and 2.12) and, in these cases, we will be focused on the different smoothness conditions on K.

Related with the singular integral operator T, we can formally define the commutator with symbol $b \in L^1_{\text{loc}}(\mathbb{R}^n)$, by

$$[b,T]f = bTf - T(bf).$$

The higher order commutator of order $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ of T is defined by

$$T_b^0 = T, \quad T_b^m = [b, T_b^{m-1}].$$

Let $0 < \delta < 1$. We say that a function b belongs to the space $\Lambda(\delta)$ if there exists a positive constant C such that, for every x, $y \in \mathbb{R}^n$

$$|b(x) - b(y)| \leq C|x-y|^\delta. \tag{2.3}$$

The smallest of such constants will be denoted by $\|b\|_{\Lambda(\delta)}$. We shall be dealing with commutators with symbols belonging to this class of functions.

Given a weight w, that is, a non-negative and locally integrable function, we say that a measurable function f belongs to $L^p_w(\mathbb{R}^n)$ for some $1 < p < \infty$, if $fw \in L^p(\mathbb{R}^n)$.

We are interested in studying the boundedness properties of the commutators T_b^m on weighted spaces, where the symbol $b \in \Lambda(\delta)$. We shall first consider their continuity on weighted Lebesgue spaces of the type defined previously. We shall also analyze the boundedness of T_b^m from weighted Lebesgue spaces into certain weighted version of Lipschitz spaces. For a weight w

and $0 \leq \delta < 1$, these spaces are denoted by $\mathbb{L}_w(\delta)$ and collect the functions $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ that satisfy

$$\sup_B \frac{\|w\chi_B\|_\infty}{|B|^{1+\delta/n}} \int_B |f(x) - f_B| dx < \infty,$$

where $\|g\|_\infty$ denotes the essential supremum of a measurable function g. The case $\delta = 0$ of the space above was introduced in [29] as a weighted version of the space of functions with bounded mean oscillation. It is well known that, when $w = 1$ and $0 < \delta < 1$, this space coincides with the space $\Lambda(\delta)$ defined in (2.3) and, if $w = 1$ and $\delta = 0$, this is the well known BMO space.

The classes of weights we will be dealing with are the well-known $A_{p,q}$ classes of Muckenhoupt and Wheeden ([29]). For $1 \leq p, q < \infty$ these classes are defined as the weights w such that

$$\sup_B \left(\frac{1}{|B|} \int_B w(x)^q dx \right)^{1/q} \left(\frac{1}{|B|} \int_B w(x)^{-p'} dx \right)^{1/p'} < \infty.$$

When $q = \infty$, we understand that $w \in A_{p,\infty}$ as $w^{-p'} \in A_1$.

We classify the operators defined in (2.1) into two different types, according to the smoothness conditions satisfied by K.

2.1 Singular integral operators with Lipschitz regularity

In addition to the properties of the kernel K given above, we shall first suppose that it satisfies the smoothness condition H^*_∞, which is given by

$$\left|K(x-y) - K(x'-y)\right| + \left|K(y-x) - K(y-x')\right| \leq C \frac{|x-x'|^\eta}{|x-y|^{n+\eta}},$$

for some positive constant C and some $0 < \eta \leq 1$, whenever $|x - y| \geq 2\,|x - x'|$.

We now give the boundedness results between weighted Lebesgue spaces for the higher order commutators of T with Lipschitz symbols. Recall that, in this result, no condition on the size of the kernel K is imposed. The corresponding result for $b \in BMO$ was proved in [34]. In order to simplify the hypothesis we shall suppose that $m \in \mathbb{N}_0$ with the convention that $\beta/0 = \infty$ if $\beta > 0$.

Theorem 2.1. *Let $0 < \delta < \min\{\eta, n/m\}$. Let $1 < p < n/(m\delta)$, $1/q = 1/p - m\delta/n$ and $b \in \Lambda(\delta)$. If $w \in A_{p,q}$, then there exists a positive constant C such that*

$$\left(\int_{\mathbb{R}^n} |T_b^m f(x)|^q w(x)^q \, dx \right)^{1/q} \leq C \, \|b\|^m_{\Lambda(\delta)} \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)^p dx \right)^{1/p}$$

for every $f \in L^p_w(\mathbb{R}^n)$.

Remark 2.2. When $m = 0$ it is well known that the result above holds (see, for example [16] and, for the unweighted case, [28]). Notice that there are no symbols or parameters δ in the hypothesis in this case.

The next result gives the continuity properties of T_b^m between weighted Lebesgue spaces and $\mathbb{L}_w(\widetilde{\delta})$ spaces. We additionally suppose that the kernel K satisfies the size condition (2.2). When the symbol b of the commutator belongs to BMO, H is the Hilbert transform and $m = 1$, it was proved in [20] that the boundedness of $[b, H]$ from $L^\infty(\mathbb{R})$ into BMO implies that b is a constant function. So, in this sense, our next result is an important contribution.

Theorem 2.3. *Let $0 < \delta < \min\{\eta, n/m\}$. Let $n/(m\delta) \le r < n/((m-1)\delta)$, if $m \ge 1$ or $r = \infty$, if $m = 0$. Let $\widetilde{\delta} = m\delta - n/r$ and $b \in \Lambda(\delta)$. If $w \in A_{r,\infty}$ if $r < \infty$ or $w^{-1} \in A_1$, if $r = \infty$, then there exists a positive constant C such that*

$$\|T_b^m f\|_{\mathbb{L}_w(\widetilde{\delta})} \le C \|b\|_{\Lambda(\delta)}^m \|fw\|_{L^r}$$

for every $f \in L_w^r(\mathbb{R}^n)$.

Remark 2.4. When $m = 1$, $w = 1$, the result above was proved in [28] in the general context of non-doubling measures. On the other hand, when $m = 0$ this result is a generalization of that obtained in [30] for the Hilbert transform.

Remark 2.5. If $r = n/(m\delta)$, then $\widetilde{\delta} = 0$ and the space $\mathbb{L}_w(\widetilde{\delta})$ is the weighted version of the BMO space introduced in [29]. Thus, this is the endpoint value from which the Lebesgue spaces change into BMO and then into Lipschitz spaces, when the operator T_b^m acts.

For the extreme value $r = n/((m-1)\delta)$, $m \in \mathbb{N}$ and $0 < \delta < \min\{\eta, n/m\}$, we obtain the following endpoint result in order to characterize the symbol b in $\Lambda(\delta)$ in terms of the boundedness of T_b^m in the sense of Theorem 2.3. In order to give this result we introduce some previous notation. For $k = 0, 1, ..., m$ we denote $c_k = m!/(k!(m-k)!)$. In addition, if x, $u \in \mathbb{R}^n$, we denote $S(x, u, k) = (b(x) - b_B)^{m-k} T((b - b_B)^k f_2)(u)$, where $f_2 = f\chi_{\mathbb{R}^n \setminus B}$ for a given ball B and a locally integrable function f.

Theorem 2.6. *Let $m \in \mathbb{N}$, $0 < \delta < \min\{\eta, n/m\}$ and $r = n/((m-1)\delta)$. If $w \in A_{n/(m\delta),\infty}$ and $b \in \Lambda(\delta)$, the following statements are equivalent.*

(i) $T_b^m : L_w^r(\mathbb{R}^n) \hookrightarrow \mathbb{L}_w(\delta)$;

(ii) There exists a positive constant C such that

$$\frac{\|w\chi_B\|_\infty}{|B|^{1+\frac{\delta}{n}}} \int_B \left| \sum_{k=0}^m c_k \left[S(x, u, k) - (S(\cdot, u, k))_B \right] \right| dx \le C \|fw\|_r, \tag{2.4}$$

for every ball $B \subset \mathbb{R}^n$, $u \in B$ and $f \in L^1_{\text{loc}}(\mathbb{R}^n)$.

Remark 2.7. The same result holds if we take $\delta = 0$, $r = \infty$, $b \in BMO$ and $w^{-1} \in A_1$ in the hypothesis of the previous result.

Remark 2.8. In the unweighted case, when $m = 1$, and consequently, $r = \infty$, the result above was proved in [28] in a more general context of non-homogeneous spaces. Certainly, their result was inspired in the article of [20], where the same result is proved for $m = 1$, $w = 1$ and $b \in BMO$.

Remark 2.9. As we said previously, in [20] the authors obtained that, if H is the Hilbert transform, $b \in BMO$, $n = 1$, $m = 1$ and $w = 1$, the boundedness of the commutator $[b, H]$ from $L^\infty(\mathbb{R})$ into BMO implies that b is a constant function. Let us recall that the Hilbert transform H is defined by

$$Hf(x) = \text{p.v.} \int_{\mathbb{R}} \frac{f(y)}{x - y}\, dy.$$

In our case, in the same situation on m and w but taking $b \in \Lambda(\delta)$, by (2.4) we can deduce that, if $[b, H]$ is bounded from L^∞ into $\mathbb{L}(\delta)$, then

$$\frac{1}{|B|^{1+\frac{\delta}{n}}} \int_B \left| \sum_{k=0}^{1} c_k \left[S(x, u, k) - (S(\cdot, u, k))_B \right] \right| dx \le C \|f\|_\infty .$$

It is not difficult to see that

$$\frac{1}{|B|^{1+\frac{\delta}{n}}} \int_B \left| \sum_{k=0}^{1} c_k \left[S(x, u, k) - (S(\cdot, u, k))_B \right] \right| dx = \frac{1}{|B|^{1+\frac{\delta}{n}}} \int_B |b(x) - b_B|\, dx \left| \int_{(2B)^c} \frac{f(y)}{u - y} dy \right|,$$

so that

$$\frac{1}{|B|^{1+\frac{\delta}{n}}} \int_B |b(x) - b_B|\, dx \left| \int_{(2B)^c} \frac{f(y)}{u - y} dy \right| \le C \|f\|_\infty .$$

Following the same arguments as in [20] with $f_N(y) = \chi_{B(0,N)}(u - y)\text{sig}(u - y)$ for $N \in \mathbb{N}$, we obtain that

$$\frac{1}{|B|^{1+\frac{\delta}{n}}} \int_B |b(x) - b_B|\, dx \int_{(2B)^c \cup \{|u-y|<N\}} \frac{dy}{|u - y|} \le C.$$

Due to the fact that $\int_{(2B)^c \cup \{|u-y|<N\}} \frac{dy}{|u-y|} \to \infty$ when $N \to \infty$, we have $b(x) = b_B$ almost everywhere, for every ball B, which yields that b is essentially constant.

