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ORIGINAL PAPER

Factor congruences in BCK-algebras

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Abstract In this paper, we characterize factor congruences in the quasivariety of BCK-algebras. As an application we prove that the free algebra over an infinite set of generators is indecomposable in any subvariety of BCK-algebras. We also study the decomposability of free algebras in the variety of hoop residuation algebras ($\mathcal{HBCK}$) and its subvarieties. We prove that free algebras in a non k-potent subvariety of $\mathcal{HBCK}$ are indecomposable while finitely generated free algebras in k-potent subvarieties have a unique non-trivial decomposition into a direct product of two factors, and one of them is the two-element implication algebra.

Keywords Factor congruences · Implicative filters · BCK-algebras · Pocrims · Hoops · Free algebras · Decomposability

1 Preliminaries

A BCK-algebra is an algebra $\boldsymbol{A} = \langle A, \to, 1\rangle$ of type $\langle 2, 0\rangle$ satisfying the following identities and quasi-identity:

(1) $(x \to y) \to ((y \to z) \to (x \to z)) \approx 1$,
(2) $x \to 1 \approx 1$,
(3) $1 \to x \approx x$,
(4) if $x \to y \approx 1$ and $y \to x \approx 1$, then $x \approx y$.

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BCK-algebras were introduced by K. Iséki as algebraic models of C. A. Meredith's BCK-calculus, a non-classical calculus containing implication as the only propositional connective. They form a quasivariety $\mathcal{BCK}$ that is not a variety.

The $\{\to, 1\}$-subreducts of hoops form a subvariety of $\mathcal{BCK}$, denoted $\mathcal{HBCK}$ (Ferreirim 1992, 2001), that can be defined within the quasivariety $\mathcal{BCK}$ by the identity

$$(x \to y) \to (x \to z) \approx (y \to x) \to (y \to z).$$

If $\boldsymbol{A} \in \mathcal{BCK}$, A is a partially ordered set by means of $a \leq b$ if and only if $a \to b = 1$, for $a, b \in A$. The least upper bound of a and b, if there exists, will be denoted by $a \vee b$. Similarly, $a \wedge b$ will denote the greatest lower bound of a and b, in case it exists.

In this paper, we present a general characterization of factor congruences in BCK-algebras and use this characterization to determine the direct decomposability or indecomposability of free algebras for some quasivarieties of BCK-algebras. It is a continuation of the investigation initiated in Díaz Varela and Torrens (2003, 2006), and we cite these two papers for some of the proofs. Nevertheless, in the varieties of Tarski algebras and Łukasiewicz implication algebras studied in Díaz Varela and Torrens (2003, 2006), respectively, the join operation $x \vee y$ is given by the term $(x \to y) \to y$ and if a set $\{a, b\}$ has lower bounds, there exists the infimum $a \wedge b$. Some of the results of Díaz Varela and Torrens (2003, 2006) strongly rely on these two properties that do not hold for BCK-algebras in general. This is the main difference between Díaz Varela and Torrens (2003, 2006) and the present paper.

We summarize some basic properties of the arrow operation in BCK-algebras in the following lemma.

Lemma 1 (Iséki and Tanaka 1978) *In a BCK-algebra $\boldsymbol{A}$, for $x, y \in A$,*

(5) $x \to x \approx 1$,
(6) $x \to (y \to z) \approx y \to (x \to z)$,
(7) $(x \to y) \leq (y \to z) \to (x \to z)$,
(8) *if* $x \leq y$ *then* $y \to z \leq x \to z$,
(9) $x \leq (x \to y) \to y$,
(10) $((x \to y) \to y) \to y = x \to y$,
(11) $y \leq x \to y$.

A subset F of a BCK-algebra $\boldsymbol{A}$ is called a *filter* of $\boldsymbol{A}$ if $1 \in F$ and whenever $a \in F$ and $a \to b \in F$ then $b \in F$.

