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Vibration of non-homogeneous rectangular membranes with arbitrary interfaces

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Abstract

The study deals with the generalized solution of the title problem. The free vibration problem of a rectangular membrane with partial domains each of uniform density and arbitrary interface is tackled. Previous studies by other researchers include straight parallel to the borders and oblique interfaces, and bent ones. The solution is found by means of a direct variational method with a series composed with a complete set of functions. Two alternative sets are explored: trigonometric and power series. Such series are uniformly convergent to the exact solution. The approach is straightforward and very efficient from the computational viewpoint. A determinant-factorization method is employed to automatically eliminate eventual spurious frequencies. The well-known analogy between plates and membranes does not hold in this problem and a demonstration is included. Diverse illustrations are worked out as the cases of an oblique straight line interface and an open curve line which divides the membrane in two domains each of different density are first presented. Also a rectangular membrane with an interior closed domain is stated and the numerical example of a circular interior zone is included. Comparison between the two alternatives and with other authors' results show excellent agreement. In all cases the computational cost is very low.

1. Introduction

Many researchers have addressed the vibrational problem of a great variety of membranes with regions of different density such as rectangular, circular and annular membranes (e.g. Refs. [1–10]). However, many of them deal with continuous variation of the density or variation of density in steps with interfaces parallel to one of the edges. More recently, Kang [8] has solved the case of a straight oblique interface and lately the problem of a bent interface [9].

In the present paper the natural vibration problem of rectangular membranes with an arbitrary interface which separates domains with different densities is tackled through a variational method with two alternatives approaches, i.e. Two sets of base functions, a trigonometric set and power series. A variational direct

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approach developed before by the authors and applied to various structural problems that makes use of an extended trigonometric (complete) set (see for instance Refs. [11–13]) is employed to solve the title problem. As is shown the core problem is in someway included in a previous study of vibration of plates [13]. Also algebraic sets have been previously applied to solve other mechanical problems (e.g. Ref. [14]). To the authors knowledge no analytical solution is reported in the open literature of a rectangular membrane with regions of different densities with curve interfaces, either open or closed.

First the vibrational problem of a rectangular membrane with regions of different densities is stated. Then the direct variational method and the two different approaches are presented. One is a trigonometric set that identically satisfies the boundary conditions of the membrane and constitutes a complete subset and the other one is the algebraic set. In the latter the boundary conditions are accounted for by means of Lagrange multipliers. A determinant-factorization method is employed to automatically eliminate eventual spurious frequencies. This result becomes a relevant feature since other reported techniques appeal to comparisons with known results to discard those values. Additionally the statement and use of both methodologies is straightforward and the results are of arbitrary precision.

The well-known analogy between the frequency parameter of membranes and simply supported plates is usually employed (e.g. Refs. [15,16]). This analogy is not valid—ingeneral—when the density is not uniform. This theorem has been demonstrated by the authors and is summarized in Appendix A.

As is known the differential problem of the vibrating membrane is governed by the Helmholtz equation. This also represents the eigenvalue problem of cavities and the results herein presented may be applied to such problems.

2. Problem statement

The linear natural vibration problem of the membrane (Fig. 1) is governed by the following partial differential equation (also known as Helmoltz equation) in $w = w(X, Y)$ (the transverse modal shape),

$$\nabla^2 w + \Omega^{*2} w = 0 \tag{1}$$

under the condition

$$w_{(\Gamma)} = 0, \tag{2}$$

where

$$\nabla^2(\cdot) \equiv w_{XX} + w_{YY}, \tag{3}$$

$$\Omega^{*2} \equiv \frac{\rho(x, y)}{T}\omega^2 \tag{4}$$

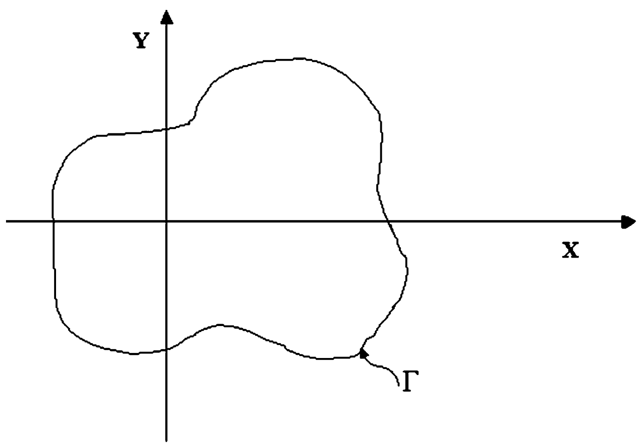

Fig. 1. General configuration of a membrane of domain A.

in which ω is the natural circular frequency, T is the uniform tension per unit length applied over the membrane and, $\rho(x, y)$ is the variable density per unit of area. The derivatives are denoted by subscripts. The energy functional U^* related with this problem writes

$$2U^* = \int\int_{(A)} (w_X^2 + w_Y^2 - \Omega^{*2} w^2)\, \mathrm{d}X\, \mathrm{d}Y. \tag{5}$$

