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Lagrangian systems with higher order constraints

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A class of mechanical systems subject to higher order constraints (i.e., constraints involving higher order derivatives of the position of the system) are studied. We call them *higher order constrained systems* (HOCSs). They include simplified models of elastic rolling bodies, and also the so-called *generalized nonholonomic systems* (GNHSs), whose constraints only involve the velocities of the system (i.e., first order derivatives in the position of the system). One of the features of this kind of systems is that D'Alembert's principle (or its nonlinear higher order generalization, the Chetaev's principle) is not necessarily satisfied. We present here, as another interesting example of HOCS, systems subjected to friction forces, showing that those forces can be encoded in a second order kinematic constraint. The main aim of the paper is to show that every HOCS is equivalent to a GNHS with linear constraints, in a canonical way. That is to say, systems with higher order constraints can be described in terms of one with linear constraints in velocities. We illustrate this fact with a system with friction and with Rocard's model [*Dynamique Générale des Vibrations* (1949), Chap. XV, p. 246 and *L'instabilité en Mécanique; Automobiles, Avions, Ponts Suspendus* (1954)] of a pneumatic tire. As a by-product, we introduce some applications on higher order tangent bundles, which we expect to be useful for the study of intrinsic aspects of the geometry of such bundles. [DOI: 10.1063/1.2740470]

I. INTRODUCTION

Systems with several types of generalized nonholonomic constraints have been studied, specially in the control and mechanics literature, using different approaches. Inspired by simplified models of pneumatic tires[1–4] and elastic rolling bodies, in Ref. 5 a class of dynamical systems with higher order constraints were defined (see also Refs. 6 and 7). They include, as a particular case, Lagrangian systems with generically nonlinear constraints in velocities, studied in Ref. 8 under the name of *generalized nonholonomic systems* (GNHSs). The main aim of Ref. 5 was to study constrained mechanical systems for which D'Alembert's principle, or its nonlinear (and higher order) generalization, the Chetaev's principle,[9–12] does not necessarily apply. That paper has adopted a Lagrangian, or more precisely, a variational-like approach, generalizing previous works on the subject. See Ref. 13 and references therein, where the problem is approached from a Hamiltonian point of view. The two approaches, the Lagrangian and the Hamiltonian one, were compared in Ref. 8, for the subclass of GNHSs.

The main feature of the above mentioned systems is that, in order to write down their equations of motion, it is not enough to give the kinematic constraints to which they are subjected, but

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also the subspace where the constraints forces take their values, i.e., the directions of constraint forces. Note that, for systems with linear constraints (in velocities) and satisfying D'Alembert's principle, the subspace of constraints forces is derived from kinematic constraints: *constraint forces are orthogonal to (or more precisely, annihilate) admissible velocities*.

In this work we give a new (but equivalent) definition of the systems studied in Ref. 5, using a notation that emphasizes the intrinsic character of involved objects. We call them *higher order constrained systems* (HOCSs). They are given by triples (L, C_K, F_V), or equivalently by triples (L, C_K, C_V), where

- $L: TQ \to \mathbb{R}$ is a Lagrangian function on a configuration space Q;
- C_K is a submanifold of some higher order tangent bundle $T^{(k)}Q$,[14,15] for some $k \in \mathbb{N}$, representing the kinematic constraints of the system;
- F_V is a collection of subspaces $F_V[x] \subset T^*Q$, with $x \in T^{(l)}Q$ (for some $l \in \mathbb{N}$), representing the possible directions of constraint forces;
- C_V is a collection of subspaces $C_V[x] \subset TQ$, with $x \in T^{(l)}Q$ (for some $l \in \mathbb{N}$), playing the role of *virtual displacements*. C_V is called the space of *virtual* or *variational constraints*.

C_V and F_V are related by annihilation: $C_V^o[x] = F_V[x]$, for all $x \in T^{(l)}Q$. This gives rise to a generalization of the principle of virtual works: constraint forces do *not* do work along the directions of virtual displacements.

When kinematic constraints are linear, i.e., C_K is a distribution on Q, D'Alembert's principle says that F_V is a codistribution such that $F_V^o = C_K$, or equivalently, $C_V = C_K$. Then, in the D'Alembertian situation, variational constraints are derived from the kinematic ones. But in general, kinematic and variational constraints must be taken as independent notions, and one should not attempt to derive one from the other by a universal procedure.

We emphasize in the present paper the point of view that external forces should, in principle, be encoded in the variational constraints, and not just added on the right hand side of the equations. As an example, we consider Lagrangian systems subjected to friction forces, the latter defined by a *Rayleigh dissipation function*[16,17] $\mathcal{F}: TQ \to \mathbb{R}$. We show that each one of such systems can be described as a HOCS with second order kinematic constraints, i.e., C_K is a submanifold of $T^{(2)}Q$. (However, one must recognize that completely arbitrary forces do not seem always conveniently represented by suitable and natural constraint forces.) This point of view is consistent with the idea behind D'Alembert's principle, where the constraint forces are *encoded in the geometry of the system*, given by the nonholonomic distribution. An important point is that, in presence of a group of symmetry, the resulting equations of motion will be *automatically covariant*, provided that the group respects the constraints. Then the covariance of the resulting constraint forces follows at once. This approach has also the consequence that reduced equations for systems with generalized constraints could be written (planned for a future work) in the spirit of Refs. 18 and 19, by studying the geometry of the constraints and the variations. Also, numerical integrators like the ones considered, for instance, in Ref. 20 for D'Alambertian systems, could be written for the case of generalized constraints like those considered in this paper.

A particularly interesting subclass of HOCSs are the previously mentioned GNHSs. Among them, we have the linear (affine) GNHSs, given by triples (L, C_K, C_V) such that C_K and C_V are (generalized) distributions on Q (C_K is an affine subbundle of TQ). One of the goals of the present work is to show that every HOCS is equivalent to a linear GNHS. More precisely, given a HOCS (L, C_K, C_V) on a configuration space Q, there exists a manifold $\tilde{Q}$ and a linear GNHS defined there, such that the trajectories of the former are (essentially) in bijection with that of the latter. This is done in two steps. We first show that every HOCS is equivalent to a GNHS with affine constraints, and then that every affine GNHS is equivalent to a linear one. The basic idea in both steps is to add new degrees of freedom appropriately. Thus, we have a process of *order-lowering* of constraints, and a process of *linearization* of them. These procedures enable us to apply to HOCSs, in presence of symmetry, the reduction technics developed in Ref. 21 for GNHSs.

It is worth mentioning that original HOCS and corresponding GNHS are canonically related, in the sense that no further structure is needed (beyond the data defining the HOCS) in order to establish the mentioned equivalence.

The work is organized as follows. In Sec. II we introduce the coordinate-free (or intrinsic) terminology that will be used in the paper. We recall the variational formulation of a (standard) Lagrangian system in terms of infinitesimal variations, using the canonical involution $\kappa: TTQ \to TTQ$, and then we derive the Euler-Lagrange equations in a coordinate-free expression by using an affine connection on Q and a related vector bundle isomorphism $\beta: TTQ \to TQ \oplus TQ \oplus TQ$. In Sec. III, we define the HOCSs, we describe some relevant examples, and show that their equations of motion can be seen as a generalized version of the Lagrange-D'Alembert equations. Before we do that we give a brief survey of higher order tangent bundles, its local description, and some useful structures on them. In Sec. IV, we study dissipative systems, showing that systems with friction forces can be studied in terms of HOCSs with second order kinematic constraints. Then, we define the notion of HOCS with external forces, giving as an illustrative example the control of the inverted planar pendulum on a cart with friction. Finally, in Sec. V, we develop the above mentioned order-lowering and linearization procedures, and apply them to a couple of simple mechanical systems: a massive particle moving in an anisotropic viscous fluid, and the Rocard model of a pneumatic tire (as presented in Ref. 5).

II. LAGRANGIAN SYSTEMS

Let us denote (Q,L) a Lagrangian system defined on a configuration space Q with Lagrangian function $L: TQ \to \mathbb{R}$. In this section we first recall how the trajectories of (Q,L) are usually characterized in the variational formalism, in terms of infinitesimal variations. After that, introducing a connection on the configuration space, we construct a global expression of the Euler-Lagrange equations. We will work in the C^∞ category and follow a notation close to that of Ref. 22.

Basic notation and terminology. Let Q be an n-dimensional manifold. Consider a chart (U,φ) of Q, with $\varphi: U \to \mathbb{R}^n$. Given $q \in U$, we write

$$\varphi(q) = (q^1, \ldots, q^n).$$

Sometimes, we shall also use q to denote the n-uple $(q^1, \ldots, q^n)$. Given a curve $\gamma:[t_1,t_2] \to Q$ such that $U \cap range(\gamma) \neq \emptyset$, we write

$$\varphi(\gamma(t)) = (q^1(t), \ldots, q^n(t)) \equiv q(t), \tag{1}$$

inside the open interval where $\varphi \circ \gamma$ is defined. The *velocity* of γ is given by an application $\gamma' : (t_1,t_2) \to TQ$ such that

$$\gamma'(t) = \frac{d}{dt}\gamma(t) = \gamma_*(d/dt|_t).$$

Note that $\gamma'(t) \in T_{\gamma(t)}Q$ for all t. For the tangent bundle of Q we use the induced coordinate charts (TU,φ_*), being $\varphi_*: TU \to T\mathbb{R}^n$ the differential of φ. Given $X \in T_qQ$, $q \in U$, we sometimes adopt the notation

$$\varphi_*(X) = (q^1, \ldots, q^n; \dot{q}^1, \ldots, \dot{q}^n) \quad \text{or } (q^1, \ldots, q^n; \delta q^1, \ldots, \delta q^n),$$

or, for short, $\varphi_*(X) = (q,\dot{q})$ or $(q,\delta q)$. Given γ as above, we write for its velocity

$$\varphi_*(\gamma'(t)) = (q^1(t), \ldots, q^n(t); \dot{q}^1(t), \ldots, \dot{q}^n(t)) \equiv (q(t),\dot{q}(t)).$$

For the bundle TTQ we shall consider the induced charts (TTU,φ_{**}) and we will write its local coordinates as follows:

$$(q,\delta q,\dot q,\delta\dot q) \quad \text{or } (q,\dot q,\delta q,\delta\dot q), \tag{2}$$

whenever it is convenient.

A. Variational formulation

Let $\gamma:[t_1,t_2]\to Q$ be a curve on Q, and $\gamma':(t_1,t_2)\to TQ$ its velocity. Recall that the action S evaluated at γ is

$$S[\gamma]=\int_{t_1}^{t_2} L(\gamma'(t))dt;$$

and a deformation of γ is an application

$$\Delta\gamma:[t_1,t_2]\times(-\varepsilon,\varepsilon)\to Q:(t,\lambda)\mapsto\Delta\gamma(t,\lambda),$$

such that

a. $\Delta\gamma(t_1,\lambda)=\gamma(t_1)$ and $\Delta\gamma(t_2,\lambda)=\gamma(t_2)$ for all $\lambda\in(-\varepsilon,\varepsilon)$;
b. $\Delta\gamma(t,0)=\gamma(t)$ for all $t\in[t_1,t_2]$.

Related to any deformation and any $\lambda\in(-\varepsilon,\varepsilon)$ we have a curve on Q

$$\Delta\gamma_\lambda:[t_1,t_2]\to Q:t\mapsto\Delta\gamma(t,\lambda),$$

with velocity

$$\Delta\gamma_\lambda':(t_1,t_2)\to TQ:t\mapsto\Delta\gamma_*(\,\partial/\partial t|_{(t,\lambda)}).$$

In terms of $\Delta\gamma_\lambda$, properties (a) and (b) translate, respectively, into $\Delta\gamma_\lambda(t_1)=\gamma(t_1)$ and $\Delta\gamma_\lambda(t_2)=\gamma(t_2)$, for all λ, and $\Delta\gamma_0=\gamma$.

Finally, by the very definition of a Lagrangian system, γ is a trajectory of (Q,L) if and only if for all deformations $\Delta\gamma$ of γ we have

$$\left.\frac{d}{d\lambda}S[\Delta\gamma_\lambda]\right|_{\lambda=0}=0, \tag{3}$$

or equivalently

$$\int_{t_1}^{t_2}\left.\frac{d}{d\lambda}L(\Delta\gamma_\lambda'(t))\right|_{\lambda=0}dt=0.$$

We want to write Eq. (3) in terms of infinitesimal variations of γ. Recall that an infinitesimal variation of a curve $\gamma:[t_1,t_2]\to Q$ is another curve

$$\delta\gamma:[t_1,t_2]\to TQ,$$

such that

a. $\delta\gamma(t)\in T_{\gamma(t)}Q$, $\forall t\in[t_1,t_2]$;
b. $\delta\gamma(t_1)$ and $\delta\gamma(t_2)$ belong to the zero subbundle of TQ.

Remark 1: *It is worth mentioning that the symbols* Δ *and* δ *(and also the* prime *that appears in velocities) should not be interpreted as being operators acting on spaces of curves, but just as a part of another symbol.*

Given a chart (U,φ) of Q, we shall write, on the induced chart (TU,φ_*) of TQ,

$$\varphi_*(\delta\gamma(t))=(q(t),\delta q(t)).$$

By $\delta\gamma'$ we shall denote the velocity of $\delta\gamma$, namely,

$$\delta\gamma':(t_1,t_2) \to TTQ : t \mapsto \delta\gamma'(t) = \delta\gamma_*(d/dt|_t). \tag{4}$$

In the chart (TTU,φ_{**}) [recall Eq. (2)], the local expression for $\delta\gamma'(t)$ will be

$$\varphi_{**}(\delta\gamma'(t)) = \left(q(t),\delta q(t),\frac{d}{dt}q(t),\frac{d}{dt}\delta q(t)\right) = (q(t),\delta q(t),\dot{q}(t),\delta\dot{q}(t)).$$

Remark 2: *Using the canonical projection $\tau_Q:TQ \twoheadrightarrow Q$, condition (a) can be rewritten $\tau_Q \circ \delta\gamma = \gamma$. Considering also $\tau_{TQ}:TTQ \twoheadrightarrow TQ$, we have*

$$\tau_{TQ} \circ \delta\gamma' = \delta\gamma \quad \text{and } \tau_{Q*} \circ \delta\gamma' = \gamma', \tag{5}$$

being $\tau_{Q*}:TTQ \twoheadrightarrow TQ$ the differential of τ_Q.

Note that, by formula

$$\delta\gamma(t) = \Delta\gamma_*(\partial/\partial\lambda|_{(t,0)}), \tag{6}$$

every deformation $\Delta\gamma$ gives rise to an infinitesimal variation $\delta\gamma$, and every infinitesimal variation gives rise to a deformation. In fact, fixing a Riemannian structure on Q, for each infinitesimal variation $\delta\gamma$ we have a related deformation $\Delta\gamma(t,\lambda)=\exp_{\gamma(t)}(\lambda\delta\gamma(t))$ satisfying Eq. (6), where $\exp_q:T_qQ \to Q$ is the exponential map related to the given Riemannian structure. From this relation we have the following theorem.

