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Algebraic sequences for $\zeta(3)$ and a hybrid Catalan's constant†

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We give two examples of algebraic sequences arising from modular forms which are good approximations for $\zeta(3)$ and the hybrid constant $8C + (\pi^2/\sqrt{2})$ where $C = \sum_0^\infty (-1)^n/(2n+1)^2$ is Catalan's constant.

Keywords: special sequences; modular forms

AMS Subject Classication: 11K31; 11F11

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1. Introduction

Apery was the first to prove the irrationality of $\zeta(3)$ in a very striking way (see [11]). For an elegant proof of this, see [1]. In [2] F. Beukers used modular forms in a surprising way to prove irrationality results for certain constants obtaining Apery's results among others. Tanguy Rivoal was the first to obtain general results concerning irrationalities of the zeta function at odd integers $n \geq 5$ and later he proved that among $\zeta(5), \zeta(7), \dots, \zeta(21)$ at least one is irrational. Zudilin went further to prove that among $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ one is irrational ([8,9,14]).

Our aim in this note is to give two examples of 'interesting' sequences in the spirit of Beukers' paper [2].

We will use Beukers' ideas heavily: one generates a function $\sum_1^\infty (a_n - \xi b_n)t^n$ which has an apparent singularity at t_1. It is shown that such singularity does not exist and that the radius of convergence is t_2 (at least): the point where the next singularity appears, $0 < |t_1| < |t_2|$. The construction of such function is done using modular forms and functions and the 'jump' from t_1 to t_2 yields the algebraic numbers a_n, b_n which are rapidly convergent sequences to ξ.

This paper is organized as follows. In Section 2, we give the general ideas for the construction of the above function $\sum_1^\infty (a_n - \xi b_n)t^n$. Then in Section 3, we give our main results, namely, Theorems 3.1 and 3.2.

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For example in Theorem 3.2, we show for the hybrid constant $8C + (\pi^2/\sqrt{2})$ that non-trivial sequences a_n, b_n exist such that

$$a_n - \left(8C + \frac{\pi^2}{\sqrt{2}}\right) b_n = O\left(\frac{1}{33.7^n}\right)$$

and that for an infinite number of natural n one has

$$|b_n| \gg \frac{1}{0.252^n}$$

where $b_n, a_n\ lcm\{1, \ldots, n\}^2$ are numbers of the form $m + p\sqrt{2}$ with m, p integers.

Similarly in Theorem 3.1, we show that non-trivial sequences a_n, b_n exist such that

$$a_n - \zeta(3) b_n = O\left(\frac{1}{67.9^n}\right)$$

and that for an infinite number of natural n one has

$$|b_n| \gg \frac{1}{0.0148^n}$$

where $b_n, a_n\ lcm\{1, \ldots, n\}^3$ are numbers of the form $m + p\sqrt{2}$ with m, p integers.

The sequences for $\zeta(3)$ are given neatly, as a generating function in t (up to a constant) by

$$\int_0^1 \int_0^1 \int_0^1 \frac{\mathrm{d}x\mathrm{d}y\mathrm{d}z}{\{1-(1-xy)z - t(2+t(-17+12\sqrt{2}))xyz(1-x)(1-y)(1-z)\}}.$$

2. Geometrical ideas

Here we outline the general ideas and the tools used in the construction.

Denote by $\mathcal{H} = Im(\tau = \tau_1 + i\tau_2) > 0$, the upper half plane and by R_1 an open hyperbolic polygonal region in the upper half plane bounded by the lines $i\tau_2$ and $1/2 + i\tau_2$, $0 < \tau_2 < \infty$, and a finite number of arcs of circles with centres on the real line as shown in Figure 1. We number the vertices (or cusps) of this region R_1 counted in an counterclockwise way with $i\infty = c_0$, c_1, c_2, …, $1/2 + i\infty = c_{n+1} = c_0$. We assume that the arc of circle joining c_1 and c_2 belongs to the circle $|\tau| = 1/\sqrt{N}$ for some positive number N, that is, the line joining $i\infty$ with c_1 meets the arc of this circle at an angle of $\pi/2$.

Assume that:

(i) $t(\tau) : R_1 \to \mathcal{H}$ is a conformal mapping and $t(c_0 = i\infty) = 0$.
(ii) $0 < t(c_1) < t(c_2) \leqslant \min_{i=3,\ldots,n} |t(c_i)|$.
(iii) Assume that we have functions $E(\tau)$, $f(\tau)$ holomorphic in $|q| < 1$, $q = \mathrm{e}^{i2\pi\tau}$, and a number $\xi \in \mathbb{C}$ such that $E(\tau)(f(\tau) - \xi) = E(-1/N\tau)(f(-1/N\tau) - \xi)$.
(iv) $E(\tau)(f(\tau) - \xi) = E(\tau + 1)(f(\tau + 1) - \xi)$.
(v) $E(\tau)(f(\tau) - \xi) = \sum_1^\infty \gamma_n q^n$, in some neighbourhood of $q = 0$ with $\gamma_1 \neq 0$.

