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Conditions for Permutability of Congruences in Implication Algebras

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Abstract In this paper we give conditions on an implication algebra **A** so that two congruences θ_1, θ_2 on **A** permute, i.e. $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$. We also provide simpler conditions for permutability in finite implication algebras. Finally we present some applications of these characterizations.

Keywords Implication algebras · Congruences · Permutability

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1 Introduction

In this section we will introduce some basic properties of implication algebras and fix the notation used throughout the article. In the next section we will present the main result of the article, namely, a characterization for the permutability of two congruences on an implication algebra. The conditions found may be simplified in the case of finite implication algebras; this will be carried out in Section 3. Finally, in Section 4, we will turn to some applications of the preceding results. Specifically we will derive as a simple consequence the characterization given by Cornish in [4] of congruence-permutable implication algebras. We will also give a simpler proof of a characterization of factor congruences in implication algebras that appears in [5] and find those congruences in finitely generated free implication algebras that permute with every other congruence.

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An *implication algebra* is an algebra $\mathbf{A} = \langle A, \rightarrow, 1\rangle$ of type $\langle 2, 0\rangle$ that satisfies the equations:

(1) $x \rightarrow x = 1$
(2) $(x \rightarrow y) \rightarrow x = x$,
(3) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$,
(4) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$.

The theory of implication algebras was developed by Abbott in [1, 2] (see also [8]) and he showed that there is a bijective correspondence from the class of all implication algebras onto the class of all semi-boolean algebras, i.e. join semi-lattices in which every principal filter is a Boolean algebra.

On an implication algebra $\mathbf{A}$, the relation given by $a \leq b$ if and only if $a \rightarrow b = 1$ is a partial order, called the *natural order of* $\mathbf{A}$, with 1 as its greatest element, that is, $a \rightarrow 1 = 1$ for every $a \in A$. Every pair of elements $x, y \in A$ has a join $x \vee y$ given by the term $(x \rightarrow y) \rightarrow y$. Moreover, if $a, b, c \in A$ and $a, b \geq c$, then the meet of a and b exists and it holds that $p(a, b) = a \wedge b$ where $p(x, y)$ is the polynomial $(x \rightarrow (y \rightarrow c)) \rightarrow c$.

We now summarize some properties that hold in any implication algebra and will be used throughout the article.

- $1 \rightarrow x = x$,
- $y \leq x \rightarrow y$,
- $x \vee (x \rightarrow y) = 1$,
- if $x \leq y$, then $z \rightarrow x \leq z \rightarrow y$ and $y \rightarrow z \leq x \rightarrow z$,
- $x \vee y = 1$ iff $x \rightarrow y = y$,
- $x \leq y \rightarrow z$ iff $y \leq x \rightarrow z$,
- $x \rightarrow (x \rightarrow y) = x \rightarrow y$.

In addition, the lattice operations satisfy the following properties for every $x, y, z \in A$:

- $(x \rightarrow z) \wedge (y \rightarrow z)$ exists and $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$,
- $z \rightarrow (x \vee y) = (z \rightarrow x) \vee (z \rightarrow y)$,
- if $x \wedge y$ exists, $(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$,
- if $x \wedge y$ exists, then $(z \rightarrow x) \wedge (z \rightarrow y)$ also exists and $z \rightarrow (x \wedge y) = (z \rightarrow x) \wedge (z \rightarrow y)$.

The class $\mathbb{I}$ of all implication algebras is a locally finite, 3-permutable, 3-distributive variety, but it is not permutable (see [7]). All of its members are semisimple; in fact, they are the semisimple Hilbert algebras (see [2] and [6]), and the two-element implication algebra $\mathbf{2} = \langle\{0, 1\}, \rightarrow, 1\rangle$ is, up to isomorphism, the unique subdirectly irreducible implication algebra. As an immediate consequence of this last result, implication algebras may be also characterized as the $\{\rightarrow, 1\}$-subreducts of Boolean algebras.

Implication algebras are 1-regular, i.e. every congruence is determined by the congruence class to which 1 belongs. For each congruence relation θ on an implication algebra $\mathbf{A}$, $1/\theta$ is an implicative filter, i.e. it contains 1 and if $a, a \rightarrow b \in 1/\theta$, then $b \in 1/\theta$; equivalently, $1/\theta$ is a nonempty set which is upwardly closed with respect to the natural order and if $a, b \in 1/\theta$ and $a \wedge b$ exists, then $a \wedge b \in 1/\theta$. Conversely, for any implicative filter F of $\mathbf{A}$ the relation $\theta_F = \{(a, b) \in A^2 : a \rightarrow b, b \rightarrow a \in F\}$ is a congruence on $\mathbf{A}$ such that $F = 1/\theta_F$. In fact the correspondence $\theta \mapsto 1/\theta$ gives an order isomorphism from the lattice $Con(\mathbf{A})$ of all congruence relations on $\mathbf{A}$ onto the set of all implicative filters of $\mathbf{A}$, ordered by inclusion. We use the symbols Δ and ∇ for the least and greatest elements in $Con(\mathbf{A})$, respectively. For convenience we write $a \overset{\theta}{\equiv} b$ instead of $(a, b) \in \theta$. Observe that although $\wedge$ is not a term operation, if $a \overset{\theta}{\equiv} b$, $c \overset{\theta}{\equiv} d$, and both $a \wedge c$ and $b \wedge d$ exist, then $a \wedge c \overset{\theta}{\equiv} b \wedge d$.