Theorem 3: *A curve $\gamma:[t_1,t_2]\to Q$ is a trajectory of the Lagrangian system (Q,L), i.e., it satisfies Eq. (3), if and only if for all its infinitesimal variations $\delta\gamma$ we have*

$$\int_{t_1}^{t_2} \langle dL(\gamma'(t)),\kappa(\delta\gamma'(t))\rangle dt = 0. \tag{7}$$

Before the proof let us recall the definition of the *canonical involution* $\kappa:TTQ\to TTQ$ (see, for instance, Ref. 23), that appears in Eq. (7), and some of its properties. In local terms, given a chart (U,φ) of Q, we have [recall Eq. (2) again]

$$[\varphi_{**}\circ\kappa\circ\varphi_{**}^{-1}](q,\delta q,\dot{q},\delta\dot{q}) = (q,\dot{q},\delta q,\delta\dot{q}). \tag{8}$$

For an intrinsic definition consider an application

$$\Gamma:(-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon) \to Q : (t,\lambda) \mapsto \Gamma(t,\lambda), \tag{9}$$

and interpret the quantities

$$\Gamma_*(\partial/\partial\lambda|_{(t,0)}) \quad \text{and } \Gamma_*(\partial/\partial t|_{(0,\lambda)})$$

as the values of applications

$$(-\varepsilon,\varepsilon) \to TQ:t \mapsto \Gamma_*(\partial/\partial\lambda|_{(t,0)})$$

and

$$(-\varepsilon,\varepsilon) \to TQ:\lambda \mapsto \Gamma_*(\partial/\partial t|_{(0,\lambda)}),$$

respectively. Every element $V\in TTQ$ can be written as the second derivative

$$V = [\Gamma_*(\partial/\partial\lambda|_{(t,0)})]_*(d/dt|_0)$$

of some Γ as above. Then κ is defined by

$$\kappa(V) = [\Gamma_*(\partial/\partial t|_{(0,\lambda)})]_*(d/d\lambda|_0). \tag{10}$$

Note that κ satisfies

$$\kappa \circ \kappa = id_{TTQ} \quad \text{and} \quad \tau_{TQ} \circ \kappa = (\tau_Q)_*, \tag{11}$$

and accordingly

$$(\tau_Q)_* \circ \kappa = \tau_{TQ}. \tag{12}$$

Proof of Theorem 3: For a given curve γ consider a deformation $\Delta\gamma$, its related family of curves $\Delta\gamma_\lambda$, and their corresponding velocities

$$\Delta\gamma'_\lambda : (t_1, t_2) \to TQ,$$

given by

$$\Delta\gamma'_\lambda(t) = (\Delta\gamma_\lambda)_*(d/dt|_t) = \Delta\gamma_*(\partial/\partial t|_{(t,\lambda)}). \tag{13}$$

Consider also the related infinitesimal variation

$$\delta\gamma(t) = \Delta\gamma_*(\partial/\partial\lambda|_{(t,0)}) \tag{14}$$

[recall Eq. (6)]. Clearly, in order to prove the theorem, since

$$\left.\frac{dS[\Delta\gamma_\lambda]}{d\lambda}\right|_{\lambda=0} = \int_{t_1}^{t_2} \left.\frac{d}{d\lambda} L(\Delta\gamma'_\lambda(t))\right|_{\lambda=0} dt,$$

it is enough to show that the equality

$$\left.\frac{d}{d\lambda} L(\Delta\gamma'_\lambda(t))\right|_{\lambda=0} = \langle dL(\gamma'(t)), \kappa(\delta\gamma'(t))\rangle \tag{15}$$

holds. On one hand, we have

$$\left.\frac{d}{d\lambda} L(\Delta\gamma'_\lambda(t))\right|_{\lambda=0} = \langle dL(\gamma'(t)), (\Delta\gamma'_\lambda(t))_*(d/d\lambda|_0)\rangle = \langle dL(\gamma'(t)), [\Delta\gamma_*(\partial/\partial t|_{(t,\lambda)})]_*(d/d\lambda|_0)\rangle,$$

where we used equality $\Delta\gamma'_0(t) = \gamma'(t)$, and the differential $(\Delta\gamma'_\lambda(t))_*$ has been constructed by regarding $\Delta\gamma'_\lambda(t)$ as a function of λ only [see Eq. (13) for the second identity]. On the other hand, using the definition of κ [see Eq. (10)], and Eqs. (4) and (14),

$$[\Delta\gamma_*(\partial/\partial t|_{(t,\lambda)})]_*(d/d\lambda|_0) = \kappa([\Delta\gamma_*(\partial/\partial\lambda|_{(t,0)})]_*(d/dt|_t)) = \kappa(\delta\gamma_*(d/dt|_t)) = \kappa(\delta\gamma'(t)). \tag{16}$$

This ends the proof. ■

B. Euler-Lagrange equations

Local expression. It is well known that a curve γ is a trajectory of (Q,L) if and only if for every local chart (U,φ) of Q the curve $q(t) = \varphi \circ \gamma(t)$ satisfies (in the open interval where the composition $\varphi \circ \gamma$ is defined)

$$\frac{d}{dt}\left(\frac{\partial(L \circ \varphi_*^{-1})}{\partial \dot{q}^i}(q(t), \dot{q}(t))\right) - \frac{\partial(L \circ \varphi_*^{-1})}{\partial q^i}(q(t), \dot{q}(t)) = 0, \tag{17}$$

for $i = 1, \ldots, n$. Equations (17) are the local expression of the Euler-Lagrange (EL) equations for (Q,L). Choosing an affine connection on Q we can write a coordinate-free expression of them. In order to do that, we need some definitions and results about affine connections.

The isomorphism β. Let us consider a vector bundle $\Pi: \mathcal{U} \twoheadrightarrow M$, with base manifold M and projection Π, and fix an affine connection $\nabla: \mathfrak{X}(M) \times \Gamma(\mathcal{U}) \to \Gamma(\mathcal{U})$. Related to the latter we can define a vector bundle isomorphism

$$\beta : T\mathcal{U} \rightarrow \mathcal{U} \oplus TM \oplus \mathcal{U},$$

given as follows. Denote, as usual, by $\tau_{\mathcal{U}} : T\mathcal{U} \twoheadrightarrow \mathcal{U}$ the canonical projection and by $\Pi_* : T\mathcal{U} \twoheadrightarrow TM$ the differential of Π. Given $V \in T\mathcal{U}$, consider a curve $W : (-\varepsilon, \varepsilon) \rightarrow \mathcal{U}$ passing through $\tau_{\mathcal{U}}(V)$ and with tangent V at $t=0$, i.e.,

$$W_*(d/dt|_0) = V.$$

Now define

$$\beta(V) = \tau_{\mathcal{U}}(V) \oplus \Pi_*(V) \oplus \frac{D}{Dt} W(0). \tag{18}$$

For the inverse

$$\beta^{-1} : \mathcal{U} \oplus TM \oplus \mathcal{U} \rightarrow T\mathcal{U},$$

given $X \oplus Y \oplus Z$ belonging to $\mathcal{U} \oplus TM \oplus \mathcal{U}$, let us fix $W : (-\varepsilon, \varepsilon) \rightarrow \mathcal{U}$ such that

$$W(0) = X, \quad (\Pi \circ W)_*(d/dt|_0) = Y \quad \text{and} \quad \frac{D}{Dt} W(0) = Z,$$

and define

$$\beta^{-1}(X \oplus Y \oplus Z) = W_*(d/dt|_0).$$

It can be shown that the definitions of β and β^{-1} are independent of W.

Fixing a vector $X \in \mathcal{U}$, we have applications

$$\beta_X : T_X\mathcal{U} \rightarrow TM \oplus \mathcal{U} \quad \text{and} \quad \beta_X^{-1} : TM \oplus \mathcal{U} \rightarrow T_X\mathcal{U},$$

such that

$$\beta_X(V) = \Pi_*(V) \oplus \frac{D}{Dt} W(0) \quad \text{and} \quad \beta_X^{-1}(Y \oplus Z) = W_*(d/dt|_0), \tag{19}$$

and their transpose

$$\beta_X^* : T^*M \oplus \mathcal{U}^* \rightarrow T_X^*\mathcal{U},$$

$$\beta_X^{*-1} : T_X^*\mathcal{U} \rightarrow T^*M \oplus \mathcal{U}^*.$$

Definition 4: *Let $\mathcal{V}$ be another fiber bundle on M and $f : \mathcal{U} \rightarrow \mathcal{V}$ a fiber-preserving map. We define the fiber derivative of f as the application $\mathbb{F}f : \mathcal{U} \rightarrow \mathcal{U}^* \times_M \mathcal{V}^*$ such that*

$$\mathbb{F}f(X)(Y,Z) = \left\langle \left. \frac{D}{Ds} F(X + sY) \right|_{s=0}, Z \right\rangle, \quad X, Y \in \mathcal{U}, \quad Z \in \mathcal{V}, \tag{20}$$

and the base derivative $\mathbb{B}f : \mathcal{U} \rightarrow T^*M \times_M \mathcal{V}^*$ by

$$\mathbb{B}F(X)(Y,Z) = \left\langle \left. \frac{D}{Ds} F(W(s)) \right|_{s=0}, Z \right\rangle, \quad X \in \mathcal{U}, \quad Y \in TM, \quad Z \in \mathcal{V}, \tag{21}$$

where W is a curve such that

$$W(0) = X, \quad (\Pi \circ W)_*(d/dt|_0) = Y \quad \text{and} \quad \frac{D}{Dt} W(0) = 0.$$

In definition above if $\mathcal{V}$ is the trivial line bundle $M\times\mathbb{R}$, we can think of f as a function $f:\mathcal{U}\to\mathbb{R}$, and the fiber and base derivatives can be described as fiber-preserving applications $\mathbb{F}f:\mathcal{U}\to\mathcal{U}^*$ and $\mathbb{B}f:\mathcal{U}\to T^*M$ given by

$$\langle \mathbb{F}f(X),Z\rangle = \left.\frac{df(X+sZ)}{ds}\right|_{s=0} \tag{22}$$

and

$$\langle \mathbb{B}f(X),Y\rangle = \left.\frac{df(W(s))}{ds}\right|_{s=0}, \tag{23}$$

respectively. Note that, in this case, $\mathbb{F}f$ is independent of ∇ but $\mathbb{B}f$ is not. The next result follows immediately.

Lemma 5: *Given $f:\mathcal{U}\to\mathbb{R}$ and $X\in\mathcal{U}$ we have*

$$\beta_X^{*-1}(df(X)) = \mathbb{B}f(X)\oplus\mathbb{F}f(X). \tag{24}$$

Coordinate-free expression. We are now in position to write an explicit global expression for Eq. (17).

From an affine connection ∇ on Q we can define, as above, a linear bundle isomorphism

$$\beta:TTQ\to TQ\oplus TQ\oplus TQ,$$

taking $\mathcal{U}=TQ$ and $\Pi=\tau_Q$. It is immediate that for the velocity $\delta\gamma'$ of a variation we have

$$\beta(\delta\gamma'(t)) = \delta\gamma(t)\oplus\gamma'(t)\oplus\frac{D}{Dt}\delta\gamma(t). \tag{25}$$

The following result describes the action of κ on $TQ\oplus TQ\oplus TQ$.

Lemma 6: *If $T:\mathfrak{X}(Q)\times\mathfrak{X}(Q)\to\mathfrak{X}(Q)$ denotes the torsion of ∇ then*

$$\kappa_\beta\equiv\beta\circ\kappa\circ\beta^{-1}:TQ\oplus TQ\oplus TQ\to TQ\oplus TQ\oplus TQ$$

is given by

$$\kappa_\beta(X\oplus Y\oplus Z) = Y\oplus X\oplus(Z+T(X,Y)). \tag{26}$$

For simplicity, we shall assume from now on that ∇ is torsion-free.

The following theorem will be demonstrated later (see Theorem 17), in a more general context.

Theorem 7: *A curve $\gamma:[t_1,t_2]\to Q$ is a trajectory of (Q,L) if and only if the EL equation*

$$-\frac{D}{Dt}\mathbb{F}L(\gamma'(t))+\mathbb{B}L(\gamma'(t))=0 \tag{27}$$

holds.

Remark 8: *Equation (27) can be obtained as a particular case of the Lagrange-Poincaré equation, where the symmetry group is the trivial Lie group $G=\{e\}$ (see Ref. 18). If we relax the torsion-free condition for ∇, then EL equation takes the form*

$$-\frac{D}{Dt}\mathbb{F}L(\gamma'(t))+\mathbb{B}L(\gamma'(t))+\langle\mathbb{F}L(\gamma'(t)),T(\gamma'(t),\cdot)\rangle=0,$$

as it has been shown in Ref. 24.

C. EL operator

As it is well known the EL operator $\mathcal{EL}(L)$ is canonically defined as an application from $T^{(2)}Q$ to T^*Q (see Ref. 18). It can be easily shown that, upon the choice of an affine connection, we can also write $\mathcal{EL}(L)$ as an application

$$\mathcal{EL}(L):TQ \oplus TQ \to T^*Q,$$

given by

$$\mathcal{EL}(L)(X,Y) = -\langle \mathbb{F}(\mathbb{F}L)(X),Y\rangle - \langle \mathbb{B}(\mathbb{F}L)(X),X\rangle + \mathbb{B}L(X), \tag{28}$$

for all $X,Y \in TQ$ [see Eqs. (20) and (21) for the definition of fiber and base derivatives of an application from TQ to T^*Q, as $\mathbb{F}L$].

Now, we shall write equations of motion of a Lagrangian system in terms of the operator $\mathcal{EL}(L)$, as given in Eq. (28). In order to do that let us recall that, by Lemma 6.6 of Ref. 25, given a curve Γ on TQ with $\tau_Q \circ \Gamma = \gamma$, we have

$$\frac{D}{Dt}F(\Gamma(t)) = \left\langle \mathbb{F}F(\Gamma(t)), \frac{D}{Dt}\Gamma(t) \right\rangle + \langle \mathbb{B}F(\Gamma(t)), \gamma'(t)\rangle. \tag{29}$$

Theorem 9: *A curve* $\gamma:[t_1,t_2]\to Q$ *is a trajectory of* (Q,L) *if and only if*

$$\mathcal{EL}(L)\left(\gamma'(t), \frac{D}{Dt}\gamma'(t)\right) = 0, \qquad \forall\, t \in (t_1,t_2).$$

In other terms γ *is a trajectory if and only if (iff), for all* $t \in (t_1,t_2)$,

$$-\left\langle \mathbb{F}(\mathbb{F}L)(\gamma'(t)), \frac{D}{Dt}\gamma'(t) \right\rangle - \langle \mathbb{B}(\mathbb{F}L)(\gamma'(t)),\gamma'(t)\rangle + \mathbb{B}L(\gamma'(t)) = 0.$$

Proof: It is enough to use Eq. (29) for $F = \mathbb{F}L$ and $\Gamma = \gamma'$. ■

III. HIGHER ORDER CONSTRAINED LAGRANGIAN SYSTEMS

In Ref. 5 a class of dynamical systems has been defined such that the principles of D'Alembert or Chetaev do not apply. They are Lagrangian systems with higher order kinematic constraints, which in this paper will be called higher order constrained systems (HOCSs). This kind of systems includes rubber wheels and other viscoelastic rolling bodies. In this section we shall recall the definition given in Ref. 5, using terminology of the previous section, emphasizing the intrinsic nature of involved objects.

First, we will introduce some technical tools for higher order tangent bundles. For instance, we consider operators

$$\kappa_l:T^{(l)}[TQ] \to T[T^{(l)}Q] \quad \text{and } S_l:T[T^{(l)}Q] \to T[T^{(l)}Q],$$

generalizing the canonical involution κ and the canonical endomorphism S (corresponding to the $l=1$ case), as well as a related lth order version of the Liouville vector field. Operators κ_l (or similar to them) appear also in Refs. 26 and 27. Such objects will be used in the last section, where order-lowering and linearization procedures are developed.

We also study identifications of the type $T^{(l)}Q = TQ \oplus \ldots \oplus TQ$, defined upon the choice of an affine connection on Q, which have been previously studied in Ref. 28.

Finally, we show that the equations of motion for HOCSs can be written as a generalized version of Lagrange-D'Alembert equations.

A. Bundles $T^{(k)}Q$

For $k \geq 0$, let us denote by $T^{(k)}Q$ the kth *order tangent bundle*[14,15] of Q. This is a fiber bundle

$$\tau_Q^{(k)}:T^{(k)}Q \twoheadrightarrow Q$$

such that for each $q \in Q$ the fiber $T_q^{(k)}Q$ is a set of equivalence classes $[\gamma]^{(k)}$ of curves $\gamma:(-\varepsilon,\varepsilon) \to Q$ satisfying $\gamma(0)=q$. The equivalence relation says that $\gamma_1 \sim \gamma_2$ iff, for every chart (U,φ) containing q, the equations

$$\left.\frac{d^s}{dt^s}\right|_{t=0}(\varphi \circ \gamma_1) = \left.\frac{d^s}{dt^s}\right|_{t=0}(\varphi \circ \gamma_2) \quad \text{for } s=0,\ldots,k,$$

are fulfilled. From the definitions we have $T^{(0)}Q=Q$, $T^{(1)}Q=TQ$, $\tau_Q^{(0)}=id_Q$ (the identity map) and $\tau_Q^{(1)}=\tau_Q$ (the canonical projection of TQ onto Q).