In the case of a rectangular membrane with constant T and variable density in regions (density ρ_n is constant in each region n of a total of NR regions of area A_n) (e.g. the case $NR = 3$ is shown in Fig. 2), the energy is written as follows:

$$2U^* = \int_0^a \int_0^b (w_X^2 + w_Y^2)\, \mathrm{d}X\, \mathrm{d}Y - \frac{\omega^2}{T} \sum_{n=1}^{NR} \rho_n \int\int_{(A_n)} w^2\, \mathrm{d}X\, \mathrm{d}Y. \tag{6}$$

It is convenient to non-dimensionalize the problem using the following variables:

$$\lambda = \frac{a}{b}, \tag{7}$$

$$x \equiv \frac{X}{a}; \quad y \equiv \frac{\lambda Y}{a}, \tag{8}$$

$$r_n \equiv \frac{\rho_n}{\rho_1} \quad (n = 1, 2, \ldots, NR) \ (r_1 = 1). \tag{9}$$

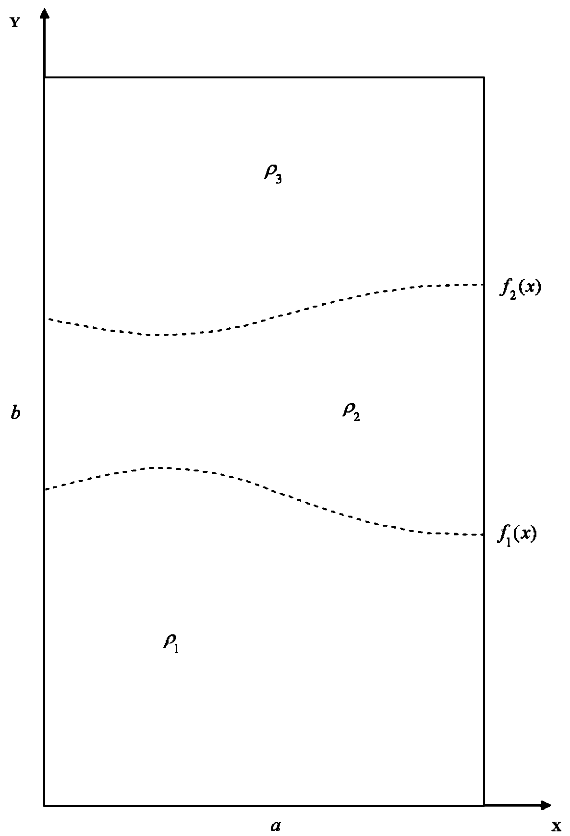

Fig. 2. Non-homogenous membrane.

Let us denote the derivatives with respect to the non-dimensionalized variables as

$$(\cdot)' \equiv \frac{\partial(\cdot)}{\partial x}; \quad \bar{(\cdot)} \equiv \frac{\partial(\cdot)}{\partial y} \tag{10}$$

and so on. The functional (6) becomes (within a constant)

$$2U = \int_0^1 \int_0^1 (w'^2 + \lambda^2 \bar{w}^2)\,\mathrm{d}x\,\mathrm{d}y - \Omega^2 \sum_{n=1}^{NR} r_n \int\int_{(An)} w^2\,\mathrm{d}x\,\mathrm{d}y, \tag{11}$$

where the non-dimensionalized frequency parameter Ω is

$$\Omega = \sqrt{\frac{\rho_1}{T}}\omega a. \tag{12}$$

The arbitrary curves (interfaces) that separate the regions should be given in the form $y = f_m(x)$ with $m = 0, 1, 2, \ldots, NR$. Evidently (Fig. 2) $f_0(x) = 0$ and $f_{NR}(x) = 1$.

The two alternatives in the use of the direct method to solve the title problem are presented in the following sections. In both the modal shapes will be represented by complete subsets. Finally, the extreme condition that should be imposed to the functional, i.e. $\delta U = 0$, yields

$$\delta U = \int_0^1 \int_0^1 (w'\delta w' + \lambda^2 \bar{w}\delta\bar{w})\,\mathrm{d}x\,\mathrm{d}y - \Omega^2 \sum_{n=1}^{NR} r_n \int\int_{(An)} w\delta w\,\mathrm{d}x\,\mathrm{d}y = 0. \tag{13}$$

3. Selected base sets

In this section the two generalized solutions are presented. First the so-called Whole Element Method (WEM) and second the power series expansion.

3.1. Variational solution WEM

First the title problem is tackled by means of a direct method using an extended trigonometric base set. A complete set is employed to represent the main unknowns with uniform convergence and satisfying the boundary condition $w_{(\Gamma)}$. The authors have developed a methodology named WEM (MEC in Spanish). This method (see for instance Refs. [11–13]) makes use of a complete series of extended trigonometric functions in a unitary domain which are systematically stated for one-, two- and three-dimensional domains and also for spatial or time problems, and other complexities. In particular two possible extended trigonometric series composed by a complete set in the two-dimensional domain, among infinite possibilities, are

$$\begin{aligned} w = w(x,y) = &\sum_{i_1}^{\infty}\sum_{j_1}^{\infty} A_{ij}s_i s_j + y\left(\sum_{i_1}^{\infty} A_{i0}s_i + b_0\right) \\ &+ x\left(\sum_{j_1}^{\infty} A_{0j}s_j + a_0\right) + \sum_{i_1}^{\infty} a_{i0}s_i + \sum_{j_1}^{\infty} a_{0j}s_j + k, \end{aligned} \tag{14}$$

$$w = w(x,y) = \sum_{i_1}^{\infty}\sum_{j_1}^{\infty} B_{ij}c_i c_j + \sum_{i_1}^{\infty} B_{i0}c_i + \sum_{j_1}^{\infty} B_{0j}c_j + B_{00} \tag{15}$$

in which the following notation was introduced: $s_i = \sin(i\pi x)$, $s_j = \sin(j\pi y)$, $c_i = \cos(i\pi x)$ and, $c_j = \cos(j\pi y)$. Also i_1 denotes $i = 1$ and $j_1, j = 1$.