2.2 Singular integral operators with Hörmander type regularity

Before introducing the smoothness conditions on the kernel that we shall consider in this section, we give some previous notation.

By a Young function we mean a function $\Phi : [0,\infty) \to [0,\infty)$ that is increasing, convex and verifies $\Phi(0) = 0$ and $\Phi(t) \to \infty$ when $t \to \infty$. The Φ-Luxemburg average over a ball B is defined, for a locally integrable function f, by

$$\|f\|_{\Phi,B} = \inf\left\{\lambda > 0 : \frac{1}{|B|}\int_B \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\right\}.$$

It is well-known that the commutators of singular integral operators can be controlled, in some sense, by maximal type operators associated to Young functions that involve these averages. More precisely, if $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we define the maximal operator associated to a Young function Φ, by

$$M_\Phi f(x) = \sup_{B \ni x} \|f\|_{\Phi,B},$$

where the supremum is taken over every ball B that contains $x \in \mathbb{R}^n$. The fractional version of this operator is given, for $0 < \alpha < n$, by

$$M_{\alpha,\Phi} f(x) = \sup_{B \ni x} |B|^{\alpha/n} \|f\|_{\Phi,B}.$$

Given a Young function Φ, the following Hölder's type inequality holds for every pair of measurable functions f, g

$$\frac{1}{|B|}\int_B |f(x)g(x)|dx \le 2\|f\|_{\Phi,B}\|g\|_{\widetilde{\Phi},B},$$

where $\widetilde{\Phi}$ is the complementary Young function of Φ, defined by

$$\widetilde{\Phi}(t) = \sup_{s>0}\{st - \Phi(s)\}.$$

It is easy to see that $t \le \Phi^{-1}(t)\widetilde{\Phi}^{-1}(t) \le 2t$ for every $t > 0$.

Moreover, given Φ, Ψ and Θ Young functions verifying that $\Phi^{-1}(t)\Psi^{-1}(t) \lesssim \Theta^{-1}(t)$ for every $t > 0$, the following generalization holds

$$\|fg\|_{\Theta,B} \lesssim \|f\|_{\Phi,B}\|g\|_{\Psi,B}.$$

The expression $A \lesssim B$ means that there exists a positive constant C such that $A \le CB$. With $A \approx B$ we mean $A \lesssim B$ and $B \lesssim A$.

We are now in position to define the smoothness condition on K. These definitions were introduced in [27]. We say that $K \in H_\Phi$ if there exist $c \ge 1$ and $C > 0$ such that for every $y \in \mathbb{R}^n$ and $R > c|y|$

$$\sum_{j=1}^{\infty}(2^j R)^n \| \left(K(\cdot - y) - K(\cdot)\right) \chi_{|\cdot| \sim 2^j R} \|_{\Phi, B(0, 2^{j+1}R)} \le C, \tag{2.5}$$

where $|\cdot| \sim s$ means the set $\{x \in \mathbb{R}^n : s < |x| \leq 2s\}$.

For example, when $\Phi(t) = t^q$, $1 \leq q < \infty$, we denote this class by H_q and it can be written as

$$\sum_{j=1}^{\infty}(2^jR)^n \left(\frac{1}{(2^jR)^n}\int_{|x|\sim 2^jR}|K(x-y)-K(x)|^q dx\right)^{1/q} \leq C.$$

We say that $K \in H_\infty$ if K satisfies condition (2.5) with $\|.\|_{L^\infty,B(0,2^{j+1}R)}$ in place of $\|.\|_{\Phi,B(0,2^{j+1}R)}$.

The kernels given above are, a priori, less regular than the kernel of the singular integral operator T defined previously and they have been studied by several authors. For example, in [26], the author studied singular integrals given by a multiplier. If $\mathsf{m} : \mathbb{R}^n \to \mathbb{R}$ is a function, the multiplier operator T_{m} is defined, through the Fourier transform, as $\widehat{T_{\mathsf{m}}f}(\zeta) = \mathsf{m}(\zeta)\widehat{f}(\zeta)$ for f in the Schwartz class. Under certain conditions on the derivatives of m, the multiplier operator T_{m} can be seen as the limit of convolution operators T_{m}^N, having a simpler form. Their corresponding kernels K^N belong to the class H_r with constant independent of N, for certain values of $r > 1$ given by the regularity of the function m.

The classes H_q, $1 \leq q < \infty$, appeared implicit in [24] where it is shown that the classical L^q-Dini condition for K implies $K \in H_q$ (see also [37] and [38]).

Other examples of this type of operators are singular integrals operators with rough kernels, that is, with kernel $K(x) = \Omega(x)|x|^{-n}$ where Ω is a function defined on the unit sphere S^{n-1} of $\mathbb{R}^n$, extended to $\mathbb{R}^n \setminus \{0\}$ radially. The function Ω is an homogeneous function of degree 0. In [26, Proposition 4.2], the authors showed that $K \in H_\Phi$, for certain Young function Φ, provided that $\Omega \in L^\Phi(S^{n-1})$ with

$$\int_0^1 \omega_\Phi(t)\frac{dt}{t} < \infty,$$

where ω_Φ is the L^Φ-modulus of continuity of Ω given by

$$\omega_\Phi(t) = \sup_{|y|\leq t} ||\Omega(\cdot + y) - \Omega(\cdot)||_{\Phi,S^{n-1}} < \infty,$$

for every $t \geq 0$.

Let T^+ be the differential transform operator studied in [3], [22] and [26] and defined by

$$T^+f(x) = \sum_{j\in\mathbb{Z}}(-1)^j\left(D_jf(x) - D_{j-1}f(x)\right),$$

where

$$D_jf(x) = \frac{1}{2^j}\int_x^{x+2^j} f(t)\,dt$$

The operator above appears when dealing with the rate of convergence of the averages D_jf, and it is a one-sided singular integral of convolution type with a kernel K supported in $(0,\infty)$ given by

$$K(x)=\sum_{j\in\mathbb{Z}}(-1)^j\left(\frac{1}{2^j}\chi_{(-2^j,0)}(x)-\frac{1}{2^{j-1}}\chi_{(-2^{j-1},0)}(x)\right).$$

In [26] the authors proved that $K\in\bigcap_{r\geq 1}H_r$ but $K\notin H_\infty$. Moreover, $K\in H_\psi$, where $\psi(t)=\exp t^{1/1+\epsilon}-1$.

As we said previously, we are interested in studying the higher order commutators of T. Since we are dealing with symbols of Lipschitz type, the smoothness condition associated to these commutators is defined as follows.

Definition 2.10. Let $m\in\mathbb{N}_0$, $0\leq\delta<\min\{1,n/m\}$ and let Φ be a Young function. We say that $K\in H_{\Phi,m}(\delta)$ if

$$\sum_{j=1}^{\infty}(2^j)^{m\delta}(2^jR)^n\|\left(K(\cdot-y)-K(\cdot)\right)\chi_{|\cdot|\sim 2^jR}\|_{\Phi,B(0,2^{j+1}R)}\leq C.$$

for some constants $c\geq 1$ and $C>0$ and for every $y\in\mathbb{R}^n$ with $R>c|y|$.

Clearly, when $\delta=0$ or $m=0$, $H_{\Phi,m}(\delta)=H_\Phi$.

Remark 2.11. It is easy to see that $H_{\Phi,m}(\delta_2)\subset H_{\Phi,m}(\delta_1)\subset H_\Phi$ whenever $0\leq\delta_1<\delta_2<\min\{1,n/m\}$.

As we have mentioned above, Fourier multipliers and singular integrals with rough kernels are examples of singular integral operators with $K\in H_\Phi$ for certain Young function Φ. By assuming adequate conditions depending on δ on the multiplier m, or on the L^Φ-modulus of continuity ω_Φ, we can obtain kernels $K\in H_{\Phi,m}(\delta)$. This fact can be proved by adapting Proposition 4.2 and 6.2 in [26].

We shall also deal with a class of Young functions that arises in connection with the boundedness of the fractional maximal operator M_Ψ on weighted Lebesgue spaces (see §4). Given $0<\theta<n$, $1\leq\beta<p<n/\theta$ and a Young function Ψ, we shall say that $\Psi\in\mathcal{B}_{\theta,\beta}$ if $t^{-\theta/n}\Psi^{-1}(t)$ is the inverse of a Young function and $\Psi^{1+\frac{\rho\theta}{n}}\in B_\rho$ for every $\rho>n\beta/(n-\theta\beta)$, that is, there exists a positive constant c such that

$$\int_c^\infty\frac{\Psi^{1+\frac{\rho\theta}{n}}}{t^\rho}\frac{dt}{t}<\infty$$

for each of those values of ρ.

We now state the following generalizations of Theorems 2.1 and 2.3. We shall consider again $m\in\mathbb{N}_0$. Recall that, as in Theorem 2.1, no condition on the size of K is assumed.

Theorem 2.12. *Let $0 < \delta < \min\{1, n/m\}$. Let $1 < p < n/(m\delta)$, $1/q = 1/p - m\delta/n$ and $b \in \Lambda(\delta)$. Assume that T has a kernel $K \in H_\Phi$ for a Young function Φ such that its complementary function $\widetilde{\Phi} \in \mathcal{B}_{m\delta,\beta}$ for some $1 \leq \beta < p$. Then, if w is a weight verifying $w^\beta \in A_{\frac{p}{\beta},\frac{q}{\beta}}$, there exists a positive constant C such that*

$$\left(\int_{\mathbb{R}^n} |T_b^m f(x)|^q w(x)^q dx\right)^{1/q} \leq C\|b\|_{\Lambda(\delta)}^m \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)^p dx\right)^{1/p}$$

for every $f \in L_w^p(\mathbb{R}^n)$.