For each chart (U,φ) of Q, we have the naturally related charts

$$(T^{(k)}U,T^{(k)}\varphi),$$

with $T^{(k)}\varphi:T^{(k)}U \to \mathbb{R}^{n(k+1)}$ given by

$$T^{(k)}\varphi([\gamma]^{(k)}) = (q^{(0)},\ldots,q^{(k)}),$$

being

$$q^{(j)} = \left.\frac{d^j(\varphi \circ \gamma)}{dt^j}\right|_0 = (q^{1,(j)},\ldots,q^{n,(j)}).$$

For $k=0$ and $k=1$ it follows that $T^{(0)}\varphi=\varphi$ and $T^{(1)}\varphi=T\varphi=\varphi_*$, respectively. Sometimes we shall write, for $i=1,\ldots,n$,

$$q^{i,(0)}=q^i, \quad q^{i,(1)}=\dot{q}^i \text{ or } \delta q^i, \quad \text{and } q^{i,(2)}=\ddot{q}^i.$$

Then,

$$q^{(0)}=q, \quad q^{(1)}=\dot{q} \text{ or } \delta q, \quad \text{and } q^{(2)}=\ddot{q}.$$

Given a curve $\gamma:[t_1,t_2] \to Q$ its *k-lift* is the curve

$$\gamma^{(k)}:(t_1,t_2) \to T^{(k)}Q : t \mapsto [\gamma_t]^{(k)},$$

where $\gamma_t:(-\varepsilon_t,\varepsilon_t) \to Q$ is given by

$$\gamma_t(s) = \gamma(s+t) \quad \text{and } \varepsilon_t = \min\{t-t_1,t_2-t\}.$$

The 1-lift of γ is precisely its velocity $\gamma':(t_1,t_2) \to TQ$.

Remark 10: *Sometimes we will write $\gamma^{(k)}:[t_1,t_2] \to T^{(k)}Q$. We must understand $\gamma^{(k)}(t_1)$ and $\gamma^{(k)}(t_2)$ as the limits of $\gamma^{(k)}(t)$ when t tends to t_1 and t_2, respectively.*

Consider now an application $f:N \to M$. Its *k-lift*

$$T^{(k)}f:T^{(k)}N \to T^{(k)}M$$

is defined by

$$T^{(k)}f([\gamma]^{(k)}) = [f \circ \gamma]^{(k)}.$$

Of course, we have $T^{(1)}f=f_*$.

Some natural applications. Let us indicate by $\tau_Q^{(l,k)}$ the natural submersions

$$\tau_Q^{(l,k)}:T^{(k)}Q \twoheadrightarrow T^{(l)}Q : [\gamma]^{(k)} \mapsto [\gamma]^{(l)} \quad \text{for } l \leq k,$$

given locally as

$$(q^{(0)}, \ldots, q^{(l)}, \ldots, q^{(k)}) \mapsto (q^{(0)}, \ldots, q^{(l)}).$$

Of course, since $T^{(1)}Q=TQ$, we have $\tau_Q^{(0,1)}=\tau_Q$. On the other hand, $\tau_Q^{(l,l)}=id_{T^{(l)}Q}$.

By $\kappa_l : T^{(l)}[TQ] \to T[T^{(l)}Q]$ and $S_l : T[T^{(l)}Q] \to T[T^{(l)}Q]$, with $l \geqslant 1$, we denote the lth *order analog of canonical involution* $\kappa=\kappa_1$ and *canonical endomorphism* $S=S_1$, given in a local chart by

$$(T^{(l)}\varphi)_* \circ \kappa_l \circ [T^{(l)}(\varphi_*)]^{-1} \quad \text{and } (T^{(l)}\varphi)_* \circ S_l \circ (T^{(l)}\varphi)_*^{-1},$$

or equivalently, in coordinates, by

$$((q^{(0)}, \delta q^{(0)}), \ldots, (q^{(l)}, \delta q^{(l)})) \mapsto (q^{(0)}, \ldots, q^{(l)}; \delta q^{(0)}, \ldots, \delta q^{(l)}) \tag{30}$$

and

$$(q^{(0)}, \ldots, q^{(l)}; \delta q^{(0)}, \ldots, \delta q^{(l)}) \mapsto (q^{(0)}, \ldots, q^{(l)}; 0, \delta q^{(0)}, \ldots, \delta q^{(l-1)}), \tag{31}$$

respectively. For $l=0$ we write $\tau_Q^{(k)}=\tau_Q^{(0,k)}$ and take $S_0=\kappa_0=id_{TQ}$. We leave to the reader the proof of the next result.

Lemma 11: *Let $f:Q\to Q$ be a smooth application. Then, for all $l \geqslant 0$*

$$(T^{(l)}f)_* \circ \kappa_l = \kappa_l \circ T^{(l)}(f_*).$$

Let us call j_n the natural embedding

$$j_n : T^{(n)}Q \hookrightarrow T[T^{(n-1)}Q] : [\gamma]^{(n)} \mapsto [\gamma^{(n-1)}]^{(1)}, \tag{32}$$

for $n \geqslant 1$, and take $j_0=id_Q$ for $n=0$. (Note that $j_1=id_{TQ}$.) In a local chart

$$j_n : (q^{(0)}, \ldots, q^{(n)}) \mapsto (q^{(0)}, \ldots, q^{(n-1)}; q^{(1)}, \ldots, q^{(n)}).$$

It can be shown that, for any curve $\gamma:[t_1,t_2]\to Q$,

$$j_2(\gamma^{(2)}(t)) = \gamma''(t), \tag{33}$$

with $\gamma'' : (t_1,t_2) \to TTQ$ given by

$$\gamma''(t) = \gamma'_*(\, d/dt|_t) \in T_{\gamma'(t)}TQ.$$

The following result will be useful later.

Lemma 12: *For all $k,l \geqslant 0$, it can be shown that*

$$(\tau_Q^{(l,k)})_* \circ \kappa_k = \kappa_l \circ \tau_{TQ}^{(l,k)} \tag{34}$$

and

$$(\tau_Q^{(l,k)})_* \circ j_{k+1} = j_{l+1} \circ \tau_{TQ}^{(l+1,k+1)}. \tag{35}$$

In particular,

$$(\tau_Q^{(k)})_* \circ \kappa_k = \tau_{TQ}^{(k)}. \tag{36}$$

Related to S_l let us define the lth *order Liouville vector field*

$$\Delta_l \in \mathfrak{X}[T^{(l)}Q],$$

which in coordinates reads

$$\Delta_l = (q^{(0)}, \ldots, q^{(l)}; 0, q^{(1)}, \ldots, q^{(l)}). \tag{37}$$

Note that Δ_0 is the null section of TQ, and $\Delta_1=\Delta$ is the standard Liouville vector field. In the same way that for $l=1$ we characterize the second order ordinary differential equations (SODEs) as those vector fields $X\in\mathfrak{X}(TQ)$ such that $S(X)=\Delta$, we can characterize the $(l+1)$th order ordinary differential equations [$(l+1)$th order ODE] as those $X \in \mathfrak{X}(T^{(l)}Q)$ such that

$$S_l(X) = \Delta_l. \tag{38}$$

In local terms, X is of the form

$$(q^{(0)}, \ldots, q^{(l)}; q^{(1)}, \ldots, q^{(l-1)}, \chi(q^{(0)}, \ldots, q^{(l-1)})), \tag{39}$$

for some function χ. It is worth mentioning that any nth order ODE X can be written $X = j_n \circ \sigma$ being σ a section of the fiber bundle $T^{(n)}Q \twoheadrightarrow T^{(n-1)}Q$, that is,

$$S_{n-1} \circ j_n \circ \sigma = \Delta_{n-1} \tag{40}$$

for all sections $\sigma: T^{(n-1)}Q \to T^{(n)}Q$.

Finally, consider the natural submersions $\mathfrak{s}_l: T^{(l)}[TQ] \twoheadrightarrow T^{(l)}Q \times_Q TQ$ given by

$$\mathfrak{s}_l(x) = (\tau_{T^{(l)}Q} \circ \kappa_l(x); \tau_{TQ}^{(l)}(x)). \tag{41}$$

In local coordinates we have

$$\mathfrak{s}_l: (q^{(0)}, \delta q^{(0)}, \ldots, q^{(l)}, \delta q^{(l)}) \mapsto (q^{(0)}, \ldots, q^{(l)}; \delta q^{(0)}).$$

In particular, given an infinitesimal variation $\delta\gamma$ of a curve γ it follows that

$$\mathfrak{s}_l(\delta\gamma^{(l)}(t)) = (\gamma^{(l)}(t), \delta\gamma(t)). \tag{42}$$

Structures associated with a given connection. Let us fix an affine connection ∇ on Q. Define the diffeomorphism

$$\zeta_k: T^{(k)}Q \to \underbrace{TQ \oplus \ldots \oplus TQ}_{k \text{ terms}}, \tag{43}$$

as

$$\zeta_k([\gamma]^{(k)}) = \gamma'(0) \oplus \frac{D\gamma'}{Dt}(0) \oplus \ldots \oplus \frac{D^k\gamma'}{Dt^k}(0). \tag{44}$$

Note that given a curve γ we have for its k-lift

$$\zeta_k(\gamma^{(k)}(t)) = \gamma'(t) \oplus \frac{D\gamma'}{Dt}(t) \oplus \ldots \oplus \frac{D^k\gamma'}{Dt^k}(t).$$

Remark 13: Via *the map ζ_k a linear bundle structure is defined on $T^{(k)}Q$.*

For any space $T^{(l)}[TQ]$ there is a diffeomorphism

$$t\zeta_l: T^{(l)}[TQ] \to T^{(l)}Q \times_Q \underbrace{TQ \oplus \ldots \oplus TQ}_{l+1 \text{ terms}}, \tag{45}$$

such that, for a given a curve $\Lambda: (-\varepsilon, \varepsilon) \to TQ$, we have

$$[\Lambda]^{(l)} \mapsto \left([\gamma]^{(l)}, \Lambda(0) \oplus \left.\frac{D\Lambda}{Dt}\right|_0 \oplus \ldots \oplus \left.\frac{D^l\Lambda}{Dt^l}\right|_0\right),$$

where $\gamma = \tau_Q \circ \Lambda$. For instance, given a variation $\delta\gamma$ of a curve γ in Q

$$t\zeta_l(\delta\gamma^{(l)}(t)) = \left(\gamma^{(l)}(t), \delta\gamma(t) \oplus \frac{D\delta\gamma}{Dt}(t) \oplus \ldots \oplus \frac{D^l\delta\gamma}{Dt^l}(t)\right).$$

Fixing a coordinate chart and taking $l=1$ we have

$$t\zeta_2[(q^{(0)},\delta q^{(0)}),(q^{(1)},\delta q^{(1)})]=[(q^{(0)},q^{(1)});\delta q^{(0)},\delta q^{(1)}+\Gamma\delta q^{(0)}q^{(1)}],$$

where Γ indicates the Christoffel symbols associated with the chosen connection. It is worth mentioning that $t\zeta_0=id_{TQ}$ and that $t\zeta_1=\beta\circ\kappa$. Now let us prove a useful result.

Lemma 14: *Given $q\in Q$, $x\in T_q^{(l)}Q$, and $V_i\in T_qQ$, $i=0,\ldots,l$, we have that*

$$\tau_{TQ}^{(l)}\circ(t\zeta_l)^{-1}(x,V_0\oplus V_1\oplus\ \ldots\ \oplus V_l)=V_0.$$

Proof: Fix $q\in Q$, $x\in T_q^{(2)}Q$, $V_i\in T_qQ$, and a local chart (U,φ) such that $q\in U$, and let us write

$$q=q^{(0)},\quad x=(q^{(0)},\ldots,q^{(l)}),\quad \text{and } V_i=(q^{(0)};\delta q^{(i)}).$$

It is easy to see that

$$(t\zeta_2)^{-1}(x,V_0\oplus\ \ldots\ \oplus V_l)=[(q^{(0)},\delta q^{(0)}),(q^{(1)},\delta q^{(1)}-\Gamma\delta q^{(0)}q^{(1)}),\ldots].$$

Then,

$$\tau_{TQ}^{(l)}\circ(t\zeta_l)^{-1}(x,V_0\oplus V_1\oplus\ \ldots\ \oplus V_l)=(q^{(0)},\delta q^{(0)})=V_0,$$

as we wanted to show. ■

B. HOCS

Definition 15: *Given a manifold Q let us consider the triples (L,C_K,C_V) with*

$$L:TQ\to\mathbb{R},\quad C_K\subset T^{(k)}Q,\quad C_V\subset T^{(l)}Q\times_Q TQ,\quad k,l\geqslant 0.$$

We assume that C_K is a submanifold of $T^{(k)}Q$. Regarding C_V, we assume that it is a subset such that, for every $q\in Q$ and $\eta\in T_q^{(l)}Q$, the subset

$$C_V(\eta)\equiv C_V\cap(\{\eta\}\times T_qQ)\subset\{\eta\}\times T_qQ,$$

naturally identified with a subset of T_qQ, is either empty or a linear subspace. We shall refer to these triples as Lagrangian systems with higher order constraints, or simply, HOCSs, with Lagrangian function L, kinematic constraints C_K and variational constraints C_V. Elements of C_V will be called virtual displacements. We will say that $\gamma:[t_1,t_2]\to Q$ is a trajectory of (L,C_K,C_V) if the following conditions are satisfied:

1. $\gamma^{(k)}(t)\in C_K$, $\forall t\in(t_1,t_2)$;
2. *for all variations $\delta\gamma$ such that $(\gamma^{(l)}(t),\delta\gamma(t))\in C_V$, or equivalently $\delta\gamma(t)\in C_V(\gamma^{(l)}(t))$, $\forall t\in(t_1,t_2)$, we have*

$$\int_{t_1}^{t_2}\langle dL(\gamma'(t)),\kappa(\delta\gamma'(t))\rangle dt=0. \tag{46}$$

Definition 16: *For each $q\in Q$ and $\eta\in T_q^{(l)}Q$, consider the annihilator*

$$F_V(\eta)=(C_V(\eta))^o\subset T_q^*Q,$$

whenever $C_V(\eta)$ is nonempty. Such subspaces give rise to a subset $F_V\subset T^{(l)}Q\times_Q T^*Q$ that we will call the space of constraint forces.

Some remarks on the definition of C_V are listed as follows.

1. Note that we are not asking for C_V to be a submanifold of $T^{(l)}Q\times_Q TQ$. This situation occurs, for instance, for systems with friction (see Sec. IV).
2. For $l=0$ we shall identify $C_V\subset Q\times_Q TQ$ with the subset of TQ defined by subspaces

$$C_V(q) = C_V \cap (\{q\} \times T_q Q).$$

If C_V is a submanifold then the above subspaces define a distribution of constant dimension. Similar remarks hold for F_V.

3. If a curve $\gamma:[t_1,t_2] \to Q$ such that $\gamma^{(k)}(t) \in C_K$ has no variations compatible with C_V, in the sense that there exists $t_o \in (t_1,t_2)$ such that $C_V(\gamma^{(l)}(t_o))$ is empty, then Eq. (46) does not impose any condition, and accordingly, γ is automatically a trajectory of the system. We can avoid this situation by imposing some additional conditions on C_K and C_V (which are fulfilled by all the examples presented in this paper), as those discussed in the Appendix.
4. If $k=0$, i.e., $C_K \subset Q$, we can replace C_K by TC_K. On the other hand, if $l=0$, i.e., $C_V \subset TQ$, we can replace C_V by $TQ \times_Q C_V$. It is clear that the resulting systems are equivalent to the original ones.

Now let us consider some examples. Fix a manifold Q and a Lagrangian function $L:TQ \to \mathbb{R}$.