In the following we show that if we have functions $E(\tau)$, $f(\tau)$, $t(\tau)$ meeting conditions (i), …, (v) and we know in an explicit way the values $t(c_1), \ldots, t(c_n)$, then we can generate the function $\sum_1^\infty (a_n - \xi b_n) t^n$ as stated in the introduction. Note that such a map $t(\tau)$ always exists by Riemann's mapping theorem.

First, we need to observe that conditions (i), (ii) imply

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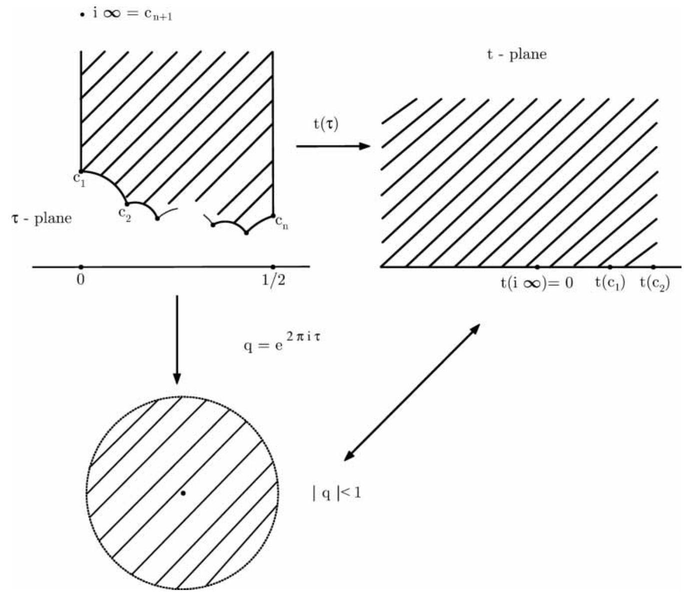

Figure 1. The conformal mapping τ.

(vi) in a neighbourhood of $t = 0$ one has $q = q(t) = \sum_1^\infty \alpha_n t^n$, $\alpha_1 \neq 0$ and $q = \mathrm{e}^{i2\pi\tau}$.

The proof of (vi) is as follows (Figure 2): extending by continuity the map $t(\tau)$ to the boundary and using Schwarz reflecting principle, one may reflect R_1 through the line $(i\infty, c_1)$ obtaining a conformal mapping from $R_1 \cup R_1' \cup (i\infty, c_1)$ to $\mathcal{H} \cup -\mathcal{H} \cup (0, t(c_1))$, where R_1' is the reflection of R_1 through the line $Im(\tau)$. But by construction $t(-1/2 + i\tau_2) = t(1/2 + i\tau_2)$ for large $\tau_2 > 0$ and therefore one may look at the (extended) function $t(\tau)$ as a function of $q = \mathrm{e}^{i2\pi\tau}$, that is $t(\tau) = \sum_n \beta_n q^n$. The condition $t(i\infty) = 0$ gives $\beta_n = 0$ for $n = 0, -1, -2, \ldots$. Since $t(\tau)$ has no branching at 0, we have $\beta_1 \neq 0$, therefore one may invert the series giving the desired property (vi) with $\alpha_1 = 1/\beta_1$.

Condition (iv) says that $E(\tau)(f(\tau) - \xi)$ may be viewed as a function of q. Condition (v) requires that this function be holomorphic at $q = 0$ with a simple zero.

The main point of the above construction is to look at this last function as a function of t, that is, using (v), (vi)

$$E(\tau)(f(\tau) - \xi) = \sum_1^\infty \gamma_n q(t)^n = \sum_1^\infty (a_n - \xi b_n) t^n$$

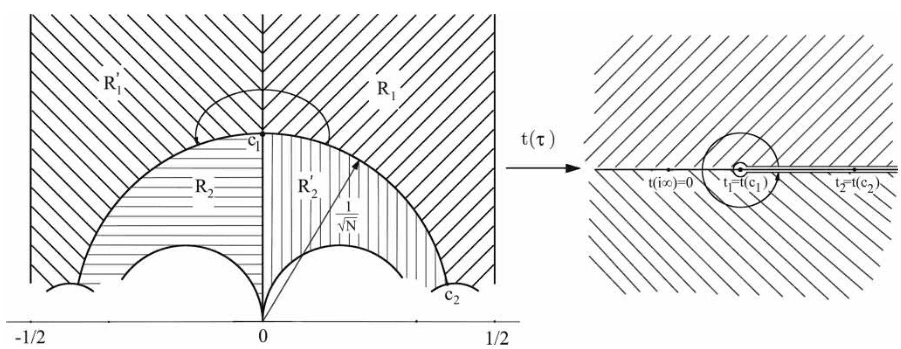

Figure 2. Extending τ by reflection.

