Active speed role in opinion formation of interacting moving agents

G. R. Terranova, J. A. Revelli and G. J. Sibona

Instituto de Fsica Enrique Gaviola (IFEG), CONICET
FaMAF, Universidad Nacional de Córdoba, Medina Allende s/n X5000HUA, Córdoba, Argentina

PACS 05.40.-a – Stochastic processes
PACS 87.23.Ge – Dynamics of social systems
PACS 05.45.-a – Nonlinear dynamics

Abstract –We propose a general non linear analytical framework to study the evolution of the opinion state of a population of moving individuals. This novel scheme allows us to study a broad range of social phenomena, like for example the influence of agent interaction dynamics in the opinion formation or the inclusion of different individual's idiosyncrasies. We consider societies composed by agents who adopt one of the n possible opinions or internal states. The opinion state may only be modified while the agent keeps contact with other one. In general, this framework could be solved numerically, and, for some special perturbative cases, it is possible to find analytical steady states. In order to check our scheme for different social conventions, we implement computational simulations of an ensemble of self-propelled agents, finding a good agreement between theory and simulation results. We found, for slow society kinetics in all the cases studied, that there exist a shift of the opinion populations towards the moderate opinion states. This suggest that active speed can be understood as a parameter measuring the social temperature of the community.

Introduction. – In the last years, statistical mechanics has extended its success in physics to other knowledge fields such as the behavioral, social, and political sciences [1–4]. In this context, tools, developed to describe systems composed by a great number of interacting elementary entities, has been applied to study the social behavior of human societies. Indeed, there are several works that analyze social collective phenomena as a result of basic agent-agent interactions [5–7, 13, 14]. As for example of such models we can mention the work of Galam et al. [7] who had studied the effects of "contrarians" (agents who adopt the opposite position to the prevailing choice of others) on the dynamics of opinion forming using a two state opinion model. Vazquez et al [14] examined the evolution of individual's opinions in an interacting population of leftists, centrists and rightists. They found, in the mean field limits, the probability to reach consensus or to reach a steady mixed state among the different opinions. Summing up, opinion dynamics modeling has become a challenging field to study the evolution of opposite ideas in communities, analyzing the emergence of a collective consensus or a steady state of mixed opinions.

Previous works in this area [2, 3, 10] generally consider closed systems, meaning that the same pool of actors could interact during the evolution. This can be interpreted by assuming that the inherent dynamical timescales (e.g. opinion convergence time) are much faster than those associated to the processes responsible for a modification of the group composition (e.g. migration, birth/death). Such an implicit assumption is certainly correct in modern society, where mean lifetime has been prolonged and communication interactions among individuals have had an explosive increase.

Recently there has been an increase in the studies of many moving individuals [9, 11, 12, 15], nevertheless there are scarce works that take into account free moving agents to study social systems [15–18]. Although there exist many opinion formation models to characterize community behavior or decision making, most of them are based on networks [19–23]; i.e considering agents that occupy fixed positions into a lattice or space. Nevertheless, more realistic situations can be studied considering models that take into account individual mobility. As an example, ideas spreading through social contacts are strongly influenced by mobility. As an example, Sousa et.al [9] modified a Sznajd model in order to incorporate agent mobility. By making synchronous updating, they found that this model always reach a consensus steady state. In a recent work [24] we show the importance of movement and agent-agent interactions dynamics in the outcome of an epidemic disease. That work could be extended to the transmission of other excitations that take place among the agents. In particular, opinion formation represents a complete different scenario due to the involved transmission dynamics. In a disease, the infected agent can only pass the disease to a susceptible, meanwhile in opinion formation both agent can modify the inner state of the other.

In this letter we present a general framework to study the opinion state evolution of a community, based on the interactions among the agents. We derive an analytical expression to compute the evolution of the population sharing a given idea from a discrete set of them. Our scheme permits to analyze many different influence scenarios, ranging from a few opinion groups to a complex set of distinct believes influencing each other in diverse ways. In this work, we find exacts results for the steady opinion state populations for some particular cases. In order to analyze the influence of spatial agent dynamics, like individual mobility and interacting time, we implement a computational model of self propelled agents. Furthermore, this model allows us to validate the theoretical and numerical results found with our scheme for different social settings.