- A Lagrangian system (Q,L) corresponds to the $k=l=0$ case with $C_K=Q$ and $C_V=TQ$ (recall the identification mentioned in the second remark on C_V). Obviously F_V is the zero subbundle of T^*Q (there is no constraint force).
- Lagrangian systems with holonomic constraints satisfying D'Alembert's principle are $k=l=0$ cases with C_K a submanifold of Q and $C_V=TC_K$ (recall the second remark above). It is easy to show that each one of these cases can be reduced to a Lagrangian system $(C_K,\, L|_{TC_K})$. If C_K is locally defined by equations

$$w^a(q) = 0, \quad a = 1, \ldots, k,$$

then C_V is given by (sum over repeated index convention is assumed)

$$\frac{\partial w^a}{\partial q^i}(q)\, \delta q^i = 0, \quad a = 1, \ldots, k,$$

with q belonging to C_K. The corresponding space $F_V \subset T^*_{C_K}Q$ is given by covectors f with coordinates

$$f_i = \lambda_a \frac{\partial w^a}{\partial q^i}(q), \quad \lambda_a \in \mathbb{R},$$

for $q \in C_K$.
- Lagrangian systems with linear constraints satisfying D'Alembert's principle are $k=1$, $l=0$ cases where C_K is a distribution along a submanifold $Q_1 = \tau_Q(C_K) \subset Q$ and

$$C_V = C_K \cap TQ_1 = C_K \cap T[\tau_Q(C_K)].$$

Note that C_V is a distribution on Q_1. If C_V is involutive then for each integral submanifold $C \subset Q_1$ we have a Lagrangian system $(C,\, L|_{TC})$. For local expressions, see example below.
- Lagrangian systems with affine constraints satisfying D'Alembert's principle are $k=1$, $l=0$ cases with C_K an affine subbundle of TQ along a submanifold $Q_1 \subset Q$, such that we can write $C_K = C_K^{\mathrm{vec}} + X$. Here C_K^{vec} and X are a distribution and a section, respectively, along Q_1 and $C_V = C_K^{\mathrm{vec}} \cap TQ_1$.

Assume that $Q_1 = Q$. Then C_K is given locally by equations

$$w_i^a(q)\dot{q}^i = \gamma^a(q), \quad a = 1, \ldots, k,$$

while C_V is a distribution on Q defined by

$$w_i^a(q)\, \delta q^i = 0, \quad a = 1, \ldots, k.$$

Of course the linear case corresponds to $\gamma^a = 0$, for all a.

- Lagrangian systems with (generically) nonlinear constraints satisfying Chetaev's principle are $k=l=1$ cases with C_K a submanifold of TQ. In order to construct C_V consider the distribution along

$$C_K \cap TQ_1 \subset TQ$$

[where again $Q_1=\tau_Q(C_K)$] defined as

$$\mathcal{C}_V = S^{-1}[T(C_K \cap TQ_1)],$$

being $S=S_1:TTQ\to TTQ$ the vertical endomorphism [see Eq. (31)]. The latter is given in coordinates by

$$(q,\dot{q};\delta q,\delta\dot{q}) \mapsto (q,\dot{q};0,\delta q).$$

Finally, consider the natural submersion [see Eq. (41) for $l=1$]

$$\mathfrak{s}_1:TTQ \twoheadrightarrow TQ\times_Q TQ,$$

such that $\mathfrak{s}_1(x)=(\tau_{TQ}(x),\tau_{Q*}(x))$. In coordinates $\mathfrak{s}_1(q,\dot{q},\delta q,\delta\dot{q})=(q,\dot{q},\delta q)$. Now define

$$C_V = \mathfrak{s}_1(\mathcal{C}_V) \subset (C_K \cap TQ_1)\times_Q TQ.$$

Assuming that $Q_1=Q$, if kinematic constraints C_K are

$$w^a(q,\dot{q}) = 0, \quad a=1,\ldots,k, \tag{47}$$

then variational constraints $C_V\subset C_K\times_Q TQ$ are given by δq such that

$$\frac{\partial w^a}{\partial \dot{q}^i}(q,\dot{q})\,\delta q^i = 0, \quad a=1,\ldots,k.$$

Obviously the affine case corresponds to

$$w^a(q,\dot{q}) = w_i^a(q)\dot{q}^i - \gamma^a(q).$$

Constraint forces $F_V\subset C_K\times_Q T^*Q$ are given, for each $(q,\dot{q})\in C_K$, by covectors of components

$$f_i = \lambda_a \frac{\partial w^a}{\partial \dot{q}^i}(q,\dot{q}), \quad \lambda_a \in \mathbb{R},$$

which in the affine case take the form $f_i=\lambda_a w_i^a(q)$.

The previous examples define a class of systems whose virtual displacements C_V are derived from kinematic constraints C_K. For these systems, one only needs the data (L,C_K). Systems that do not belong to this class appear, for instance, in servomechanisms, as we illustrate below.

- The controlled inverted cart-pendulum, with a control strategy as explained in Refs. 8, 13, and 29, is a simple Lagrangian system with $Q=\mathbb{R}\times S^1$ as configuration space. The subset $C_K\subset TQ$ is an affine subbundle and $C_V\subset TQ=T\mathbb{R}\times TS^1$ a distribution

$$C_V = \mathbb{O}_{\mathbb{R}} \times TS^1,$$

being $\mathbb{O}_{\mathbb{R}}$ the zero subbundle of $T\mathbb{R}$. Locally, if we take x and θ as being the coordinates for the factors $\mathbb{R}$ and S^1 of configuration space, respectively, then C_V is given by the equation $\delta x=0$. The latter does not depend at all on the form of C_K.

This example will be developed in more detail in Sec. IV D.

- Generalized nonholonomic systems (GNHSs), defined in Ref. 8, constitute a subclass of HOCSs with $k=1$ and $l=0$ or 1. All examples given in this section are GNHSs.

Interesting examples of higher order constraints, which are not GNHSs, are given by Rocard's model of a pneumatic tire,[5] and by an alternative description of systems with friction (see Sec. IV).

C. Generalized Lagrange-D'Alembert equations

Choose a connection ∇ on Q and assume for simplicity that it is torsion-free. As we have explained before, one has a bundle isomorphism

$$\beta: TTQ \to TQ \oplus TQ \oplus TQ.$$

Theorem 17: *A curve* $\gamma:[t_1,t_2]\to Q$ *is a trajectory of a HOCS given by the triple* (L, C_K, C_V) *if and only if* $\gamma^{(k)}(t)\in C_K$ *and*

$$\left\langle -\frac{D}{Dt}\mathbb{F}L(\gamma'(t)) + \mathbb{B}L(\gamma'(t)), \delta\gamma(t)\right\rangle = 0, \quad \forall\, t \in (t_1,t_2), \tag{48}$$

for all variations $\delta\gamma$ *such that* $\delta\gamma(t)\in C_V(\gamma^{(l)}(t))$. *We shall call* Eq. (48) *generalized Lagrange-D'Alembert equations*.

Proof: From Eq. (46), introducing in the second argument of the pairing the identity $\beta^{-1}\circ\beta$, we have that a trajectory $\gamma:[t_1,t_2]\to Q$ must satisfy

$$\int_{t_1}^{t_2}\langle dL(\gamma'(t)), \beta^{-1}_{\gamma'(t)}\circ\kappa_\beta\circ\beta(\delta\gamma'(t))\rangle dt = 0,$$

for all variation $\delta\gamma$ in C_V. Transposing $\beta^{-1}_{\gamma'(t)}$ and using Eqs. (24)–(26),

$$\langle dL_{\gamma'}, \beta^{-1}_{\gamma'}\circ\kappa_\beta\circ\beta(\delta\gamma')\rangle = \left\langle \beta^{*-1}_{\gamma'}(dL_{\gamma'}), \delta\gamma \oplus \frac{D}{Dt}\delta\gamma\right\rangle = \left\langle \mathbb{B}L(\gamma')\oplus\mathbb{F}L(\gamma'), \delta\gamma\oplus\frac{D}{Dt}\delta\gamma\right\rangle$$
$$= \langle \mathbb{B}L(\gamma'), \delta\gamma\rangle + \left\langle \mathbb{F}L(\gamma'), \frac{D}{Dt}\delta\gamma\right\rangle,$$

where we have omitted t for simplicity. On the other hand,

$$\frac{d}{dt}\langle\mathbb{F}L(\gamma'), \delta\gamma\rangle = \left\langle \mathbb{F}L(\gamma'), \frac{D}{Dt}\delta\gamma\right\rangle + \left\langle \frac{D}{Dt}\mathbb{F}L(\gamma'), \delta\gamma\right\rangle, \tag{49}$$

by definition of the covariant derivative in T^*Q. Since $\delta\gamma(t_1)$ and $\delta\gamma(t_2)$ belong to the zero subbundle of TQ, we have

$$\int_{t_1}^{t_2}\frac{d}{dt}\langle\mathbb{F}L(\gamma'), \delta\gamma\rangle dt = \langle\mathbb{F}L(\gamma'(t_1)), \delta\gamma(t_1)\rangle - \langle\mathbb{F}L(\gamma'(t_2)), \delta\gamma(t_2)\rangle = 0.$$

As a consequence γ satisfies

$$\int_{t_1}^{t_2}\left\langle -\frac{D}{Dt}\mathbb{F}L(\gamma'(t)) + \mathbb{B}L(\gamma'(t)), \delta\gamma(t)\right\rangle dt = 0 \tag{50}$$

for all variations $\delta\gamma$ such that

$$\delta\gamma(t) \in C_V(\gamma^{(l)}(t)),$$

what clearly implies Eq. (48). ■

Remark 18: *If* C_K and C_V *satisfy the assumption in the Appendix* (*see Proposition 39), and if in addition* C_V *is a submanifold, then Eq.* (48) *is equivalent to*

$$-\frac{D}{Dt}\mathbb{F}L(\gamma'(t)) + \mathbb{B}L(\gamma'(t)) \in F_V(\gamma^{(l)}(t)), \quad \forall\, t \in (t_1,t_2). \tag{51}$$

This is because, on one hand, for all $t_o \in (t_1,t_2)$ the subset $C_V(\gamma^{(l)}(t_o))$ is nonempty (by the mentioned assumption), and, on the other hand, given $v \in C_V(\gamma^{(l)}(t_o))$ there exists a variation $\delta\gamma$ in C_V such that $\delta\gamma(t_0)=v$ (since C_V is a submanifold). As a consequence, for each t_o the quantity

$$D(\mathbb{F}L(\gamma'(t_o))Dt) - \mathbb{B}L(\gamma'(t_o))$$

must annihilate all of $C_V(\gamma^{(l)}(t_o))$.

In local terms, it is easy to show that a curve γ is a trajectory of (L,C_K,C_V) if and only if for every chart (U,φ) of Q the curve $q(t)=\varphi\circ\gamma(t)$ satisfies (in the open interval where the composition $\varphi\circ\gamma$ is defined)

$$(q^{(0)}(t), \ldots ,q^{(k)}(t)) \in T^{(k)}\varphi(\,C_K|_U) \tag{52}$$

and

$$\left(\frac{d}{dt}\left(\frac{\partial(L\circ\varphi_*^{-1})}{\partial\dot{q}^i}(q(t),\dot{q}(t))\right) - \frac{\partial(L\circ\varphi_*^{-1})}{\partial q^i}(q(t),\dot{q}(t))\right)\delta q^i(t) = 0,$$

for all curves $\delta q(t)$ such that

$$(q^{(0)}(t), \ldots ,q^{(l)}(t), \delta q(t)) \in (T^{(l)}\varphi \times \varphi_*)(\,C_V|_U), \tag{53}$$

or equivalently

$$\delta q(t) \in \varphi_*[C_V((T^{(l)}\varphi)^{-1}(q^{(0)}(t), \ldots ,q^{(l)}(t)))]. \tag{54}$$

In a simpler form, conditions (52) and (54) can be written as

$$(q^{(0)}(t), \ldots ,q^{(k)}(t)) \in C_K \quad \text{and}\ \delta q(t) \in C_V((q^{(0)}(t), \ldots ,q^{(l)}(t))).$$

In terms of the EL operator [see Eq. (28)] we have the following.

Theorem 19: *A curve $\gamma:[t_1,t_2]\to Q$ is a trajectory of L,C_K,C_V if and only if $\gamma^{(k)}(t) \in C_K$ and*

$$\left\langle \mathcal{EL}(L)\left(\gamma'(t),\frac{D}{Dt}\gamma'(t)\right),\delta\gamma(t)\right\rangle = 0, \quad \forall\, t \in (t_1,t_2),$$

or equivalently (omitting t, just for brevity),

$$\left\langle -\left\langle \mathbb{F}(\mathbb{F}L)(\gamma'),\frac{D}{Dt}\gamma'\right\rangle - \langle \mathbb{B}(\mathbb{F}L)(\gamma'),\gamma'\rangle + \mathbb{B}L(\gamma'),\delta\gamma\right\rangle = 0$$

for all $\delta\gamma$ such that $\delta\gamma(t) \in C_V(\gamma^{(l)}(t))$.

Under the conditions of Remark 18 the equations above can be rewritten as

$$\mathcal{EL}(L)\left(\gamma'(t),\frac{D}{Dt}\gamma'(t)\right) \in F_V(\gamma^{(l)}(t)), \quad \forall\, t \in (t_1,t_2).$$

As we have said before, $\mathcal{EL}(L)$ is actually defined as an application with domain $T^{(2)}Q$. In order to relate both expressions of $\mathcal{EL}(L)$ we only must identify bundles $T^{(2)}Q$ and $TQ\oplus TQ$ through the isomorphism [see Eqs. (43) and (44)]

$$\zeta_2:T^{(2)}Q \to TQ \oplus TQ : [\gamma]^{(2)} \mapsto \gamma'(0) \oplus \left.\frac{D\gamma'}{Dt}\right|_0 .$$

IV. SYSTEMS WITH FRICTION FORCES

Consistent with the point of view of the present paper, outlined in the Introduction, we will show in this section how to encode friction forces as being constraint forces. Also, HOCSs, with external forces are considered. A particle moving in a viscous fluid and the inverted cart-pendulum with friction are studied.

A. Friction forces as constraints

Let us consider a Lagrangian system (Q,L) subjected to friction forces. The latter are given by the fiber derivative of the so-called Rayleigh dissipation function[16,17] $\mathcal{F}:TQ\to\mathbb{R}$, which in turn is defined by the formula

$$\mathcal{F}(v)=\frac{1}{2}\mathcal{R}(v,v),$$

where $\mathcal{R}:TQ\times TQ\to\mathbb{R}$ is a positive-semidefinite tensor (i.e., $\mathcal{R}(v,v)\geqslant 0$ for all $v\in TQ$). Sometimes $\mathcal{R}$, instead of $\mathcal{F}$, is taken as being the basic data (see Ref. 22). Note that $\mathcal{F}$ satisfies

$$\mathcal{F}(v)=\frac{1}{2}\langle\mathbb{F}\mathcal{F}(v),v\rangle,\qquad\forall\, v\in TQ,\tag{55}$$

or equivalently

$$\mathbb{F}\mathcal{F}=\mathcal{R}^{\flat},$$

where $\mathcal{R}^{\flat}:TQ\to T^{*}Q$ is, as usual, defined by

$$\langle\mathcal{R}^{\flat}(v),\varpi\rangle=\mathcal{R}(v,\varpi),\qquad\forall\, v,\varpi\in TQ.$$

In local coordinates $\mathcal{F}$ can be written as

$$\mathcal{F}(q,\dot{q})=\frac{1}{2}A_{ij}(q)\dot{q}^i\dot{q}^j,$$

with $A_{ij}(q)$ a positive-semidefinite matrix.

By definition of systems with friction a curve $\gamma:[t_1,t_2]\to Q$ is a trajectory if and only if for all infinitesimal variations $\delta\gamma:[t_1,t_2]\to TQ$ we have

$$\int_{t_1}^{t_2}(\langle dL(\gamma'(t)),\kappa(\delta\gamma'(t))\rangle-\langle\mathbb{F}\mathcal{F}(\gamma'(t)),\delta\gamma(t)\rangle)dt=0.\tag{56}$$

Fixing a connection and using Theorem 7 the last equations can be rewritten as

$$-\frac{D}{Dt}\mathbb{F}L(\gamma'(t))+\mathbb{B}L(\gamma'(t))-\mathbb{F}\mathcal{F}(\gamma'(t))=0.\tag{57}$$

In this section we are going to show that each one of these systems can be described in terms of a related HOCS. In other words, solutions to Eq. (57) can be obtained by studying trajectories of an appropriate HOCS.