Remark 2.13. If we consider, for example, $\Phi(t) = e^{t^{1/\gamma}} - e$ with $\gamma > 0$, then $\widetilde{\Phi}(t) \approx t(1+\log^+ t)^\gamma$ and this function verifies condition $\mathcal{B}_{m\delta,1}$. Thus, Φ satisfies the hypothesis of the theorem above and, in this case, we can take $w \in A_{p,q}$. As we have mentioned before, condition $\mathcal{B}_{m\delta,\beta}$ is related with the boundedness of the corresponding fractional maximal operator $M_{m\delta,\widetilde{\Phi}}$ between L_w^p and L_w^q when $w^\beta \in A_{\frac{p}{\beta},\frac{q}{\beta}}$ (see Theorem 4.5 below). When $\beta > 1$, a typical example is $\widetilde{\Phi}(t) = t^\beta(1+\log^+ t)^\gamma$ for $\gamma \geq 0$. In this case, the Young function Φ related with the smoothness condition on the kernel K given in the theorem above is $\Phi(t) = t^{\beta'}(1 + \log^+ t)^{-\gamma/(\beta-1)}$, where $\beta' = \beta/(\beta - 1)$.

Theorem 2.14. *Let $0 < \delta < \min\{1, n/m\}$, $n/(m\delta) \leq r < n/((m-1)\delta)$ and $\widetilde{\delta} = m\delta - n/r$. Let w be a weight such that $w^\beta \in A_{r/\beta,\infty}$ for some $1 < \beta < r$. Assume that T has a kernel $K \in H_{\Phi,m}(\delta)$ for a Young function Φ such that $\Phi^{-1}(t) \lesssim t^{\frac{\beta-1}{r}}$ for every $t > 0$. If $b \in \Lambda(\delta)$, then there exists a positive constant C such that*

$$\|T_b^m f\|_{\mathbb{L}_w(\widetilde{\delta})} \leq C\,\|b\|_{\Lambda(\delta)}^m\,\|fw\|_{L^r}$$

for every $f \in L_w^r(\mathbb{R}^n)$.

Theorem 2.15. *Let $m \in \mathbb{N}$, $0 < \delta < \min\{1, n/m\}$ and $r = n/((m-1)\delta)$. Let w be a weight such that $w^\beta \in A_{r/\beta,\infty}$ for some $1 < \beta < r$. Let T be a singular integral operator with kernel $K \in H_{\Phi,m}(\delta)$ where Φ is a Young function verifying $\Phi^{-1}(t) \lesssim t^{\frac{\beta-1}{r}}$ for every $t > 0$, and $\widetilde{\Phi} \in \mathcal{B}_{m\delta,\beta}$. If $b \in \Lambda(\delta)$, the following statements are equivalent,*

(i) $T_b^m : L_w^r(\mathbb{R}^n) \hookrightarrow \mathbb{L}_w(\delta)$;

(ii) There exists a positive constant C such that

$$\frac{\|w\chi_B\|_\infty}{|B|^{1+\frac{\delta}{n}}} \int_B \left|\sum_{k=0}^m c_k\left[S(x,u,k) - (S(\cdot,u,k))_B\right]\right| dx \leq C\,\|fw\|_r\,, \tag{2.6}$$

for every ball $B \subset \mathbb{R}^n$, $u \in B$ and $f \in L_{\mathrm{loc}}^1(\mathbb{R}^n)$.

3 Extrapolation to variable Lebesgue spaces

We are now interested in obtaining results of the type described above in the variable Lebesgue space context by using extrapolation techniques. In order to establish the main theorems we give some definitions and notations.

Let $p(\cdot):\mathbb{R}^n\to[1,\infty)$ be a measurable function. For $A\subset\mathbb{R}^n$ we define

$$p_A^- = \operatorname*{ess\,inf}_{x\in A} p(x) \qquad p_A^+ = \operatorname*{ess\,sup}_{x\in A} p(x).$$

For simplicity we denote $p^+ = p^+_{\mathbb{R}^n}$ and $p^- = p^-_{\mathbb{R}^n}$.

We say that $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ if $1<p^-\le p(\cdot)\le p^+<\infty$ and we say that $p(\cdot)\in\mathcal{P}^{\log}(\mathbb{R}^n)$ if $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ and it satisfies the following inequalities

$$|p(x)-p(y)|\le\frac{C}{\log(e+1/|x-y|)}, \qquad \text{for every } x,y\in\mathbb{R}^n.$$

and

$$|p(x)-p(y)|\le\frac{C}{\log(e+|x|)}, \qquad \text{with } |y|\ge|x|.$$

The variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is the set of the measurable functions f defined on $\mathbb{R}^n$ such that, for some positive λ, the convex functional modular

$$\varrho(f/\lambda)=\int_{\mathbb{R}^n}|f(x)/\lambda|^{p(x)}\,dx$$

is finite. A Luxemburg type norm can be defined in $L^{p(\cdot)}(\mathbb{R}^n)$ by taking

$$\|f\|_{L^{p(\cdot)}}=\|f\|_{p(\cdot)}=\inf\{\lambda>0:\varrho(f/\lambda)\le 1\}.$$

These spaces are special cases of Museliak-Orlicz spaces (see [31]), and generalize the classical Lebesgue spaces. For more information see, for example [23], [12], [15].

Let $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ be an exponent such that $1<\beta\le p^-\le p(\cdot)\le p^+<\frac{n\beta}{(n-\beta)^+}$ and let $\frac{\delta(\cdot)}{n}=\frac{1}{\beta}-\frac{1}{p(\cdot)}$. The space $\mathbb{L}(\delta(\cdot))$ is defined by the set of the measurable functions f such that

$$|||f|||_{\mathbb{L}(\delta(\cdot))}=\sup_B\frac{1}{|B|^{\frac{1}{\beta}}\|\chi_B\|_{p'(\cdot)}}\int_B|f-m_Bf|<\infty.$$

When $p(\cdot)$ is equal to a constant p, this space coincide with the space $\mathbb{L}_1(n/\beta-n/p)$, which is also the same as $\Lambda(n/\beta-n/p)$.

The spaces $\mathbb{L}(\delta(\cdot))$ were $\frac{\delta(\cdot)}{n} = \frac{1}{\beta} - \frac{1}{p(\cdot)}$ were introduced in [36]. In that article, the authors give conditions on the exponent $p(\cdot)$ that guarantee the boundedness of the fractional integral operator I_α from $L^{p(\cdot)}$ spaces into $\mathbb{L}(\delta(\cdot))$ spaces.

We say that $(p(\cdot), v)$ is an M-pair if and only if the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}_v(\mathbb{R}^n)$ and $L^{p'(\cdot)}_{v^{-1}}(\mathbb{R}^n)$, where $L^{p(\cdot)}_w(\mathbb{R}^n)$ denotes the space of all measurable functions f such that $fw \in L^{p(\cdot)}(\mathbb{R}^n)$.

Let $0 \leq \theta < n$ and $p(\cdot),\ q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $1/q(\cdot) = 1/p(\cdot) - \theta/n$, $p^+ < n/\theta$. We say that a weight $w \in A_{p(\cdot),q(\cdot)}$ if there exists a positive constant C such that for every ball B

$$\|w\chi_B\|_{q(\cdot)}\|w^{-1}\chi_B\|_{p'(\cdot)} \leq C|B|^{1-\theta/n}.$$

When $p(\cdot) = q(\cdot)$ we denote this class by $A_{p(\cdot)}$. It is well know that $w \in A_{p(\cdot)}$ if and only if $M : L^{p(\cdot)}_w \hookrightarrow L^{p(\cdot)}_w$ ([11]).

In [14] the authors proved the following extrapolation results.

Theorem 3.1 ([14]). *Suppose that for some p_0, q_0, $1 < p_0 \leq q_0 < \infty$, and every $w_0 \in A_{p_0,q_0}$, the inequality*

$$\|fw_0\|_{q_0} \leq C\,\|gw_0\|_{p_0},$$

holds for some positive constant C.

Given $p(\cdot),\ q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, suppose that

$$\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \frac{1}{p_0} - \frac{1}{q_0} = \frac{1}{\sigma'}.$$

If $w \in A_{p(\cdot),q(\cdot)}$ and $(q(\cdot)/\sigma,\ w^\sigma)$ is an M-pair, then

$$\|fw\|_{L^{q(\cdot)}} \leq C\,\|gw\|_{L^{p(\cdot)}}.$$

The theorem holds for $p_0 = 1$ if we assume only that the maximal operator is bounded on $L^{(q(\cdot)/q_0)'}_{w^{-q_0}}(\mathbb{R}^n)$.

It is easy to see that, if $0 < \delta < \min\{\eta, n/m\}$, $1/p(\cdot) - 1/q(\cdot) = m\delta/n$, then $w \in A_{p(\cdot),q(\cdot)}$ is equivalent to $w^\sigma \in A_{q(\cdot)/\sigma}$, with $\sigma = n/(n - m\delta)$. This fact allows us to say that $(q(\cdot)/\sigma,\ w^\sigma)$ is an M-pair. Thus, as a consequence of Theorem 2.1, if T is defined as in this theorem, we have that the pair $(g, T^m_b g)$ satisfies the hypothesis of Theorem 3.1. Therefore we obtain the following result.

Theorem 3.2. *Let T be the operator defined in* (2.1) *with kernel K satisfying condition H_∞^*. Let $0<\delta<\min\{\eta, n/m\}$ and $b\in\Lambda(\delta)$. Given $p(\cdot),\ q(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ such that*

$$\frac{1}{p(\cdot)}-\frac{1}{q(\cdot)}=\frac{m\delta}{n},$$

if $w\in A_{p(\cdot),q(\cdot)}$ then

$$\|T_b^m fw\|_{L^{q(\cdot)}}\leq C\,\|fw\|_{L^{p(\cdot)}}\,.$$

An analogous result can be obtained by extrapolation when the kernel K satisfies a Hörmander type condition. Thus, by Theorem 2.12 we obtain the following theorem.