Let us use the series as in Eq. (14). When dealing with the vibration of rectangular membranes and after the boundary conditions are imposed, fortunately many terms of the series (14) cancel. Still, it constitutes a complete subset. Then the following complete series are used to represent the variables in the domain $\{D : 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$

$$
\begin{aligned}
w &= \sum_{i_1}^{M} \sum_{j_1}^{N} A_{ij} s_i s_j,\\
w' &= \sum_{i_1}^{M} \sum_{j_1}^{N} (i\pi) A_{ij} c_i s_j,\\
\bar{w} &= \sum_{i_1}^{M} \sum_{j_1}^{N} (j\pi) A_{ij} s_i c_j.
\end{aligned}
\tag{16}
$$

The three expansions (16) are of uniform convergence. In theory, $M, N \to \infty$. In practice M, N are finite though arbitrary precision may be obtained. These expansions are introduced in Eq. (13) and after finding its extreme, the eigenvalue problem is reduced to the solution of the following equation (the equation is particularized to the case of two regions with densities ρ_1 and ρ_2)

$$\sum_{i_1} \sum_{j_1} \delta A_{ij} \sum_{p_1} \sum_{q_1} A_{pq} \Phi_{ijpq} = 0 \tag{17}$$

with

$$
\begin{aligned}
\Phi_{ijpq} &= \pi^2 ip\varepsilon_{ijpq} + \lambda\pi^2 jq\phi_{ijpq} - \Omega^2[\mu^{(1)}_{ijpq} + (\rho_2/\rho_1 - 1)\mu^{(2)}_{ijpq}],\\
\varepsilon_{ijpq} &= \int_0^1 c_i c_p \,\mathrm{d}x \int_0^1 s_j s_q \,\mathrm{d}y; \quad \phi_{ijpq} = \int_0^1 s_i s_p \,\mathrm{d}x \int_0^1 c_j c_q \,\mathrm{d}y,\\
\mu^{(1)}_{ijpq} &= \int_0^1 s_i s_p \,\mathrm{d}x \int_0^1 s_j s_q \,\mathrm{d}y; \quad \mu^{(2)}_{ijpq} = \int_\alpha^\beta s_i s_p \,\mathrm{d}x \int_{f_1(x)}^{f_2(x)} s_j s_q \,\mathrm{d}y,
\end{aligned}
$$

where α, β and $f_1(x)$, $f_2(x)$ are the limits of the region with different density in the x and y directions, respectively.

Numerical examples that illustrate the methodology will be shown in section Numerical illustrations bellow. They include membranes of two regions of different density with straight or curve interface and a closed inner region.

3.2. Direct method with power series

A set of algebraic functions—power series—is used as base functions in the direct method. In order to satisfy the boundary condition, Lagrange multipliers are introduced. The coordinates are set as in Fig. 3. Instead of using expressions (16) the following power expansions are introduced:

$$w = \sum_{i=0} \sum_{j=0} A_{ij} x^i y^j. \tag{18}$$

Evidently, this expansion constitutes a complete set for continuous functions—as is the case of the modal shapes—although the boundary conditions of the membrane are not satisfied (as happened before with functions (16)). Recall that $w(x, y)$ should be null at the boundaries $(\pm 0.5, y)$, and $(x, \pm 0.5)$ (see Fig. 3). These conditions

$$\sum_{i_0} A_{ij}(0.5)^i = 0 \text{ (a)}; \quad \sum_{i_0} A_{ij}(-0.5)^i = 0 \text{ (b)} \ \ (\forall j), \tag{19}$$

$$\sum_{j_0} A_{ij}(0.5)^j = 0 \text{ (c)}; \quad \sum_{j_0} A_{ij}(-0.5)^j = 0 \text{ (d)} \ \ (\forall i) \tag{20}$$