The rest of this paper is organized as follows: In Sec. II we introduce the analytical model and, in Sec. III, we present exact results for special cases. In Sec. IV we introduce the self propelled agents model, comparing in Sec V the computational results of the simulations with the numerical solution of the equations introduced in Sec. II. Finally, in Sec. VI we conclude with a short discussion of the results.

Analytical model. – We consider a society composed of N individuals (agents), each of them having an inner state representing its opinion. In order to fix ideas, we assume only two possible opposite doctrines or beliefs, A and B, but the scheme could be extended to any number of ideas. All agents subscribe to only one of these ideas but with a certain degree of confidence, from m possible. Then, the set of available agent inner states could be represented in a one-dimensional discrete space of $2m$ different opinion states OS. The number of individuals sharing a specific degree of confidence in an opinion is defined by P_i, $(i = 1, ..., 2m)$, corresponding the OS 1 and $2m$ to the strongest conviction on opinions A and B respectively, and m and $m+1$ to the weakest positions (see Fig.1. Adjacent sites in the lattice corresponds to close OS. In other words agents in the states i and $i+1$ have different but similar OS, being possible for them to be convinced to change from one state to the other. To facilitate the comprehension, in what follows, we will refer to this lattice

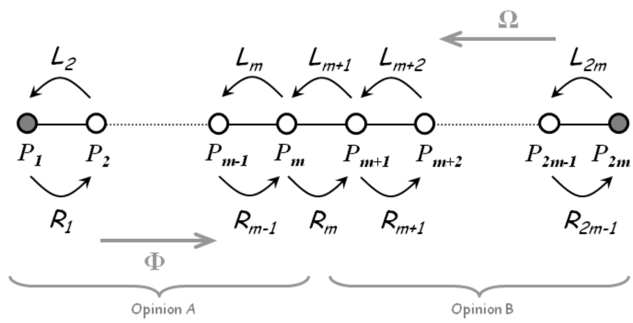

Figure 1: One-dimensional representation of the opinion states corresponding to populations P_i. The arrows R_i (L_i) represent the fraction of agents belonging to P_i changing its opinion to the right (left). The gray arrows indicate the directions of the change of opinions represented by the matrices $\mathbf{\Phi}$ and $\mathbf{\Omega}$.

(Fig. 1) when we talk about first "neighboring" OS, or opinion change to the "right" or to the "left" (no matter which doctrines are A and B). We also assume that the population is conservative, i.e. the $\sum_i P_i = N$ at all times. Furthermore, this means that all N individuals of the system are always in a given opinion state.

Agents moves in a finite 2D space, interacting with others by direct contact. This interaction could not only change its movement dynamics but also its OS. We will only consider jumps to neighboring states in the opinion space assuming that, in general, individuals change their opinion smoothly and the characteristic time to change opinion is larger than space dynamics characteristic time. In other words, the new opinion of an agent after an encounter can not be radically different from the previous one.

Furthermore, we suppose that the probability that an agent changes its opinion during an encounter depends on the opinion of both interacting agents. Generalizing the process, we consider that changes of opinion can be in either direction, nearing or separating the opinion state of the interacting agents. This allows us to model not only a positive exchange of ideas (persuasion), but also an aversion among them (rejection). Then, we define two $2m \times 2m$ matrices, $\mathbf{\Phi}$ and $\mathbf{\Omega}$, with coefficients ϕ_{ij} (ω_{ij}) that represent the frequency of opinion change to the right (left), i.e. the probability per unit time that an agent with OS j changes it to $j+1$ ($j-1$), while interacting with another in state i. Accordingly, the agents at the radical populations, P_1 and P_{2m}, can only change its opinion in one direction (see Fig. 1), weakening its confidence. Then, the coefficients $\phi_{i,2m} = \omega_{i,1} = 0$, $\forall i$; representing the impossibility to have a more radical belief in that idea.