2 Permutability of Congruences

Two congruences θ_1, θ_2 on an algebra $\mathbf{A}$ are said to *permute* if $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$. The following result will be useful to characterize the permutability of two congruences θ_1, θ_2 because it allows us to restrict our attention to the case in which $\theta_1 \cap \theta_2 = \Delta$. The proof is straightforward.

Lemma 2.1 *Let θ_1, θ_2 be two congruences on an algebra $\mathbf{A}$ and let $\theta = \theta_1 \cap \theta_2$. Then $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$ if and only if $\theta_1/\theta \circ \theta_2/\theta = \theta_2/\theta \circ \theta_1/\theta$.*

The following lemma gives a useful characterization for congruence classes in implication algebras.

Lemma 2.2 *Let $\theta \in Con(\mathbf{A})$, where $\mathbf{A} \in \mathbb{I}$. Then for all $x \in A$ we have that*

$$x/\theta = \big\{\alpha_1 \wedge (\alpha_2 \to x) : \alpha_1, \alpha_2 \in 1/\theta \text{ and } \alpha_1 \wedge (\alpha_2 \to x) \text{ exists}\big\}.$$

Equivalently $y \in x/\theta$ if and only if there exist $\alpha_1, \alpha_2 \in 1/\theta$ such that $y = \alpha_1 \wedge (\alpha_2 \to x)$.

Proof Let $y \in x/\theta$. Then $\alpha_1 = x \to y \in 1/\theta$ and $\alpha_2 = y \to x \in 1/\theta$. Observe that $\alpha_2 \to x = (y \to x) \to x = (x \to y) \to y = \alpha_1 \to y$. Hence $y \le \alpha_2 \to x$. As $y \le \alpha_1$, $\alpha_1 \wedge (\alpha_2 \to x)$ exists and we have

$$\alpha_1 \wedge (\alpha_2 \to x) = \big((\alpha_2 \to x) \to (\alpha_1 \to y)\big) \to y = 1 \to y = y.$$

Conversely, given $\alpha_1, \alpha_2 \in 1/\theta$, such that $\alpha_1 \wedge (\alpha_2 \to x)$ exists, we have that $\alpha_1 \wedge (\alpha_2 \to x) \overset{\theta}{\equiv} 1 \wedge (1 \to x) = x$, thus $\alpha_1 \wedge (\alpha_2 \to x) \in x/\theta$. □

Theorem 2.3 *Let $\mathbf{A} \in \mathbb{I}$ and $\theta_1, \theta_2 \in Con(\mathbf{A})$ such that $\theta_1 \cap \theta_2 = \Delta$. We have that $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$ if and only if the following conditions are satisfied:*

(1) *If $\alpha \in 1/\theta_1$ and $\beta \in 1/\theta_2$, then $\alpha \wedge \beta$ exists,*
(2) *Assume that $\alpha_1, \alpha_2 \in 1/\theta_1$, that $\beta_1, \beta_2 \in 1/\theta_2$, that $x \in A$ and that $\alpha_1 \wedge (\alpha_2 \to x)$ and $\beta_1 \wedge (\beta_2 \to x)$ both exist. Then $(\alpha_1 \wedge \beta_1) \wedge ((\alpha_2 \wedge \beta_2) \to x)$ exists.*

Proof Suppose $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$.

To prove (1) let $\alpha \in 1/\theta_1$ and $\beta \in 1/\theta_2$. Then $\alpha \overset{\theta_1}{\equiv} 1 \overset{\theta_2}{\equiv} \beta$, i.e. $\alpha \overset{\theta_1 \circ \theta_2}{\equiv} \beta$, so $\alpha \overset{\theta_2 \circ \theta_1}{\equiv} \beta$. Consequently, there exists $c \in A$ such that $\alpha \overset{\theta_2}{\equiv} c \overset{\theta_1}{\equiv} \beta$.