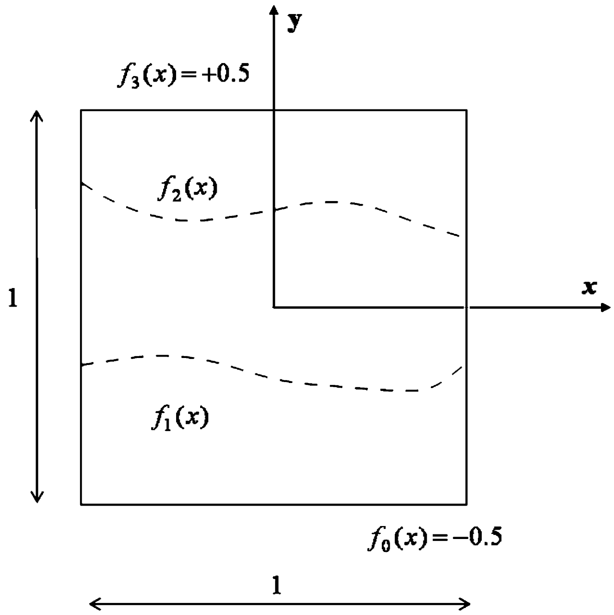

Fig. 3. Non-homogeneous membrane. Non-dimensionalized coordinates used in the power series technique.

are handled with the use of Lagrange multipliers ($\mu_j^{(1)}, \mu_j^{(2)}, v_i^{(1)}, v_i^{(2)}$) to expand the governing functional (11) (with the corresponding change of coordinates) and the following extended functional U_e results:

$$2U_e = 2U - \sum_{i_0}^{M} \sum_{j_0}^{N} A_{ij}[\mu_j^{(1)}(0.5)^i + \mu_j^{(2)}(-0.5)^i + v_i^{(1)}(0.5)^j + v_i^{(2)}(-0.5)^j]. \tag{21}$$

Thus the restrictions to be fulfilled by the unknowns A_{ij} are made evident. The characteristic equation is found after imposing the extreme condition to the functional (21)

$$\begin{aligned}\delta U_e = \delta U - \sum_{i_0}^{M} \sum_{j_0}^{N} \{&\delta A_{ij}[\mu_j^{(1)}(0.5)^i + \mu_j^{(2)}(-0.5)^i + v_i^{(1)}(0.5)^j \\ &+ v_i^{(2)}(-0.5)^j] + A_{ij}[\delta\mu_j^{(1)}(0.5)^i + \delta\mu_j^{(2)}(-0.5)^i \\ &+ \delta v_i^{(1)}(0.5)^j + \delta v_i^{(2)}(-0.5)^j]\} = 0.\end{aligned} \tag{22}$$

Let us denote the unknowns vector by $\mathbf{V}' = [\mathbf{A}' \vdots \mathbf{L}']$ with

$$\begin{aligned}\mathbf{A}' &= [A_{00} \cdots A_{0N} A_{10} \cdots A_{1N} \cdots A_{M0} \cdots A_{MN}], \\ \mathbf{L}' &= [\mu_0^{(1)} \cdots \mu_N^{(1)} \mu_0^{(2)} \cdots \mu_N^{(2)} v_0^{(1)} \cdots v_M^{(1)} v_0^{(2)} \cdots v_M^{(2)}].\end{aligned} \tag{23}$$

Based on the fundamental lemma of the calculus of variation the following linear homogeneous system yields:

$$\begin{aligned}\mathbf{BA} + \mathbf{K}^{\mathrm{T}}\mathbf{L} &= \mathbf{0} \\ \mathbf{KA} &= \mathbf{0}.\end{aligned} \tag{24}$$

B of components b_{IJ} is a square matrix of order $(M+1)\cdot(M+1)$ while **K** of components k_{RS} is a rectangular matrix of order $2(M+N)\cdot[(M+1)(N+1)]$. Also $I \equiv (N+1)i + j + 1\ \forall ij$, $J \equiv (N+1)p + q + 1\ \forall pq$, ($i, p = 0, 1, \ldots, M$) and ($j, q = 0, 1, \ldots, N$). The b_{IJ}'s write

$$\begin{aligned}b_{IJ} = {}&(0.5)^{i+j+p+q}[1-(-1)^{i+p+1}][1-(-1)^{j+q-1}] \\ &\times\left[\frac{ip}{(i+p-1)(j+q+1)} + \lambda^2 \frac{jq}{(i+p+1)(j+q-1)}\right] - \Omega^2 \sum_{n=1}^{NR} r_n \varphi_{ijpq}^{(n)}\end{aligned} \tag{25}$$

where

$$\varphi_{ijpq}^{(n)} \equiv \int_{-0.5}^{0.5} x^{i+p}\,\mathrm{d}x \int_{f_{n-1}(x)}^{f_n(x)} y^{j+q}\,\mathrm{d}y \tag{26}$$

and $f_n(x)$ are the limits of the successive regions. Additionally the Lagrange conditions lead to (with $S=(N+1)i+j+1$)

$$k_{RS}\begin{cases} =(0.5)^i;\ R=j+1 \\ \quad (i=0,1,\ldots,M),\ (j=0,1,\ldots,N-1) \\ =(-0.5)^i;\ R=j+N \\ \quad (i=0,1,\ldots,M),\ (j=0,1,\ldots,N-2) \\ =(0.5)^j;\ R=i+2(N-1)+1 \\ \quad (i=0,1,\ldots,M),\ (j=0,1,\ldots,N) \\ =(-0.5)^j;\ R=i+(M+1)+2(N-1)+1 \\ \quad (i=0,1,\ldots,M),\ (j=0,1,\ldots,N). \end{cases} \tag{27}$$

