Competition between surface relaxation and ballistic deposition models in scale free networks.

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Abstract – In this paper we study the scaling behavior of the fluctuations in the steady state $W_S$ with the system size $N$ for a surface growth process given by the competition between the surface relaxation (SRM) and the Ballistic Deposition (BD) models on degree uncorrelated Scale Free networks (SF), characterized by a degree distribution $P(k) \sim k^{-\lambda}$, where $k$ is the degree of a node. It is known that the fluctuations of the SRM model above the critical dimension ($d_c = 2$) scales logarithmically with $N$ on euclidean lattices. However, Pastore y Piontti et. al. [A. L. Pastore y Piontti et. al., Phys. Rev. E 76, 046117 (2007)] found that the fluctuations of the SRM model in SF networks scale logarithmically with $N$ for $\lambda < 3$ and as a constant for $\lambda \geq 3$. In this letter we found that for a pure ballistic deposition model on SF networks $W_S$ scales as a power law with an exponent that depends on $\lambda$. On the other hand when both processes are in competition, we find that there is a continuous crossover between a SRM behavior and a power law behavior due to the BD model that depends on the occurrence probability of each process and the system size. Interestingly, we find that a relaxation process contaminated by any small contribution of ballistic deposition will behave, for increasing system sizes, as a pure ballistic one. Our findings could be relevant when surface relaxation mechanisms are used to synchronize processes that evolve on top of complex networks.

In the last decade the study of complex networks received much attention because many processes take place on top of these kinds of structures. Here, we consider networks with scale free (SF) topologies characterized by a degree distribution $P(k) \sim k^{-\lambda}$, where $k$ is the degree or number of connections that a node can have with $k_{max} \geq k \geq k_{min}$, where $k_{max}$ and $k_{min}$ are the maximal and minimal degree respectively, and $\lambda$ measures the heterogeneity of the distribution [1,2]. Historically, the research on complex networks was mainly focused on how the topology affects processes that evolve on top of them, such as epidemic spreading [3], traffic flow [1,5], cascading failures [6,7] and synchronization [8,10]. Recently, it was shown that the scaling behavior of processes, such as synchronization and jamming [11,12], are fully governed by the topology of the underlying network. Synchronization processes are very important in many real situations such as supply-chain networks based on electronic transactions [13], neuronal networks [14] and financial transaction between traders [15]. One

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of the most successful attempts to model synchronization phenomenon is to map them into a non equilibrium surface growth \[16\] where the relevant magnitude that represents the departure from synchronization is the dispersion or fluctuations of some scalar field \(h\) over the nodes of a network given by \(W = \left\{1/N \sum_{i=1}^{N} (h_i - \langle h \rangle)^2 \right\}^{1/2},\) where \(h_i\) represents the scalar on node \(i, \langle h \rangle\) is the mean value, \(N\) is the system size and \{\} denotes an average over network configurations. Synchronization problems deal with the optimization of the fluctuations in the steady state, being the system optimally synchronized when those fluctuations are minimized. On the other hand a system is called scalable if their fluctuations do not depend, or depend weakly, on the system size.

Many dynamic synchronization processes can be described by simple growth models. One of the most successful models of surface growth with relaxation is the Family model \[17\] (SRM) that represents very well film growth either by vapor or chemical deposition. Other very important model is the Ballistic Deposition (BD) \[18\] that represents sedimentation (SRM) that represents very well film growth either by vapor or chemical deposition. Other of the most successful models of surface growth with relaxation is the Family model \[17\] or depend weakly, on the system size.

Meanwhile for the BD model the evolution rules are

\[
\begin{align*}
\text{(1)} & \text{ if } h_i \leq h_j \forall j \in v_i \Rightarrow h_i = h_i + 1, \text{ else } ; \\
\text{(2)} & \text{ if } h_j < h_n \forall n \neq j \in v_i \Rightarrow h_j = h_j + 1 .
\end{align*}
\]

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\text{(2)} & \text{ if } h_n \geq h_j \forall j \neq n \in v_i \Rightarrow h_i = h_n .
\end{align*}
\]
As initial condition we choose \( \{h_i\} = 0 \) for \( i = 1, ..., N \) and the time step increases by \( 1/N \). The SF networks with different values of \( \lambda \) were constructed using the configurational model \cite{26}.

Fig. 1: Fluctuations \( W_S \) as function of \( N \) in log-log scale for pure BD model (\( p = 0 \)) in SF networks for \( \lambda = 2.5 \) (\( \circ \)), 2.8 (\( \square \)), 3.0 (\( \diamond \)), 3.2 (\( \triangle \)) and 3.5 (\( * \)). The dashed lines correspond to a power law fitting for large system sizes \( N \) with \( W_S \sim N^{\alpha} \).