Thus we have $c \to \alpha \in 1/\theta_2$. Moreover, as $c \to \alpha \ge \alpha \in 1/\theta_1$, $c \to \alpha \in 1/\theta_1$. Hence $c \to \alpha \in 1/\theta_1 \cap 1/\theta_2$. But $1/\theta_1 \cap 1/\theta_2 = \{1\}$ because $\theta_1 \cap \theta_2 = \Delta$. We conclude that $c \to \alpha = 1$, i.e. $c \le \alpha$.

In the same way we can show that $c \le \beta$, therefore $\alpha \wedge \beta$ exists.

Assuming the conditions in (2), consider $a = \alpha_1 \wedge (\alpha_2 \to x)$ and $b = \beta_1 \wedge (\beta_2 \to x)$. Hence

$$a = \alpha_1 \wedge (\alpha_2 \to x) \overset{\theta_1}{\equiv} x \overset{\theta_2}{\equiv} \beta_1 \wedge (\beta_2 \to x) = b\,.$$

Since θ_1 and θ_2 permute, there exists $y \in A$ such that $a \overset{\theta_2}{\equiv} y \overset{\theta_1}{\equiv} b$.

By (1), $\alpha_1 \wedge \beta_1$ and $\alpha_2 \wedge \beta_2$ both exist. Thus

$$y \to (\alpha_1 \wedge \beta_1) \overset{\theta_2}{\equiv} y \to \alpha_1\,.$$

As $a \leq \alpha_1$, $y \to a \leq y \to \alpha_1$, so the equation above, together with the fact that $y \to a \in 1/\theta_2$, implies that $y \to (\alpha_1 \wedge \beta_1) \in 1/\theta_2$.

Likewise,

$$y \to (\alpha_1 \wedge \beta_1) \overset{\theta_1}{\equiv} y \to \beta_1,$$

and since $b \leq \beta_1$, then $y \to b \leq y \to \beta_1$ and $y \to b \in 1/\theta_1$, so we get $y \to (\alpha_1 \wedge \beta_1) \in 1/\theta_1$. We have thus proved that $y \to (\alpha_1 \wedge \beta_1) \in 1/\theta_1 \cap 1/\theta_2 = \{1\}$. Therefore $y \leq \alpha_1 \wedge \beta_1$.

We also have that

$$y \to \big((\alpha_2 \wedge \beta_2) \to x\big) \overset{\theta_2}{\equiv} y \to (\alpha_2 \to x) \geq y \to a \in 1/\theta_2,$$

$$y \to \big((\alpha_2 \wedge \beta_2) \to x\big) \overset{\theta_1}{\equiv} y \to (\beta_2 \to x) \geq y \to b\, \in 1/\theta_1.$$

As a consequence, $y \to ((\alpha_2 \wedge \beta_2) \to x) \in 1/\theta_1 \cap 1/\theta_2 = \{1\}$, i.e. $y \leq (\alpha_2 \wedge \beta_2) \to x$.

Then y is a lower bound of $\alpha_1 \wedge \beta_1$ and $(\alpha_2 \wedge \beta_2) \to x$. Therefore, the meet of these two elements must exist. This completes the proof of condition (2).

Conversely, suppose conditions (1) and (2) hold. Let $a \overset{\theta_1}{\equiv} x \overset{\theta_2}{\equiv} b$.

As $a \in x/\theta_1$ and $b \in x/\theta_2$, by Lemma 2.2 there exist $\alpha_1, \alpha_2 \in 1/\theta_1$ and $\beta_1, \beta_2 \in 1/\theta_2$ such that

$$a = \alpha_1 \wedge (\alpha_2 \to x), \qquad b = \beta_1 \wedge (\beta_2 \to x).$$

By (1) and (2), $(\alpha_1 \wedge \beta_1) \wedge ((\alpha_2 \wedge \beta_2) \to x)$ exists. Thus we have

$$\begin{aligned} a &= \alpha_1 \wedge (\alpha_2 \to x) \\ &\overset{\theta_2}{\equiv} (\alpha_1 \wedge \beta_1) \wedge \big((\alpha_2 \wedge \beta_2) \to x\big) \\ &\overset{\theta_1}{\equiv} \beta_1 \wedge (\beta_2 \to x) \\ &= b \end{aligned}$$

therefore $a \overset{\theta_2 \circ \theta_1}{\equiv} b$. □

The general characterization of permutability now follows from Lemma 2.1 and the previous theorem.

Corollary 2.4 *Let* $\mathbf{A} \in \mathbb{I}$ *and* $\theta_1, \theta_2 \in Con(\mathbf{A})$, $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$ *if and only if the congruences* $\theta_1/(\theta_1 \cap \theta_2)$ *and* $\theta_2/(\theta_1 \cap \theta_2)$ *satisfy conditions* (1) *and* (2) *of Theorem* 2.3 *in the implication algebra* $\mathbf{A}/(\theta_1 \cap \theta_2)$.