In order to obtain analytical results we only consider binary interactions between agents. This way we can represent with $R_i = \sum_k \phi_{ki} P_k$ and $L_i = \sum_k \omega_{ki} P_k$ the fractions of agents belonging to opinion i that change their opinions to the right and left, respectively, in an encounter with any other agent. Thus, the master equation for P_i

can be written as

$$\dot{P}_i = R_{i-1}P_{i-1} + L_{i+1}P_{i+1} - (R_i + L_i)\, P_i, \qquad (1)$$

If we define $\mathbf{R} = (R_1, ..., R_{2m-1}, 0)$ and $\mathbf{L} = (0, L_2, ..., L_{2m})$, then it is straightforward that $\mathbf{R} = \mathbf{P}^T\mathbf{\Phi}$ and $\mathbf{L} = \mathbf{P}^T\mathbf{\Omega}$, where $\mathbf{P} = (P_1, ..., P_{2m})$ and $\mathbf{P}^T$ is the transposed vector of $\mathbf{P}$.

Then, the master equation, having all information needed to compute the evolution of the system, can be written as:

$$\dot{\mathbf{P}} = \mathbf{T} \times \mathbf{P}, \qquad (2)$$

where $\mathbf{T}$ is a 3-diagonal square matrix of dimension $2m$ with coefficients $T_{i,i} = -(R_i + L_i)$ (main diagonal), $T_{i,i+1} = L_{i+1}$ (upper diagonal), $T_{i+1,i} = R_i$ (lower diagonal) and $T_{ij} = 0$ elsewhere. The matrix $\mathbf{T}$ has just one lower and one upper diagonal because we only allowed changes of opinions to neighboring sites. If we want to study more radical changes of the OS, at a "distance" d in the opinion lattice (to second neighbors, etc), then the matrix $\mathbf{T}$ will have d lower and upper non-zero diagonals.

Analytical Results. – We can always find numerical results of Eq.2, nevertheless it is possible to obtain exact solutions for some special situations. We present here some of them:

Case 1: The simplest possible situation is that the probability to change opinion in an interaction is always the same, independent of the OS of the agents involved. Then, for all values of i: $\phi_{ij} = \phi$ for $j \neq 2m$, and, $\omega_{ij} = \omega$ for $j \neq 1$ (remember that by definition, $\phi_{i2m} = \omega_{i1} = 0$). If $\phi = \omega$, the probability that an agent changes its opinion in any direction is the same. Then each OS will have the same mean number of agents, i.e., $P_i = N/2m$.

If $\phi > \omega$ ($\phi < \omega$) the probability to change the OS to the right (left) is increased by a factor ϕ/ω. In this case, the steady state populations ensue that $P_i = (\phi/\omega)P_{i-1}$. Indeed, it is just like the OS of agents are influenced by an uniform bias in a given direction. This situation would occur when one of the ideas is much easier to explain than the another, or there exists a cultural bias that promotes the acceptance of one belief over the other.

Case 2: An interesting situation is to consider that the sum of the rates for all possible reactions during an interaction, $\Sigma = \phi_{ij} + \phi_{ji} + \omega_{ij} + \omega_{ji}$, is constant regardless of the agent states i and j. Then, considering for simplicity $m = 2$ (two different levels of confidence for each opinion), the matrices $\mathbf{\Phi}$ and $\mathbf{\Omega}$ are defined as

$$\mathbf{\Phi} = \begin{pmatrix} 1/2 & 1/3 & 1/3 & 0 \\ 1/3 & 1/4 & 1/4 & 0 \\ 1/3 & 1/4 & 1/4 & 0 \\ 1/2 & 1/3 & 1/3 & 0 \end{pmatrix},$$

$$\mathbf{\Omega} = \begin{pmatrix} 0 & 1/3 & 1/3 & 1/2 \\ 0 & 1/4 & 1/4 & 1/3 \\ 0 & 1/4 & 1/4 & 1/3 \\ 0 & 1/3 & 1/3 & 1/2 \end{pmatrix}, \qquad (3)$$

For the chosen parameters we find $\mathbf{P} = 1/58(12, 17, 17, 12)$, i.e., the extreme or radical populations, P_1 and P_4, are lower than those with moderate ideas. This fact can be easily observed from the matrices by noting a larger probability to reach a moderate opinion than to leave it (17/6 to 14/6), assuming an equal opinion agent population. The opposite behavior occurs for extreme opinions (7/6 to 10/6). These difference produces an increase in the moderate opinion populations to equilibrate the flux among opinions. Interestingly the steady states do not depend on the value of Σ, it just introduces a rescaling of the transient time to reach the asymptotic regime.