Theorem 3.3. *Let T be the operator defined in* (2.1). *Let $0<\delta<\min\{1, n/m\}$ and $b\in\Lambda(\delta)$. Let us also suppose that $K\in H_\Phi$ for a Young function Φ such that its complementary function $\widetilde{\Phi}\in\mathcal{B}_{m\delta,1}$. Given $p(\cdot),\ q(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ such that*

$$\frac{1}{p(\cdot)}-\frac{1}{q(\cdot)}=\frac{m\delta}{n},$$

if $w\in A_{p(\cdot),q(\cdot)}$ then

$$\|T_b^m fw\|_{L^{q(\cdot)}}\leq C\,\|fw\|_{L^{p(\cdot)}}\,.$$

In [4] the authors proved the following theorem, which exhibit an extrapolation result starting from a -hypothesis that involves inequalities of the type $L_w^s-\mathbb{L}_w(\delta)$, and obtaining unweighted estimates in the variable context of the type $L^{p(\cdot)}-\mathbb{L}(\tilde{\delta}(\cdot))$.

Theorem 3.4 ([4])**.** *Let $1<\theta<\infty$, $0\leq\delta<1$, and let s be such that $\delta/n=1/\theta-1/s$. Suppose that $p(\cdot)\in\mathcal{P}^{log}(\mathbb{R}^n)$ and $\tilde{\delta}(\cdot)/n=1/\theta-1/p(\cdot)$ with $\delta\leq\tilde{\delta}(\cdot)$. If f and g are two measurable functions such that the inequality*

$$||f||_{\mathbb{L}_w(\delta)}\leq C\,\|gw\|_s\,,$$

holds for every weight w in $A_{s,\infty}$ and some positive constant $C=C(w)$, then there exits a positive constant C such that the inequality

$$||f||_{\mathbb{L}(\tilde{\delta}(\cdot))}\leq C\,\|g\|_{p(\cdot)}$$

holds.

Thus, from Theorem 2.3 we can derive the following result in the variable exponent context.

Theorem 3.5. *Let T be the operator defined in* (2.1) *with kernel K satisfying conditions* (2.2) *and H_∞^*. Let $0<\delta<\min\{\eta, n/m\}$ and $b\in\Lambda(\delta)$. Let $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ such that $n/m\delta\leq p^-\leq p(\cdot)\leq p^+<n/((m-1)\delta)$ and $\tilde{\delta}(\cdot)/n=m\delta/n-1/p(\cdot)$.Then there exists a positive constant C such that*

$$||T_b^m f||_{\mathbb{L}(\tilde{\delta}(\cdot))}\leq C\,\|f\|_{p(\cdot)}\,.$$

4 Auxiliary results

In this section we give some previous results. We begin with some inequalities involving functions in $\Lambda(\delta)$.

Lemma 4.1. *Let $0 < \delta < 1$ and $B \subset \mathbb{R}^n$ a ball. If $b \in \Lambda(\delta)$, then*

(i) for every $y \in \lambda B$, $\lambda \geq 1$,

$$|b(y) - b_B| \leq C\|b\|_{\Lambda(\delta)}|\lambda B|^{\frac{\delta}{n}}.$$

(ii) for every $j \in \mathbb{N}$

$$|b_{2^{j+1}B} - b_{2B}| \leq 2^n j|2^{j+1}B|^{\frac{\delta}{n}}\|b\|_{\Lambda(\delta)}.$$

In order to obtain the boundedness result between Lebesgue spaces, we prove the following key estimate, which shows how can we control the higher order commutators of T by a fractional maximal function via the sharp maximal operator $M^{\sharp}_{0,\gamma}$, $0 < \gamma < 1$, given by $M^{\sharp}_{0,\gamma}f := M^{\sharp}_0(|f|^{\gamma})^{1/\gamma}$ where

$$M^{\sharp}_0 f(x) = \sup_{B \ni x} \inf_{a \in \mathbb{R}} \frac{1}{|B|}\int_B |f(y) - a|\, dy.$$

Lemma 4.2. *Let $m \in \mathbb{N}$, $0 < \gamma < 1/m$ and $0 < \delta < \min\{1, n/m\}$. Let $b \in \Lambda(\delta)$ and T a singular integral operator with kernel K. Then, there exists a positive constant C such that*

(i) if $K \in H^*_{\infty}$,

$$M^{\sharp}_{0,\gamma}(T^m_b f)(x) \lesssim \|b\|^m_{\Lambda(\delta)}\left(\sum_{j=0}^{m-1} M_{\theta_j,\gamma}(|T^j_b f|)(x) + M_{\theta_0}f(x)\right),$$

where $\theta_j = \delta(m-j)$, $j = 0, \ldots, m$.

(ii) if $K \in H_{\Phi}$ for some Young function Φ,

$$M^{\sharp}_{0,\gamma}(T^m_b f)(x) \lesssim \|b\|^m_{\Lambda(\delta)}\left(\sum_{j=0}^{m-1} M_{\theta_j,\gamma}(|T^j_b f|)(x) + M_{\theta_0,\widetilde{\Phi}}f(x)\right),$$

where $\theta_j = \delta(m-j)$, $j = 0, \ldots, m$, and $\widetilde{\Phi}$ is the complementary function of Φ.

Remark 4.3. For $0 < \delta < 1$, $m = 1$ and $K \in H^*_{\infty}$ and homogeneous of degree $-n$, the proof of *(i)* can be found in [35] for a larger class of Lipschitz spaces with variable parameter.

Proof of Lemma 4.2: Fix B a ball containing x, and decompose the commutator in the following way (see, for instance, [17] or [33])

$$T_b^m f(x) = \sum_{j=0}^{m-1} C_{j,m}(b(x) - b_{2B})^{m-j} T_b^j f(x) + T((b - b_{2B})^m f)(x).$$

If we split $f = f_1 + f_2$ where $f_1 = f\chi_{2B}$, it is sufficient to estimate, for $0 < \gamma < 1/m$, the average

$$\left(\frac{1}{|B|}\int_B |T_b^m f(y) - T((b - b_{2B})^m f_2)(x_B)|^\gamma dy\right)^{1/\gamma} \leq I + II + III,$$

where x_B denotes the center of B, and

$$I = \sum_{j=0}^{m-1} \left(\frac{1}{|B|}\int_B (b(y) - b_{2B})^{(m-j)\gamma} |T_b^j f(y)|^\gamma dy\right)^{\frac{1}{\gamma}},$$

$$II = \left(\frac{1}{|B|}\int_B |T((b - b_{2B})^m f_1)(y)|^\gamma dy\right)^{\frac{1}{\gamma}},$$

$$III = \left(\frac{1}{|B|}\int_B |T((b - b_{2B})^m f_2)(y) - T((b - b_{2B})^m f_2)(x_B)|^\gamma dy\right)^{\frac{1}{\gamma}}.$$

For simplicity, we will assume $\|b\|_{\Lambda(\delta)} = 1$. We shall first estimate I. From Lemma 4.1 *(i)* we have

$$\begin{aligned}
I &\lesssim \sum_{j=0}^{m-1} \|b\|_{\Lambda(\delta)}^{m-j} |B|^{\frac{\delta(m-j)}{n}} \left(\frac{1}{|B|}\int_B |T_b^j f(y)|^\gamma dy\right)^{\frac{1}{\gamma}} \\
&= C\sum_{j=0}^{m-1} \left(\frac{1}{|B|^{1-\frac{(m-j)\delta\gamma}{n}}}\int_B \left|T_b^j f(y)\right|^\gamma\right)^{1/\gamma} \\
&\lesssim \sum_{j=0}^{m-1} M_{\theta_j,\gamma}(|T_b^j f|)(x)
\end{aligned}$$

where $\theta_j = (m-j)\delta$. Note that the last maximal operator is of fractional-type since $0 < \theta_j < (m-j)n/m \leq n$ for every $0 \leq j \leq m-1$.

We will now estimate II. Since T is of weak type $(1,1)$ and $0 < \gamma < 1$, from Kolmogorov inequality and the fact that $y, z \in B$ we obtain

$$\begin{aligned}
II &\leq \frac{1}{|B|}\int_{2B} |b(z) - b_{2B}|^m |f(z)| dz \\
&\leq |B|^{m\delta/n} \frac{1}{|B|}\int_{2B} |f(z)| dz \\
&\lesssim M_{\theta_0} f(x).
\end{aligned}$$

Since $0 < \delta < n/m$, it is clear that $0 < \theta_0 < n$, so M_{θ_0} is a fractional-type maximal operator.

In order to estimate III, we first observe that, by Jensen's inequality

$$III \leq \frac{1}{|B|} \int_B |T_b^m \left((b - b_{2B})^m f_2\right)(y) - T((b - b_{2B})^m f_2)(x_B)| dy,$$

and, setting $B_j = 2^j B$, the integrand can be estimated, using Lemma 4.1 *(i)*, as follows

$$\begin{aligned} &|T_b^m \left((b - b_{2B})^m f_2\right)(y) - T((b - b_{2B})^m f_2)(x_B)| \\ &\qquad \leq \sum_{j=1}^{\infty} \int_{B_{j+1} \setminus B_j} |K(y - z) - K(x_B - z)| |b(z) - b_{2B}|^m |f(z)| dz \\ &\qquad \lesssim \|b\|_{\Lambda(\delta)}^m \sum_{j=1}^{\infty} |B_{j+1}|^{\frac{m\delta}{n}} \int_{B_{j+1} \setminus B_j} |K(y - z) - K(x_B - z)| |f(z)| dz. \end{aligned} \tag{4.1}$$

Here, we must distinguish the cases $K \in H_\infty^*$ and $K \in H_\Phi$.