3 The Finite Case

In this section we simplify the result in Theorem 2.3 when the implication algebra **A** is finite. We begin by showing some useful properties about congruence classes.

Lemma 3.1 *Let* $\mathbf{A} \in \mathbb{I}$ *and* $\theta \in Con(\mathbf{A})$. *Then each congruence class of* θ *is closed under join.*

Proof If $a \overset{\theta}{\equiv} b$, then $a \vee a \overset{\theta}{\equiv} a \vee b$, then $a \overset{\theta}{\equiv} a \vee b$. □

Corollary 3.2 *Let* $\theta \in Con(\mathbf{A})$ *where* $\mathbf{A}$ *is a finite algebra in* $\mathbb{I}$. *Then each congruence class modulo* θ *has a greatest element.*

Proof Let $x \in A$. As A is finite, x/θ is finite. Then there exists $y = \bigvee x/\theta$ and by the preceding lemma $y \in x/\theta$. Thus y is the greatest element in x/θ. □

Lemma 3.3 *Let* $\theta \in Con(\mathbf{A})$ *where* $\mathbf{A}$ *is a finite algebra in* $\mathbb{I}$. *Let* $x \in A$ *be such that* $x = \bigvee x/\theta$, *then:*

(a) $x \vee \alpha = 1$ *for all* $\alpha \in 1/\theta$.
(b) $x/\theta = \{x \wedge \alpha : \alpha \in 1/\theta$ *such that* $x \wedge \alpha$ *exists*$\}$.
(c) *If* $y \in x/\theta$, *then there exists a unique* $\alpha \in 1/\theta$ *such that* $y = x \wedge \alpha$.
(d) *If* $x \leq y$, *then* $y = \bigvee y/\theta$.

Proof

(a) Let $\alpha \in 1/\theta$, then $\alpha \to x \overset{\theta}{\equiv} 1 \to x = x$, therefore $\alpha \to x \in x/\theta$. Since $x = \bigvee x/\theta$, we have that $\alpha \to x \leq x$, so $\alpha \to x = x$. This in turn implies that $x \vee \alpha = 1$.
(b) Let $y \in x/\theta$. Then $\alpha = x \to y \in 1/\theta$.
As $y \leq x$ and $y \leq \alpha$, the meet between x and α exists. We have $x \wedge \alpha = (\alpha \to (x \to y)) \to y = (\alpha \to \alpha) \to y = 1 \to y = y$.
Conversely, if $x \wedge \alpha$ exists for some $\alpha \in 1/\theta$, we have that $x \wedge \alpha \overset{\theta}{\equiv} x \wedge 1 = x$, therefore $x \wedge \alpha \in x/\theta$.
(c) We only need to prove the uniqueness of α. To do this, we note that if $y = x \wedge \alpha$, it is necessary that $\alpha = x \to y$. In fact,

$$x \to y = x \to (x \wedge \alpha) = (x \to x) \wedge (x \to \alpha) = 1 \wedge \alpha = \alpha.$$

Observe that we used the fact that $x \to \alpha = \alpha$ given in (a) by the equivalent condition $x \vee \alpha = 1$.
(d) Let $x \leq y$ and consider $u = \bigvee y/\theta$. Then $u \to y \in 1/\theta$ and $(u \to y) \to x \overset{\theta}{\equiv} 1 \to x = x$, i.e. $(u \to y) \to x \in x/\theta$. Since $x = \bigvee x/\theta$, we have $(u \to y) \to x \leq x$ and in fact $(u \to y) \to x = x$.
Hence $x \to (u \to y) = u \to y$, and then $u \to (x \to y) = u \to y$. As $x \leq y$, it follows that $u \leq y$.
In addition, since $u = \bigvee y/\theta$, $y \leq u$. This shows that $y = u = \bigvee y/\theta$. □

Lemma 3.4 *Let* $\mathbf{A}$ *be a finite algebra in* $\mathbb{I}$ *and let* $\theta_1, \theta_2 \in Con(\mathbf{A})$ *be such that* $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$. *Let* $x \in A$ *be such that* $x = \bigvee x/\theta_1 = \bigvee x/\theta_2$, *then* $x = \bigvee x/(\theta_1 \circ \theta_2)$.

Proof Let $y \in x/(\theta_1 \circ \theta_2) = x/(\theta_2 \circ \theta_1)$. Then there exists $z \in A$ such that $y \stackrel{\theta_2}{\equiv} z \stackrel{\theta_1}{\equiv} x$. Let $s = \bigvee z/\theta_2 = \bigvee y/\theta_2$. Then $z \leq s$ and $y \leq s$.

As $s \stackrel{\theta_2}{\equiv} z$, we have that $s \vee x \stackrel{\theta_2}{\equiv} z \vee x = x$ because $x = \bigvee x/\theta_1$ and $z \in x/\theta_1$.