The eigenvalues may be stated by imposing the nullity condition to the determinant of the complete matrix

$$\begin{vmatrix} \mathbf{B} & \mathbf{K}^{\mathrm{T}} \\ \mathbf{K} & \mathbf{0} \end{vmatrix} = 0. \tag{28}$$

It is known (e.g. Ref. [17]) that if **B** is not singular the determinant (28) is equivalent to

$$|\mathbf{B}||\mathbf{K}\mathbf{B}^{-1}\mathbf{K}^{\mathrm{T}}| = 0. \tag{29}$$

Then the next result (theorem) is demonstrated: any of the two involved factors may be null but since the eigenvalues obtained with $|\mathbf{B}|$ do not involve the boundary restrictions, the frequencies are to be found with

$$|\mathbf{K}\mathbf{B}^{-1}\mathbf{K}^{\mathrm{T}}| = 0. \tag{30}$$

Instead if determinant (28) is directly solved spurious frequencies would yield and have to be selected by additional means such as comparison with already known results (see for instance Refs. [8,9]). This determinant-factorization method becomes an interesting feature since the spurious frequencies are automatically discarded. Numerical examples are presented in the next section.

4. Numerical illustrations

First a rectangular membrane with $a=1$ m, $b=1.8$ m (i.e. $\lambda=a/b=0.555..$) is addressed. As a reference the values of the frequency parameter Ω (Eq. (12)) of a rectangular membrane with homogeneous density are reported in Table 1 found with the direct method with two proposed sets. In the case of the homogeneous membrane the trigonometric solution is reduced to just one term with $M=1$ or $M=2$ or $M=3$, etc., for the

Table 1
Frequency parameter Ω of the homogeneous rectangular membrane

Order	Exact	Algebraic set			Trigonometric set
		$M=N=4$	$M=N=9$	$M=N=11$	Exact
1	3.5939	3.5939	3.5939	3.5939	3.5939
2	4.6962	4.7784	4.6962	4.6962	4.6962
3	6.1062	6.4336	6.1062	6.1062	6.1062
4	6.5211	6.7116	6.5211	6.5211	6.5211
5	7.1877	7.4137	7.1877	7.1877	7.1877
6	7.6556	8.5745	7.6565	7.6556	7.6556

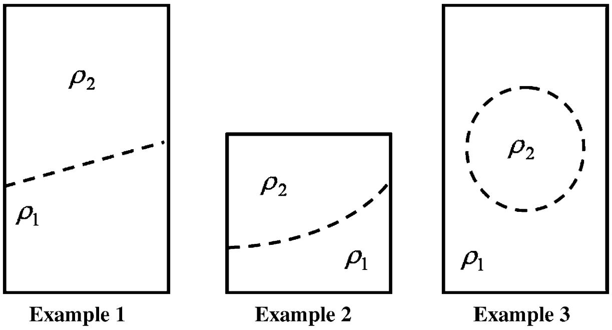

Fig. 4. Geometrical scheme of the three examples of non-homogeneous membranes.

Table 2
Frequency parameter Ω of a rectangular membrane with straight interface (Eq. (31)). Trigonometric series. Example 1. $\rho_2 = 2\rho_1$

Order	Frequency parameter				
	$M = N = 5$	$M = N = 8$	$M = N = 10$	$M = N = 12$	Kang and Lee [8]
1	2.7686	2.7681	2.7681	2.7680	2.768
2	3.9686	3.9664	3.9661	3.9659	3.966
3	4.8466	4.8452	4.8450	4.8449	4.845
4	5.0089	5.0062	5.0061	5.0060	5.005
5	5.8535	5.8445	5.8432	5.8427	5.842

different modes. Now the non-homogeneous case with two regions of different density ($\rho_2 = 2\rho_1$) separated by an oblique interface is tackled (see Fig. 4, Example 1). The interface is given by the following line:

$$Y = 0.3X + 0.7 \tag{31}$$

with $0 \leqslant X \leqslant a$ and $0 \leqslant Y \leqslant b$. The values obtained using the direct method with trigonometric functions (Eq. (16)) are depicted in Table 2 for various number of terms of the series. The values reported in Kang and Lee [8] are depicted in the rightmost column. In this Ref. [8] a superposition of two semi-infinite membranes is proposed and a sum of trigonometric functions employed to solve the governing differential equation. The methodology is somewhat involved and some spurious frequencies are found which are discarded by comparison with other methods (e.g. FEM). On the other hand, the herein presented direct method with a complete trigonometric set yields all the frequencies without being necessary such an analysis of the results.

The case of a square membrane ($a = 1\,\text{m},\ b = 1\,\text{m}$) with a curve interface was also studied. Fig. 4, Example 2, shows the case of the membrane of two regions with different density ($\rho_2 = 2\rho_1$) and a curve interface. In particular, the parabola

$$Y = 0.8X^2 - 0.5X + 0.4 \tag{32}$$

with $0 \leqslant X \leqslant a$ and $0 \leqslant Y \leqslant b$, is chosen as the interface curve. The values of the first five frequency parameters are reported in Table 3, found with the direct method using both the trigonometric and the algebraic sets.