Case 3: Another interesting situation is to consider that the probabilities to change the OS depends on the belief of the agent. When the agents, i and j, has the same opinion (for example both A or B in Fig. 1) $\phi_{ij} = \omega_{ij} = \eta$. If not, there are different rates of persuasion toward each opinion: $\phi_{ij} = \eta + \delta$, and $\omega_{ij} = \eta + \epsilon$ for moderate i OS, and $\phi_{ij} = \eta + \epsilon$, and $\omega_{ij} = \eta + \delta$ for radical i OS. In other words, the communication among individuals of different beliefs induce a different opinion change, depending on the OS of agents. For this case with $m = 2$ the populations splits into two states:

$$P_1 = P_2 = \frac{2\eta+\epsilon}{2(4\eta+\epsilon+\delta)} \quad P_3 = P_4 = \frac{2\eta+\delta}{2(4\eta+\epsilon+\delta)}\ , \qquad (4)$$

It is interesting to observe that if $\epsilon = \delta$ we return to case 1 (all populations equivalent), besides the different communication that there still exists.

Computational model. – In order to check the accuracy of our analytical approach considering just binary interactions, and to analyze more complex and interesting situations, we have performed agent-based simulations for the evolution of the opinion dynamics in a moving population. We follow the same ideas used in a previous work [24] to simulate the kinetic dynamics of an excitable mobile agent system. The opinion (excitation) is an inner state, that can only be modified by keeping physical contact with other individuals for a finite period of time. We consider agents as self-propelled disks of radius r, moving in 2D space with periodic boundary conditions. In absence of interactions, agents advance at a constant speed in the direction of the active movement, that changes at Poissonian distributed times. The collisions between two or more agents are mediated by a repulsive soft-core two-body potential that penalizes agent overlapping. During each interaction (a relatively slow process that depends on the agent propulsion) agents keep physical contact for a non-vanishing period of time, allowing a communication, or ideas exchange, among the agents which can modify its inner OS.

It is worth remarking here that we have allowed only one change of OS per encounter. The goal was to simulate real world situations of individual interactions, where only one person "wins" a discussion by influencing the other with its speech.

We had previously shown that in mobile agent systems the coupling between mean contact time and agent-agent collision rate is crucial to understand the excitation dynamics [24]. This is also the case for opinion formation. We have found previously [25] that the mean contact time is inversely proportional to the agent propulsion speed, v. Since v also rules out the collision rate among the agents, then the active speed is a crucial parameter which governs the asymptotic opinion regime.

In order to compare numerical and analytical results, we have to write the coefficients ϕ_{ij} and ω_{ij} in terms of the agent-agent collision rate $\alpha = v\sigma_0\rho$. Here, $\sigma_0 = 4r$ is the scattering cross section of agents, and $\rho = N/V$ is the agent density, considering $V = L \times L$ the moving area of characteristic size L. In general, α could depend on the opinion states of the interacting agents, due to a different kinetics depending on the OS, but for simplicity we assumed it to be independent of i and j. Then, the rates to change the OS are $\phi_{ij} = \alpha \times \phi'_{ij}$ and $\omega_{ij} = \alpha \times \omega'_{ij}$, where ϕ'_{ij} and ω'_{ij} are the probabilities that an agent changes its opinion in a given direction during a contact event. Following a previous work [24] we can write ϕ'_{ij} and ω'_{ij} as a function of v as:

$$\phi'_{ij} = \left[1 - e^{(-\lambda\Sigma_{ij})}\right] \times \frac{\phi''_{ij}}{\Sigma_{ij}}, \qquad \omega'_{ij} = \left[1 - e^{(-\lambda\Sigma_{ij})}\right] \times \frac{\omega''_{ij}}{\Sigma_{ij}}, \tag{5}$$

where λ denotes the mean duration of the collision event, ϕ''_{ij} and ω''_{ij} represent the probability per unit time that an agent changes its opinion during an encounter, and $\Sigma_{ij} = \phi''_{ij} + \omega''_{ij} + \phi''_{ji} + \omega''_{ji}$.