If $K \in H_\infty^*$,

$$\begin{aligned} &|T_b^m((b - b_{2B})^m f_2)(y) - T((b - b_{2B})^m f_2)(x_B)| \\ &\qquad \lesssim \|b\|_{\Lambda(\delta)}^m \sum_{j=1}^{\infty} |B_{j+1}|^{\frac{\delta m}{n}} \int_{B_{j+1} \setminus B_j} \frac{|y - x_B|^\eta}{|y - z|^{n+\eta}} |f(z)| dz \\ &\qquad \lesssim \|b\|_{\Lambda(\delta)}^m \sum_{j=1}^{\infty} |B_{j+1}|^{\frac{m\delta}{n}} 2^{-j\eta} \frac{1}{|B_{j+1}|} \int_{B_{j+1}} |f(z)| dz \\ &\qquad \approx \|b\|_{\Lambda(\delta)}^m \sum_{j=1}^{\infty} 2^{-j\eta} \frac{1}{|B_{j+1}|^{1 - \frac{\theta_0}{n}}} \int_{B_{j+1}} |f(y)| dy \\ &\qquad \leq \|b\|_{\Lambda(\delta)}^m M_{\theta_0} f(x) \sum_{j=1}^{\infty} 2^{-j\eta} \lesssim \|b\|_{\Lambda(\delta)}^m M_{\theta_0} f(x), \end{aligned}$$

since $\eta > 0$. Therefore

$$III \lesssim \|b\|_{\Lambda(\delta)}^m M_{\theta_0} f(x).$$

Let us now consider the case $K \in H_\Phi$. Applying Hölder's inequality with Φ and $\widetilde{\Phi}$ in (4.1), we obtain

$$\begin{aligned} &|T_b^m((b - b_{2B})^m f_2)(y) - T((b - b_{2B})^m f_2)(x_B)| \\ &\quad \lesssim \|b\|_{\Lambda(\delta)}^m \sum_{j=1}^{\infty} |B_{j+1}|^{\frac{m\delta}{n}+1} \| \left(K(\cdot - (y - x_B)) - K(\cdot)\right) \chi_{|\cdot| \sim 2^j R} \|_{\Phi, B_{j+1}} \|f\|_{\widetilde{\Phi}, B_{j+1}} \\ &\quad \lesssim \|b\|_{\Lambda(\delta)}^m \sum_{j=1}^{\infty} |B_{j+1}| \| \left(K(\cdot - (y - x_B)) - K(\cdot)\right) \chi_{|\cdot| \sim 2^j R} \|_{\Phi, B_{j+1}} |B_{j+1}|^{\frac{m\delta}{n}} \|f\|_{\widetilde{\Phi}, B_{j+1}} \end{aligned}$$

$$
\begin{aligned}
&\lesssim \|b\|_{\Lambda(\delta)}^m M_{\theta_0,\widetilde{\Phi}} f(x) \sum_{j=1}^{\infty} (2^j R)^n \| \left(K(\cdot - (y - x_B)) - K(\cdot)\right) \chi_{|\cdot|\sim 2^j R} \|_{\Phi, B_{j+1}} \\
&\lesssim \|b\|_{\Lambda(\delta)}^m M_{\theta_0,\widetilde{\Phi}} f(x).
\end{aligned}
$$

Therefore,

$$
III \lesssim \|b\|_{\Lambda(\delta)}^m M_{\theta_0,\widetilde{\Phi}} f(x).
$$

Combining all these estimates, we obtain the desired pointwise inequalities. □

The following result is a variant of the well-known Fefferman-Stein's inequality (see [16]) and it will be a key estimate to prove Theorem 2.1.

Lemma 4.4 ([34]). *Let $0 < p < \infty$ and $0 < \gamma < 1$. Let w be a weight in the A_∞ class. Then, there exists a positive constant C such that*

$$
\int_{\mathbb{R}^n} M_\gamma f(x)^p w(x) dx \leq C[w]_{A_\infty} \int_{\mathbb{R}^n} M_{0,\gamma}^{\sharp} f(x)^p w(x) d \tag{4.2}
$$

holds for every function f for which the left hand side is finite.

We shall also need two results involving the boundedness of fractional maximal operators associated with Young functions, that can be found in [2].

Theorem 4.5 ([2]). *Let $0 < \alpha < n$, $1 \leq \beta < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. Let w be a weight such that $w^\beta \in A_{p/\beta, q/\beta}$. Let Ψ be a Young function that satisfies $\Psi \in \mathcal{B}_{\alpha,\beta}$. Then, $M_{\alpha,\Psi}$ is bounded from $L^p(w^p, \mathbb{R}^n)$ into $L^q(w^q, \mathbb{R}^n)$.*

Note that if $\Psi = t^\beta(1 + \log^+ t)^\gamma$ for any $\gamma \geq 0$, then $\Psi \in \mathcal{B}_{\alpha,\beta}$ and the following result holds.

Theorem 4.6 ([2]). *Let $0 < \alpha < n$, $1 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. Let w be a weight and $\Psi(t) = t^\beta(1 + \log^+ t)^\gamma$ where $1 \leq \beta < p$ and $\gamma \geq 0$. Then, $M_{\alpha,\Psi}$ is bounded from $L^p(w^p, \mathbb{R}^n)$ into $L^q(w^q, \mathbb{R}^n)$ if and only if $w^\beta \in A_{p/\beta, q/\beta}$.*

In order to prove Theorem 2.6, we shall need the following estimate.

Lemma 4.7. *Let $0 < \delta < \min\{\eta, n/(m-1)\}$, for $0 < \eta \leq 1$. Let $r = n/((m-1)\delta)$, $w \in A_{r,\infty}$, $b \in \Lambda(\delta)$ and $f \in L_w^r(\mathbb{R}^n)$. Let $B \subset \mathbb{R}^n$ be a ball and $f_2 = f\chi_{\mathbb{R}^n \setminus 2B}$. If T is a singular integral operator with kernel $K \in H_\infty^*$, then, for every $x, u \in B$,*

$$
\left| T((b - b_B)^k f_2)(x) - T((b - b_B)^k f_2)(u) \right| \lesssim \frac{\|b\|_{\Lambda(\delta)}^k \|fw\|_r |B|^{\frac{\delta(k-m+1)}{n}}}{\|w\chi_B\|_\infty}
$$

for each $k = 0, ..., m$.

Proof of Lemma 4.7. If $K \in H^*_\infty$, by taking $x, u \in B$, and $0 \le k \le m$, and setting $B_j = 2^j B$, we have from Lemma 4.1 *(i)* that

$$\begin{aligned}
&|T((b-b_B)^k f_2)(x) - T((b-b_B)^k f_2)(u)| \\
&\qquad \le \int_{(2B)^c} |K(x-y) - K(u-y)|\,|b(y)-b_B|^k\,|f(y)|\,dy \\
&\qquad \lesssim \|b\|^k_{\Lambda(\delta)} \sum_{j=1}^{\infty} |B_{j+1}|^{\frac{\delta k}{n}} \int_{B_{j+1}\setminus B_j} \frac{|x-u|^\eta}{|y-u|^{n+\eta}}\,|f(y)|\,dy \\
&\qquad \lesssim \|b\|^k_{\Lambda(\delta)} \sum_{j=1}^{\infty} \frac{|B_{j+1}|^{\frac{\delta k}{n}}\,|B|^{\frac{\eta}{n}}}{|B_{j+1}|^{\frac{n+\eta}{n}}} \int_{B_{j+1}\setminus B_j} |f(y)|\,dy.
\end{aligned}$$

Now by Hölder's inequality and the fact that $w \in A_{r,\infty}$ with $r = n/((m-1)\delta)$, we get

$$\begin{aligned}
&|T((b-b_B)^k f_2)(x) - T((b-b_B)^k f_2)(u)| \\
&\qquad \lesssim \|b\|^k_{\Lambda(\delta)}\,\|fw\|_r \sum_{j=1}^{\infty} \frac{|B_{j+1}|^{\frac{\delta k}{n}}}{|B_{j+1}|\,2^{j\eta}} \left\|w^{-1}\chi_{B_{j+1}}\right\|_{r'} \\
&\qquad \lesssim \|b\|^k_{\Lambda(\delta)}\,\|fw\|_r\,\|w\chi_B\|^{-1}_\infty \sum_{j=1}^{\infty} \frac{|B_{j+1}|^{\frac{\delta(k-m+1)}{n}}}{2^{j\eta}} \\
&\qquad \lesssim \|b\|^k_{\Lambda(\delta)}\,\|fw\|_r\,\|w\chi_B\|^{-1}_\infty\,|B|^{\frac{\delta(k-m+1)}{n}} \sum_{j=1}^{\infty} 2^{j(\delta(k-m+1)-\eta)} \\
&\qquad \lesssim \|b\|^k_{\Lambda(\delta)}\,\|fw\|_r\,\|w\chi_B\|^{-1}_\infty\,|B|^{\frac{\delta(k-m+1)}{n}}
\end{aligned}$$

where the series is summable since $0 \le k \le m$ and $\delta < \eta$. $\square$

Lemma 4.8. *Let $m \in \mathbb{N}$, $0 < \delta < \min\{1, n/(m-1)\}$, $r = n/((m-1)\delta)$, $b \in \Lambda(\delta)$ and $f \in L^r_w(\mathbb{R}^n)$ where w is a weight such that $w^\beta \in A_{r/\beta,\infty}$ for some $1 < \beta < r$. Let $B \subset \mathbb{R}^n$ be a ball and $f_2 = f\chi_{\mathbb{R}^n\setminus 2B}$. If T is a singular integral operator with kernel $K \in H_{\Phi,m}(\delta)$, where Φ is a Young function verifying $\Phi^{-1}(t) \lesssim t^{\frac{\beta-1}{r}}$ for every $t > 0$, then, for every $x, u \in B$,*

$$\left|T((b-b_B)^k f_2)(x) - T((b-b_B)^k f_2)(u)\right| \lesssim \frac{\|fw\|_r\,\|b\|^k_{\Lambda(\delta)}\,|B|^{\frac{\delta(k-m+1)}{n}}}{\|w\chi_B\|_\infty}$$

for each $k = 0, ..., m$.