The radius of convergence of this last series is at least $\min_{i=1,\dots,n}\ |t(c_i)|$, but at $t(c_1)$ there is no branching because of (iii). Indeed, we can still extend the map $t(\tau)$ to the regions R_2, R_2' as shown in Figure 2 (they are the images of regions R_1, R_1' by the transformation $\tau \to -(1/N\tau)$). This extended map $t(\tau)$ is a 2 to 1 covering of the t-plane branched at $t(c_1)$. The series in t of the function $E(\tau)(f(\tau) - \xi)$ can be extended by analytic continuation around $t(c_1)$ and eventually may have branching there. However, taking a curve encircling $t(c_1)$ once it lifts to an arc in $R_1 \cup R_1'$ and we reach the same value for $E(\tau)(f(\tau) - \xi)$. That is, this function is uniform in the t-plane and therefore $t(c_1)$ is not a branching point. Thus, the radius of convergence of the series is at least $\min_{i=2,\dots,n}\ |t(c_i)| = t(c_2)$ by (ii). This suggests that the sequences a_n, b_n may be 'good' approximations for the number ξ if the 'jump' from $t(c_1)$ to $t(c_2)$ is large.

To generate functions $E(\tau)$, $f(\tau)$ and a constant ξ with properties (iii), (iv), (v), one uses the following lemma proved by Beukers ([2], p. 273), classical modular forms and theta functions whose definitions we recall later.

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LEMMA 2.1 *Let $F(\tau) = \sum_1^\infty a_n q^n$, $q = e^{2\pi i\tau}$ be a Fourier series convergent for $|q| < 1$, such that for some $k, n \in N$, then*

$$F\left(-\frac{1}{N\tau}\right) = \epsilon(-i\tau\sqrt{N})^k F(\tau)$$

where $\epsilon = \pm 1$. Let $f(\tau)$ be the Fourier series

$$f(\tau) = \sum_1^\infty \frac{a_n}{n^{k-1}} q^n.$$

Let

$$L(F, s) = \sum_1^\infty \frac{a_n}{n^s}$$

and finally,

$$h(\tau) = f(\tau) - \sum_{0\leqslant r<(k-2)/2} \frac{L(F, k-r-1)}{r!}(2\pi i\tau)^r.$$

Then

$$h(\tau) - D = (-1)^{(k-1)}\epsilon(-i\tau\sqrt{N})^{k-2} h\left(-\frac{1}{N\tau}\right)$$

where $D = 0$ if k is odd, and $D = L(F, k/2)(2\pi i\tau)^{1/2(k-1)}/((1/2)k-1)!$ if k is even. Moreover, $L(F, k/2) = 0$ if $\epsilon = -1$.

Recall the classical modular forms defined by

$$E_2(\tau) = 1 - 24\sum_{n=1}^\infty \sigma_1(n)q^n$$

$$E_4(\tau) = 1 + 240\sum_{n=1}^\infty \sigma_3(n)q^n$$

($\sigma_i(n) = \sum_{d/n} d^i$), and the theta functions [7] defined by $\theta_3(\tau) = \sum_{m\in Z} q^{m^2/2}$, $\theta_2(\tau) = \sum_{m\in Z} q^{(m+1/2)^2/2}$, $\theta_4(\tau) = \sum_{m\in Z}(-1)^m q^{m^2/2}$ (recall that $q = \mathrm{e}^{2\pi i\tau}$), with the properties

$$\theta_j(\tau+2) = \theta_j(\tau),\quad j = 3,4;\qquad \theta_2(\tau+2) = i\theta_2(\tau)$$
$$\theta_3\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\theta_3(\tau)$$
$$\theta_2\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\theta_4(\tau)$$
$$\theta_4\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\theta_2(\tau).$$

If N is a natural number with many divisors, one takes the test function $F(\tau) = \sum_{d/N} \mathrm{const}(d) E_k(\mathrm{d}\tau)$, where $E_k(\tau)$ are the normalized modular forms. Recall that $E_k(d\tau)$ is invariant under $\Gamma_0(N)$ (defined below) for d a divisor of N. Then one looks for constants const(d) subject to the conditions $F(-1/N\tau) = \pm(-i\tau\sqrt{N})^k F(\tau)$, $F(\tau) = q + \ldots$ (or what is the same $F(i\infty) = 0$) and $L(F, k-r-1) = 0$ for $r = 1, 2, \ldots < (k-2)/2$, $L(F, k-1) = \xi$. If N has many divisors, then the number of constants can be enough to meet conditions (iii), (iv), (v).

For the function $E(\tau)$, one does the same but requiring only that $E(-1/N\tau) = \pm(-i\tau\sqrt{N})^{k-2}E(\tau)$.

In example 3.2, we will use combinations of theta functions.

2.1. *An important observation*

The mapping $t(\tau)$ will be constructed from the Hauptmoduln of a certain discrete subgroup Γ_S of the modular group. An element of the modular group is a fractional linear transformation T of the Riemann sphere: $T\tau = (a\tau+b)/(c\tau+d)$, with $a, b, c, d \in Z$ and $ad - bc = 1$. We will loosely speak of T as the transformation $(a\tau+b)/(c\tau+d)$. As usual, such elements are identified with matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ modulo $\pm\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, namely PSL$(2, \mathbb{Z})$.

We will restrict to the following case which is enough for our situation. Here, as usual, $\Gamma_0(N)$ is the subgroup of the modular group generated by the matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, with $c = 0$ (mod$N$), $a, b, c, d \in Z$ and $ad - bc = 1$.