Finally a rectangular membrane ($a = 1\,\text{m},\ b = 1.8\,\text{m}$) with an inner closed region of different density (Fig. 4, Example 3) was analyzed and the frequency parameter Ω results depicted in Table 4. In particular, the density of a circular region, with origin in the center of the plate and radius 0.4 m is twice, three and five times the density of the membrane (i.e. $r = \rho_2/\rho_1$, $r = 2, 3, 5$).

As expected, the effect of a region with larger density is to lower the frequencies. Fig. 5 shows the influence of the change of density on the successive frequency parameters.

With respect to the methodology, the direct method offers a very simple and reliable tool to solve this type of problem. Given the simplicity of the boundaries the trigonometric set satisfies the end conditions previously and constitutes a complete subset. On the other hand, the algebraic set (power series) includes the use of

Table 3
Frequency parameter Ω of a square membrane with curve interface (Eq. (32)). Example 2. $\rho_2 = 2\rho_1$

Order	Frequency parameter	
	Trigonometric set	Algebraic set
	$M = N = 10$	$M = N = 8$
1	3.3373	3.3374
2	5.2591	5.2595
3	5.6367	5.6377
4	7.0885	7.0904
5	7.4718	7.4729

Table 4
Frequency parameter Ω of a rectangular membrane with an inner closed region of different densities. Example 3

Order	Trigonometric set ($M = N = 10$)			Algebraic set ($M = N = 11$)		
	$r = 2$	$r = 3$	$r = 5$	$r = 2$	$r = 3$	$r = 5$
1	2.7386	2.2856	1.7995	2.7389	2.2860	1.7999
2	4.0156	3.4927	2.8318	4.0168	3.4948	2.8345
3	5.0200	4.1793	3.2807	5.0210	4.1806	3.2821
4	5.4314	4.9317	4.1367	5.4386	4.9614	4.2051
5	6.2559	5.3985	4.3270	6.2600	5.4070	4.3384
6	6.6140	6.1391	4.8361	6.6225	6.1442	4.8414

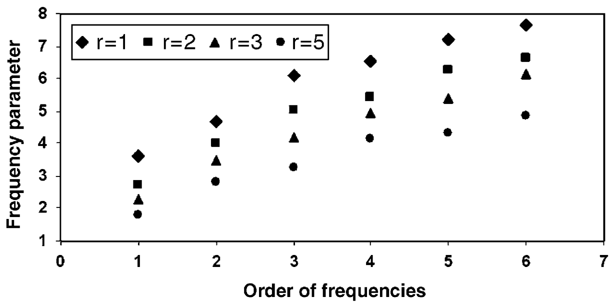

Fig. 5. Rectangular membrane with a circular inner region. Influence of density ratio on the frequency parameter.

Lagrange multiplier to fulfill those conditions. However, other techniques could have been employed to satisfy the boundary conditions. The convergence is found to be better with the trigonometric set and has the particularity of remaining constant at higher frequencies. In the comparison with results from other authors (Example 1), though all methods attain the same values, the advantage of the present methodologies is that no "selection" of results have to be performed. Additionally the use of complete sets in the direct method ensures the convergence of the solution to the exact one, i.e. the results are of arbitrary precision. For instance if one needs frequencies accurate to three decimal places, the number of terms should be increased until the significant digits remain constant. Thus the resulting eigenvalue is exact.

5. Final comments

A direct method approach is herein employed to solve the natural vibrational problem of non-homogeneous rectangular membranes. Two different base functions that belong to complete sets are inserted in the governing functional. Then the variational approach gives the necessary equation to solve the eigenvalue problem. In particular the density is varied in steps and three examples were numerically solved: an oblique interface, a curve interface and a closed inner region. Both the trigonometric and the algebraic sets exhibited an excellent performance, giving between 4 and 5 digits of accuracy with only 10 terms. It should be pointed out that when dealing with the power series the increment of the number of terms eventually leads to ill-conditioning of the system. A determinant-factorization method is employed to automatically eliminate eventual spurious frequencies. This is a relevant feature since other reported techniques appeal to comparisons with known results to discard those values.

As mentioned before, the well-known analogy between plates and membranes that is valid with homogeneous membranes, does not hold in the case of non-homogeneous membranes (see Appendix A). Consequently—in general—the frequencies of membranes with different densities may not be found from the similar plate results or viceversa.

Acknowledgments

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Appendix A. The analogy between membranes and plates does not hold with non-homogeneous domains

The analogy for natural frequencies and critical loads, between membranes and simply supported plates of homogeneous polygonal domains, is well known (see Refs. [15,16], for instance). In a recent work Ref. [18] a demonstration is presented though with other complexities. In this Appendix the demonstration that the analogy is not valid for rectangular membranes with partial regions of different densities is shown. Let us tackle a rectangular domain with an interface ($\mathscr{C}$) that divides it in two regions of densities ρ_1 and $\rho_2 = r\rho_1$ ($r \neq 1$), respectively. The differential system that governs the natural vibration problem of a simply supported plate with $w_j(x, y)$ being the transverse modal shapes of each region j (to fix ideas, two regions are dealt with,