In Fig. 1 we show the fluctuations \( W_S \) as a function of \( N \) for the BD model (\( p = 0 \)) in log-log scale for different values of \( \lambda \). We can see that for all the values of \( \lambda \) there exists a power law behavior denoted by the dashed lines that represent the fitting of the data with \( W_S \sim N^{\alpha} \) for large system sizes, where the exponent \( \alpha \) depends on \( \lambda \). Clearly, we can observe that this behavior is very different than the one found for the SRM model on SF networks (see Eq. (1)). The power law behavior of the fluctuations in the steady state of the BD model seems to be an inherent property of the model regardless of the topology of the complex network over which the process evolves, although the exponent of the power law depends on the heterogeneity of the network. We find that as the heterogeneity of the network decreases, above \( \lambda = 2.5 \), finite size effects become more relevant modifying the power law behavior found for large \( N \). Thus, we propose that

\[
W_S \sim N^{\alpha} (1 + AN^{-\delta}) ,
\]

where the term \( N^{-\delta} \) is the correction to finite size scaling with \( \delta > 0 \) and \( A \) a constant. We can measure the finite size effects taking the derivative of \( W_S/N^{\alpha} \) with respect to \( N \)

\[
\chi(N) = \frac{\partial}{\partial N} \left( \frac{W_S}{N^{\alpha}} \right) \sim N^{-(\delta+1)} .
\]

In Fig. 2 a) we plot \( \chi(N) \) as a function of \( N \) up to \( N = 25000 \), for two values of \( \lambda > 2.5 \). The dashed lines represent the fitting with a power law from where we obtain the values of \( \delta \). In order to show that our proposed scaling (see Eq. (2)) is correct in Fig. 2 b) we plot \( W_S/N^{\alpha} \) as function of \( N^{-\delta} \). We can see that for large system sizes \( W_S/N^{\alpha} \) does not depend on \( N \), but as \( N \) decreases we observe a linear behavior, which shows the agreement between the proposed corrections to scaling and the simulations.

Next, we study the competition between the SRM and the BD models, i.e., the scaling behavior of \( W_S \) with \( N \) when we increase the parameter \( p \). Although the conclusions reached
Fig. 2: a) Corrections to the scaling $\chi(N)$ as function of $N$ in log-log scale for $\lambda = 2.8$ ($\bigcirc$) and $3.5$ ($\Box$). The dashed lines correspond to the power law fitting from which we obtain $\delta$. b) Linear-linear plot of $W_S/N^\alpha$ as function of $N^{-\delta}$ for the same values of $\lambda$ than in a).

from the competition between the two models are qualitatively the same for all the values of $\lambda$, we will consider only the case $\lambda = 2.5$ because the effects found are more pronounced in this case and because for this value of $\lambda$ finite size effects are negligible as we mention above. In Fig. 3 we plot $W_S$ as function of $N$ in log-log scale for different values of $p$. We can see that for $p = 0$ there is a pure power law with exponent 0.186 (see Fig 1) and for $p = 1$ the scaling behavior of $W_S$ with $N$ corresponds to a logarithm as expected for $\lambda < 3$ [24, 25]. For values $0 < p < 1$ we find that there is a crossover between both regimes. The corresponding curves become steeper as $p$ increases leading to expect that they will cross at some point. However, as the system size is increased we observe that the curves change their slope, avoiding the crossing between them. This is can be expected since the fluctuations in the competition with the SRM model, which tend to smooth out the interface via diffusion, cannot overcome the fluctuations of the pure BD model. In fact, we expect that for large sizes ($N \to \infty$), the behavior of all the curves with $p < 1$ will tend to follow a power law with the same exponent that for the case $p = 0$. However from our data we cannot show our expectation even for the system sizes as big as the ones that we have achieved ($N \sim 10^6$). Note that each point in Figs. 1 and 3 corresponds to $10^4$ realizations of the networks for
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fix \( N \) where each value of \( W_S \) was obtained by a linear fitting on the steady state. Thus to reach bigger system sizes is very time consuming and goes beyond our computational capabilities. For this reason from the results shown until here we can only conclude that in the competition between both models, even for \( p \) close to one, the BD dominates the behavior of the fluctuations.