Proof. Fix $x, u \in B$ and $0 \le k \le m$. Setting $B_u = B(u, R)$ which satisfies $B \subset 2B_u \subset 4B$, and using Lemma 4.1 *(i)* we have

$$|T((b-b_B)^k f_2)(x) - T((b-b_B)^k f_2)(u)|$$

$$
\begin{aligned}
&\lesssim \int_{\mathbb{R}^n\setminus B_u} |b(y)-b_{B_u}|^k |K(x-y)-K(u-y)||f(y)|dy\\
&\lesssim \sum_{j=0}^{\infty}\int_{2^{j+1}B_u\setminus 2^jB_u} |b(y)-b_{B_u}|^k |K(x-y)-K(u-y)||f(y)|dy\\
&\lesssim \|b\|_{\Lambda(\delta)}^k \sum_{j=0}^{\infty} |2^{j+1}B_u|^{\frac{\delta k}{n}} \int_{2^{j+1}B_u\setminus 2^jB_u} |K(x-y)-K(u-y)||f(y)|dy.
\end{aligned}
$$

Since $1/r+1/(r/\beta)'=1-(\beta-1)/r$, we can use Hölder's inequality with $\Phi^{-1}(t)t^{1/r}t^{1/(r/\beta)'}\lesssim t$ and the fact that $w^\beta\in A_{r/\beta,\infty}$, to get

$$
\begin{aligned}
&|T((b-b_B)^kf_2)(x)-T((b-b_B)^kf_2)(u)| \qquad (4.3)\\
&\lesssim \|b\|_{\Lambda(\delta)}^k\|fw\|_r \sum_{j=0}^{\infty} |2^{j+1}B_u|^{\frac{\delta k}{n}+\frac{1}{r'}} \||K(\cdot-(u-x))-K(\cdot)|\chi_{|\cdot|\sim 2^jR}\|_{\Phi,2^{j+1}B_u} \frac{\|w^{-1}\chi_{2^{j+1}B_u}\|_{(r/\beta)'}}{|2^{j+1}B_u|^{1/(r/\beta)'}}\\
&\lesssim \frac{\|b\|_{\Lambda(\delta)}^k |B|^{\frac{\delta k}{n}-\frac{1}{r}}\|fw\|_r}{\|w\chi_{2B_u}\|_\infty} \sum_{j=1}^{\infty} (2^jR)^n 2^{j(\delta k-\frac{n}{r})} \||K(\cdot-(u-x))-K(\cdot)|\chi_{|\cdot|\sim 2^jR}\|_{\Phi,2^{j+1}B_u}\\
&\le \frac{\|b\|_{\Lambda(\delta)}^k |B|^{\frac{\delta(k-m+1)}{n}}\|fw\|_r}{\|w\chi_{B}\|_\infty} \sum_{j=1}^{\infty} 2^{jm\delta}(2^jR)^n \||K(\cdot-(u-x))-K(\cdot)|\chi_{|\cdot|\sim 2^jR}\|_{\Phi,2^{j+1}B_u}\\
&\lesssim \frac{\|b\|_{\Lambda(\delta)}^k |B|^{\frac{\delta(k-m+1)}{n}}\|fw\|_r}{\|w\chi_{B}\|_\infty},
\end{aligned}
$$

where we have used that $\delta k - n/r \le m\delta$ for $m\in\mathbb{N}$, and that $K\in H_{\Phi,m}(\delta)$. □

5 Proofs of main results

Proof of Theorem 2.1: The proof will be done by induction. Notice that when $m=0$, $p=q$ and it is known that the boundedness result holds for $A_{p,p}$ weights (see, for example, [16]). By homogeneity we can also supposse that $\|b\|_{\Lambda(\delta)}=1$.

Fix $m\in\mathbb{N}$ and define the following auxiliary exponents

$$\frac{1}{p_j}=\frac{1}{q}+\frac{\delta(m-j)}{n}=\frac{1}{p}-\frac{j\delta}{n}, \quad j=0,\dots,m.$$

Clearly, $p_m=q$ and, if $\theta_j=(m-j)\delta$, we have that

$$\frac{1}{p_j}=\frac{1}{q}+\frac{\theta_j}{n}=\frac{1}{p}-\frac{j\delta}{n}, \quad j=0,\dots,m.$$

Notice also that $p\le p_j\le p_l\le q$ for every $0\le j\le l\le m$.

It is easy to see that $w \in A_{p,q}$ yields $w^\gamma \in A_{\frac{p}{\gamma},\frac{q}{\gamma}}$ for every $0<\gamma<1$, for. Moreover, from the properties of these classes, we have that $w^\gamma \in A_{\frac{p_j}{\gamma},\frac{p_l}{\gamma}}$ for every $0 \le j \le l \le m$.

By applying Fefferman-Stein's inequality (4.2) with $w^q \in A_{1+q/p'} \subset A_\infty$, we get

$$\|wT_b^m f\|_q \le \|wM_\gamma(T_b^m f)\|_q \lesssim \|wM^\sharp_{0,\gamma}(T_b^m f)\|_q.$$

Now, by taking $0<\gamma<1/m$ and since $K \in H^*_\infty$, from Lemma 4.2 we have that

$$\|wT_b^m f\|_q \lesssim \sum_{j=0}^{m-1} \|wM_{\theta_j,\gamma}(|T_b^j f|)\|_q + \|wM_{\theta_0} f\|_q.$$

Since $w \in A_{p,q}$ and $1/q = 1/p - \theta_0/n$, we have that

$$\|wT_b^m f\|_q \lesssim \sum_{j=0}^{m-1} \|wM_{\theta_j,\gamma}(|T_b^j f|)\|_q + \|fw\|_p.$$

On the other hand, since $w^\gamma \in A_{\frac{p_j}{\gamma},\frac{q}{\gamma}}$ for every $j=1,\dots,m-1$, then the fractional maximal operator $M_{\theta_j\gamma}$ is bounded from $L^{\frac{p_j}{\gamma}}(\mathbb{R}^n)$ to $L^{\frac{q}{\gamma}}(\mathbb{R}^n)$. Thus, we have that

$$\begin{aligned}
\|wT_b^m f\|_q &\lesssim \sum_{j=0}^{m-1} \|w^\gamma M_{\theta_j\gamma}(|T_b^j f|^\gamma)\|_{q/\gamma}^{1/\gamma} + \|wf\|_p \\
&\lesssim \sum_{j=0}^{m-1} \|w^\gamma (T_b^j f)^\gamma\|_{p_j/\gamma}^{1/\gamma} + \|wf\|_p \\
&\lesssim \sum_{j=0}^{m-1} \|wT_b^j f\|_{p_j} + \|wf\|_p.
\end{aligned}$$

Since $1/p_j = 1/p - (j\delta)/n$ and $w \in A_{p,p_j}$, we apply the inductive hypothesis to get

$$\|wT_b^m f\|_q \lesssim \sum_{j=0}^{m-1} \|wf\|_p + \|wf\|_p \lesssim \|wf\|_p. \qquad \square$$

Proof of Theorem 2.3: Fix $f \in L^r_w(\mathbb{R}^n)$. For a ball $B \subset \mathbb{R}^n$, set $f_1 = f\chi_{2B}$, $f_2 = f - f_1$ and $a_B = \frac{1}{|B|}\int_B T_b^m f_2$. Then,

$$\begin{aligned}
\frac{\|w\chi_B\|_\infty}{|B|}\int_B |T_b^m f(x) - a_B|\,dx &\le \frac{\|w\chi_B\|_\infty}{|B|}\int_B |T_b^m f_1(x)|\,dx + \frac{\|w\chi_B\|_\infty}{|B|}\int_B |T_b^m f_2(x) - a_B|\,dx \\
&= I_1 + I_2.
\end{aligned}$$

Let us first notice that, since $w \in A_{r,\infty}$, there exists $1 < s' < r$ such that $w \in A_{s',\infty}$ and we can choose $1 < p < n/(m\delta) \le r$ such that $\frac{1}{s} = \frac{1}{p} - \frac{m\delta}{n}$.

Thus, from theorem 2.1, the estimate of I_1 is as follows

$$\begin{aligned}
I_1 &= \frac{\|w\chi_B\|_\infty}{|B|}\int_B |T_b^m f_1(x)|\,dx \\
&\leq \frac{\|w\chi_B\|_\infty}{|B|}\|T_b^m f_1 w\|_s \|w^{-1}\chi_B\|_{s'} \\
&\leq C\|b\|_{\Lambda(\delta)}^m \frac{1}{|B|^{1/s}}\|fw\chi_B\|_p \\
&\leq C\|b\|_{\Lambda(\delta)}^m \frac{|B|^{m\delta/n}}{|B|^{1/p}}\|fw\chi_B\|_p \\
&\leq C\|b\|_{\Lambda(\delta)}^m |B|^{m\delta/n-1/r}\|fw\|_r \\
&\leq C\|b\|_{\Lambda(\delta)}^m |B|^{\tilde{\delta}/n}\|fw\|_r.
\end{aligned}$$

For I_2, we first estimate the difference $|T_b^m f_2(x) - (T_b^m f_2)_B|$ for every $x \in B$. Since

$$|T_b^m f_2(x) - (T_b^m f_2)_B| \leq \frac{1}{|B|}\int_B |T_b^m f_2(x) - T_b^m f_2(y)|\,dy,$$

we analyze $A = |T_b^m f_2(x) - T_b^m f_2(y)|$. If $x, y \in B$

$$\begin{aligned}
A &\leq \int_{(2B)^c} |(b(x)-b(z))^m K(x-z) - (b(y)-b(z))^m K(y-z)|\,|f(z)|\,dz \\
&\leq \int_{(2B)^c} |b(x)-b(z)|^m\,|K(x-z)-K(y-z)|\,|f(z)|\,dz \\
&\quad + \int_{(2B)^c} |(b(x)-b(z))^m - (b(y)-b(z))^m|\,|K(y-z)|\,|f(z)|\,dz \\
&\leq \int_{(2B)^c} |b(x)-b(z)|^m\,|K(x-z)-K(y-z)|\,|f(z)|\,dz \\
&\quad + |b(x)-b(y)|\sum_{k=0}^{m-1}\int_{(2B)^c} |b(x)-b(z)|^{m-1-k}\,|b(y)-b(z)|^k\,|K(y-z)|\,|f(z)|\,dz \\
&= I_3 + I_4.
\end{aligned}$$