Every filter of a BCK-algebra is an order filter, that is, if $a \in F$ and $a \leq b$ then $b \in F$. Since $b \leq a \to b$ for any a, b in a BCK-algebra $\boldsymbol{A}$, filters of $\boldsymbol{A}$ are always subuniverses.

Let us write $x \to^0 y =_{def} y$, and for $n \geq 0$, $x \to^{n+1} y =_{def} x \to (x \to^n y)$. If $\boldsymbol{A}$ is a BCK-algebra and $X \subseteq A$, then the filter generated by X is the set $Fg(X) = \{b \in A :$ there exists $n < \omega$ and $a_1, a_2, \ldots, a_n \in X : a_1 \to (a_2 \to (\ldots(a_n \to b)\ldots)) = 1\}$. In particular, if $X = \{a\}$ then $Fg(a) = \{b \in A :$ there exists $n < \omega : a \to^n b = 1\}$.

Filters of a BCK-algebra form an algebraic lattice, denoted $\mathcal{F}(\boldsymbol{A})$, and hence pseudocomplemented, with $F \wedge G = F \cap G$ and $F \vee G = Fg(F \cup G)$. For every filter F of $\boldsymbol{A}$, let $F^{\perp} = \{b \in A : (b \to a) \to a = 1$ for all $a \in F\}$ and observe that $F^{\perp} = \{b \in A : b \to a = a$ for all $a \in F\}$. $F^{\perp}$ is the pseudocomplement of F in $\mathcal{F}(\boldsymbol{A})$ (Radomir Halas 2003). Moreover, $F^{\perp} = \{b \in A : b \vee a = 1$ for every $a \in F\}$ (the join $b \vee a$ exists and is equal to 1). Indeed, it is clear that $\{b \in A : b \vee a = 1$ for every $a \in F\} \subseteq F^{\perp}$, since $(a \to b) \to b$ is an upper bound of $\{a, b\}$. Conversely, let $b \in F^{\perp}$ and $a \in F$. If c is an upper bound of $\{a, b\}$, then $c \in F \cap F^{\perp} = \{1\}$. Thus $c = 1$. Then $b \vee a = 1$.

Given a filter F of a BCK-algebra $\boldsymbol{A}$, the binary relation $\theta_F = \{(a, b) \in A \times A : a \to b, b \to a \in F\}$ is not only a congruence on $\boldsymbol{A}$ but also $\boldsymbol{A}/\theta_F$ is a BCK-algebra (relative congruence). Conversely, given any congruence θ on $\boldsymbol{A}$, the set $1/\theta = \{a \in A : (a, 1) \in \theta\}$ is a filter of $\boldsymbol{A}$. There is an order-isomorphism between $\mathcal{F}(\boldsymbol{A})$ and the set of congruences θ on $\boldsymbol{A}$ such that $\boldsymbol{A}/\theta$ is a BCK-algebra (Blok and Raftery 1995, Proposition 1; Cornish 1982, p. 108). A class $\mathcal{K}$ of BCK-algebras is called a *relative subvariety* of $\mathcal{BCK}$ if there exists a variety $\mathcal{V}$ such that $\mathcal{K} = \mathcal{V} \cap \mathcal{BCK}$.

An important subclass of the variety $\mathcal{HBCK}$ is the class of $\{\to, 1\}$-subreducts of Wajsberg hoops (and therefore of MV-algebras). We will refer to these algebras as Łukasiewicz BCK-algebras (called Łukasiewicz implication algebras in Díaz Varela and Torrens (2006) and Łukasiewicz residuation algebras in Berman and Blok 2004). They form a variety that is characterized relative to $\mathcal{HBCK}$ by the equation $(x \to y) \to y \approx (y \to x) \to x$. This variety will be denoted by $\mathcal{L}$. The subdirectly irreducible algebras in $\mathcal{L}$ are linearly ordered, and the finite ones are the algebras $\boldsymbol{Ł}_n = \langle\{e^0, e, \ldots, e^n\}, \to, 1\rangle$, where $1 = e^0 > e > e^2 > \cdots > e^n$, and

$$e^i \to e^j = \begin{cases} 1 & \text{if } i \geq j, \\ e^{j-i} & \text{otherwise.} \end{cases}$$