Numerical Results. – We present here both numerical solutions of the equations presented in Sec.II and computational simulations performed with the model described in Sec.IV. We considered $m = 2$, studying the different cases presented in Sec.III as a function of the active speed v. In Fig.2 are shown results for the simplest case, where all the probabilities per unit time that an agent changes its opinion during an interaction are the same constant, regardless the opinion of the other agent (Case 1). The steady state populations are depicted as a function of the active speed for a system with parameters given in [26]. There exists a good agreement between simulations and the results obtained solving numerically Eq. 2. As was already mentioned $\lambda \sim 1/v$, existing a strong dependence of the coefficients ϕ_{ij} and ω_{ij} with active speed. For a very energetic agent regime ($v >> max(\Sigma_{ij})$) the matrix coefficients are $\phi_{ij} = \alpha\lambda\phi''_{ij}$ and $\omega_{ij} = \alpha\lambda\omega''_{ij}$. In particular, for the system studied, the agent dynamics parameters α and λ do not depend on the OS. Then the steady state populations reduces to the analytical results already obtained for Case 1 with $\phi = \omega$ (all populations goes to 0.25). The mentioned parameters just introduce a rescaling of the transient time before reaching the asymptotic values for the different opinion populations. In the opposite v limit, for $v << min(\Sigma_{ij})$, the matrix coefficients tend to $\phi_{ij} = \alpha\phi''_{ij}/\Sigma_{ij}$ and $\omega_{ij} = \alpha\omega''_{ij}/\Sigma_{ij}$. Each coefficient depends on Σ_{ij}, the sum of all different opinion changes that can occur in a collision between agents i and j. Because in this case all opinion change rates have the same value, the matrices $\mathbf{\Phi}$ and $\mathbf{\Omega}$ are proportional to Eq.3 of Case 2 in Sec.II. As was mentioned for the analytical results, the steady state populations presents a shift favoring the moderate opinions at the expense of the radical ones due to an increase in the probability to reach moderate opinions than to leave them.

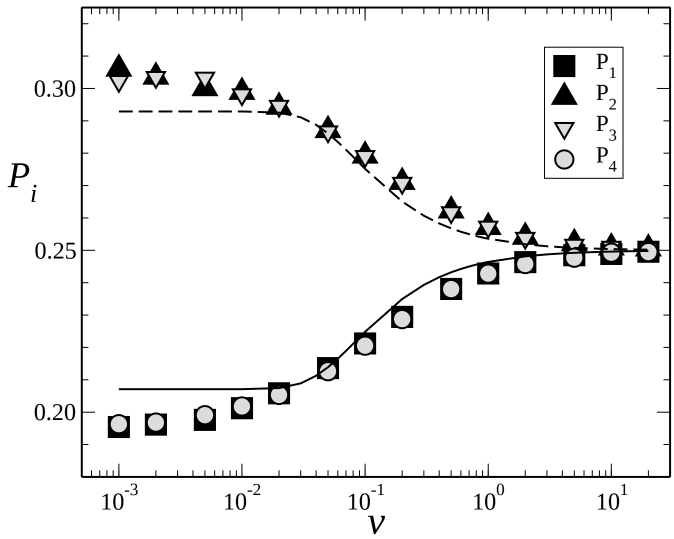

Figure 2: Steady state agent populations as a function of the agents active speed for the case in which the opinion change rate is a constant. The rest of parameters are as [26]. Lines correspond to numerical results obtained from solving Eq. (2).

Another interesting situation is analyzed in Fig.3, where the probability to change opinion in an interaction is a constant depending on the direction of the opinion change. As in case 1, for all values of i: $\phi_{ij} = \phi$ and $\omega_{ij} = \omega$ with $\phi \neq \omega$. The parameters are as in Fig.2 [26] except that $\omega = 0.1$. The analytical result found previously agrees with the simulation at high active speeds, where there is a bias towards populations with OS to left in the opinion space. In the low speed limit the effect observed in Fig.2 repeats, increasing the moderate opinions populations at the expense of the extreme positions.