By the definition of $\Lambda(\delta)$, we get that

$$\begin{aligned}
I_3 &\lesssim \|b\|_{\Lambda(\delta)}^m \int_{(2B)^c} |x-z|^{\delta m}\,|K(x-z)-K(y-z)|\,|f(z)|\,dz \\
&\lesssim \|b\|_{\Lambda(\delta)}^m \sum_{j=1}^{\infty}\int_{2^{j+1}B\setminus 2^jB} |x-z|^{\delta m}\,\frac{|x-y|^\eta}{|x-z|^{n+\eta}}\,|f(z)|\,dz \\
&\lesssim \|b\|_{\Lambda(\delta)}^m \sum_{j=1}^{\infty}\frac{2^{j\delta m}|B|^{\frac{\delta m}{n}}}{2^{j(n+\eta)}|B|}\int_{2^{j+1}B\setminus 2^jB} |f(z)|\,dz \\
&\lesssim \|b\|_{\Lambda(\delta)}^m \,\|fw\|_r \sum_{j=1}^{\infty}\frac{2^{j\delta m}|B|^{\frac{\delta m}{n}}}{2^{j\eta}|2^jB|}\,\left\|w^{-1}\chi_{2^{j+1}B}\right\|_{r'}
\end{aligned}$$

Then, by Hölder's inequality, the definition of $\tilde{\delta}$ and the fact that $w \in A_{r,\infty}$, we deduce that

$$I_3 \lesssim \|b\|_{\Lambda(\delta)}^m \|fw\|_r |B|^{\frac{\tilde{\delta}}{n}} \sum_{j=1}^{\infty} 2^{j(\tilde{\delta}-\eta)} \frac{\left\|w^{-1}\chi_{2^{j+1}B}\right\|_{r'}}{|2^{j+1}B|^{1/r'}} \lesssim [w]_{A_{r,\infty}} \frac{\|b\|_{\Lambda(\delta)}^m \|fw\|_r |B|^{\frac{\tilde{\delta}}{n}}}{\|w\chi_B\|_\infty}.$$

In order to estimate I_4, we use that $b \in \Lambda(\delta)$ to get that

$$\begin{aligned}
I_4 &\lesssim \|b\|_{\Lambda(\delta)} |x-y|^{\delta} \|b\|_{\Lambda(\delta)}^{m-1} \sum_{k=0}^{m-1} \sum_{j=1}^{\infty} \int_{2^{j+1}B \setminus 2^j B} |x-z|^{\delta(m-1-k)} |y-z|^{\delta k} |K(x-z)| |f(z)| \, dz \\
&\lesssim \|b\|_{\Lambda(\delta)}^m |B|^{\frac{\delta}{n}} \sum_{j=1}^{\infty} |2^{j+1}B|^{\frac{\delta(m-1)}{n}-1} \int_{2^{j+1}B \setminus 2^j B} |f(z)| \, dz \\
&\lesssim \|b\|_{\Lambda(\delta)}^m \|fw\|_r |B|^{\frac{\delta}{n}} \sum_{j=1}^{\infty} |2^{j+1}B|^{\frac{\delta(m-1)}{n}-1} \|w^{-1}\chi_{2^{j+1}B}\|_{r'} \\
&\lesssim \frac{\|b\|_{\Lambda(\delta)}^m \|fw\|_r |B|^{\frac{\tilde{\delta}}{n}}}{\|w\chi_B\|_\infty} \sum_{j=1}^{\infty} 2^{j(\tilde{\delta}-\delta)} \\
&\lesssim \frac{\|b\|_{\Lambda(\delta)}^m \|fw\|_r |B|^{\frac{\tilde{\delta}}{n}}}{\|w\chi_B\|_\infty}.
\end{aligned}$$

We are done. □

Proof of Theorem 2.6 . Let $B \subset \mathbb{R}^n$ be a ball and $x \in B$. Let $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$. Then,

$$\begin{aligned}
T_b^m f(x) - (T_b^m f)_B =& T_b^m f_1(x) - (T_b^m f_1)_B \\
&+ \sum_{k=0}^{m} c_k \Big[(b(x)-b_B)^{m-k} T((b-b_B)^k f_2)(x) \\
&- \frac{1}{|B|} \int_B (b(z)-b_B)^{m-k} T((b-b_B)^k f_2)(z) dz \Big].
\end{aligned}$$

We can rewrite the above identity in the following form

$$\begin{aligned}
T_b^m f(x) - (T_b^m f)_B &= \sigma_1(x) - (\sigma_1)_B \\
&+ \sum_{k=0}^{m} c_k \left[\sigma_2(x,u,k) - (\sigma_2(\cdot,u,k))_B + \sigma_3(x,u,k) - (\sigma_3(\cdot,u,k))_B \right],
\end{aligned}$$

where

$$\sigma_1(x) = T_b^m f_1(x),$$

$$\sigma_2(x,u,k) = (b(x)-b_B)^{m-k} \Big(T((b-b_B)^k f_2)(x) - T((b-b_B)^k f_2)(u) \Big),$$

$$\sigma_3(x,u,k) = (b(x)-b_B)^{m-k} T((b-b_B)^k f_2)(u).$$

For σ_1, since $w \in A_{\frac{n}{m\delta},\infty}$, there exists $1<p<\frac{n}{m\delta}$ such that $w \in A_{p,\infty}$. We take $\frac{1}{q} = \frac{1}{p} - \frac{m\delta}{n}$, so $q > p$ and $w \in A_{q,\infty}$ and, moreover, $w \in A_{p,q}$. By applying Hölder's inequality and the boundedness of T_b^m from $L_w^p(\mathbb{R}^n)$ to $L_w^q(\mathbb{R}^n)$ (Theorem 2.1) we obtain that

$$\begin{aligned}\frac{1}{|B|}\int_B |\sigma_1(x)|dx &\leq \frac{C}{|B|}\left(\int_B |T_b^m f_1(x) w(x)|^q\right)^{1/q} \left\|w^{-1}\chi_B\right\|_{q'} \\ &\lesssim \|b\|_{\Lambda(\delta)}^m \frac{\|fw\chi_B\|_p}{|B|} \left\|w^{-1}\chi_B\right\|_{q'}.\end{aligned}$$

Since $\frac{1}{p} = \frac{(m-1)\delta}{n} + \frac{1}{q} + \frac{\delta}{n}$, we can apply again Hölder's inequality and the fact that $w \in A_{q,\infty}$ to get

$$\begin{aligned}\frac{1}{|B|}\int_B |\sigma_1(x)|dx &\lesssim \|b\|_{\Lambda(\delta)}^m \|fw\|_{\frac{n}{(m-1)\delta}} |B|^{\delta/n} \frac{\left\|w^{-1}\chi_B\right\|_{q'}}{|B|^{1/q'}} \\ &\lesssim \|b\|_{\Lambda(\delta)}^m \|fw\|_{\frac{n}{(m-1)\delta}} |B|^{\delta/n} \|w\chi_B\|_\infty^{-1}.\end{aligned}$$

In order to estimate σ_2 we use the inequality

$$\frac{1}{|B|}\int_B |b(x)-b_B|^{m-k}\,dx \leq \|b\|_{\Lambda(\delta)}^{m-k} |B|^{\frac{\delta(m-k)}{n}}$$

and Lemma 4.7 to obtain

$$\begin{aligned}\frac{1}{|B|}\int_B |\sigma_2(x,u,k)|\,dx &\lesssim \|b\|_{\Lambda(\delta)}^k \|fw\|_{\frac{n}{(m-1)\delta}} \|w\chi_B\|_\infty^{-1} \frac{|B|^{\frac{\delta(k-m+1)}{n}}}{|B|} \int_B |b(x)-b_B|^{m-k}\,dx \\ &\lesssim \|b\|_{\Lambda(\delta)}^m \|fw\|_{\frac{n}{(m-1)\delta}} \|w\chi_B\|_\infty^{-1} |B|^{\delta/n}.\end{aligned}$$

Consequently, since

$$\begin{aligned}\sum_{k=0}^m c_k\left[\sigma_3(x,u,k) - (\sigma_3(\cdot,u,k))_B\right] &= [T_b^m f(x) - (T_b^m f)_B] - [\sigma_1(x) - (\sigma_1)_B] \\ &\quad - \sum_{k=0}^m c_k\left[\sigma_2(x,u,k) - (\sigma_2(\cdot,u,k))_B\right]\end{aligned}$$

by first assuming that $T_b^m f : L_w^{\frac{n}{(m-1)\delta}} \hookrightarrow \mathbb{L}_w(\delta)$, then

$$\begin{aligned}&\frac{1}{|B|}\int_B \left|\sum_{k=0}^m c_k\left[\sigma_3(x,u,k) - (\sigma_3(\cdot,u,k))_B\right]dx\right| \\ &\quad\leq \frac{1}{|B|}\int_B |T_b^m f(x) - (T_b^m f)_B|\,dx + \frac{2}{|B|}\int_B |\sigma_1(x)|dx \\ &\quad\quad + \sum_{k=0}^m c_k \frac{2}{|B|}\int_B |\sigma_2(x,u,k)|\,dx\end{aligned}$$

$$\lesssim \|b\|_{\Lambda(\delta)}^{m} \|fw\|_{\frac{n}{(m-1)\delta}} \|w\chi_B\|_{\infty}^{-1} |B|^{\delta/n}.$$

On the other hand, if we suppose that (2.4) holds, it is easy to see that $T_b^m f : L_w^{\frac{n}{(m-1)\delta}}(\mathbb{R}^n) \hookrightarrow \mathbb{L}_w(\delta)$. □

Proof of Theorem 2.12: We proceed by induction. By homogeneity we shall supposse that $\|b\|_{\Lambda(\delta)} = 1$. We must point out that the case $m = 0$ was already proved in [2]. As in the proof of Theorem 2.1 we have that

$$\|wT_b^m f\|_q \lesssim \|wM_{0,\gamma}^{\sharp}(T_b^m f)\|_q.$$

We shall now use the second part of Lemma 4.2, since we have that $K \in H_\Phi$. Thus, we obtain that

$$\|wT_b^m f\|_q \lesssim \sum_{j=0}^{m-1} \|wM_{\theta_j,\gamma}(|T_b^j f|)\|_q + \|wM_{m\delta,\widetilde{\Phi}} f\|_q.$$

From the hypothesis on the weight w and the Young function $\widetilde{\Phi}$, by Theorem 4.5 we know that $\|wM_{m\delta,\widetilde{\Phi}} f\|_q \lesssim \|fw\|_p$.