Theorem 20: *The trajectories of a system with friction, defined by the data*

$$L,\mathcal{F}:TQ\to\mathbb{R}$$

[i.e., the solutions to Eq. (57)], are trajectories of a HOCS $(L,C_K^{\mathcal{F}},C_V^{\mathcal{F}})$ *with*

$$C_K^{\mathcal{F}}=\{a\in T^{(2)}Q:\langle dE(\tau_Q^{(1,2)}(a)),j_2(a)\rangle=-2\mathcal{F}\circ\tau_Q^{(1,2)}(a)\}\tag{58}$$

and

$$C_V^{\mathcal{F}} = \{(v,\varpi) \in TQ \times_Q TQ : \langle \mathbb{F}\mathcal{F}(v), \varpi \rangle = 0\}, \tag{59}$$

where

$$E{:}TQ \to \mathbb{R} : v \mapsto \langle \mathbb{F}L(v), v \rangle - L(v)$$

is the energy function *associated with L. Moreover, if* $\gamma{:}[t_1,t_2] \to Q$ *is a trajectory of the HOCS given above, such that the set of zeros of the function* $\mathbb{F}\mathcal{F}\circ\gamma' : (t_1,t_2) \to T^*Q$ *is empty or is given by the union of a finite number of closed intervals (what eventually includes isolated points), then* γ *is also a trajectory of the system with friction.*

Some remarks are listed as follows.

1. In the analytic category a system with friction and its related HOCS, defined in theorem above, are equivalent. This is because the condition on the set of zeros of $\mathbb{F}\mathcal{F}\circ\gamma'$ stated in the last part of the theorem is automatically satisfied. We will not prove this assertion here.
2. As it is well known, the energy function can also be written in terms of the Liouville vector field Δ. In fact, it is easy to show that

$$\Delta \cdot L(v) = \langle \mathbb{F}L(v), v \rangle, \tag{60}$$

   and accordingly $E = \Delta \cdot L - L$.
3. In local coordinates, $C_K^{\mathcal{F}}$ is formed by points $(q,\dot{q},\ddot{q})$ such that

$$\frac{\partial E}{\partial q^i}(q,\dot{q})\dot{q}^i + \frac{\partial E}{\partial \dot{q}^i}(q,\dot{q})\ddot{q}^i = -2\mathcal{F}(q,\dot{q})(= -A_{ij}(q)\dot{q}^i\dot{q}^j), \tag{61}$$

   and $C_V^{\mathcal{F}}$ is formed by $(q,\delta q,\dot{q},\delta\dot{q})$ satisfying

$$0 = \frac{\partial \mathcal{F}}{\partial \dot{q}^i}(q,\dot{q})\,\delta q^i (= A_{ij}(q)\dot{q}^i\delta q^j). \tag{62}$$

4. Note that

$$C_V^{\mathcal{F}}(v) = \langle \mathbb{F}\mathcal{F}(v) \rangle^o,$$

   i.e., $C_V^{\mathcal{F}}(v)$ is the annihilator of the space generated by the vector $\mathbb{F}\mathcal{F}(v)$. Then, the dimension of $C_V^{\mathcal{F}}(v)$ is *not* constant.

To prove the last theorem we need the following lemmas.

Lemma 21: *For a positive-semidefinite tensor* $\varphi{:}TQ \times TQ \to \mathbb{R}$ *and its related linear bundle morphism* $\varphi^\flat : TQ \to T^*Q$, *we have that* $\varphi(v,v)=0$, *for some* $v \in TQ$, *if and only if* $\varphi^\flat(v)=0$. *This implies that*, $\mathcal{F}(v)=0$, *for some* v, *if and only if* $\mathbb{F}\mathcal{F}(v)=0$.

The proof of this lemma is elementary.

Lemma 22: *For any curve* $\gamma{:}[t_1,t_2] \to Q$ *we have*

$$\left\langle -\frac{D}{Dt}\mathbb{F}L(\gamma'(t)) + \mathbb{B}L(\gamma'(t)), \gamma'(t) \right\rangle = -\langle dE(\gamma'(t)), \gamma''(t) \rangle. \tag{63}$$

Proof: Since on one hand

$$\frac{d}{dt}\langle \mathbb{F}L(\gamma'(t)), \gamma'(t) \rangle = \left\langle \mathbb{F}L(\gamma'(t)), \frac{D}{Dt}\gamma'(t) \right\rangle + \left\langle \frac{D}{Dt}\mathbb{F}L(\gamma'(t)), \gamma'(t) \right\rangle$$

[recall Eq. (49)] and on the other hand

$$\frac{d}{dt}\langle \mathbb{F}L(\gamma'(t)), \gamma'(t)\rangle = \frac{d}{dt}[\Delta \cdot L(\gamma'(t))] = \langle d(\Delta \cdot L)_{\gamma'(t)}, \gamma''(t)\rangle$$

[see (60)], it follows that

$$\left\langle -\frac{D}{Dt}\mathbb{F}L(\gamma'(t)), \gamma'(t)\right\rangle = \left\langle \mathbb{F}L(\gamma'(t)), \frac{D}{Dt}\gamma'(t)\right\rangle - \langle d(\Delta \cdot L)_{\gamma'(t)}, \gamma''(t)\rangle.$$

Then

$$\left\langle -\frac{D}{Dt}\mathbb{F}L(\gamma'(t)) + \mathbb{B}L(\gamma'(t)), \gamma'(t)\right\rangle \tag{64}$$

is equal to

$$\left\langle \mathbb{B}L(\gamma'(t)) \oplus \mathbb{F}L(\gamma'(t)), \gamma'(t) \oplus \frac{D}{Dt}\gamma'(t)\right\rangle - \langle d(\Delta \cdot L)_{\gamma'(t)}, \gamma''(t)\rangle.$$

In addition, using β for the first term of the right-hand side, since

$$\beta^{*}_{\gamma'(t)}(\mathbb{B}L(\gamma'(t)) \oplus \mathbb{F}L(\gamma'(t))) = dL(\gamma'(t))$$

and

$$\beta^{-1}_{\gamma'(t)}\left(\gamma'(t) \oplus \frac{D}{Dt}\gamma'(t)\right) = \gamma''(t),$$

we have that Eq. (64) is

$$\langle dL(\gamma'(t)), \gamma''(t)\rangle - \langle d(\Delta \cdot L)_{\gamma'(t)}, \gamma''(t)\rangle = \langle d(L - \Delta \cdot L)_{\gamma'(t)}, \gamma''(t)\rangle = -\langle dE(\gamma'(t)), \gamma''(t)\rangle,$$

which is the claim of the lemma. ∎

Now we can demonstrate the theorem.

Proof of Theorem 20: If $\gamma:[t_1,t_2]\to Q$ is a solution to Eq. (56) then, clearly,

$$\int_{t_1}^{t_2} \langle dL(\gamma'(t)), \kappa(\delta\gamma'(t))\rangle dt = 0 \tag{65}$$

for all $\delta\gamma$ such that

$$\langle \mathbb{F}\mathcal{F}(\gamma'(t)), \delta\gamma(t)\rangle = 0, \tag{66}$$

i.e., for all $\delta\gamma$ such that $\delta\gamma(t) \in C_V^{\mathcal{F}}(\gamma'(t))$ [see Eq. (59)]. We have seen in Theorem 17 that the equations above are equivalent to the condition

$$\left\langle -\frac{D}{Dt}\mathbb{F}L(\gamma'(t)) + \mathbb{B}L(\gamma'(t)), \delta\gamma(t)\right\rangle = 0 \tag{67}$$

for all $\delta\gamma$ such that $\delta\gamma(t) \in C_V^{\mathcal{F}}(\gamma'(t))$. On the other hand, using Eqs. (55) and (57) and the Lemma 22, our curve γ must satisfy

$$\langle dE(\gamma'(t)), \gamma''(t)\rangle = -2\mathcal{F}(\gamma'(t)). \tag{68}$$

Also, since [recall Eq. (33)]

$$\gamma''(t) = j_2(\gamma^{(2)}(t)) \text{ and } \gamma'(t) = \tau_Q^{(1,2)}(\gamma^{(2)}(t)),$$

we have

$$\langle dE(\tau_Q^{(1,2)}(\gamma^{(2)}(t))), j_2(\gamma^{(2)}(t))\rangle = -2\mathcal{F}\circ\tau_Q^{(1,2)}(\gamma^{(2)}(t)),$$

which implies that $\gamma^{(2)}(t)$ belongs to the submanifold $C_K^{\mathcal{F}}$ defined by Eq. (58). All this means that γ is a trajectory of $(L, C_K^{\mathcal{F}}, C_V^{\mathcal{F}})$.

Now we will prove the last statement. Let γ be a trajectory of $(L, C_K^{\mathcal{F}}, C_V^{\mathcal{F}})$ such that $\mathbb{F}\mathcal{F}\circ\gamma'$ satisfies the hypothesis of the theorem. In particular, γ satisfies Eq. (67) for all variations satisfying Eq. (66). For t belonging to the interior $J\subset(t_1,t_2)$ of the set of zeros of $\mathbb{F}\mathcal{F}\circ\gamma'$ (if $J\neq\emptyset$), it is easy to show [from Eqs. (67) and (66)] that

$$-\frac{D}{Dt}\mathbb{F}L(\gamma'(t)) + \mathbb{B}L(\gamma'(t)) = 0 = \mathbb{F}\mathcal{F}(\gamma'(t));$$

thus γ coincides, on J, with a trajectory of the system with friction. Consider now the interior $I\subset(t_1,t_2)$ of the support of $\mathbb{F}\mathcal{F}\circ\gamma'$, i.e., the set

$$I = \{t \in (t_1,t_2): \mathbb{F}\mathcal{F}(\gamma'(t)) \neq 0\},$$

or equivalently, from Lemma 21,

$$I = \{t \in (t_1,t_2): \mathcal{F}(\gamma'(t)) \neq 0\}.$$

Note that, by the hypothesis of the theorem, $I\cup J$ coincides with (t_1,t_2) up to a finite number of isolated points. Since $C_V^{\mathcal{F}}(\gamma'(t))=\langle\mathbb{F}\mathcal{F}(\gamma'(t))\rangle^o$, and it has constant rank for $t\in I$, it easily follows that for t belonging to I,

$$-\frac{D}{Dt}\mathbb{F}L(\gamma'(t)) + \mathbb{B}L(\gamma'(t)) \in [C_V^{\mathcal{F}}(\gamma'(t))]^o.$$

In other words there must exist a function $\lambda: I\to\mathbb{R}$ (a *Lagrange multiplier*) such that

$$-\frac{D}{Dt}\mathbb{F}L(\gamma'(t)) + \mathbb{B}L(\gamma'(t)) = \lambda(t)\mathbb{F}\mathcal{F}(\gamma'(t)).$$

Contracting this equation with $\gamma'(t)$ we have, using Eqs. (63) and (55),

$$-\langle dE(\gamma'(t)), \gamma''(t)\rangle = 2\lambda(t)\mathcal{F}(\gamma'(t)).$$

On the other hand, since γ also satisfies Eq. (68), we must have

$$(\lambda(t)-1)\mathcal{F}(\gamma'(t)) = 0,$$

and accordingly $\lambda(t)=1$. Summing up, γ satisfies

$$-\frac{D}{Dt}\mathbb{F}L(\gamma'(t)) + \mathbb{B}L(\gamma'(t)) = \mathbb{F}\mathcal{F}(\gamma'(t))$$

for all $t\in I\cup J$. By continuity, we have that γ is a trajectory of the system with friction. ■

B. A particle moving in a viscous fluid

Consider a particle of mass m immersed in an anisotropic viscous fluid. The configuration space of the system is $Q=\mathbb{R}^3$. Let us denote its points by $\mathbf{x}$. The dissipation function of the system, using matrix notation, is given by

$$\mathcal{F}(\mathbf{x},\dot{\mathbf{x}}) = \frac{1}{2}\dot{\mathbf{x}}^t\mathbb{F}(\mathbf{x})\dot{\mathbf{x}},$$

where $\mathbb{F}(\mathbf{x})$ is a real 3×3 matrix, and t indicates transposition. Suppose in addition that the particle is subjected to conservative forces, with related potential energy $V(\mathbf{x})$. Thus, the Lagrangian is

$$L(\mathbf{x},\dot{\mathbf{x}}) = \frac{1}{2}m\dot{\mathbf{x}}^t\dot{\mathbf{x}} - V(\mathbf{x})$$

and the energy function is

$$E(\mathbf{x},\dot{\mathbf{x}}) = \frac{1}{2}m\dot{\mathbf{x}}^t\dot{\mathbf{x}} + V(\mathbf{x}).$$

As a consequence, the related kinematic and variational constraints are [recall Eqs. (58), (59), (61), and (62)]

$$C_K^{\mathcal{F}} = \{(\mathbf{x},\dot{\mathbf{x}},\ddot{\mathbf{x}}): m\dot{\mathbf{x}}^t\ddot{\mathbf{x}} + \dot{\mathbf{x}}^t\,\nabla\,V(\mathbf{x}) = -\,\dot{\mathbf{x}}^t\mathbb{F}(\mathbf{x})\dot{\mathbf{x}}\}$$

and

$$C_V^{\mathcal{F}} = \{(\mathbf{x},\dot{\mathbf{x}},\delta\mathbf{x}): \dot{\mathbf{x}}^t\mathbb{F}(\mathbf{x})\,\delta\mathbf{x} = 0\}.$$

C. HOCS with external forces

Definition 23: *For a given a manifold Q a HOCS with external forces is a 4-uple (L,C_K,C_V,u) such that (L,C_K,C_V) defines a HOCS and $u:TQ\to T^*Q$ is a fiber preserving map, which will be called the external force of the system. We shall say that $\gamma:[t_1,t_2]\to Q$ is a trajectory of (L,C_K,C_V,u) if (1) $\gamma^{(k)}(t)\in C_K$, $\forall t\in(t_1,t_2)$;* or *(2) for all variations $\delta\gamma$ such that $(\gamma^{(l)}\times(t),\delta\gamma(t))\in C_V$, $\forall t\in(t_1,t_2)$, we have the following:*

$$\int_{t_1}^{t_2}(\langle dL(\gamma'(t)),\kappa(\delta\gamma'(t))\rangle + \langle u(\gamma'(t)),\delta\gamma(t)\rangle)dt = 0.$$

Theorem 24: *A curve $\gamma:[t_1,t_2]\to Q$ is a trajectory of a HOCS with external forces (L,C_K,C_V,u) if and only if $\gamma^{(k)}(t)\in C_K$ and*

$$\left\langle -\frac{D}{Dt}\mathbb{F}L(\gamma'(t)) + \mathbb{B}L(\gamma'(t)) - u(\gamma'(t)),\delta\gamma(t)\right\rangle = 0, \qquad \forall\; t\in(t_1,t_2);$$

for all variations $\delta\gamma$ such that $\delta\gamma(t)\in C_V(\gamma^{(l)}(t))$.