A similar situation occurs in case that the probabilities to change the OS depends on the belief of the agent (case 3 in Sec.II) presented in Fig.4. Here $\eta = 0.05$, $\delta = 0.01$ and $\epsilon = 0.1$, implying a tendency to the left in the opinion diagram when there is an encounter between agents of different ideas. As is observed in the previous cases, there is a good agreement between simulations and analytical results at high active speeds, but, at the opposite kinetic limit, there is an increase (decrease) in the population of a moderate (radical) opinion.

This situation is more interesting even if we consider a the opposite situation of the case 3, i.e. there is a difference

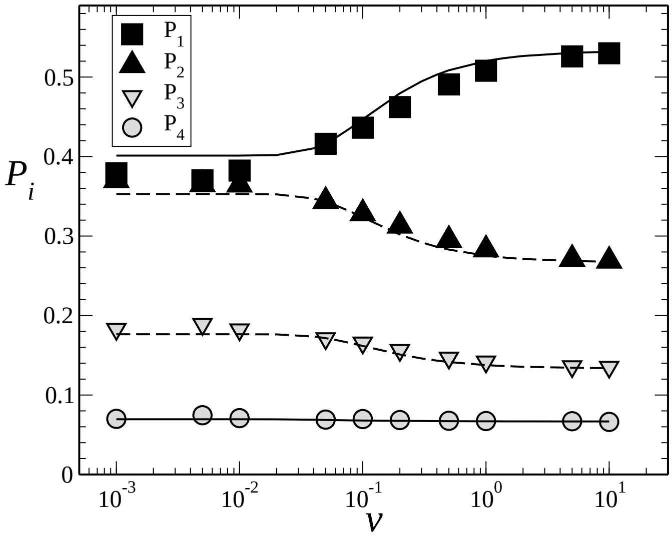

Figure 3: Steady state agent populations as a function of the agents active speed for the cese in which the change rate depends only on the direction. $\omega = 0.1$ and the rest of parameters are as [26]. Lines correspond to numerical results obtained from solving Eq. (2).

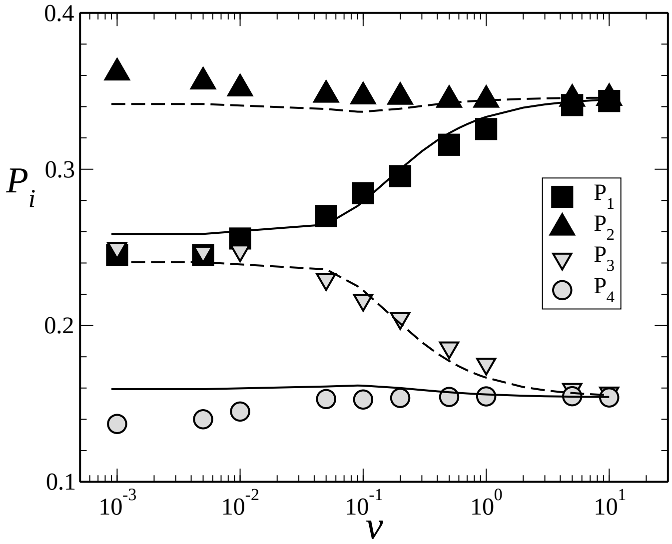

Figure 4: Steady state agent populations as a function of the agents active speed for the case that the opinion rate depends on the belief of the agents (Case 3). Here $\eta = 0.05$, $\delta = 0.01$ and $\epsilon = 0.1$, and the rest of parameters are as [26]. Lines correspond to numerical results obtained from solving Eq. (2).

in the communication among individuals sharing the same belief , showing a tendency toward one of them (Case 4). Then, if there is an interaction of two agents, i and j, with the same opinion (both A or B) $\phi_{ij} = \phi$ and $\omega_{ij} = \omega$. If they do not share the belief then the rates are the same, $\phi_{ij} = \omega_{ij} = \eta$. If $\phi = 0.09$, $\omega = 0.05$ and $\eta = 0.15$ there is a competition among the opinion depending on how well mixed is the population. In Fig.5 it is clear that the tendency to the right is reduced at low active speed. Moreover, the shift toward moderate opinion observed in previous cases change the majority population in case 4.