Denote by Γ_S a group such that:

(a) Γ_S is a finite index subgroup of $\Gamma_0(N)$.
(b) $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_S$.
(c) Γ_S has a finite set of generators $\left(\begin{smallmatrix} a & b \\ c & a \end{smallmatrix}\right)$ such that the involution $\iota_N\left(\begin{smallmatrix} a & b \\ c & a \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & c/N \\ b.N & a \end{smallmatrix}\right)$ is a bijection on that set.

From (b) and (c), it is clear that $\left(\begin{smallmatrix} 1 & 0 \\ N & 1 \end{smallmatrix}\right) \in \Gamma_S$. Moreover, Γ_S is invariant under ι_N.

We will need the following

LEMMA 2.2 *If $y(\tau)$ is any function invariant under Γ_S i.e. $y(\tau) = y(a\tau+b/c\tau+d)$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_S$ then $y(-1/N\tau)$ is invariant under Γ_S. Moreover if $y(\tau)$ is a Hauptmoduln, then $y(-1/N\tau) = Ay(\tau) + B/Cy(\tau) + D$ for some numbers A, B, C, D.*

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Proof If $\left(\begin{smallmatrix} a & b \\ c & a \end{smallmatrix}\right) \in \Gamma_S$, then also $\left(\begin{smallmatrix} a & -c/N \\ -b.N & a \end{smallmatrix}\right) \in \Gamma_S$. Let $y(\tau)$ be invariant by Γ_S. We have

$$y\left(-\frac{1}{N\,(a\tau+b/c\tau+a)}\right) = y\left(\frac{-c/N - a/(N\tau)}{a+(bN)/(\tau N)}\right)$$
$$= y\left(\frac{a(-1/(N\tau)) - c/N}{-bN(-1/(N\tau))+a}\right) = y\left(-\frac{1}{N\tau}\right). \qquad \blacksquare$$

Finally we recall Dedekind's eta function $\eta(\tau) = q^{1/24}\Pi_1^{\infty}(1-q^n)$ and $\Delta(\tau) = \eta(\tau)^{24}$, $q = \mathrm{e}^{2\pi i\tau}$.

3. Examples

We recall that in the examples below, the arcs of circles are all centred at the real line in the τ-plane.

3.1. *Example 1*

Denote Γ_S the subgroup of $\Gamma_1(6)$ generated by the transformations $\tau+1$, $\tau/6\tau+1$, $11\tau+4/30\tau+11$, $49\tau+20/120\tau+49$, $11\tau+5/24\tau+11$. Recall [2] that $\Gamma_1(6)$ is generated by transformations $\tau+1$, $\tau/6\tau+1$ and $5\tau+2/12\tau+5$, with Hauptmoduln given by

$$y(\tau) = \frac{\Delta(\tau)^{1/6}\Delta(6\tau)^{1/3}}{\Delta(2\tau)^{1/3}\Delta(3\tau)^{1/6}} \tag{1}$$

and values at the cusps $y(i\infty)=0$, $y(0)=1/9$, $y(1/3)=1$, $y(1/2)=\infty$. Then $\tau \to y(\tau)$ maps in a conformal way the shaded region R_1 in Figure 3 onto $\mathcal{H}$.

To obtain a Hauptmoduln for Γ_S, we apply Schwartz reflection principle to the arc of Figure 3 going from 1/3 to 1/2, that is the isometric circle of the transformation $5\tau-2/-12\tau+5$.

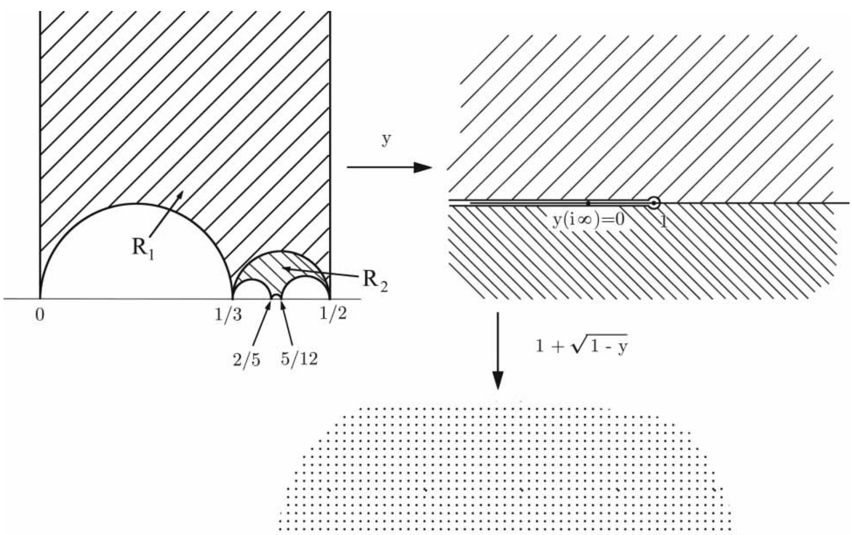

Figure 3. Conformal map $\tau \to y_1(\tau)$ in Example 1.