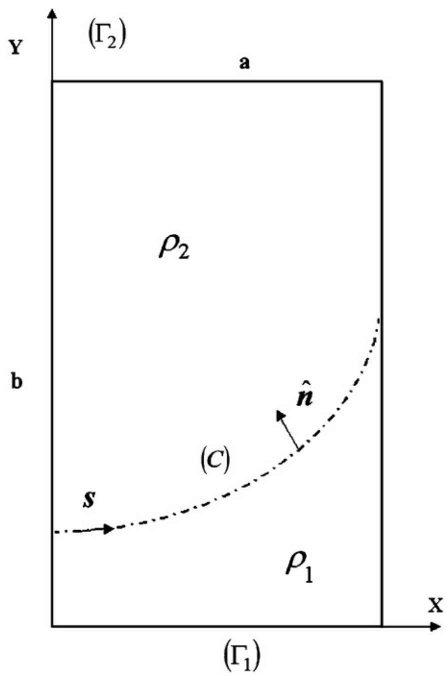

Fig. A.1. Rectangular domain with interface curve $\mathscr{C}$ and external boundaries Γ_j. Particular case of two regions of different densities.

as in Fig. A.1), is

$$\nabla^2\nabla^2 w_j - \Omega_{pj}^2 w_j = 0 \quad (j = 1, 2). \tag{A.1}$$

$x = X/a$ and $y = \lambda Y/a$ and the following notation was introduced:

$$\nabla^2(\cdot) \equiv \frac{\partial^2(\cdot)}{\partial x^2} + \lambda^2 \frac{\partial^2(\cdot)}{\partial y^2}, \tag{A.2a}$$

$$\Omega_{pj}^2 = \frac{\rho_j h}{D}\omega^2 a^4. \tag{A.2b}$$

in which h is the thickness and D its flexural rigidity and the following boundary conditions (Γ_j is the boundary of region j and $\mathscr{C}$ is the interface curve):

$$|w_j|_{(\Gamma_j)} = |M_n|_{(\Gamma_j)} = 0 \tag{A.3a1,A.3a2}$$

$$|w_1 = w_2|_{(\mathscr{C})} \tag{A.3b}$$

$$\left|\frac{\partial w_1}{\partial n} = \frac{\partial w_2}{\partial n}\right|_{(\mathscr{C})} \tag{A.3c}$$

$$\left|\frac{\partial^2 w_1}{\partial n^2} = \frac{\partial^2 w_2}{\partial n^2}\right|_{(\mathscr{C})} \tag{A.3d}$$

By M_n we denote the bending moment along the normal to the boundary. In general (Fig. A.2)

$$M_n = -D\left[\frac{\partial^2 w}{\partial n^2} + \nu\left(\frac{\partial^2 w}{\partial s^2} + \frac{d\alpha}{ds}\frac{\partial w}{\partial n}\right)\right], \tag{A.4}$$

where ν is the Poisson's coefficient and $d(\cdot)/ds$ is the derivative along s. Also the Laplacian (A.2a) in n, s coordinates writes

$$\nabla^2(\cdot) = a^2\left[\frac{\partial^2(\cdot)}{\partial n^2} + \frac{\partial(\cdot)}{\partial s^2} + \frac{d\alpha}{ds}\frac{\partial(\cdot)}{\partial n}\right]. \tag{A.5}$$

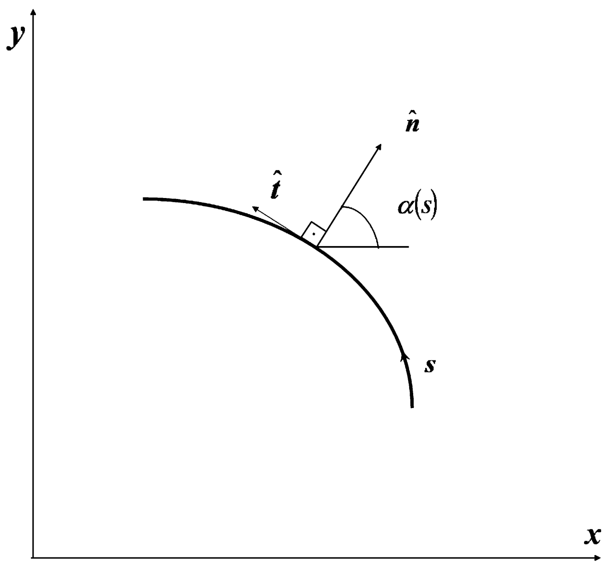

Fig. A.2. Directions over the boundary.

It is important to recall a consequence of the expressions (A.3a–d) that are written in what follows

$$\left|\frac{\partial^m w_j}{\partial s^m}\right|_{(\Gamma_j)} = 0, \tag{A.6a}$$

$$\left|\frac{\partial^m w_1}{\partial s^m} = \frac{\partial^m w_2}{\partial s^m}\right|_{(\mathscr{C})}, \tag{A.6b}$$

$$\left|\frac{\partial^{m+1} w_1}{\partial n \partial s^m} = \frac{\partial^{m+1} w_2}{\partial n \partial s^m}\right|_{(\mathscr{C})}, \tag{A.6c}$$

$$\left|\frac{\partial^{m+2} w_1}{\partial n^2 \partial s^m} = \frac{\partial^{m+2} w_2}{\partial n^2 \partial s^m}\right|_{(\mathscr{C})} \quad (m = 1, 2, \ldots). \tag{A.6d}$$