The variety $\mathcal{L}$ is generated by the algebra $\boldsymbol{C}_\omega$ whose universe is the set $\{(0, y) : y \in \mathbb{N}\} \cup \{(1, -y) : y \in \mathbb{N}\}$, where $\mathbb{N}$ is the set of non-negative integers, and

$$(x, y) \to (z, u) = \begin{cases} (1, 0) & \text{if } z > x, \\ (1, \min(0, u - y)) & \text{if } z = x, \\ (1 - x + z, u - y) & \text{otherwise.} \end{cases}$$

$\boldsymbol{C}_\omega$ is the $\{\to, 1\}$-reduct of the Chang's algebra (see Chang 1958, p. 474).

The set $Ł_\omega = \{(1, -y) : y \in \mathbb{N}\}$ is the unique maximal filter of $\boldsymbol{C}_\omega$, with $\boldsymbol{C}_\omega/Ł_\omega \cong \boldsymbol{Ł}_1$. Its associated subalgebra $\boldsymbol{Ł}_\omega$ is not finitely generated, and any infinite subalgebra of $\boldsymbol{Ł}_\omega$ is isomorphic to a copy of it. Moreover, every non-trivial finite subalgebra of $\boldsymbol{Ł}_\omega$ is isomorphic to $\boldsymbol{Ł}_n$, for some $n > 0$. In addition, $\boldsymbol{C}_\omega$ is generated by $\{(0, 1), (0, 0)\}$, and for each n, $\boldsymbol{Ł}_n$ is generated by $\{e^n, e^d\}$, with $(d, n) = 1$. Every finitely generated subalgebra of $\boldsymbol{Ł}_\omega$ is isomorphic to $\boldsymbol{Ł}_n$, for some $n > 0$. In particular, $\boldsymbol{Ł}_n$ is a subalgebra of $\boldsymbol{Ł}_m$ for all $n \leq m$, and every infinite $\mathcal{L}$-chain contains a copy of $\boldsymbol{Ł}_n$ for all $n \geq 0$ (see Komori 1978). Finally, it is easy to see that any simple algebra in $\mathcal{L}$ is isomorphic to $\boldsymbol{Ł}_\alpha$ for some $\alpha \in \omega \cup \{\omega\}$ (see again Komori 1978).

We say that a BCK-algebra is *k-potent*, $0 < k < \omega$, if it satisfies the identity

$$(\epsilon_k) \qquad x \to^k y = x \to^{k+1} y.$$

The class of k-potent algebras forms a relative subvariety of $\mathcal{BCK}$ that is a variety. In particular, the variety of idempotent BCK-algebras is the variety of Hilbert algebras Diego (1965). We have that $\boldsymbol{Ł}_n$ is k-potent if and only if $n \leq k$. If $\boldsymbol{A}$ is k-potent, then $Fg(a) = \{b \in A : a \to^k b = 1\}$.

An especially important result to the present paper concerns the structure of the subdirectly irreducible algebras in $\mathcal{HBCK}$ (Blok and Ferreirim 2000). Given two BCK-algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ such that $A \cap B = \{1\}$, let $\boldsymbol{A} \oplus \boldsymbol{B}$ be the algebra $\langle A \cup B, \to, 1\rangle$, where $1 = 1^{\boldsymbol{A}} = 1^{\boldsymbol{B}}$, and

$$x \to y = \begin{cases} x \to^{\boldsymbol{A}} y & \text{if } x, y \in A, \\ x \to^{\boldsymbol{B}} y & \text{if } x, y \in B, \\ y & \text{if } x \in B, y \in A, \\ 1 & \text{if } x \in A, x \neq 1, y \in B. \end{cases}$$

$\boldsymbol{A} \oplus \boldsymbol{B}$ is a BCK-algebra, and it is k-potent if and only if both $\boldsymbol{A}$ and $\boldsymbol{B}$ are. The following theorem characterizes the subdirectly irreducible members of the variety $\mathcal{HBCK}$ (Blok and Ferreirim 2000; Ferreirim 1992):