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Therefore

$$y_1(\tau) = 1 + \sqrt{1 - y(\tau)} \tag{2}$$

with $\sqrt{z}$ taken with a cut in the positive real axis, maps in a conformal way the region $R_1 \cup R_2$ shown in Figure 3 onto $\mathcal{H}$. But $R_1 \cup R_2 \cup R_1' \cup R_2'$, where $'$ denotes the reflection through the line $i\tau_2$ for τ_2 real on the τ-plane, is the fundamental region of Γ_S. Thus using Poincare's theorem ([5,6]), one checks that $y_1(\tau)$ is the Hauptmoduln of Γ_S.

Values at the cusps are given by (using analytic continuation) $y_1(i\infty) = 0$, $y_1(0) = 1 - 2\sqrt{2}/3$, $y_1(1/3) = 1$, $y_1(2/5) = 1 + 2\sqrt{2}/3$, $y_1(5/12) = 2$, $y_1(1/2) = \infty$. Notice the expansion $y_1(\tau) = q/2 - 15q^2/8 + 65q^3/16 - \dots$ in the neighborhood of $q = 0$.

Using Lemma 2.2, one sees that $y_1(-1/6\tau)$ is invariant under Γ_S and therefore $y_1(-1/6\tau) = (y_1(\tau) - 1 + 2\sqrt{2}/3)/(y_1(\tau) - 1)$. For later use, notice that $z = (z - 1 + 2\sqrt{2}/3)/(z - 1)$ has solutions $z = 1 \pm (8/9)^{1/4}$ (*). Moreover the function

$$t(\tau) = y_1(\tau)\frac{(y_1(\tau) - 1 + 2\sqrt{2}/3)}{(y_1(\tau) - 1)} \cdot 3(3 + 2\sqrt{2}) \tag{3}$$

is invariant for $-1/6\tau$, that is,

$$t(\tau) = t(-1/6\tau) \tag{4}$$

and therefore maps the region shown in Figure 4 onto $\mathcal{H}$. Values at $c_1 = i/\sqrt{6}$ and $c_2 = (120 + i\sqrt{6})/294$, where c_2 is the intersection of the circle of radius $1/\sqrt{6}$ centred at zero and the isometric circle of the transformation $(49\tau - 20)/(-120\tau + 49)$, can be calculated using (3), (4) and (*) to give $t(i\infty) = 0$, $t(c_1) = 3(3 + 2\sqrt{2})(1 - 2^{3/4}/\sqrt{3})^2 = 0.01472\dots$, $t(c_2) = 3(3 + 2\sqrt{2})(1 + 2^{3/4}/\sqrt{3})^2 = 67.926\dots$, $t(1/2) = \infty$.

Recall Beukers' formulae (see [2]),

$$\begin{aligned} 40F(\tau) &= E_4(\tau) - 36E_4(6\tau) - 7(4E_4(2\tau) - 9E_4(3\tau)) \\ 24E(\tau) &= -5(E_2(\tau) - 6E_2(6\tau)) + 2E_2(2\tau) - 3E_2(3\tau)) \end{aligned} \tag{5}$$

These functions transform like $E(-1/6\tau) = -6\tau^2 E(\tau)$, $F(-1/6\tau) = -6^2.\tau^4 F(\tau)$, $F(i\infty) = 0$.

The Dirichlet series of $F(\tau)$, in the sense of Lemma 2.1, is (see [2,10]) $L(F,s) = 6(1 - 6^{2-s} - 7.2^{2-s} + 7.3^{2-s})\zeta(s)\zeta(s-3)$ and therefore $L(F,3) = \zeta(3)$. Again, applying Lemma 2.1

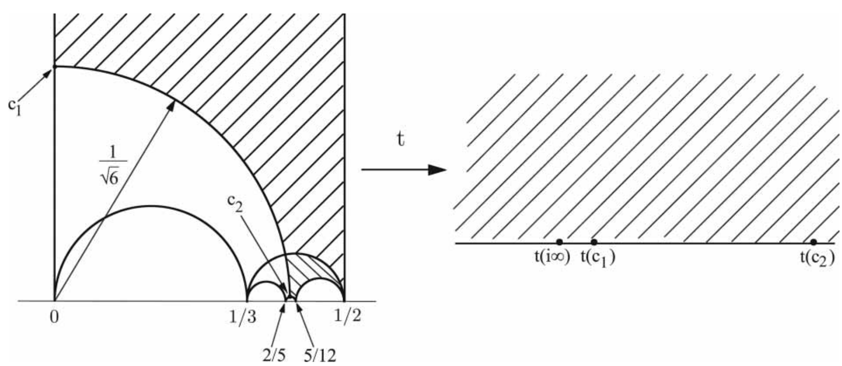

Figure 4. The map t in Example 1.