Proof: We must repeat the proof of Theorem 17. ■

Remark 25: *If C_V is a submanifold the equation above is equivalent to*

$$-\frac{D}{Dt}\mathbb{F}L(\gamma'(t)) + \mathbb{B}L(\gamma'(t)) - u(\gamma'(t)) \in F_V(\gamma^{(l)}(t)), \qquad \forall\; t\in(t_1,t_2).$$

In local terms γ is a trajectory of (L,C_K,C_V,u) if and only if for every chart (U,φ) of Q the curve $q(t)=\varphi\circ\gamma(t)$ satisfies, in local coordinates,

$$(q^{(0)}(t),\ldots,q^{(k)}(t)) \in C_K \tag{69}$$

and for all curves $\delta q(t)$ such that

$$\delta q(t) \in C_V((q^{(0)}(t), \ldots, q^{(l)}(t))) \tag{70}$$

we have

$$\left(\frac{d}{dt}\left(\frac{\partial(L \circ \varphi_*^{-1})}{\partial \dot{q}^i}(q(t),\dot{q}(t))\right) - \frac{\partial(L \circ \varphi_*^{-1})}{\partial q^i}(q(t),\dot{q}(t))\right)\delta q^i(t) + u_i(t)\,\delta q^i(t) = 0. \tag{71}$$

Here each $u_i(t)$ is the ith component of the vector

$$(\varphi^*_{\gamma(t)})^{-1}[u(\gamma'(t))] \in \mathbb{R}^n.$$

Suppose that we have a Lagrangian system (L,Q) with friction forces given by $\mathcal{F}$. If kinematic and variational constraints $C_K \subset T^{(k)}Q$ and $C_V \subset T^{(l)}Q \times_Q TQ$ are added, the trajectories of the resulting mechanical system are those of the 4-uple $(L, C_K, C_V, -\mathbb{F}\mathcal{F})$. We shall show that $(L, C_K, C_V, -\mathbb{F}\mathcal{F})$ can be described in terms of a HOCS. To see that let us first suppose, without lost of generality (see the forth remark below Definition 15), that $k, l \geqslant 1$, and consider the following definitions.

a. If $l=1$, define $S_K^{\mathcal{F}} \subset T^{(2)}Q$ as being the set of all points $\alpha \in T^{(2)}Q$ such that

$$\langle dE, j_2(\alpha)\rangle + 2\mathcal{F} \circ \tau_Q^{(1,2)}(\alpha) = \langle f, \tau_Q^{(1,2)}(\alpha)\rangle,$$

for some $f \in F_V(\tau_Q^{(1,2)}(\alpha))$.

b. If $l \geqslant 2$, define $S_K^{\mathcal{F}} \subset T^{(l)}Q$ as being the set of all points $\alpha \in T^{(l)}Q$ satisfying

$$\langle dE, j_2(\tau_Q^{(2,l)}(\alpha))\rangle + 2\mathcal{F} \circ \tau_Q^{(1,l)}(\alpha) = \langle f, \tau_Q^{(1,l)}(\alpha)\rangle,$$

for some $f \in F_V(\alpha)$.

For any of the cases (a) or (b) above, define

$$S_V^{\mathcal{F}} = \{(v, \varpi) \in T^{(l)}Q \times_Q TQ : \langle \mathbb{F}\mathcal{F} \circ \tau_Q^{(1,l)}(v), \varpi\rangle = 0\}.$$

Note that

$$S_V^{\mathcal{F}}(v) = \langle \mathbb{F}\mathcal{F} \circ \tau_Q^{(1,l)}(v)\rangle^o, \tag{72}$$

where $\langle \mathbb{F}\mathcal{F} \circ \tau_Q^{(1,l)}(v)\rangle$ means the subspace generated by the vector

$$\mathbb{F}\mathcal{F} \circ \tau_Q^{(1,l)}(v).$$

The following result generalizes Theorem 20.

Theorem 26: *The trajectories of a HOCS with friction forces*

$$(L, C_K, C_V, -\mathbb{F}\mathcal{F}),$$

such that C_V is a submanifold are trajectories of the triple $(L, C_K^{\mathcal{F}}, C_V^{\mathcal{F}})$ with

$$C_K^{\mathcal{F}} = \begin{cases} C_K \cap (\tau_Q^{(n,k)})^{-1}(S_K^{\mathcal{F}}) \subset T^{(k)}Q & \text{if } k \geqslant n \\ (\tau_Q^{(k,n)})^{-1}(C_K) \cap S_K^{\mathcal{F}} \subset T^{(n)}Q & \text{if } k < n; \end{cases} \tag{73}$$

where $n = \max\{2, l\}$ and

$$C_V^{\mathcal{F}} = C_V \cap S_V^{\mathcal{F}}. \tag{74}$$

Moreover, if $\gamma:[t_1,t_2] \to Q$ is a trajectory of the triple given above, such that the set of zeros of the function $\mathbb{F}\mathcal{F} \circ \gamma' : (t_1,t_2) \to T^*Q$ is empty or is given by the union of a finite number of closed intervals (including isolated points), then γ is also a trajectory of $(L, C_K, C_V, -\mathbb{F}\mathcal{F})$.

Proof: For simplicity consider the case in which $k \geqslant l = 1$ (the other cases are left to the reader). If $\gamma:[t_1,t_2]\to Q$ is a trajectory of $(L,C_K,C_V,-\mathbb{F}\mathcal{F})$ then $\gamma^{(k)}(t)\in C_K$ and

$$\int_{t_1}^{t_2}(\langle dL(\gamma'(t)),\kappa(\delta\gamma'(t))\rangle-\langle\mathbb{F}\mathcal{F}(\gamma'(t)),\delta\gamma(t)\rangle)dt=0 \tag{75}$$

for all $\delta\gamma$ such that $\delta\gamma(t)\in C_V(\gamma'(t))$. As a direct consequence we have

$$\int_{t_1}^{t_2}\langle dL(\gamma'(t)),\kappa(\delta\gamma'(t))\rangle dt=0$$

for all $\delta\gamma$ such that

$$\delta\gamma(t)\in C_V(\gamma'(t)) \text{ and } \langle\mathbb{F}\mathcal{F}(\gamma'(t)),\delta\gamma(t)\rangle=0,$$

i.e., $\delta\gamma(t)\in C_V^{\mathcal{F}}(\gamma'(t))$. If we show that $\gamma^{(k)}(t)\in C_K^{\mathcal{F}}$ then γ will be a trajectory of $(L,C_K^{\mathcal{F}},C_V^{\mathcal{F}})$. To do this let us first note that, since C_V is a submanifold, Eq. (75) is equivalent to

$$-\frac{D}{Dt}\mathbb{F}L(\gamma'(t))+\mathbb{B}L(\gamma'(t))-\mathbb{F}\mathcal{F}(\gamma'(t))\in F_V(\gamma'(t)),\qquad\forall\, t\in(t_1,t_2).$$

Contracting equation above with $\gamma'(t)$ we have

$$\left\langle-\frac{D}{Dt}\mathbb{F}L(\gamma'(t))+\mathbb{B}L(\gamma'(t))-\mathbb{F}\mathcal{F}(\gamma'(t)),\gamma'(t)\right\rangle=\langle f(t),\gamma'(t)\rangle$$

for some $f(t)\in F_V(\gamma'(t))$. From Lemma 22 we know that

$$\left\langle-\frac{D}{Dt}\mathbb{F}L(\gamma'(t))+\mathbb{B}L(\gamma'(t)),\gamma'(t)\right\rangle=-\langle dE(\gamma'(t)),j_2(\gamma^{(2)}(t))\rangle.$$

Besides, from the fact that $\gamma'(t)=\tau_Q^{(1,2)}(\gamma^{(2)}(t))$ and using Eq. (55) we have

$$\langle\mathbb{F}\mathcal{F}(\gamma'(t)),\gamma'(t)\rangle=2\mathcal{F}\circ\tau_Q^{(1,2)}(\gamma^{(2)}(t)).$$

Then

$$\langle dE(\tau_Q^{(1,2)}(\gamma^{(2)}(t))),j_2(\gamma^{(2)}(t))\rangle+2\mathcal{F}\circ\tau_Q^{(1,2)}(\gamma^{(2)}(t))=\langle f(t),\tau_Q^{(1,2)}(\gamma^{(2)}(t))\rangle,$$

and accordingly $\gamma^{(2)}(t)\in S_K^{\mathcal{F}}$. This implies that $\gamma^{(k)}(t)\in C_K^{\mathcal{F}}$ as we wanted to show.

To prove the last statement let γ be a trajectory of $(L,C_K^{\mathcal{F}},C_V^{\mathcal{F}})$ such that $\mathbb{F}\mathcal{F}(\gamma'(t))\neq 0$ for all t (the other possible situations can be worked out as in Theorem 20). Then $\gamma^{(k)}(t)\in C_K^{\mathcal{F}}$ and

$$-\frac{D}{Dt}\mathbb{F}L(\gamma'(t))+\mathbb{B}L(\gamma'(t))\in[C_V^{\mathcal{F}}(\gamma'(t))]^o,\qquad\forall\, t\in(t_1,t_2).$$

Since $C_K^{\mathcal{F}}=C_K\cap(\tau_Q^{(n,k)})^{-1}(S_K^{\mathcal{F}})$ then $\gamma^{(k)}(t)\in C_K$. On the other hand, since

$$C_V^{\mathcal{F}}(\gamma'(t))=C_V(\gamma'(t))\cap S_V^{\mathcal{F}}(\gamma'(t)),$$

it follows that [recall Eq. (72)]

$$[C_V^{\mathcal{F}}(\gamma'(t))]^o=F_V(\gamma'(t))+\langle\mathbb{F}\mathcal{F}(\gamma'(t))\rangle.$$

Accordingly, there must exist a function $\lambda:(t_1,t_2)\to\mathbb{R}$ such that

$$-\frac{D}{Dt}\mathbb{F}L(\gamma'(t)) + \mathbb{B}L(\gamma'(t)) - \lambda(t)\mathbb{F}\mathcal{F}(\gamma'(t)) \in F_V(\gamma'(t)).$$

Following the same steps as in Theorem 20 it can be shown that $\lambda(t)=1$ and, consequently, γ is a trajectory of $(L, C_K, C_V, -\mathbb{F}\mathcal{F})$. ■

D. The controlled inverted cart-pendulum with friction

As briefly mentioned in the previous section, the inverted cart-pendulum with control strategies given in Refs. 8, 13, and 29 defines a GNHS, in particular, a HOCS. Let us describe in more detail this system and then let us add friction forces.

The inverted cart-pendulum consists of a straight rod which remains in a vertical plane, such that one of its tips, say O, can move only along a straight horizontal line contained in that plane. The configuration space for this system is $Q=\mathbb{R}\times S^1$. Given $(x,\theta)\in\mathbb{R}\times S^1$, x represents the position of the point O and θ represents the angle of rotation of the rod with respect to the vertical line, measured in the counter-clockwise sense (Warning: this way of measuring θ is different from the one adopted in Ref. 8). The Lagrangian of the system is

$$L(x,\theta,\dot{x},\dot{\theta}) = \frac{1}{2}m\dot{x}^2 - ml\dot{\theta}\dot{x}\cos\theta + \frac{1}{2}I\dot{\theta}^2 - mgl\cos\theta, \tag{76}$$

where m is the mass of the rod, l is the distance from O to its center of mass, I is its moment of inertia with respect to O, and g is the acceleration of gravity. To compare with expressions that appear in Ref. 8 we must replace there θ by $\theta+\pi/2$.

It is worth remarking that the system has an unstable fixed point at $\theta=0$. A classical control problem is, by applying a force in the horizontal x direction, to take the rod from any position to the upright position, which should be converted into a stable fixed point. In order to do that, we can impose an affine kinematic constraint

$$C_K = \{(x,\theta,\dot{x},\dot{\theta}) : \dot{x} + b\dot{\theta} = a\sin\theta\} \subset TQ, \quad a,b \in \mathbb{R}, \tag{77}$$

with a space of constraint forces $F_V \subset T^*Q$ spanned by dx (this gives the direction of the control signal), or equivalently, with variational constraints

$$C_V = \{(x,\theta,\delta x,\delta\theta) : \delta x = 0\}; \tag{78}$$

and then look for the values of a,b that give rise [for the resulting GNHS (L,C_K,C_V)] to the desired behavior, as we have done in Ref. 8.

Here we want to solve this problem in the presence of friction forces given by a dissipation function

$$\mathcal{F}(x,\dot{x},\theta,\dot{\theta}) = \frac{1}{2}(\mu\dot{x}^2 + \nu\dot{\theta}^2), \quad \mu,\nu > 0.$$

Thus we have a GNHS with external forces $(L,C_K,C_V,-\mathbb{F}\mathcal{F})$ whose equations of motion are [see Eqs. (69)–(71)]

$$\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} + \frac{\partial \mathcal{F}}{\partial \dot{x}}\right)\delta x + \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} + \frac{\partial \mathcal{F}}{\partial \dot{\theta}}\right)\delta\theta = 0,$$

$$\dot{x} + b\dot{\theta} = a\sin\theta, \quad \delta x = 0.$$

From the explicit form of L the equations are

$$I\ddot{\theta} - ml\ddot{x}\cos\theta - mgl\sin\theta + \nu\dot{\theta} = 0, \quad \dot{x} + b\dot{\theta} = a\sin\theta,$$

or, by replacing $\ddot{x}$ in the first equation using the second one,

$$(I + mlb\cos\theta)\ddot{\theta} + (\nu - mla\cos^2\theta)\dot{\theta} - mgl\sin\theta = 0,$$

$$\dot{x} + b\dot{\theta} = a\sin\theta. \tag{79}$$

Remark 27: *Of course, we can study the problem by considering the related HOCS* $(L, C_K^{\mathcal{F}}, C_V^{\mathcal{F}})$, *with* $C_K^{\mathcal{F}} \subset T^{(2)}Q$ *given by points* $(x, \theta, \dot{x}, \dot{\theta}, \ddot{x}, \ddot{\theta})$ *such that [see Eqs. (73) and (74)]*

$$\dot{x} + b\dot{\theta} = a\sin\theta \text{ and } \frac{\partial E}{\partial x}\dot{x} + \frac{\partial E}{\partial \theta}\dot{\theta} + \frac{\partial E}{\partial \dot{x}}\ddot{x} + \frac{\partial E}{\partial \dot{\theta}}\ddot{\theta} = -(\mu\dot{x}^2 + \nu\dot{\theta}^2),$$

being

$$E = \frac{1}{2}m\dot{x}^2 - ml\dot{\theta}\dot{x}\cos\theta + \frac{1}{2}I\dot{\theta}^2 + mgl\cos\theta,$$

and $C_V^{\mathcal{F}} \subset TQ \times_Q TQ$ *defined as*

$$C_V^{\mathcal{F}} = \{(x, \theta, \dot{x}, \dot{\theta}, \delta x, \delta\theta) : \delta x = 0, \mu\dot{x}\delta x + \nu\dot{\theta}\delta\theta = 0\} = \{(x, \theta, \dot{x}, \dot{\theta}, \delta x, \delta\theta) : \delta x = 0, \dot{\theta}\delta\theta = 0\}.$$

But we shall focus on Eq. (79).

The equation for θ, i.e.,

$$(I + mlb\cos\theta)\ddot{\theta} + (\nu - mla\cos^2\theta)\dot{\theta} - mgl\sin\theta = 0$$

[see Eq. (79)], becomes near $\theta = 0$ the following equation:

$$(I + mlb)\ddot{\theta} + (\nu - mla)\dot{\theta} - mgl\theta = 0.$$

It is easy to see that for $a > \nu/ml$ and $b < -I/ml$ the upright position is stable (a node or a focus, depending on the values of b), and the basin of attraction is defined by the interval $(-\theta_o, \theta_o)$, with $\cos\theta_o = -I/mlb$. Hence, in that range of parameters a and b the constraints Eq. (77) lead us to the desired behavior.

V. ORDER-LOWERING AND LINEARIZATION

One of the main results in the present paper is described in this section. We will show that any HOCS (L, C_K, C_V) on a configuration space Q can be described by another one $(\tilde{L}, \tilde{C}_K, \tilde{C}_V)$ on a canonically related manifold $\tilde{Q}$, with kinematics and variational constraints given by generalized distributions on $\tilde{Q}$ (or along a submanifold of it). Note that the triple $(\tilde{L}, \tilde{C}_K, \tilde{C}_V)$ defines a GNHS with linear constraints, but not satisfying, in general, $\tilde{C}_K = \tilde{C}_V$, that is, D'Alembert principle. We shall proceed in two steps as follows.

1. Any HOCS is equivalent to a GNHS with affine kinematic constraints.
2. Any GNHS with affine kinematic constraints can be described as one with linear constraints.