Theorem 2 *An algebra $\boldsymbol{A} \in \mathcal{HBCK}$ is subdirectly irreducible if and only if $\boldsymbol{A} = \boldsymbol{B} \oplus \boldsymbol{C}$, where $\boldsymbol{B} \in \mathcal{HBCK}$ and $\boldsymbol{C}$ is a subdirectly irreducible algebra in $\mathcal{L}$.*

Corollary 3 *A k-potent algebra $\boldsymbol{A} \in \mathcal{HBCK}$ is subdirectly irreducible if and only if $\boldsymbol{A} = \boldsymbol{B} \oplus \boldsymbol{L}_n$, where $\boldsymbol{B}$ is a k-potent algebra in $\mathcal{HBCK}$ and $1 \leq n \leq k$.*

2 Factor congruences

In this section we provide a characterization of factor congruences (factor filters) for any algebra $\boldsymbol{A}$ in $\mathcal{BCK}$.

Recall that a complemented congruence relation θ on an algebra $\boldsymbol{A}$ with complement θ' is called a *factor congruence* if it permutes with θ'; $\{\theta, \theta'\}$ is called a pair of factor congruences. If $\{\theta, \theta'\}$ is a pair of factor congruences on $\boldsymbol{A}$, then $\boldsymbol{A} \cong \boldsymbol{A}/\theta \times \boldsymbol{A}/\theta'$ (Burris and Sankappanavar 1981). An algebra $\boldsymbol{A}$ is *directly indecomposable* if and only if the only pair of factor congruences of $\boldsymbol{A}$ is the trivial $\{\Delta, \nabla\}$. A congruence relation θ is a factor congruence if and only if $\theta \circ \theta' = \nabla$, where θ' denotes the pseudocomplement of θ in the algebraic lattice of congruence relations on $\boldsymbol{A}$.

We say that $\{F, F^{\perp}\}$ is a pair os *factor filters* if and only if the associated congruences θ_F and $\theta_{F^{\perp}}$ form a pair of factor congruences.

Observe that if F_1 and F_2 are filters of $\boldsymbol{A}$ such that $\boldsymbol{A} \cong \boldsymbol{A}/F_1 \times \boldsymbol{A}/F_2$ then $\{F_1, F_2\}$ is a pair of factor filters of $\boldsymbol{A}$, and $\alpha : \boldsymbol{A} \to \boldsymbol{A}/F_1 \times \boldsymbol{A}/F_2, \alpha(a) = (a/F_1, a/F_2)$ is an isomorphism. It is easy to see that, in this case, $\boldsymbol{F_1} \cong \alpha(\boldsymbol{F_1}) \cong \{1\} \times \boldsymbol{A}/F_2 \cong \boldsymbol{A}/F_2$ and $\boldsymbol{F_2} \cong \alpha(\boldsymbol{F_2}) \cong \boldsymbol{A}/F_1 \times \{1\} \cong \boldsymbol{A}/F_1$.

The next theorem characterizes pairs of factor filters in a BCK-algebra $\boldsymbol{A}$. It is quite similar to Díaz Varela and Torrens (2003, Lemma 2) and Díaz Varela and Torrens (2006, Lemma 4), the only difference being item (2)(i). The proof is also different. It is not difficult to see that this characterization can be extended to the class of all pocrims.

Theorem 4 *Let F be a filter of an algebra $\boldsymbol{A}$ in BCK. Then the following conditions are equivalent*:

(1) *$\{F, F^{\perp}\}$ is a pair of factor filters of $\boldsymbol{A}$.*
(2) (i) *$\boldsymbol{A}/F \cong \boldsymbol{F}^{\perp}$, $\boldsymbol{A}/F^{\perp} \cong \boldsymbol{F}$,*
   (ii) *For every $a \in A$ there are unique $c_a \in F$ and $b_a \in F^{\perp}$ such that $a = c_a \wedge b_a$,*
   (iii) *For every $c \in F$ and $b \in F^{\perp}$ there is $a \in A$ such that $c \wedge b = a$.*

Proof (1) $\Rightarrow$ (2). (i) is a consequence of the observation preceding this theorem.