Thus, as $x = \bigvee x/\theta_2$ and $s \vee x \in x/\theta_2$, we have that $s \vee x \leq x$, i.e. $s \leq x$. Then $y \leq s \leq x$.

This shows that $x = \bigvee x/(\theta_1 \circ \theta_2)$. □

We are now ready to prove the main theorem of this section.

Theorem 3.5 *Let* $\mathbf{A}$ *be a finite algebra in* $\mathbb{I}$ *and let* $\theta_1, \theta_2 \in Con(\mathbf{A})$ *be such that* $\theta_1 \cap \theta_2 = \Delta$. *Then* $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$ *if and only if given* $x \in A$ *such that* $x = \bigvee x/\theta_1 = \bigvee x/\theta_2$ *we have that for all* $a \in x/\theta_1$ *and* $b \in x/\theta_2$, *the meet* $a \wedge b$ *exists.*

Proof Suppose $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$. Let $x \in A$ be such that $x = \bigvee x/\theta_1 = \bigvee x/\theta_2$, $a \in x/\theta_1$ and $b \in x/\theta_2$.

Since $a \stackrel{\theta_1}{\equiv} x \stackrel{\theta_2}{\equiv} b$, we have $a \stackrel{\theta_1 \circ \theta_2}{\equiv} b$, therefore $a \stackrel{\theta_2 \circ \theta_1}{\equiv} b$. Hence there exists $c \in A$ such that $a \stackrel{\theta_2}{\equiv} c \stackrel{\theta_1}{\equiv} b$.

Observe that $c \stackrel{\theta_1}{\equiv} b \stackrel{\theta_2}{\equiv} x$, i.e. $c \stackrel{\theta_1 \circ \theta_2}{\equiv} x$. By the preceding lemma $x = \bigvee x/(\theta_1 \circ \theta_2)$, so $c \leq x$. Hence $x \to a \leq c \to a$. Now since $x \to a \in 1/\theta_1$, we conclude that $c \to a \in 1/\theta_1$. Since $a \stackrel{\theta_2}{\equiv} c$, we have that $c \to a \in 1/\theta_2$. Therefore $c \to a \in 1/\theta_1 \cap 1/\theta_2 = \{1\}$ because $\theta_1 \cap \theta_2 = \Delta$. Then $c \to a = 1$ and $c \leq a$.

Analogously, $c \leq b$. Consequently, the meet of a and b exists.

Conversely, suppose $a \stackrel{\theta_1}{\equiv} c \stackrel{\theta_2}{\equiv} b$ and consider the following elements of A:

$$s = \bigvee a/\theta_1 = \bigvee c/\theta_1, \quad t = \bigvee b/\theta_2 = \bigvee c/\theta_2, \quad u = \bigvee a/\theta_2, \quad v = \bigvee b/\theta_1.$$

Since $c \stackrel{\theta_2}{\equiv} t$, we have $s \vee c \stackrel{\theta_2}{\equiv} s \vee t$, i.e. $s \stackrel{\theta_2}{\equiv} s \vee t$.

Since $a \stackrel{\theta_2}{\equiv} u$, we have $a \vee s \stackrel{\theta_2}{\equiv} u \vee s$, i.e. $s \stackrel{\theta_2}{\equiv} u \vee s$.

Therefore, $u \vee s \stackrel{\theta_2}{\equiv} s \vee t$.

Now, as $u = \bigvee u/\theta_2$ and $t = \bigvee t/\theta_2$, by Lemma 3.3 (d),

$$u \vee s = \bigvee (u \vee s)/\theta_2 = \bigvee (s \vee t)/\theta_2 = s \vee t.$$

Analogously, we can prove that $t \vee v = s \vee t$.

Since $a \stackrel{\theta_1}{\equiv} s$, we get $u \vee a \stackrel{\theta_1}{\equiv} u \vee s$, i.e. $u \stackrel{\theta_1}{\equiv} u \vee s = s \vee t$. Hence $u \in (s \vee t)/\theta_1$.

Since $b \stackrel{\theta_2}{\equiv} t$, we get $b \vee v \stackrel{\theta_2}{\equiv} t \vee v$, i.e. $v \stackrel{\theta_2}{\equiv} t \vee v = s \vee t$. Thus $v \in (s \vee t)/\theta_2$.

By Lemma 3.3 (d) we know that $s \vee t = \bigvee (s \vee t)/\theta_1 = \bigvee (s \vee t)/\theta_2$. Consequently, by hypothesis, we conclude that $u \wedge v$ exists.

Since $s \vee t \stackrel{\theta_2}{\equiv} v$, we get $u \wedge (s \vee t) \stackrel{\theta_2}{\equiv} u \wedge v$, i.e. $u \stackrel{\theta_2}{\equiv} u \wedge v$.