The first step alone enables us to apply to HOCSs, in the presence of symmetry, the reduction technics developed in Ref. 21 for GNHSs. We shall study this in a forthcoming paper.

We shall illustrate these procedures with the example of systems with friction forces and Rocard's simplified model of pneumatic tires.

A. Order-lowering procedure

Let (L,C_K,C_V) be a HOCS with $C_K \subset T^{(k)}Q$ and $C_V \subset T^{(l)}Q \times_Q TQ$. We want to build up another HOCS with configuration space $\tilde{Q}$ and constraints

$$\tilde{C}_K \subset T\tilde{Q} \text{ and } \tilde{C}_V \subset T\tilde{Q}.$$

Fix $n \geqslant \max\{k,l\}$ and consider the natural applications

$$\tau_Q^{(k,n)}:T^{(n)}QT^{(k)}Q \text{ and } j_n:T^{(n)}Q \hookrightarrow T[T^{(n-1)}Q].$$

Let $\tilde{Q}=T^{(n-1)}Q$ and define

$$\tilde{C}_K = j_n([\tau_Q^{(k,n)}]^{-1}(C_K)) \subset T[T^{(n-1)}Q]. \tag{80}$$

This implies that

$$C_K = \tau_Q^{(k,n)}[(j_n)^{-1}(\tilde{C}_K)]. \tag{81}$$

Locally, the points of $\tilde{C}_K$ are given by $2n$-uple

$$(q^{(0)}, \ldots ,q^{(n-1)};\dot{q}^{(0)}, \ldots ,\dot{q}^{(n-1)}),$$

such that

$$(q^{(0)}, \ldots ,q^{(k)}) \in C_K \tag{82}$$

and

$$\dot{q}^{(0)} = q^{(1)},\dot{q}^{(1)} = q^{(2)}, \ldots ,\dot{q}^{(n-2)} = q^{(n-1)}. \tag{83}$$

In particular, they are of the form

$$(q^{(0)}, \ldots ,q^{(n-1)};q^{(1)}, \ldots ,q^{(n-1)},\alpha). \tag{84}$$

From the above local expressions the following lemma follows immediately.

Lemma 28: *A curve* $\tilde{\gamma}:[t_1,t_2] \to \tilde{Q}$ *satisfies* $\tilde{\gamma}'(t) \in \tilde{C}_K$ *if and only if* $\tilde{\gamma}(t)=\gamma^{(n-1)}(t)$, *where* $\gamma(t)=\tau_Q^{(n-1)}(\tilde{\gamma}(t))$ *and* $\gamma^{(k)}(t) \in C_K$.

This $\tilde{C}_K$ will be part of our new system. Note that if $n>k$ the submanifold $\tilde{C}_K$ is defined by conditions on *positions* and linear nonhomogeneous conditions (83) on *velocities* (in the sense of the new configuration space $\tilde{Q}$). For instance, for $k=l=0$, if we take $n=1$, it easily follows that $\tilde{C}_K=T_{C_K}Q$. For k or l bigger than zero and if $n>k$, it can be shown, as we will do at the end of this section, that $\tilde{C}_K$ is an affine subbundle of $T\tilde{Q}$ along

$$\tilde{Q}_1 = \tau_{\tilde{Q}}(\tilde{C}_K) = [\tau_Q^{(k,n-1)}]^{-1}(C_K) \subset \tilde{Q}.$$

Now let us define the variational constraints $\tilde{C}_V$ and the Lagrangian function $\tilde{L}$. Consider the natural surjections

$$\tau_Q^{(l,n-1)}:T^{(n-1)}Q \twoheadrightarrow T^{(l)}Q$$

and the applications

$$\kappa_l:T^{(l)}[TQ] \to T[T^{(l)}Q].$$

Define

$$\widetilde{C}_V = (\tau_Q^{(l,n-1)})_*^{-1}[\kappa_l((\mathfrak{s}_l)^{-1}(C_V))] \subset T[T^{(n-1)}Q] = T\widetilde{Q}. \tag{85}$$

In local terms $\widetilde{C}_V$ is formed by points

$$(q^{(0)}, \ldots, q^{(n-1)}, \delta q^{(0)}, \ldots, \delta q^{(n-1)}),$$

such that

$$(q^{(0)}, \ldots, q^{(l)}, \delta q^{(0)}) \in C_V,$$

while $\delta q^{(1)}, \ldots, \delta q^{(n-1)}$ are arbitrary. Using these local expressions we have the following result.

Lemma 29: *A variation $\delta\widetilde{\gamma}:[t_1,t_2]\to T\widetilde{Q}$ of a curve $\widetilde{\gamma}:[t_1,t_2]\to\widetilde{Q}$ satisfies $\delta\widetilde{\gamma}(t)\in\widetilde{C}_V$ if and only if*

$$(\tau_Q^{(n-1)})_*(\delta\widetilde{\gamma}(t)) \in C_V(\tau_Q^{(l,n-1)}(\widetilde{\gamma}(t))).$$

Finally, define the Lagrangian function on $\widetilde{Q}$ by

$$\widetilde{L} = L \circ (\tau_Q^{(n-1)})_*. \tag{86}$$

In a local chart,

$$\widetilde{L}(q^{(0)}, \ldots, q^{(n-1)}, \dot{q}^{(0)}, \ldots, \dot{q}^{(n-1)}) = L(q^{(0)}, \dot{q}^{(0)}). \tag{87}$$

Definition 30: *Given a HOCS (L,C_K,C_V) and some $n \geqslant k,l$ we call the triple $(\widetilde{L},\widetilde{C}_K,\widetilde{C}_V)$, with $\widetilde{L}$, $\widetilde{C}_K$ and $\widetilde{C}_V$ given by Eqs. (86), (80), and (85), respectively, the canonical n-lift of (L,C_K,C_V).*

Theorem 31: *Let (L,C_K,C_V) be a HOCS and let $(\widetilde{L},\widetilde{C}_K,\widetilde{C}_V)$ be its canonical n-lift. If $\gamma:[t_1,t_2]\to Q$ is a trajectory of (L,C_K,C_V) then $\widetilde{\gamma}=\gamma^{(n-1)}:[t_1,t_2]\to\widetilde{Q}$ is a trajectory of its n-lift. (Note that $\widetilde{\gamma}$ satisfies $\tau_Q^{(n-1)}\circ\widetilde{\gamma}=\gamma$.) Conversely, if $\widetilde{\gamma}$ is a trajectory of the n-lift then $\widetilde{\gamma}=\gamma^{(n-1)}$, being $\gamma=\tau_Q^{(n-1)}\circ\widetilde{\gamma}$ a trajectory of the original system.*

Proof: Let $\gamma:[t_1,t_2]\to Q$ be a trajectory of (L,C_K,C_V) and consider the curve $\widetilde{\gamma}=\gamma^{(n-1)}:[t_1,t_2]\to\widetilde{Q}$. We want to show that $\widetilde{\gamma}$ is a trajectory of $(\widetilde{L},\widetilde{C}_K,\widetilde{C}_V)$. It is clear that $\widetilde{\gamma}'(t)\in\widetilde{C}_K$. In fact,

$$\widetilde{\gamma}'(t) = j_n(\gamma^{(n)}(t))$$

and

$$\tau_Q^{(k,n)}(\gamma^{(n)}(t)) = \gamma^{(k)}(t) \in C_K.$$

It remains to see that

$$\int_{t_1}^{t_2} \langle d\widetilde{L}(\widetilde{\gamma}'(t)), \widetilde{\kappa}(\delta\widetilde{\gamma}'(t))\rangle dt = 0$$

for all variations $\delta\widetilde{\gamma}:[t_1,t_2]\to T\widetilde{Q}$ such that $\delta\widetilde{\gamma}(t)\in\widetilde{C}_V$. (Here, $\widetilde{\kappa}$ is the canonical involution related to $\widetilde{Q}$.) Fix a variation $\delta\widetilde{\gamma}$ of $\widetilde{\gamma}$ satisfying $\delta\widetilde{\gamma}(t)\in\widetilde{C}_V$. From Eq. (86) it follows that

$$\langle d\widetilde{L}(\widetilde{\gamma}'(t)), \widetilde{\kappa}(\delta\widetilde{\gamma}'(t))\rangle = \langle dL(\gamma'(t)), ((\tau_Q^{(n-1)})_{**}\circ\widetilde{\kappa})(\delta\widetilde{\gamma}'(t))\rangle.$$

It is easy to show that

$$(\tau_Q^{(n-1)})_{**}\circ\widetilde{\kappa} = \kappa\circ(\tau_Q^{(n-1)})_{**}.$$

Then,

$$\langle d\tilde{L}(\tilde{\gamma}'(t)),\tilde{\kappa}(\delta\gamma'(t))\rangle = \langle dL(\gamma'(t)),\kappa((\tau_Q^{(n-1)})_{**}(\delta\tilde{\gamma}'(t)))\rangle = \langle dL(\gamma'(t)),\kappa(\delta\gamma'(t))\rangle,$$

where $\delta\gamma(t)=(\tau_Q^{(n-1)})_*(\delta\tilde{\gamma}(t))$. Using Lemma 29 we have that

$$\delta\gamma(t) = (\tau_Q^{(n-1)})_*(\delta\tilde{\gamma}(t)) \in C_V(\gamma^{(l)}(t)).$$

Accordingly,

$$\int_{t_1}^{t_2}\langle d\tilde{L}(\tilde{\gamma}'(t)),\tilde{\kappa}(\delta\tilde{\gamma}'(t))\rangle dt = \int_{t_1}^{t_2}\langle dL(\gamma'(t)),\kappa(\delta\gamma'(t))\rangle dt = 0,$$

since γ is a trajectory of (L,C_K,C_V).

Now suppose we have a trajectory $\tilde{\gamma}:[t_1,t_2]\to\tilde{Q}$ of $(\tilde{L},\tilde{C}_K,\tilde{C}_V)$. We want to show that

$$\gamma = \tau_Q^{(n-1)}\circ\tilde{\gamma}:[t_1,t_2]\to Q$$

is a trajectory of (L,C_K,C_V). Since $\tilde{\gamma}'(t)\in\tilde{C}_K$, from Lemma 28 we can deduce that $\tilde{\gamma}(t)$ must be the $(n-1)$-lift of the curve $\tau_Q^{(n-1)}\circ\tilde{\gamma}$, i.e., $\tilde{\gamma}=\gamma^{(n-1)}$. Using this it easily follows that $\gamma^{(k)}(t)\in C_K$. Now, fix a variation $\delta\gamma$ of γ such that $\delta\gamma(t)\in C_V(\gamma^{(l)}(t))$. We need to prove that

$$\int_{t_1}^{t_2}\langle dL(\gamma'(t)),\kappa(\delta\gamma'(t))\rangle dt = 0.$$

Consider an affine connection on Q and the related diffeomorphism [see Eq. (45)]

$$t\zeta_{n-1}:T^{(n-1)}[TQ]\to T^{(n-1)}Q\times_Q\underbrace{TQ\oplus\ \dots\ \oplus TQ}_{n\ \text{terms}}.$$

Define

$$\delta\tilde{\gamma}(t) = (\kappa_{n-1}\circ(t\zeta_{n-1})^{-1})(\gamma^{(n-1)}(t),\delta\gamma(t)\oplus 0\oplus\ \dots\ \oplus 0).$$

From Lemma 12, Eq. (36), we know that

$$(\tau_Q^{(n-1)})_*\circ\kappa_{n-1} = \tau_{TQ}^{(n-1)}.$$

On the other hand, using Lemma 14 we have

$$(\tau_Q^{(n-1)})_*(\delta\tilde{\gamma}(t)) = \delta\gamma(t),$$

which implies that $\delta\tilde{\gamma}$ is a variation of $\tilde{\gamma}$ inside $\tilde{C}_V$. Following the same steps as above we have that

$$\int_{t_1}^{t_2}\langle dL(\gamma'(t)),\kappa(\delta\gamma'(t))\rangle dt = \int_{t_1}^{t_2}\langle d\tilde{L}(\tilde{\gamma}'(t)),\tilde{\kappa}(\delta\tilde{\gamma}'(t))\rangle dt,$$

and the theorem is proved. ■

Now let us show that $\tilde{C}_K$ is an affine subbundle if $n>k\geqslant 1$ (the $k=0$ case has been previously discussed).

Lemma 32: *For* $n>1$

$$j_n(T^{(n)}Q) = \ker(\tau_Q^{(n-2,n-1)})_* + X,$$

being $X\in\mathfrak{X}(T^{(n-1)}Q)$ *an nth order ODE [recall Eqs. (38) and (40)].*

Proof: Let us fix an nth order ODE X and a coordinate chart on Q. From Eq. (39)

$$X=(q^{(0)},\ldots,q^{(n-1)};q^{(1)},\ldots,q^{(n-1)},\chi(q^{(0)},\ldots,q^{(n-1)})),$$

for some function χ. In the same coordinates the elements of $j_n(T^{(n)}Q)$ and those of $\ker(\tau_Q^{(n-2,n-1)})_*$ are of the form

$$(q^{(0)},\ldots,q^{(n-1)};q^{(1)},\ldots,q^{(n-1)},\alpha) \tag{88}$$

and

$$(q^{(0)},\ldots,q^{(n-1)};0,\ldots,0,\beta),$$

respectively. Then

$$(q^{(0)},\ldots,q^{(n-1)};q^{(1)},\ldots,q^{(n-1)},\alpha)=(q^{(0)},\ldots,q^{(n-1)};0,\ldots,0,\alpha-\chi)$$
$$+(q^{(0)},\ldots,q^{(n-1)};q^{(1)},\ldots,q^{(n-1)},\chi),$$

and the lemma is proved. ■

Theorem 33: *If $k\geq 1$, then for $n>k$ we have*

$$\widetilde{C}_K=[\ker(\tau_Q^{(n-2,n-1)})_*+X]|_{\widetilde{Q}_1}.$$

Proof: From Eqs. (84) and (88), $\widetilde{C}_K$ coincide with the subbundle $j_n(T^{(n)}Q)\subset T[T^{(n-1)}Q]$ along

$$\widetilde{Q}_1=\tau_{\widetilde{Q}}(\widetilde{C}_K)=[\tau_Q^{(k,n-1)}]^{-1}(C_K)\subset T^{(n-1)}Q.$$

Then the theorem follows from the lemma above. ■

Remark 34: *For $\widetilde{C}_K$ to be an affine subbundle it is enough to take, in some particular cases, $n=k$ (if $k=\max\{k,l\}$). We shall illustrate this point at the end of the paper, with a couple of examples.*

B. Linearizing affine constraints

Let (L,C_K,C_V) be a GNHS with $C_K=C_K^{\mathrm{vec}}+X$, being C_K^{vec} a generalized distribution along a submanifold $Q_1\subset Q$ and $X:q\mapsto X_q$ a section of TQ along Q_1. Suppose that $C_V\subset TQ$ is a distribution along a submanifold Q_2. We shall call such a triple (L,C_K,C_V) an affine GNHS. If $Q_1=Q_2=Q$, then C_K and C_V are given locally by equations

$$w_i^a(q)\dot{q}^i=\gamma^a(q),\quad a=1,\ldots,k \tag{89}$$

and

$$v_i^a(q)\,\delta q^i=0,\quad a=1,\ldots,k. \tag{90}$$

Now we shall construct the *linearized* version of (L,C_K,C_V).