In order to prove (ii), observe that $\boldsymbol{A} \cong \boldsymbol{A}/F \times \boldsymbol{A}/F^{\perp} \cong \boldsymbol{F}^{\perp} \times \boldsymbol{F}$, and so for each $a \in A$ there exists a unique $b_a \in F^{\perp}$ and a unique $c_a \in F$ such that $a/F = b_a/F$ and $a/F^{\perp} = c_a/F^{\perp}$. Thus every $a \in A$ can be represented in a unique way as $j(a) = (b_a, c_a)$ in $\boldsymbol{F}^{\perp} \times \boldsymbol{F}$, with $a/F = b_a/F$ and $a/F^{\perp} = c_a/F^{\perp}$. In particular, $j(b_a) = (b_a, 1)$ and $j(c_a) = (1, c_a)$. Let us see that $a = b_a \wedge c_a$. Since $j(a) \leq j(b_a), j(c_a)$ and j is an isomorphism, we have that $a \leq b_a, c_a$. Suppose that $g \leq b_a, c_a$. Then $j(g) = (b_g, c_g) \leq (b_a, 1) = j(b_a)$ and $j(g) = (b_g, c_g) \leq (1, c_a) = j(c_a)$, thus $b_g \leq b_a$ and $c_g \leq c_a$. So $j(g) = (b_g, c_g) \leq (b_a, c_a) = j(a)$. Therefore $g \leq a$ and consequently $a = b_a \wedge c_a$.

For (iii), if $b \in F^{\perp}$ and $c \in F$, $j(b) \wedge j(c) = (b, 1) \wedge (1, c) = (b, c) \in F^{\perp} \times F$. If $a = j^{-1}((b, c))$ then $a = b \wedge c$. (2) $\Rightarrow$ (1). As $F \cap F^{\perp} = \{1\}$, $\boldsymbol{A} \hookrightarrow \boldsymbol{A}/F \times \boldsymbol{A}/F^{\perp}$, where $a \mapsto (a/F, a/F^{\perp})$ is an injective homomorphism. By (i) the mapping $j : \boldsymbol{A} \to \boldsymbol{F}^{\perp} \times \boldsymbol{F}$, $j(a) = (b_a, c_a)$ where b_a (c_a) is the unique element in $F^{\perp}$ (F) such that $a/F = b_a/F$ ($a/F^{\perp} = c_a/F^{\perp}$) is an injective homomorphism. Let us see that j is onto. For $(b, c) \in F^{\perp} \times F$, by (iii), let $a = b \wedge c$. Let $j(a) = (f, g)$, $f \in F^{\perp}$, $g \in F$. Then $j(a) = (f, g) \leq (f, 1) = j(f)$ and $j(a) = (f, g) \leq (1, g) = j(g)$. Hence $a \leq f, g$ and so $a \leq f \wedge g$. On the other hand, $a \leq b, c$, and $j(a) = (f, g) \leq j(b) = (b, 1)$ and $j(a) = (f, g) \leq j(c) = (1, c)$, then $f \leq b$ and $g \leq c$. Therefore $a \leq f \wedge g \leq b \wedge c = a$, and, by the uniqueness, $f = b$, $g = c$. Thus $j(a) = (b, c)$. □

Observe that if $\{F, F^{\perp}\}$ is a pair of factor filters of a BCK-algebra $\boldsymbol{A}$ and a, c_a and b_a are as in Theorem 4, then it is easy to see that a is minimal in $\boldsymbol{A}$ if and only if c_a is minimal in F and b_a is minimal in $F^{\perp}$.

3 Decomposability in free algebras

As an application of the characterization of pairs of factor filters in a BCK-algebra, we study now the direct decomposability of free algebras in subquasivarieties of $\mathcal{BCK}$.