On the other hand, if the $v_j(x, y)$'s are the membrane mode shapes at each region, the differential system that governs the natural vibrations of a membrane reads

$$\nabla^2 v_j + \Omega_{mj}^2 v_j = 0 \quad (j = 1, 2), \tag{A.7}$$

where

$$\Omega_{mj}^2 = \frac{\rho_j h}{T} \omega^2 a^2 \tag{A.8}$$

and T is the uniform force per unit of length acting on the membrane. The boundary conditions of the membrane are

$$|v_j|_{(\Gamma_j)} = 0 \quad (j = 1, 2), \tag{A.9a}$$

$$|v_1 = v_2|_{(\mathscr{C})}, \tag{A.9b}$$

$$\left|\frac{\partial v_1}{\partial n} = \frac{\partial v_2}{\partial n}\right|_{(\mathscr{C})} \tag{A.9c}$$

It is admitted that the material of the plates is a non-standard one since the modulus of elasticity E, the Poisson coefficient v and the thickness h remain constant while the density may vary. As stated before the analogy between the membrane and simply supported plates vibration problems is lost when a complexity as the nonuniform density is present. Let us demonstrate this property. Expressions (A.1) may be identically rewritten as

$$\nabla^2(\nabla^2 w_j - \Omega_{pj} w_j) + \Omega_{pj}(\nabla^2 w_j - \Omega_{pj} w_j) = 0 \quad (j = 1, 2). \tag{A.10}$$

Now, if $\phi_j = \phi_j(x, y)$ denotes the expression between parentheses, i.e.

$$\phi_j \equiv \nabla^2 w_j - \Omega_{pj} w_j \quad (j = 1, 2) \tag{A.11}$$

then expression (A.10) (with Eq. (A.11)) may be written as follows:

$$\nabla^2 \phi_j + \Omega_{pj} \phi_j = 0 \quad (j = 1, 2). \tag{A.12}$$

If the ϕ_j's satisfy the membrane boundary conditions (A.9) and comparing (A.12) with (A.7) evidently the analogy would exist with $\Omega_{mj}^2 = \Omega_{pj}$. However it will be shown that this is not—in general—possible. Let us first observe whether (A.9a) is fulfilled by the ϕ_j's. The right member of expression (A.11) should satisfy

$$|\nabla^2 w_j - \Omega_{pj} w_j|_{(\Gamma_j)} = 0 \quad (j = 1, 2) \tag{A.13}$$

in order that $|\phi_j|_{(\Gamma_j)} = 0$.

Since (A.3a1) is verified only the fulfillment of $|\nabla^2 w_j|_{(\Gamma_j)} = 0$ $(j = 1, 2)$ should be required. Now due to condition (A.3a2), to the fact that the boundary Γ_j is a polygonal line (i.e. $\mathrm{d}\alpha/\mathrm{d}s|_{(\Gamma_j)} = 0$) and from (A.6a) the condition $M_n|_{(\Gamma_j)} = 0$ from (A.4) implies that,

$$\left.\frac{\partial^2 w}{\partial n^2}\right|_{(\Gamma_j)} = 0. \tag{A.14}$$

In consequence the Laplacian (A.5) is also null over each (Γ_j). Then condition (A.13) is verified.

Up to this stage the analogy is apparently valid. An inspection of the continuity conditions (A.9b) and (A.9c) along the interface $\mathscr{C}$ is carried out in what follows. The next equalities should hold valid along $\mathscr{C}$ if the analogy would hold

$$|\phi_1 = \phi_2|_{(\mathscr{C})} = |\nabla^2 w_1 - \Omega_{p1} w_1 = \nabla^2 w_2 - \Omega_{p2} w_2|_{(\mathscr{C})}, \tag{A.15}$$

$$\left|\frac{\partial\phi_1}{\partial n} = \frac{\partial\phi_2}{\partial n}\right|_{(\mathscr{C})} = \left|\frac{\partial}{\partial n}(\nabla^2 w_1 - \Omega_{p1} w_1) = \frac{\partial}{\partial n}(\nabla^2 w_2 - \Omega_{p2} w_2)\right|_{(\mathscr{C})}. \tag{A.16}$$

It may be easily observed that these conditions are not, in general, satisfied. In effect, observing (A.5), conditions (A.3c–d) and (A.6b) and that $\mathrm{d}\alpha/\mathrm{d}s|_{(\mathscr{C})}$ attains only one value over interface $(\mathscr{C})$ for both regions, then it is true that

$$|\nabla^2 w_1 = \nabla^2 w_2|_{(\mathscr{C})} (= 0) \tag{A.17}$$

but in order for Eq. (A.15) to be satisfied additionally and simultaneously

$$|\Omega_{p1} w_1 = \Omega_{p2} w_2|_{(\mathscr{C})} \tag{A.18}$$

should be true. However given the condition (A.3b) and the definition (A.2b) the following identity should hold

$$\rho_1 = \rho_2, \tag{A.19}$$

which are not equal by hypothesis. Then based on the above-stated, and without being necessary to analyze (A.16), the analogy does not exist for the case under study.

The authors are at present studying a generalization of the analogy for non-homogeneous domains.

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