Let us consider the manifold $\widetilde{Q}=Q\times\mathbb{R}$ and on its tangent space $T\widetilde{Q}=TQ\times T\mathbb{R}$ define the function

$$\widetilde{L}:T\widetilde{Q}\to\mathbb{R}:(q,\lambda;v,\dot{\lambda})\mapsto L(q,v)+\frac{1}{2}\dot{\lambda}^2;$$

that is

$$\widetilde{L}(q,\lambda;v,\dot{\lambda})=L(q,v)+\frac{1}{2}\dot{\lambda}^2. \tag{91}$$

Note that if L is regular so is $\widetilde{L}$. Now define the generalized distribution

$$\tilde{C}_K = i(C_K^{\text{vec}}) + \langle \tilde{X} \rangle, \tag{92}$$

with $i: TQ \hookrightarrow TQ \times T\mathbb{R}$ the canonical inclusion and $\tilde{X}$ the section of $T\tilde{Q}$ along $\tilde{Q}_1 = Q_1 \times \mathbb{R}$ given by

$$\tilde{X}:(q,\lambda) \mapsto (q,\lambda;X_q,1).$$

Finally, define

$$\tilde{C}_V = C_V \times T\mathbb{R} \subset T\tilde{Q}. \tag{93}$$

In local terms $\tilde{C}_K$ and $\tilde{C}_V$ are given by [see Eqs. (89) and (90)]

$$w_i^a(q)\dot{q}^i - \gamma^a(q)\dot{\lambda} = 0 \text{ and } v_i^a(q)\,\delta q^i = 0, \tag{94}$$

respectively. Note, in particular, that $\delta\lambda$ is arbitrary.

Definition 35: *Given an affine GNHS (L, C_K, C_V) we will call the triple*

$$(\tilde{L},\tilde{C}_K,\tilde{C}_V),$$

with $\tilde{L}$, $\tilde{C}_K$, and $\tilde{C}_V$ given by Eqs. (91)–(93), respectively, its linearized version.

Theorem 36: *Let (L, C_K, C_V) be a triple as above, and let $(\tilde{L},\tilde{C}_K,\tilde{C}_V)$ its linearized version. A curve*

$$\gamma:[t_1,t_2] \to Q : t \mapsto \gamma(t)$$

is a trajectory of (L, C_K, C_V) if and only if for some function $\lambda:[t_1,t_2]\to\mathbb{R}$ with $\lambda(t_1)=t_1$ and $\lambda(t_2)=t_2$ the curve

$$\tilde{\gamma}:[t_1,t_2] \to \tilde{Q} : t \mapsto (\gamma(t),\lambda(t))$$

is a trajectory of $(\tilde{L},\tilde{C}_K,\tilde{C}_V)$.

Proof: We shall assume, for simplicity, that $Q_1 = Q_2 = Q$. Let γ be a trajectory of (L, C_K, C_V). Fix a chart (U,φ) of Q. Then,

$$\varphi \circ \gamma(t) = q(t) = (q^1(t), \ldots, q^n(t))$$

satisfies [recall Eqs. (89) and (90)]

$$\omega_i^a(q)\dot{q}^i = \gamma^a(q), \quad 1 \leqslant a \leqslant k$$

and

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i}\right)\delta q^i = 0,$$

for all curves $\delta q(t) = (\delta q^1(t), \ldots, \delta q^n(t))$ such that

$$v_i^b(q)\,\delta q^i = 0, \quad 1 \leqslant b \leqslant l.$$

Now consider the EL equations for $\tilde{L}$,

$$\left(\frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{q}^i} - \frac{\partial \tilde{L}}{\partial q^i}\right)\delta q^i + \left(\frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{\lambda}} - \frac{\partial \tilde{L}}{\partial \lambda}\right)\delta\lambda = 0. \tag{95}$$

Since $\delta\lambda$ is arbitrary and $\tilde{L} = L + \dot{\lambda}^2/2$ they translate into

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i}\right)\delta q^i = 0 \text{ and } \ddot{\lambda} = 0.$$

Accordingly, $\lambda(t)=c_1 t+c_0$ for some constants c_1 and c_0. Imposing on λ the conditions $\lambda(t_1)=t_1$ and $\lambda(t_2)=t_2$ we have $\lambda(t)=t$. In particular, $\dot{\lambda}(t)=1$ for all t. Therefore $(q^1(t),\ldots,q^n(t),\lambda(t))$ satisfies Eq. (95) and constraints

$$\omega_i^a(q)\dot{q}^i - \gamma^a(q)\dot{\lambda} = 0, \quad 1 \leqslant a \leqslant k,$$

and consequently $\tilde{\gamma}(t)=(\gamma(t),\lambda(t))$ is a trajectory of the linearized system [recall Eq. (94)]. The converse is immediate. ■

Remark 37: *The theorem above can be rewritten as follows. Given $q_1,q_2 \in Q$, the curve $\gamma:[t_1,t_2]\to Q$ is a trajectory of (L,C_K,C_V) with $\gamma(t_1)=q_1$ and $\gamma(t_2)=q_2$ if and only if $\tilde{\gamma}:[t_1,t_2]\to\tilde{Q}$ is a trajectory of its linearized version with $\tilde{\gamma}(t_1)=(q_1,t_1)$ and $\tilde{\gamma}(t_2)=(q_2,t_2)$. We also have the equivalent statement: given $q \in Q$ and $X_q \in T_qQ$, γ is a trajectory of the original system with $\gamma'(t_1)=X_q$ if and only if $\tilde{\gamma}$ is a trajectory of the linearized one, with*

$$\tilde{\gamma}'(t_1) = (q,t_1;X_q,1).$$

C. The case of friction forces

Let us come back to the example of a massive particle immersed in an anisotropic viscous fluid, discussed in Sec. IV B. Recall that it can be described as a HOCS with configuration space $Q=\mathbb{R}^3$, Lagrangian

$$L(\mathbf{x},\dot{\mathbf{x}}) = \frac{1}{2}m\dot{\mathbf{x}}^t\dot{\mathbf{x}} - V(\mathbf{x}),$$

and kinematic and variational constraints

$$C_K^{\mathcal{F}} = \{(\mathbf{x},\dot{\mathbf{x}},\ddot{\mathbf{x}}): m\dot{\mathbf{x}}^t\ddot{\mathbf{x}} + \dot{\mathbf{x}}^t\,\nabla\,V(\mathbf{x}) = -\,\dot{\mathbf{x}}^t\mathbb{F}(\mathbf{x})\dot{\mathbf{x}}\}$$

and

$$C_V^{\mathcal{F}} = \{(\mathbf{x},\dot{\mathbf{x}},\delta\mathbf{x}): \dot{\mathbf{x}}^t\mathbb{F}(\mathbf{x})\,\delta\mathbf{x} = 0\},$$

respectively. See Sec. IV B for details. It is worth mentioning that we have a HOCS with $k=2$ and $l=1$. Let us build up a 2-lift of the system. We shall consider the new configuration space

$$\tilde{Q} = TQ = \mathbb{R}^3 \times \mathbb{R}^3;$$

and call $(\mathbf{x},\mathbf{v})$ its points.

Remark 38: *Note that we are taking $n=2=\max\{k,l\}$ instead of a number $n>\max\{k,l\}$.*

The new Lagrangian is [see Eqs. (86) and (87)]

$$\tilde{L}(\mathbf{x},\mathbf{v},\dot{\mathbf{x}},\dot{\mathbf{v}}) = L(\mathbf{x},\dot{\mathbf{x}}) = \frac{1}{2}m\dot{\mathbf{x}}^t\dot{\mathbf{x}} - V(\mathbf{x}),$$

and the constraints are

$$\tilde{C}_K^{\mathcal{F}} = \{(\mathbf{x},\mathbf{v},\dot{\mathbf{x}},\dot{\mathbf{v}}): m\mathbf{v}^t\dot{\mathbf{v}} = -\,\mathbf{v}^t\,\nabla\,V(\mathbf{x}) - \mathbf{v}^t\mathbb{F}(\mathbf{x})\mathbf{v}, \dot{\mathbf{x}} = \mathbf{v}\}$$

and

$$\tilde{C}_V^{\mathcal{F}} = \{(\mathbf{x},\mathbf{v},\delta\mathbf{x},\delta\mathbf{v}): \mathbf{v}^t\mathbb{F}(\mathbf{x})\,\delta\mathbf{x} = 0\}.$$

Finally, we shall construct the linear version of the system. We first need to extend the configuration space to

$$\hat{Q} = \tilde{Q} \times \mathbb{R} = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R},$$

whose points will be denoted $(\mathbf{x},\mathbf{v},\lambda)$. The related Lagrangian is [see Eq. (91)]

$$\hat{L}(\mathbf{x},\mathbf{v},\lambda,\dot{\mathbf{x}},\dot{\mathbf{v}},\dot{\lambda}) = \frac{1}{2}m\dot{\mathbf{x}}^t\dot{\mathbf{x}} + \frac{1}{2}\dot{\lambda}^2 - V(\mathbf{x}),$$

the kinematic constraints $\hat{C}_K^{\mathcal{F}}$ are given by points $(\mathbf{x},\mathbf{v},\lambda,\dot{\mathbf{x}},\dot{\mathbf{v}},\dot{\lambda})$ such that

$$m\mathbf{v}^t\dot{\mathbf{v}} + (\mathbf{v}^t \nabla V(\mathbf{x}) + \mathbf{v}^t\mathbb{F}(\mathbf{x})\mathbf{v})\dot{\lambda} = 0, \quad \dot{\mathbf{x}} - \mathbf{v}\dot{\lambda} = 0,$$

and the variational ones by

$$\hat{C}_V^{\mathcal{F}} = \{(\mathbf{x},\mathbf{v},\lambda,\delta\mathbf{x},\delta\mathbf{v},\delta\lambda):\mathbf{v}^t\mathbb{F}(\mathbf{x})\delta\mathbf{x} = 0\}.$$

Summing up, we are describing a particle immersed in a viscous fluid as a particle subjected to linear kinematic constraints but whose constraint forces do not satisfy D'Alembert's principle.

D. Rocard's model

In this subsection we will apply the order-lowering and linearization procedures for Rocard's model,[2,3] as described in Ref. 5. It consists in a simplified model of a pneumatic tire rolling without sliding on a horizontal plane, and it defines a HOCS with kinematic constraints of order 2. In this model it is assumed that the plane of the pneumatic tire remains vertical. The configuration space is $Q=\mathbb{R}^2\times T^3$ whose natural coordinates will be denoted $x_1,x_2,\epsilon,\theta,\psi$. The pair (x_1,x_2) corresponds to the center of mass of the pneumatic tire; θ is the angle between the vertical plane containing the latter and some fixed vertical plane; ϵ is the angle of deviation of the plane of the tire with respect to the tangent to the trajectory of the center of mass (see Ref. 5 for details); and ψ is the angle of spinning around its axes. The Lagrangian of the system is

$$L = \frac{1}{2}M(\dot{x}_1^2 + \dot{x}_2^2) + \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}I\dot{\psi}^2 - \frac{1}{2}K\epsilon^2,$$

where M, J, and I are the mass and the principal moments of inertia, respectively; and K is a constant of elasticity. The kinematic and variational constraints are

$$\dot{x}_1 = \dot{\psi}\cos(\theta-\epsilon), \quad \dot{x}_2 = \dot{\psi}\sin(\theta-\epsilon),$$

$$-\ddot{\psi}\tan\epsilon + \dot{\psi}(\dot{\theta}-\dot{\epsilon}) = (a/M)\tan\epsilon,$$

and

$$\delta x_1 - \delta\psi\cos\theta = 0, \quad \delta x_2 - \delta\psi\sin\theta = 0,$$

$$\delta\theta - \delta\epsilon = 0, \tag{96}$$

with a another constant of the model. Here we have a HOCS with $k=2$ and $l=0$. Now we shall construct a canonical n-lift with $n=2$. Then, the new configuration space is

$$\tilde{Q} = TQ = \mathbb{R}^2 \times T^3 \times \mathbb{R}^2 \times \mathbb{R}^3,$$

whose natural coordinates we will write as follows:

$$(x_1,x_2,\epsilon,\theta,\psi,v_1,v_2,\Omega_\epsilon,\Omega_\theta,\Omega_\psi).$$

The new Lagrangian and variational constraints have the same local expressions as the original ones and the kinematic constraints are

$$\dot{x}_1 = \dot{\psi}\cos(\theta-\epsilon), \quad \dot{x}_2 = \dot{\psi}\sin(\theta-\epsilon),$$

$$-\dot{\Omega}_\psi \tan\epsilon + \dot{\psi}(\dot{\theta}-\dot{\epsilon}) = (a/M)\tan\epsilon.$$

The linearized version of the latter reads

$$\dot{x}_1 = \dot{\psi}\cos(\theta-\epsilon), \quad \dot{x}_2 = \dot{\psi}\cos(\theta-\epsilon),$$

$$-\dot{\Omega}_\psi \tan\epsilon + \dot{\psi}(\dot{\theta}-\dot{\epsilon}) = (a/M)\tan\epsilon\dot{\lambda}. \tag{97}$$

Summing up, we have transformed the original HOCS into a linear GNHS on configuration space

$$\hat{Q} = \tilde{Q} \times \mathbb{R} = \mathbb{R}^2 \times T^3 \times \mathbb{R}^2 \times \mathbb{R}^3 \times \mathbb{R},$$

with coordinates $(x_1,x_2,\epsilon,\theta,\psi,v_1,v_2,\Omega_\epsilon,\Omega_\theta,\Omega_\psi,\lambda)$, Lagrangian

$$\hat{L} = L + \frac{1}{2}\dot{\lambda}^2,$$

and constraints $\hat{C}_K$ and $\hat{C}_V$ given, respectively, by Eq. (97), and by virtual variations

$$(\delta x_1,\delta x_2,\delta\epsilon,\delta\theta,\delta\psi,\delta v_1,\delta v_2,\delta\Omega_\epsilon,\delta\Omega_\theta,\delta\Omega_\psi,\delta\lambda)$$

satisfying Eq. (96).

APPENDIX

Consider a HOCS with $C_K \subset T^{(k)}Q$ and $C_V \subset T^{(l)}Q \times_Q TQ$.

Let $\gamma:[t_1,t_2]\to Q$ be a curve such that its k-lift $\gamma^{(k)}$ belong to C_K. Then, defining $C_K^{[0]} = \tau_Q^{(k)} \times (C_K)$ we have for all $t \in (t_1,t_2)$ that

$$\gamma^{(0)}(t) = \gamma(t) \in C_K^{[0]},$$

and accordingly, assuming that $C_K^{[0]} \subset Q$ is a submanifold,

$$\gamma^{(1)}(t) = \gamma'(t) \in C_K^{[1]},$$

where by definition $C_K^{[1]} = \tau_Q^{(1,k)}(C_K) \cap TC_K^{[0]}$. Assuming that $C_K^{[1]} \subset TQ$ is a submanifold and defining

$$C_K^{[2]} = \tau_Q^{(2,k)}(C_K) \cap j_2^{-1}(TC_K^{[1]}),$$

we have that

$$\gamma^{(2)}(t) \in C_K^{[2]}.$$

The last equation follows from the fact that γ'' belongs to $\tau_Q^{(2,k)}(C_K)$ and that

$$j_2(\gamma^{(2)}(t)) = \gamma''(t) = (\gamma')_*\left(\left.\frac{d}{dt}\right|_t\right) \in TC_K^{[1]},$$

because $\gamma'(t) \in C_K^{[1]}$. In general, defining

$$C_K^{[r]} = \begin{cases} \tau_Q^{(0,k)}(C_K), & r=0 \\ \tau_Q^{(r,k)}(C_K) \cap j_r^{-1}(TC_K^{[r-1]}), & 1 \leq r \leq k \\ j_r^{-1}(TC_K^{[r-1]}), & r>k, \end{cases} \tag{A1}$$

it can be shown recursively that for all $t \in (t_1, t_2)$

$$\gamma^{(r)}(t) \in C_K^{[r]}, \quad r \geq 0.$$

Proposition 39: *Assume that the subsets $C_K^{[r]}$ given in Eq. (A1) can be defined for all $r \leq l$ (which requires that they are submanifolds for all $r < l$), and that*

$$C_V(\eta) \neq \emptyset \quad \textit{for all } \eta \in C_K^{[l]}.$$

Then every curve with k-lift in C_K has an infinitesimal variation in C_V.

Proof: *If $\gamma:[t_1,t_2] \to Q$ is a curve such that $\gamma^{(k)}(t) \in C_K$, then, from the previous discussion, $\gamma^{(l)}(t) \in C_K^{[l]}$. Since $C_V(\gamma^{(l)}(t)) \neq \emptyset$ for all $t \in (t_1,t_2)$ then $\delta\gamma:[t_1,t_2] \to TQ: t \mapsto 0$ is a variation of γ inside C_V.* ■

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