Since $s \vee t \stackrel{\theta_1}{\equiv} u$, we get $(s \vee t) \wedge v \stackrel{\theta_1}{\equiv} u \wedge v$, i.e. $v \stackrel{\theta_1}{\equiv} u \wedge v$.

Finally

$$a \stackrel{\theta_2}{\equiv} u \stackrel{\theta_2}{\equiv} u \wedge v \stackrel{\theta_1}{\equiv} v \stackrel{\theta_1}{\equiv} b,$$

and thus $a \stackrel{\theta_2 \circ \theta_1}{\equiv} b$. This suffices to conclude that congruences θ_1 and θ_2 permute. □

Corollary 3.6 *Let* $\mathbf{A}$ *be an implication algebra and* $\theta_1, \theta_2 \in Con(\mathbf{A})$ *such that* $\mathbf{A}/(\theta_1 \cap \theta_2)$ *is finite. Then* $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$ *if and only if the congruences* $\theta_1/(\theta_1 \cap \theta_2)$ *and* $\theta_2/(\theta_1 \cap \theta_2)$ *satisfy the conditions of the preceding theorem in the algebra* $\mathbf{A}/(\theta_1 \cap \theta_2)$.

4 Applications

We now turn to some applications of the results given in the preceding sections. Congruence-permutable implication algebras were characterized by Cornish in [4, Theorem 3.14]. We will obtain this result as a direct consequence of the preceding characterization for permutability. In addition, we simplify the proof of the characterization of factor congruences on an implication algebra given in [5]. Finally, taking into account the results on free implication algebras presented in [5], we find all congruences on finitely-generated free implication algebras that permute with every other congruence.

4.1 Congruence-Permutable Implication Algebras

Recall that an algebra $\mathbf{A}$ is said to be *congruence-permutable* if every pair of congruences on $\mathbf{A}$ permute (see [3] for more details). The following theorem is due to Cornish (see [4, Theorem 3.14]) and characterizes congruence-permutable implication algebras. We now derive this result as a consequence of Theorem 2.3.

Theorem 4.1 $\mathbf{A} \in \mathbb{I}$ *is congruence-permutable if and only if for every pair of elements* $a, b \in A$ *the meet* $a \wedge b$ *exists.*

Proof Suppose $\mathbf{A}$ is congruence-permutable. Note that if $a, b \in A$ and $a \vee b = 1$, then $Fg(a) \cap Fg(b) = \{1\}$, where $Fg(a)$ denotes the implicative filter generated by $a \in A$. Then, by Theorem 2.3, the meet between a and b must exist. Now, given any $a, b \in A$, we have that $a \vee (a \to b) = 1$. Therefore there exists $c \in A$ such that $c \leq a$ and $c \leq a \to b$. Then $c \leq a$ and $a \leq c \to b$. Thus $c \leq c \to b$, i.e. $c \to (c \to b) = c \to b = 1$, so $c \leq b$. This shows that c is a lower bound of a and b, whence the meet between a and b exists.

Conversely, if the meet of every pair of elements of A exists, this property also holds in $\mathbf{A}/(\theta_1 \cap \theta_2)$, where $\theta_1, \theta_2 \in \mathrm{Con}(\mathbf{A})$. Hence the conditions of Theorem 2.3 are trivially satisfied and by Corollary 2.4 we conclude that θ_1 and θ_2 permute. □

Observe that, as a consequence, we have that $\mathbf{A}$ is congruence-permutable if and only if it forms a generalized Boolean algebra under the lattice operations.

4.2 Factor Congruences

A congruence θ on an algebra $\mathbf{A}$ is called a *factor congruence* if there exists a congruence θ^* such that $\theta \cap \theta^* = \Delta, \theta \vee \theta^* = \nabla$ and $\theta \circ \theta^* = \theta^* \circ \theta$. For more details about factor congruences see [3].

Factor congruences in implication algebras are characterized in [5]. We now give a simpler proof of that result as a consequence of Theorem 2.3.

Theorem 4.2 *Let $\theta \in Con(\mathbf{A})$ where $\mathbf{A} \in \mathbb{I}$. Then θ is a factor congruence if and only if the following conditions hold:*

(a) *For every $c \in 1/\theta$ and every $b \in 1/\theta^{\perp}$, $c \wedge b$ exists.*
(b) *For every $a \in A$, there are unique $c_a \in 1/\theta$ and $b_a \in 1/\theta^{\perp}$ such that $a = c_a \wedge b_a$.*

Proof Suppose θ is a factor congruence. Condition (a) follows immediately from Theorem 2.3. Now, since $\theta \vee \theta^{\perp} = \nabla$, it follows that $Fg(1/\theta \cup 1/\theta^{\perp}) = A$. Besides, since $1/\theta$ and $1/\theta^{\perp}$ are both increasing and closed by infima (when they exist), $Fg(1/\theta \cup 1/\theta^{\perp}) = \{c \wedge b : c \in 1/\theta, b \in 1/\theta^{\perp}$ such that $c \wedge b$ exists$\}$. This implies the existence part of condition (b). Uniqueness is straightforward.