For X a set of variables, let $\boldsymbol{T}(X) = \langle T(X), \to, 1\rangle$ denote the set of all terms built in the usual recursive way from the variables in X using the operation symbols $\to$ and 1. Let $\boldsymbol{F}_{\mathcal{Q}}(\overline{X})$ denote the free algebra generated by $\overline{X}$ in a subquasivariety $\mathcal{Q}$ of $\mathcal{BCK}$. The elements of $\boldsymbol{F}_{\mathcal{Q}}(\overline{X})$ can be represented as $\overline{s} = s^{\boldsymbol{F}_{\mathcal{Q}}(\overline{X})}(\overline{x}_1, \ldots, \overline{x}_n)$, and for $s, t \in T(X)$ we have $\overline{s} = \overline{t}$ if and only if $\mathcal{Q} \models s \approx t$.

It follows from Blok and Raftery (1995), Fact 0, and an argument similar to the one in Díaz Varela and Torrens (2006) that if $\mathcal{Q}$ is a subquasivariety of $\mathcal{BCK}$, then $F_{\mathcal{V}}(\overline{X}) = \bigcup_{x \in X} [\overline{x})$, where $[\overline{x})$ denotes the set of elements $\overline{s}$ such that $\overline{x} \leq \overline{s}$.

In what follows we drop the bar in the elements of the free algebra of the quasivariety $\mathcal{Q}$ and consequently we simply write $\boldsymbol{F}_{\mathcal{Q}}(X)$ for the free algebra and $s(x_1, \ldots, x_n)$ for an element in $\boldsymbol{F}_{\mathcal{Q}}(X)$.

The proofs of the following lemma and theorem are easily obtained from the proofs of similar results in Díaz Varela and Torrens (2006).

Lemma 5 *Let $\mathcal{Q}$ be a nontrivial subquasivariety of $\mathcal{BCK}$. If X is infinite, then the unique upper bound of X in $\boldsymbol{F}_{\mathcal{Q}}(X)$ is* 1.

Theorem 6 (Díaz Varela and Torrens 2006, Theorem 8) *If F is a factor filter of $\boldsymbol{F}_{\mathcal{Q}}(X)$, then there exists $\alpha \in \boldsymbol{F}_{\mathcal{Q}}(X)$ such that $x \le \alpha$ for every $x \in X$, and either $F = [\alpha)$ or $F^{\perp} = [\alpha)$.*

Observe that the element α of Theorem 6 satisfies the property $\alpha \vee (\alpha \to x) = 1$ for every $x \in X$. Indeed, we can write $x = (\beta, \alpha)$, $\beta \in F^{\perp}$ and so $\alpha \to x = (1, \alpha) \to (\beta, \alpha) = (1 \to \beta, 1) = (\beta, 1) = \beta \in F^{\perp}$. Thus $\alpha \vee (\alpha \to x) = 1$. Therefore $\alpha \vee (\alpha \to b) = 1$ for every $b \in F_{\mathcal{Q}}(X)$.

As a consequence of Lemma 5 and Theorem 6 the following theorem is immediate.

Theorem 7 *If $\mathcal{Q}$ is any subquasivariety of $\mathcal{BCK}$ and X is infinite, then $\boldsymbol{F}_{\mathcal{Q}}(X)$ is directly indecomposable.*

In what follows we determine the decomposability of free algebras in every subvariety $\mathcal{V}$ of $\mathcal{HBCK}$.

Consider a minimal subdirect representation $\boldsymbol{F}_{\mathcal{V}}(X) \hookrightarrow \prod_{i \in I} \boldsymbol{A}_i$, where $\boldsymbol{A}_i$ is subdirectly irreducible for all i. For any $i \in I$, if π_i is the projection homomorphism from $\prod_{i \in I} \boldsymbol{A}_i$ onto $\boldsymbol{A}_i$, then the algebra $\pi_i(\boldsymbol{F}_{\mathcal{V}}(X)) = \boldsymbol{A}_i$ is generated by $\pi_i(X)$. It is easy to see that there exists one, and only one, $i_0 \in I$ such that $\boldsymbol{A}_{i_0} = \boldsymbol{Ł}_1$, and $\pi_{i_0}(x) = 0$ for every $x \in X$. For the sake of simplicity we will take $i_0 = 0$.