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with $f(\tau)$ defined by $(d^3/d\tau^3)f(\tau) = (2\pi i)^3 F(\tau)$, $f(i\infty) = 0$, we obtain

$$E\left(-\frac{1}{6}\tau\right)\left(f\left(-\frac{1}{6}\tau\right) - \zeta(3)\right) = E(\tau)(f(\tau) - \zeta(3)) =: W(q) \tag{6}$$

We gather this into:

THEOREM 3.1 *Define $t(\tau)$ by (1), (2), (3). Also let $(d^3/d\tau^3)f(\tau) = (2\pi i)^3 F(\tau)$, $f(i\infty) = 0$, $E(\tau)$, $F(\tau)$ defined by (5) and $W(q)$ defined by (6).*

Then as a function of t, the function $W(q(t))$ has radius of convergence (at least) $3(3 + 2\sqrt{2})(1 + 2^{3/4}/\sqrt{3})^2 \sim 67.92\ldots$. If one writes

$$W(q(t)) = \sum_0^\infty (a_n - \zeta(3)b_n)t^n$$

then b_n and $a_n.lcm\{1, \ldots, n\}^3$ are numbers of the form $m_1 + m_2\sqrt{2}$ with $m_1, m_2 \in \mathbb{Z}$.

The function $\sum_1^\infty b_n t^n$ has (exactly) radius of convergence $3(3 + 2\sqrt{2})(1 - 2^{3/4}/\sqrt{3})^2 \sim 0.01472\ldots$ and one has

$$W(q(t)) = -\frac{1}{2}\int_0^1\int_0^1\int_0^1 \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{\{1 - (1 - xy)z - t(2 + t(-17 + 12\sqrt{2}))xyz(1 - x)(1 - y)(1 - z)\}}$$

Moreover $a_n - \zeta(3)b_n \neq 0$ for infinitely many n.

Proof The stated radius of convergence of $W(q(t))$ follows from Section 2 because conditions (i), …, (v) are met and therefore $W(q(t))$ has radius of convergence $t(c_2)$ (at least) instead of $t(c_1)$. The only thing that was not proved was that, by construction, $t(\tau)$ has locally an inverse as a function of $q = \mathrm{e}^{2\pi i\tau}$ with coefficients of the form $m_1 + m_2\sqrt{2}$ with $m_1, m_2 \in \mathbb{Z}$. The assertion follows from the fact that in (3) the inverse of y_1 as a function of t, that is

$$y_1 = \frac{1 + t - \sqrt{1 - 34t - 24\sqrt{2}t + t^2}}{2(9 + 6\sqrt{2})}$$

has, due to Lagrange's theorem [12], coefficients of the form $m + n\sqrt{2}$ with $m, n \in \mathbb{Z}$. From this and (1), (2) the assertion follows.

Observe that $\sum_1^\infty b_n t^n$ is the function $E(\tau) = E(\tau + 1)$ viewed as a function of t, which has, at least, radius of convergence $t(c_1)$. But it is seen that $t(c_1)$ must be a branching point because $E(-1/6\tau) = -6\tau^2 E(\tau)$. Therefore, $\sum_1^\infty b_n t^n$ has radius of convergence $t(c_1)$ as stated.

The integral expression for $W(q(t_0))$ with $t_0 = y(\tau)(9y(\tau) - 1)/(y(\tau) - 1)$ is well-known ([1,3,4]), so it is a matter of writing t_0 in terms of t. This is given, using (2),(3), by $t_0(t) = t(2 + t(-17 + 12\sqrt{2}))$ which gives the desired integral form for $W(q(t))$.

If $W(q(t))$ were a polynomial in t, then by an argument similar to Beukers' paper ([3], p. 276) $f(\tau) - \zeta(3)$ would be a modular form of weight -2 for Γ_S, which is impossible. ■

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Indeed one calculates

$$\begin{aligned} q(t) &= 2t + (31+12\sqrt{2})t^2 + 192(5+3\sqrt{2})t^3 + (40036+27072\sqrt{2})t^4 \\ &\quad + (1915220+1338144\sqrt{2})t^5 + 6(16424771+11577300\sqrt{2})t^6 + \ldots, \\ \sum_0^\infty b_n t^n &= 1 + 10t + (207+60\sqrt{2})t^2 + (6596+3504\sqrt{2})t^3 \\ &\quad + (275357+178296\sqrt{2})t^4 + \ldots, \\ \sum_0^\infty a_n t^n &= 12t + (249+72\sqrt{2})t^2 + (71359/9+4212\sqrt{2})t^3 \\ &\quad + \left(\frac{11915809}{36} + 214322\sqrt{2}\right)t^4 + \ldots \end{aligned}$$

3.2. *Example 2*

Denote Γ the group generated by the transformations $\tau+1$, $\tau/8\tau+1$ and $3\tau+1/8\tau+3$ where the Hauptmoduln is given by

$$y(\tau) = 1 - \sqrt{1 - 16q\,\Pi_1^\infty\left\{\frac{1+q^{2n}}{1+q^{2n-1}}\right\}^8} = 1 - \Pi_1^\infty\left(\frac{1-q^{2n-1}}{1+q^{2n-1}}\right)^4 \tag{7}$$

The values at the cusps are $y(i\infty)=0$, $y(0)=1$, $y(1/4)=2$ and $y(1/2)=\infty$, and $y(\tau)$ maps in a conformal way R_1 onto $\mathcal{H}$ as shown in Figure 5. Next denote Γ_S the subgroup of Γ generated by the transformations $\tau+1$, $\tau/8\tau+1$, $7\tau+2/24\tau+7$, $17\tau+6/48\tau+17$, $7\tau+3/16\tau+7$. If one reflects on the arc of circle going from $1/4$ to $1/2$, that is the isometric circle of the transformation $3\tau-1/-8\tau+3$, then one obtains the shadowed region $R_1\cup R_2$ of Figure 5. But $R_1\cup R_2\cup R_1'\cup R_2'$ is the fundamental region of Γ_S, where $'$ denotes the reflection through the imaginary line in the τ-plane.