The proof now follows similar arguments as in the proof of Theorem 2.1. □

Proof of Theorem 2.14: Take f, f_1, f_2 and a_B as in the proof of Theorem 2.3, and define I_1 and I_2 likewise.

Since in I_1 we have only used the size condition S_0, the estimation is the same, by taking into account that $w^\beta \in A_{r/\beta,\infty}$ yields $w \in A_{r,\infty}$ for any $\beta \geq 1$.

For I_2 we proceed similarly but we have to use now that $K \in H_{\Phi,m}(\delta)$ with $\Phi^{-1}(t) \lesssim t^{\frac{\beta-1}{r}}$ for some $1 < \beta < r$ and all $t > 0$. We split the average into I_3 and I_4 as in the proof of Theorem 2.3. The last one can be controlled in the same form. The difference will be in I_3. Recall that

$$I_3 = \int_{(2B)^c} |b(x) - b(z)|^m |K(x-z) - K(y-z)| |f(z)| \, dz,$$

for $x \in B$.

By the definition of $\Lambda(\delta)$, we get that

$$I_3 \lesssim \|b\|_{\Lambda(\delta)}^m \int_{(2B)^c} |x-z|^{\delta m} |K(x-z) - K(y-z)| |f(z)| \, dz$$

$$\lesssim \|b\|_{\Lambda(\delta)}^m \sum_{j=1}^{\infty} |2^{j+1}B|^{\frac{\delta m}{n}} \int_{2^{j+1}B\setminus 2^jB} |K(x-z)-K(y-z)|\,|f(z)|\,dz.$$

Now, since $K \in H_{\Phi,m}(\delta)$, $w^\beta \in A_{r/\beta,\infty}$ and $\Phi^{-1}(t)t^{1/r}t^{1/(r/\beta)'} \lesssim t$, we can prodeed as in (4.3) with $k=m$ to obtain

$$I_3 \lesssim \frac{\|b\|_{\Lambda(\delta)}^m\,|B|^{\frac{\tilde{\delta}}{n}}\,\|fw\|_r}{\|w\chi_B\|_\infty}. \qquad \square$$

Proof of Theorem 2.15: We proceed as in the proof of Theorem 2.6. We must only use the corresponding hypothesis on the kernel, that guarantees the validity of Theorem 2.12 and Lemma 4.8, which are immediate from the fact that $S_0 \cap H_{\Phi,m}(\delta) \subset S_0 \cap H_\Phi$ (see Remark 2.11).

$\square$

References

[1] J. Alvarez and C. Pérez. Estimates with A_∞ weights for various singular integral operators. *Boll. Un. Mat. Ital. A (7)*, 8(1):123–133, 1994.

[2] A. Bernardis, E. Dalmasso, and G. Pradolini. Generalized maximal functions and related operators on weighted Musielak-Orlicz spaces. *Ann. Acad. Sci. Fenn. Math.*, 39(1):23–50, 2014.

[3] A. L. Bernardis, M. Lorente, F. J. Martí n Reyes, M. T. Martí nez, A. de la Torre, and J. L. Torrea. Differential transforms in weighted spaces. *J. Fourier Anal. Appl.*, 12(1):83–103, 2006.

[4] A. Cabral, G. Pradolini, and W. Ramos. Extrapolation and weighted norm inequalities between Lebesgue and Lipschitz spaces in the variable exponent context. *J. Math. Anal. Appl.*, 436(1):620–636, 2016.

[5] A. P. Calderón. Commutators of singular integral operators. *Proc. Nat. Acad. Sci. U.S.A.*, 53:1092–1099, 1965.

[6] M. Christ. *Lectures on singular integral operators*, volume 77 of *Reg. Conferences Series in Math.* Amer. Math. Soc., Providence, RI, 1990.

[7] R. Coifman and Y. Meyer. *Au delà des opérateurs pseudo-différentiels*, volume 57 of *Astérisque.* Société Mathématique de France, Paris, 1978. With an English summary.

[8] R. Coifman and Y. Meyer. Commutateurs d'intégrales singulières et opérateurs multilinéaires. *Ann. Inst. Fourier (Grenoble)*, 28(3):xi, 177–202, 1978.

[9] R. R. Coifman. Distribution function inequalities for singular integrals. *Proc. Nat. Acad. Sci. USA*, 69(10):2838–2839, 1972.

[10] R. R. Coifman and C. Fefferman. Weighted norm inequalities for maximal functions and singular integrals. *Studia Math.*, 51:241–250, 1974.

[11] D. Cruz-Uribe, L. Diening, and P. Hästö. The maximal operator on weighted variable Lebesgue spaces. *Fract. Calc. Appl. Anal.*, 14(3):361–374, 2011.

[12] D. Cruz-Uribe and A. Fiorenza. *Variable Lebesgue spaces. Foundations and harmonic analysis.* Applied and Numerical Harmonic Analysis. Birkhäuser/Springer, Heidelberg, 2013.

[13] D. Cruz-Uribe, J. Martell, and C. Pérez. Sharp two-weight inequalities for singular integrals, with applications to the Hilbert transform and the Sarason conjecture. *Adv. Math.*, 216(2):647–676, 2007.

[14] D. Cruz-Uribe and L.-A. D. Wang. Extrapolation and weighted norm inequalities in the variable Lebesgue spaces. *Trans. Amer. Math. Soc.*, 369(2):1205–1235, 2017.

[15] L. Diening, P. Harjulehto, P. Hästö, and M. Růžička. *Lebesgue and Sobolev spaces with variable exponents*, volume 2017 of *Lecture Notes in Math.* Springer, Heidelberg, 2011.

[16] J. García-Cuerva and J. L. R. de Francia. *Weighted norm inequalities and related topics*, volume 116 of *North-Holland Mathematics Studies.* North-Holland Publishing Co., Amsterdam, 1985. Notas de Matemática [Mathematical Notes], 104.

[17] J. García-Cuerva, E. Harboure, C. Segovia, and J. L. Torrea. Weighted norm inequalities for commutators of strongly singular integrals. *Indiana Univ. Math. J.*, 40(4):1397–1420, 1991.

[18] W. Guo, J. He, H. Wu, and D. Yang. Boundedness and compactness of commutators associated with lipschitz functions. preprint arXiv:1801.06064.

[19] E. Harboure, O. Salinas, and B. Viviani. Orlicz boundedness for certain classical operators. *Colloq. Math.*, 91(2):263–282, 2002.

[20] E. Harboure, C. Segovia, and J. L. Torrea. Boundedness of commutators of fractional and singular integrals for the extreme values of p. *Illinois J. Math.*, 41(4):676–700, 1997.

[21] I. Holmes and B. Wick. Two weight inequalities for iterated commutators with calderón-zygmund operators. preprint arXiv:1509.03769.

[22] R. L. Jones and J. Rosenblatt. Differential and ergodic transforms. *Math. Ann.*, 323(3):525–546, 2002.

[23] O. Kováčik and J. Rákosník. On spaces $L^{p(x)}$ and $W^{k,p(x)}$. *Czechoslovak Math. J.*, 41(4):592–618, 1991.

[24] D. S. Kurtz and R. L. Wheeden. Results on weighted norm inequalities for multipliers. *Trans. Amer. Math. Soc.*, 255:343–362, 1979.

[25] A. Lerner, S. Ombrosy, and I. Rivera-Ríos. Commutators of singular integrals revisited. preprint arXiv:1709.04724.

[26] M. Lorente, J. M. Martell, M. S. Riveros, and A. de la Torre. Generalized Hörmander's conditions, commutators and weights. *J. Math. Anal. Appl.*, 342(2):1399–1425, 2008.

[27] M. Lorente, M. S. Riveros, and A. De la Torre. Weighted estimates for singular integral operators satisfying Hörmander's conditions of Young type. *J. Fourier Anal. Appl.*, 11(5):497–509, 2005.

[28] Y. Meng and D. Yang. Boundedness of commutators with Lipschitz functions in non-homogeneous spaces. *Taiwanese J. Math.*, 10(6):1443–1464, 2006.

[29] B. Muckenhoupt and R. Wheeden. Weighted norm inequalities for fractional integrals. *Trans. Amer. Math. Soc.*, 192:261–274, 1974.

[30] B. Muckenhoupt and R. L. Wheeden. Weighted bounded mean oscillation and the Hilbert transform. *Studia Math.*, 54(3):221–237, 1975/76.

[31] J. Musielak. *Orlicz spaces and modular spaces*, volume 1034 of *Lecture Notes in Math.* Springer-Verlag, Berlin, 1983.

[32] C. Pérez. Weighted norm inequalities for singular integral operators. *J. Lond. Math. Soc. (2)*, 49(2):296–308, 1994.

[33] C. Pérez. Endpoint estimates for commutators of singular integral operators. *J. Funct. Anal.*, 128(1):163–185, 1995.

[34] C. Pérez. Sharp estimates for commutators of singular integrals via iterations of the Hardy-Littlewood maximal function. *J. Fourier Anal. Appl.*, 3(6):743–756, 1997.

[35] G. G. Pradolini and W. A. Ramos. Characterization of Lipschitz functions via the commutators of singular and fractional integral operators in variable Lebesgue spaces. *Potential Anal.*, 46(3):499–525, 2017.

[36] M. Ramseyer, O. Salinas, and B. Viviani. Fractional integrals and Riesz transforms acting on certain Lipschitz spaces. *Michigan Math. J.*, 65(1):35–56, 2016.

[37] J. L. Rubio de Francia, F. J. Ruiz, and J. L. Torrea. Calderón-Zygmund theory for operator-valued kernels. *Adv. in Math.*, 62(1):7–48, 1986.

[38] D. K. Watson. Weighted estimates for singular integrals via Fourier transform estimates. *Duke Math. J.*, 60(2):389–399, 1990.