Concluding remarks. – We introduce a novel general framework to study opinion formation dynamics in a community involved into social debate. In our model, the changes of the individual opinions arise exclusively through direct interaction among them. In this way, the contact time distribution and agents mobility become the significant parameters that rule the steady state opinion. The influence dynamics can take into account rejections of opinions, in contrast to the conventional persuasion scheme.

The evolution of the different opinion populations is described through a set of coupled master equations. It is important to remark that our theoretical scheme only considers binary collisions, i.e. contacts between two agents. As a consequence, the analytical expression Eq. 2 is not a linear equation in $\mathbf{P}$ because $\mathbf{T} = \mathbf{T}(\mathbf{P})$. A generalization of the present model taking into account crowded contacts is possible, but is difficult to obtain exact solutions from the complex analytical expressions.

To simplify the matrix complexity we have considered in this work just two opposite opinions, each having an internal degree of confidence, stressing in this way the relevance of the analytical scheme. However, the model could be extended to consider many opposite ideas in order to analyze more complex situations. A generalization to many different opinions involves letting radical changes of opinion (jumps to second neighbors or still larger ones), which just implies the construction of larger matrices with many off-diagonal terms. Now the opinion space is not linear and the number of neighbor states increases accordingly the different ideas in conflict. This could be the situation of a political election with three or more candidates.

We have been able to obtain not only numerical results, but analytical ones for some particular cases. The obtained results show that, even for the case that the probabilities per unit time of an agent to change its opinion during a collision is independent of the other agent state, the society kinetics, expressed through the agent active speed, is crucial at the moment of defining the leading opinion. For all the cases studied there exist, for slow society kinetics, a shift of the opinion populations towards the intermediate states, the moderate OS, even in the special case of non biased situations. This is a consequence of a biased rescaling (Eq.5) that increase (decrease) the probability to reach a moderate (radical) OS. This suggest that active speed can be understood as a parameter sensing social temperature. For societies in crisis the social debate increase, leading to a larger number of interactions among the individuals, ending in a radical division of the community. In more stable communities, political discussions are less frequent favoring the moderate OS.

Moreover, simulations show that there exists a crucial link between agent mobility and the spacial distribution

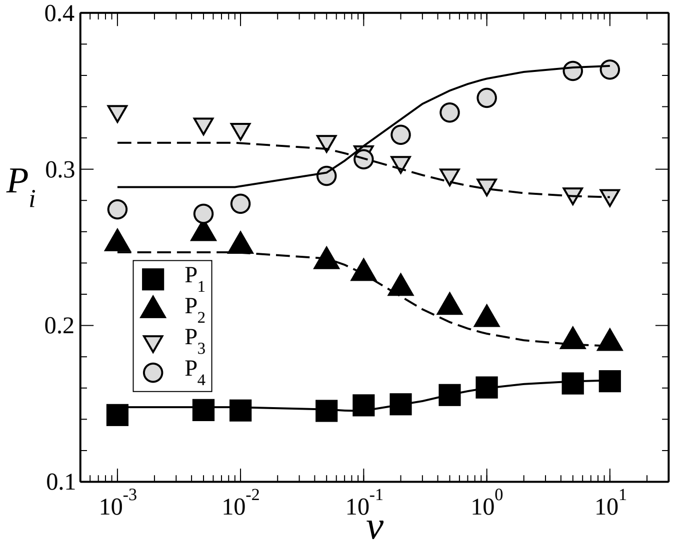

Figure 5: Steady state agent populations as a function of the agents active speed for each of the four opinion states P_i. The parameters are as [26] except that $\phi = 0.09$, $\omega = 0.05$ and $\eta = 0.15$. Lines correspond to numerical results obtained from solving Eq. (2).

of opinions. For slow mobility, the agents surroundings change slowly, creating zones with majority of one or other opinion. For high mobility, there are no leading opinion zones, and the system is completely mixed [27]. As a consequence the steady state opinion populations will follow the analytical results only at high enough active speeds. This is the reason of the difference among simulations and numerical results observed in all cases at slow active speeds. It is worth mentioning that this behavior is a consequence of letting just first opinion neighbor jumps. If we allowed agents to perform more radical changes of ideas, then the mixing of agents would be increased, leading to an equally distributed opinion population also for slow velocities.