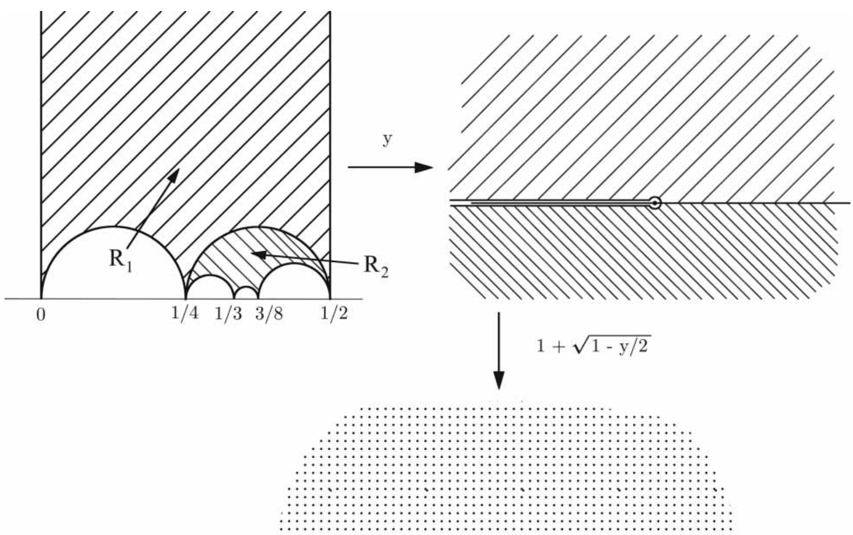

Figure 5. Conformal map $\tau \to y_1(\tau)$ in Example 2.

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Thus

$$y_1(\tau) = 1 + \sqrt{1 - \frac{y(\tau)}{2}} \tag{8}$$

(here $\sqrt{z}$ is taken with a cut in the real positive axis) is a Hauptmoduln of Γ_S (again using Poincare's theorem [6]). The values at the cusps, obtained by analytical continuation are: $y_1(i\infty) = 0$, $y_1(0) = 1 - (1/\sqrt{2})$, $y_1(1/4) = 1$, $y_1(1/3) = 1 + (1/\sqrt{2})$, $y_1(3/8) = 2$ and $y_1(1/2) = \infty$.

Now it is seen using Lemma 2.2 that $y_1(\tau)$ is invariant under $\tau \to -1/8\tau$, and therefore

$$y_1\left(-\frac{1}{8}\tau\right) = \frac{y_1(\tau) - 1 + (1/\sqrt{2})}{y_1(\tau) - 1}.$$

Notice that the equation $z = z - 1 + (1/\sqrt{2})/z - 1$ has roots $1 \pm 1/2^{1/4}$ (**).

Finally define

$$t(\tau) = y_1(\tau)\frac{(y_1(\tau) - 1 + (1/\sqrt{2}))}{(y_1(\tau) - 1)}\frac{(3 + 2\sqrt{2})}{(2 - \sqrt{2})}. \tag{9}$$

This function is invariant under Γ_S and $-1/8\tau$ and maps in a conformal way the shading area in Figure 6 onto $\mathcal{H}$.

Using (**) and the invariance properties one calculates for $c_1 = i/\sqrt{8}$ and $c_2 = 6/17 + i(1/17\sqrt{8})$ (this is the intersection of the isometric circle of the transformation $-17\tau + 6/48\tau - 17$ and a circle of radius $1/\sqrt{8}$ centred at zero) that $t(c_1) = (1 - (1/2^{1/4}))^2(3 + 2\sqrt{2})/2 - \sqrt{2} = 0.251\ldots$, $t(c_2) = (1 + (1/2^{1/4}))^2(3 + 2\sqrt{2})/2 - \sqrt{2} = 33.71\ldots$, $t(1/2) = \infty$.

Define

$$F(\tau) = \theta_2(2\tau)^4\theta_3(2\tau)^2 - 8\theta_2(4\tau)^4\theta_3(4\tau)^2 - \sqrt{8}\theta_4(2\tau)^4\theta_3(2\tau)^2 + \sqrt{8}\theta_4(4\tau)^4\theta_3(4\tau)^2,$$

$$E(\tau) = \theta_3(2\tau)^2 + \sqrt{2}\theta_3(4\tau)^2. \tag{10}$$

These functions transform like $F(\tau + 1) = F(\tau)$, $F(-1/8\tau) = i(\sqrt{8}\tau)^3F(\tau)$, $E(\tau + 1) = E(\tau)$, $E(-1/8\tau) = (-i\sqrt{8}\tau)E(\tau)$. Also $F(i\infty) = 0$.