Lemma 8 *Let $\mathcal{V}$ be a subvariety of $\mathcal{HBCK}$ and $F = [\alpha)$ a factor filter of $\boldsymbol{F}_{\mathcal{V}}(X)$. Let $\boldsymbol{A}$ be a subdirectly irreducible algebra in $\mathcal{V}$ and $f : \boldsymbol{F}_{\mathcal{V}}(X) \to \boldsymbol{A}$ an onto homomorphism. Then either $f(\alpha) = 1$ or $f(\alpha) = 0$. In the latter case $\boldsymbol{A} = \boldsymbol{Ł}_1$ and $f(x) = 0$ for every $x \in X$ (that is, $f = \pi_0$).*

Proof Suppose that $f(\alpha) \neq 1$. Since $\boldsymbol{A}$ is subdirectly irreducible, $\boldsymbol{A} = \boldsymbol{B} \oplus \boldsymbol{C}$ for some $\boldsymbol{C} \in \mathcal{Ł}$, with $|C| \ge 2$, $\boldsymbol{C}$ subdirectly irreducible (so $\boldsymbol{C}$ is an $\mathcal{Ł}$-chain).

From $x \le \alpha$ for all $x \in X$, $f(x) \le f(\alpha)$ for all $x \in X$, and then $f(\alpha) \in C$.

Let us see that from $\alpha \vee (\alpha \to x) = 1$ for all $x \in X$ we have $f(\alpha) \vee (f(\alpha) \to f(x)) = 1$. Indeed, $\alpha \vee (\alpha \to x) \le ((\alpha \to x) \to \alpha) \to \alpha$ for all $x \in X$, and so $((\alpha \to x) \to \alpha) \to \alpha = 1$. If it were the case that $f(\alpha \to x) \le f(\alpha)$ then $f(\alpha \to x) \to f(\alpha) = 1$ and $f((\alpha \to x) \to \alpha) = 1$. Thus $f((\alpha \to x) \to \alpha) \to f(\alpha) = f(\alpha)$, and hence $1 = f(((\alpha \to x) \to \alpha) \to \alpha) = f(\alpha) \neq 1$, a contradiction.

So $f(\alpha \to x) \not\le f(\alpha)$; since $f(\alpha) \in C$ it follows that $f(\alpha) \to f(x) = f(\alpha \to x) \in C$ (otherwise $f(\alpha \to x) \in B\backslash\{1\}$, whose elements are below all elements of C). Consequently, $f(x) \in C$ for every $x \in X$. Thus $\boldsymbol{A} = \boldsymbol{C} \in \mathcal{Ł}$. But in $\boldsymbol{C}$ the supremum exists and is given by $a \vee b = (a \to b) \to b$. So $f(\alpha) \vee (f(\alpha) \to f(x)) = ((f(\alpha) \to f(x)) \to f(\alpha)) \to f(\alpha) = f(((\alpha \to x) \to \alpha) \to \alpha) = f(1) = 1$.

Since 1 is join irreducible in $\boldsymbol{A}$, we have $f(\alpha) \to f(x) = 1$, that is $f(\alpha) \le f(x)$ for every $x \in X$. On the other hand $f(x) \le f(\alpha)$, and then $f(x) = f(\alpha)$ for every $x \in X$. Thus $f(X)$ is a singleton and consequently $\boldsymbol{A} = \boldsymbol{Ł}_1$ and $f(\alpha) = f(x) = 0$. ◻

Corollary 9 *If $F = [\alpha)$ is a factor filter of $\boldsymbol{F}_{\mathcal{V}}(X)$, then α is a coatom of $\boldsymbol{F}_{\mathcal{V}}(X)$.*

Lemma 10 *Let $\alpha \in F_{\mathcal{V}}(X)$, $\alpha \neq 1$, be such that $[\alpha) = \{\alpha, 1\}$, and for every $x \in X$, $x \le \alpha$ and $\alpha \vee (\alpha \to x) = 1$. Then*

(a) $\pi_0(\alpha) = 0$ *and* $\pi_i(\alpha) = 1$ *for* $i \neq 0$.