![Fig. 3: Fluctuations \( W_S \) as function of \( N \) in log-log scale for SF networks with \( \lambda = 2.5 \), \( k_{min} = 2 \), for \( p = 0 \) (○), 0.6 (□), 0.8 (○), 0.9 (△), 0.95 (▽), and 1 (*).](image)

![Fig. 4: Schematic of the formation of steps for a one-dimensional lattice where we use the threshold \( \Delta h = 2 \). The steps, represented by different colors and capital letters, are bounded by nodes whose height difference exceeds the value of 2. The numbering in the horizontal axis represents the node number and the vertical axis is the height of each node.](image)

In the competition, the BD model is responsible of generating bigger height differences (jumps) between neighboring nodes than the SRM model, which tends to smooth out the interface by diffusion (see Fig. 3). As a consequence, the big jumps that produce the BD model increase the fluctuations of the system. As \( N \) increase, the number of jumps also increases and as a consequence the BD model dominates the behavior of the system, even for \( p \) close to one. In order to verify our assumption we introduce the definition of “step”. Steps are landing structures with edges generated by the BD process and flat regions smoothed by the SRM process. Since \( W_S \) represents the dispersion of the heights around their mean
value, we can assume that most of the nodes of the network have a height \( h \approx \langle h \rangle \pm mW_S \), where \( m \) is a positive integer number that determines the degree of confidence for a Gaussian distribution, i.e., \( m = 1 \) correspond to 66% of confidence and \( m = 3 \) to 99%. We verified that the distribution of heights is a Gaussian distribution for any value of \( p \). This implies that in average the absolute value of the difference of heights between any two pair of neighboring nodes is at most

\[
\Delta h \approx \sqrt{2mW_S}.
\]

Then, if we denote the fluctuations for the pure SRM model \((p = 1)\) as \( W_S^{SRM} \), we define a step as an almost flat connected region of the interface where each pair of neighboring nodes has \( \Delta h \leq \sqrt{2mW_S^{SRM}} \). Thus, computing how many steps each model generates and how this magnitude changes with \( p \), we can measure the influence of the BD process [27].

In Fig. 4 we show for \( \Delta h > 2 \) a schematic plot for the particular case of a one-dimensional lattice where each step is marked with a different color and labeled by a letter. For example, the step C consist of nodes IV and V that have \( \Delta h < 2 \) between them, and is limited by nodes III and VI because their height difference is greater than 2. Note that a step can be formed by a single node, as in the cases A and D. Thus a step is a soft region of the interface bounded by two borders represented by nodes with an abrupt jump in their height differences \( (\Delta h > \sqrt{2mW_S^{SRM}}) \).

In Fig. 5 we plot in log-log scale the average number of steps \( n_S \) as a function of \( N \) for \( p = 0, 0.6, 0.9 \) and 1, for \( m = 3 \) and taking \( W_S^{SRM} \approx 1 \) (see Fig. 3). We can see that for \( p = 1 \), \( n_S \) is much smaller than for \( p = 0 \) for all the values of \( N \). For large \( N \) and \( 0 < p < 1 \) the curves of \( n_S \) try to reach the value of \( p = 0 \), which means that the BD is the dominating mechanism (see Fig. 3). Thus even for values of \( p \) very close to 1, corresponding to pure SRM, as \( N \) increases more steps appear and the BD model ends dominating the behavior of the fluctuations in the steady state. This explain the crossover behavior of \( W_S \) with \( N \) for all the values of \( p < 1 \). Thus a relaxation process driven by diffusion competing with a small contribution of the ballistic deposition will behave as a pure ballistic one for large \( N \). Then, if we are interested in reducing the fluctuations in order to improve the synchronization of the system when these two processes are competing, we must increase the diffusion and reduce the system size. Our findings could have a great impact when surface relaxation mechanisms are used to synchronize system on complex networks because
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In summary, we studied the scaling behavior of the fluctuations on the steady state for the ballistic deposition model in degree uncorrelated SF networks and found that $W_S \sim N^\alpha$ where the exponent $\alpha$ increases with the heterogeneity of the SF networks, i.e., as $\lambda$ decreases. Our results suggest that the power law behavior of $W_S$ is an inherent property of this process found also in euclidean lattices. We also studied the scaling of $W_S$ with $N$ for the competition between the SRM and the BD models and found that there exists a crossover between different behaviors as we increase $p$. For large $N$, we found that for $p < 1$ the values of $W_S$ approaches to the BD one, which means that the BD model always dominates the behavior of the fluctuations in the steady state. This was shown using the definition of step, where the average number of steps measure the influence of the BD process. We found that the crossover between the SRM and the BD corresponds to an abrupt increase in the average number of steps.

In a forthcoming work we will study analytically the BD model and the competition between the BD and the SRM models deriving the evolution equations for the heights from where we will be able to derive the scaling behavior of $W_S$ for all the values of $p$.

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REFERENCES

[2] As $\lambda$ decreases the heterogeneity of the network increases due to the appearance of high degree nodes called hubs.

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[27] While quantitative results depend on the value of $m$, the qualitative results do not, since the number of steps will change depending on $m$ but the crossover between the two models will be clearly presented for any $m \geq 1$.
[28] The results obtained for $\lambda \geq 3$ are very similar to the ones found for $\lambda = 2.5$, although the crossover between the two behaviors now is between a constant, consequence of the SRM model for $\lambda \geq 3$ (see Eq. 1), and a power law.