One calculates, using Jacobi's formulae as given in [13], that (here $\beta(s) = \sum_0^\infty (-1)^n/(2n + 1)^s$)

$$L(F, s) = 16\zeta(s - 2)\beta(s) - 128\frac{1}{2^s}\zeta(s - 2)\beta(s) + 4\sqrt{8}\zeta(s)\beta(s - 2) - 4\sqrt{8}\frac{1}{2^s}\zeta(s)\beta(s - 2)$$

$$L(F, 2) = 8\,C + \frac{\pi^2}{\sqrt{2}}$$

where $C = \sum_0^\infty (-1)^n/(2n + 1)^2$, is Catalan's constant.

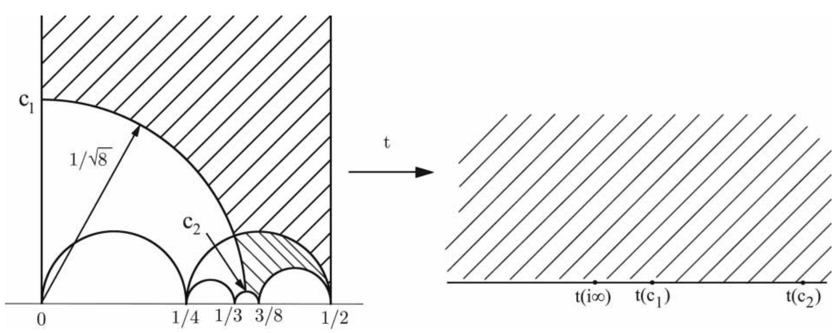

Figure 6. The map t in Example 2.

Therefore defining the unique function $f(\tau)$ as $(\mathrm{d}^2/\mathrm{d}\tau^2)f(\tau) = (2\pi i)^2 F(\tau)$, $f(i\infty) = 0$ and using Lemma 2.1 one obtains

$$E\left(-\frac{1}{8}\tau\right)\left\{f\left(-\frac{1}{8}\tau\right) - \left(8\,C + \frac{\pi^2}{\sqrt{2}}\right)\right\} = E(\tau)\left\{f(\tau) - \left(8\;C + \frac{\pi^2}{\sqrt{2}}\right)\right\} =: W(q). \quad (11)$$

We have then the following:

THEOREM 3.2 *Define $t(\tau)$ using (7–9).*

Also let $(\mathrm{d}/\mathrm{d}\tau)^2 f(\tau) = (2\pi i)^2 F(\tau)$, $f(i\infty) = 0$, where $F(\tau)$, $E(\tau)$, are defined by (10) and $W(q)$ is defined by (11).

Then as a function of t the function $W(q(t))$ has radius of convergence $(1 + 1/2^{1/4})^2(3 + 2\sqrt{2})/2 - \sqrt{2} \sim 33.71\ldots$. Moreover if one writes

$$W(q(t)) = \sum_0^\infty \left\{a_n - \left(8\;C + \frac{\pi^2}{\sqrt{2}}\right) b_n\right\} t^n$$

then b_n and $a_n.lcm\{1, \ldots, n\}^2$ are numbers of the form $m + p\sqrt{2}$ with $m, p \in \mathbb{Z}$.

Moreover $a_n - (8\;C + (\pi^2/\sqrt{2}))b_n \neq 0$ for infinitely many n and $\sum_0^\infty b_n t^n$ has radius of convergence $(1 - (1/2^{1/4}))^2(3 + 2\sqrt{2})/2 - \sqrt{2} \sim 0.251\ldots$

Proof The inverse of each function in (7–9) is a Taylor series with coefficient in $\mathbb{Z}$ or has coefficients of the $m + n\sqrt{2}$ with $m, n \in \mathbb{Z}$. Thus the function $t(\tau)$ as a function of $q = \mathrm{e}^{2\pi i\tau}$ has an inverse $q = q(t)$ with coefficients in $m + n\sqrt{2}$ with $m, n \in \mathbb{Z}$.

Again, the construction implies that conditions (i), …, (v) are met and therefore $W(q(t))$ has radius of convergence $t(c_2)$. The rest of the proof is the same as in theorem 1. ■

One calculates

$$\begin{aligned} q(t)s &= (3 - 2\sqrt{2})t + (37 - 26\sqrt{2})t^2 + (668 - 472\sqrt{2})t^3 + (15716 - 11112\sqrt{2})t^4 \\ &\quad + (408106 - 288572\sqrt{2})t^5 + \ldots, \\ \sum_0^\infty b_n t^n &= (1 + \sqrt{2}) + (12 - 8\sqrt{2})t + (120 - 84\sqrt{2})t^2 + (1960 - 1384\sqrt{2})t^3 \\ &\quad + (44504 - 31464\sqrt{2})t^4 + (1144176 - 809040\sqrt{2})t^5 + \ldots, \\ \sum_0^\infty a_n t^n &= 8\sqrt{2}t + (368 - 248\sqrt{2})t^2 + \left(\frac{13792}{9} - 1056\sqrt{2}\right)t^3 \\ &\quad - \frac{32}{9}(-1033 + 710\sqrt{2})t^4 - \frac{64}{75}(-370719 + 261893\sqrt{2})t^5 + \ldots \end{aligned}$$

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