Conversely, assume θ is a congruence satisfying (a) and (b). It suffices to show that $\theta \circ \theta^{\perp} = \nabla$. Indeed, let $a, d \in A$, then by (b) there are $c_a, c_d \in 1/\theta$ and $b_a, b_d \in 1/\theta^{\perp}$ such that $a = c_a \wedge b_a$ and $d = c_d \wedge b_d$. By (a), $c_d \wedge b_a$ exists, so $a = c_a \wedge b_a \overset{\theta}{\equiv} c_d \wedge b_a \overset{\theta^{\perp}}{\equiv} c_d \wedge b_d = d$. □

4.3 Permutability of Congruences in Free Implication Algebras

In this section we will study permutability of congruences in finitely generated free implication algebras. We will find congruences that permute with every other congruence on the algebra.

Since $\mathbb{I}$ is a congruence-distributive variety, for any algebra $\mathbf{A} \in \mathbb{I}$ the congruence lattice $\mathbf{Con}(\mathbf{A})$ is pseudocomplemented. We will denote by $\theta^{\perp}$ the pseudocomplement of a congruence θ on $\mathbf{A}$. It is not difficult to show that $1/\theta^{\perp} = \{x \in A : x \vee y = 1$ for every $y \in 1/\theta\}$.

The $|X|$-free implication algebra $\mathbf{F}_{\mathbb{I}}(X)$ generated by X is the $\{\rightarrow, 1\}$-subalgebra of the $\{\rightarrow, 1\}$-reduct of the $|X|$-free Boolean algebra $\mathbf{F}_{\mathbb{B}}(X)$ generated by X, in fact, $F_{\mathbb{I}}(X) = \{p \in F_{\mathbb{B}}(X) : p \geq x,$ for some $x \in X\}$ (see [5] and the references given there). It follows from this description that, as in free Boolean algebras, if X is infinite, then the unique upper bound of X, relative to the natural order in $\mathbf{F}_{\mathbb{I}}(X)$, is 1; and if X is finite, then the set of all upper bounds of X is $\{1, \bigvee X\}$, in which $\bigvee X$ is a coatom.

In [5] the authors find all factor congruences in $\mathbf{F}_{\mathbb{I}}(X)$. For reference, we state here the main theorem of that article.

Theorem 4.3 *If X is infinite, then $\mathbf{F}_{\mathbb{I}}(X)$ is directly indecomposable. If X is finite with at least two elements, then $\mathbf{F}_{\mathbb{I}}(X)$ has a unique non-trivial pair of factor congruences $\{\theta, \theta^{\perp}\}$ given by $1/\theta = \{1, \bigvee X\}$ and $1/\theta^{\perp} = \{a \in F_{\mathbb{I}}(X) : a \vee \bigvee X = 1\}$.*

In the proof of the following theorem, we will use the fact that the lattice of congruences $\mathbf{Con}(\mathbf{A})$ is a Boolean lattice for any finite implication algebra $\mathbf{A}$. Indeed, if we let C be the set of coatoms of $\mathbf{A}$ and for each congruence θ on $\mathbf{A}$, we let $C_\theta = \{c \in C : c \in 1/\theta\}$, it is easy to show that the mapping $\theta \mapsto C_\theta$ is an isomorphism between $\mathbf{Con}(\mathbf{A})$ and the power set of C. As a consequence we get that for each congruence θ, its pseudocomplement $\theta^{\perp}$ is, in fact, its complement.

Theorem 4.4 *Let X be a finite set with at least two elements and let $\mathbf{F}_{\mathbb{I}}(X)$ be the free implication algebra generated by X. The congruence θ given by $1/\theta = \{1, \bigvee X\}$ and its* *pseudocomplement $\theta^{\perp}$ are the only non-trivial congruences on $\mathbf{F}_{\mathbb{I}}(X)$ which permute with all congruences.*

Proof We first show that θ permutes with every congruence. To see this, let θ' be any congruence on $\mathbf{F}_{\mathbb{I}}(X)$. Let $F = 1/\theta = \{1, \bigvee X\}$ and $F' = 1/\theta'$. If $F \subseteq F'$, it is immediate that θ and θ' permute. Hence, assume that $F \not\subseteq F'$. In this case $F \cap F' = \{1\}$.

By Theorem 3.5, in order to prove that θ and θ' permute, it suffices to show that if $z = \bigvee z/\theta = \bigvee z/\theta'$ and $a \in z/\theta, b \in z/\theta'$, then $a \wedge b$ exists.