The proposed framework can be used to understand the evolution of social trends and to know the most relevant facts that spread or suppress ideas. This knowledge would become important in many social behaviors, like for example, persuading a population to evacuate a certain area due to an imminent danger, or divulging information about how avoid disease transmission.

Summing up, we present a general and simple framework that allows to tackle the study of many different aspects of a society involved in opinion formation. In particular it could be used to introduce in the description the spatial dynamics and interactions among the agents.

* * *

We thank Fernando Peruani for stimulating discussions. This work was partially supported by the National Council for Scientific and Technical Research (CONICET) of Argentina and the National University of Córdoba (Argentina).

References

[1] R. Albert and A-L. Barabasi, Rev. Mod. Phys. **74** (2002) 47.
[2] W. Weidlich, "Sociodynamics: A Systematic Approach to Mathematical Modelling in the Social Sciences", Harwood Academic Publishers 2000.
[3] S. Galam, Int. J. Mod. Phys, C **19** (2008) 409.
[4] R. Axelrod, "The Complexity of Cooperation", Princeton University Press 1997.
[5] T.M. Liggett, "Interacting Particle Systems", Springer-Verlag, New York 1985.
[6] P.L. Krapivsky and S. Redner, Phys. Rev. Lett. **90** (2003) 238701.
[7] S. Galam, Physica A **333** (2004) 453.
[8] S. Galam, B. Chopard, A. Masselot and M. Droz. Eur. Phys. J. B **4** (1998) 529.
[9] A. O. Sousa, T. Yu-Song and M. Ausloos. Eur. Phys. J. B **66** (2008) 115.
[10] S. Galam. Eur. Phys. J. B **25** (2002) Rapid note 403.
[11] F. Ginelli, F. Peruani, M. B ar and H. Chaté, Phys. Rev. Lett. **104** (2010) 184502.
[12] F. Schweitzer, "Brownian Agents and Active Particles: Collective Dynamics in the Natural and Social Sciences" Springer-Verlag, Berlin 2007.
[13] A. Pluchino, V. Latora and A. Rapisarda, Eur. Phys. J. B. **50** (2006) 169.
[14] F. Vazquez and S. Redner, J. Phys. A: Math. Gen. **37** (2004) 8479.
[15] L. Boudin, R. Monaco and F. Salvarani , Phys. Rev. E **81** (2010) 036109.
[16] D. Helbing, Physica A **196** (1993) 546.
[17] D. Helbing, J. Math. Sociol. **19** (1994) 189.
[18] D. Helbing and P. Molnár, Phys. Rev. E. **51** (1995) 4282.
[19] D.J. Watts and S.H. Strogatz, Nature **393** (1998) 440.
[20] C. Moore and M.E.J. Newman, Phys. Rev. E **61** (2000) 5678.
[21] C. Castellano, M.A. Muñoz and R. Pastor-Satorras, Phys. Rev. E **80** (2009) 041129.
[22] N. Crokidakis and C. Anteneodo, Phys. Rev. E **86** (2012) 061127.
[23] S. R. Souza and S. Gonçalves, Phys. Rev. E **85** (2012) 056103.
[24] F. Peruani and G.J. Sibona, Phys. Rev. Lett. **100** (2008) 168103.
[25] G.J. Sibona, F. Peruani and G. Terranova, In BIOMAT 2011 Int. Symp. on Mathematical and Comp. Biology. Ed. R. Mondaini, World Scientific 2011, p. 241.
[26] Simulation parameters: Number of permitted opinions $n = 4$, agent number $N = 4000$, size box $L = 500$, radius of particles $r = 1$ and $\phi''_{k4} = \omega''_{1k} = 0$ for $k = 1, 2, 3, 4$ and $\phi''_{ij} = \omega''_{ij} = 0.05$ for the other coefficients of the matrices $\mathbf{\Phi}$ and $\mathbf{\Omega}$.
[27] J.A. Revelli, G.J. Sibona and G.R. Terranova. To be published.