If $a = z$, $a \wedge b = z \wedge b = b$, so we may assume that $a \neq z$. Hence, by Lemma 3.3 (b), we know that $a/\theta = \{z, z \wedge \bigvee X\}$, so $a = z \wedge \bigvee X$.

Since there is some $x \in X$ such that $x \leq b$, $\bigvee X \wedge b$ exists and we have that

$$\left(\bigvee X \wedge b\right) \rightarrow a = \left(\bigvee X \rightarrow a\right) \vee (b \rightarrow a).$$

Since $\bigvee X \overset{\theta}{\equiv} 1$, $\bigvee X \rightarrow a \overset{\theta}{\equiv} a$. There are two possibilities, namely, $\bigvee X \rightarrow a = a$ or $\bigvee X \rightarrow a = z$. In the former case, $\bigvee X = \bigvee X \vee a = 1$, a contradiction. Hence $\bigvee X \rightarrow a = z$. This shows that

$$\left(\bigvee X \wedge b\right) \rightarrow a = z \vee (b \rightarrow a).$$

Now $z \vee (b \rightarrow a) \overset{\theta}{\equiv} z \vee (b \rightarrow z) = z \vee 1 = 1$ and $z \vee (b \rightarrow a) \overset{\theta'}{\equiv} z \vee (z \rightarrow a) = 1$. As $F \cap F' = 1$, we conclude that $z \vee (b \rightarrow a) = 1$, i.e. $(\bigvee X \wedge b) \rightarrow a = 1$. Therefore $\bigvee X \wedge b \leq a$. This shows that $a \wedge b$ exists and completes the proof that θ and θ' permute.

We now show that $\theta^{\perp}$ also permutes with all congruences. Let θ' be an arbitrary congruence on $\mathbf{F}_{\mathbb{I}}(X)$. If $\theta' \subseteq \theta^{\perp}$, it is easy to see that θ' and $\theta^{\perp}$ permute. Consequently, assume that $\theta' \not\subseteq \theta^{\perp}$. We only need to verify that $\theta^{\perp} \circ \theta' = \nabla$.

Indeed, observe that $F^{\perp} = 1/\theta^{\perp}$ is a maximal implicative filter since F is a minimal implicative filter and $\mathbf{Con}(\mathbf{F}_{\mathbb{I}}(X))$ is a Boolean lattice. It is easy to see that the two congruence classes modulo $\theta^{\perp}$ are $F^{\perp}$ and $(\bigvee X] = \{a \in F_{\mathbb{I}}(X) : a \leq \bigvee X\}$. Now, if we call $F' = 1/\theta'$, as $F' \not\subseteq F^{\perp}$, there exists $z \in F' \setminus F^{\perp}$. Since $z \notin F^{\perp}$, $z \leq \bigvee X$ and then $\bigvee X \in F'$.

Let $a, b \in F_{\mathbb{I}}(X)$. We will show that $a \overset{\theta^{\perp}\circ\theta'}{\equiv} b$. There are three different cases, namely:

- If $a, b \in F^{\perp}$ or if $a, b \in (\bigvee X]$, then $a \overset{\theta^{\perp}}{\equiv} b$ and hence $a \overset{\theta^{\perp}\circ\theta'}{\equiv} b$.
- Suppose $a \in F^{\perp}$ and $b \in (\bigvee X]$. Let $c = \bigvee b/\theta'$. By Lemma 3.3 (a), we have that $c \vee \alpha = 1$ for every $\alpha \in F'$. In particular, in the case $\alpha = \bigvee X$, we get that $c \in F^{\perp}$. Hence

$$a \overset{\theta^{\perp}}{\equiv} c \overset{\theta'}{\equiv} b.$$

- Let $a \in (\bigvee X]$ and $b \in F^{\perp}$. As $b \geq x$ for some $x \in X$, $b \wedge \bigvee X$ exists. Hence

$$a \overset{\theta^{\perp}}{\equiv} b \wedge \bigvee X \overset{\theta'}{\equiv} b \wedge 1 = b.$$

This shows that $\theta^{\perp}$ and θ' permute.

Conversely, consider a non-trivial congruence θ' on $\mathbf{F}_{\mathbb{I}}(X)$ such that θ' permutes with every congruence. We know that $\theta' \cap (\theta')^{\perp} = \Delta$ and $\theta' \vee (\theta')^{\perp} = \nabla$. Moreover, since θ' permutes with every congruence, in particular $\theta' \circ (\theta')^{\perp} = (\theta')^{\perp} \circ \theta'$. Therefore $\{\theta', (\theta')^{\perp}\}$ is a pair of factor congruences. By Theorem 4.3, either $\theta' = \theta$ or $\theta' = \theta^{\perp}$. □

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