(a) *For every $y \in F_{\mathcal{V}}(X)$, the equivalence class of y in $\boldsymbol{F}_{\mathcal{V}}(X)/[\alpha)$ is $y/[\alpha) = \{y, \alpha \to y\}$.*

(a) $[\alpha)^{\perp} = \{\alpha \to y : y \le \alpha\}$, $F_{\mathcal{V}}(X)/[\alpha) \cong [\alpha)^{\perp}$ *and* $F_{\mathcal{V}}(X)/[\alpha)^{\perp} \cong [\alpha)$.

Proof (a) can be proved as in Lemma 8.

For (b), observe that $a \not\le \alpha$ if and only if $\alpha \to a = a$. Indeed, $(\alpha \to a) \to a$ is an upper bound of both a and α, so $(\alpha \to a) \to a \in \{\alpha, 1\}$. But $a \not\le \alpha$, so $(\alpha \to a) \to a = 1$, and thus $\alpha \to a = a$. Conversely, if $a \le \alpha$ and $\alpha \to a = a$ then $\alpha \vee (\alpha \to a) = \alpha \vee a \neq 1$, a contradiction.

Now, if $y/[\alpha) = z/[\alpha)$ and $y \not\le z$, then either ($y \not\le \alpha$ and $z \le \alpha$) or ($y \le \alpha$ and $z \not\le \alpha$). Indeed, if $y, z \not\le \alpha$ then $y \to z, z \to y \not\le \alpha$. But $y \to z, z \to y \in [\alpha) = \{\alpha, 1\}$. So $y \to z = 1$ and $z \to y = 1$, that is, $y = z$, a contradiction. Similarly, if we suppose that $y, z \le \alpha$, since $y \to z \ge \alpha$ and $z \to y \ge \alpha$, it follows that $\pi_i(y) = \pi_i(z)$ for every i, so $y = z$, again a contradiction. As a consequence, each equivalence class has at most two elements.

Thus, if $y \le \alpha$, $\pi_0(y) \le \pi_0(\alpha) = 0$ and then $\pi_0(\alpha \to y) = 1$. Then $y \neq \alpha \to y$ and $y/[\alpha) = \{y, \alpha \to y\}$.

(c) is an immediate consequence of (b). ◻

Theorem 11 *Let $\alpha \in F_{\mathcal{V}}(X)$, $\alpha \neq 1$, be such that $[\alpha) = \{\alpha, 1\}$, and for every $x \in X$, $x \le \alpha$ and $\alpha \vee (\alpha \to x) = 1$. Then $\{[\alpha), [\alpha)^{\perp}\}$ is a pair of factor filters.*

Proof X is finite since otherwise $\alpha = 1$.

Let us prove condition (2) of Theorem 4. Condition (2)(i) follows from Lemma 10 (c).

In order to prove condition (2)(ii), observe that $\pi_0(\alpha \to y) = \pi_0(\alpha) \to \pi_0(y) = 1$, and for $i \neq 0$, $\pi_i(\alpha \to y) = \pi_i(\alpha) \to \pi_i(y) = 1 \to \pi_i(y) = \pi_i(y)$. So $\pi_i(\alpha \to y) = \pi_i(y)$ for every $i \in I$ if and only if $\pi_0(y) = 1$, that is, $y \in [\alpha)^{\perp}$ if and only if $\pi_0(y) = 1$. In addition, $\pi_0(y) = 1$ is equivalent to $y \not\le \alpha$. So $y \in [\alpha)^{\perp}$ if and only if $y \not\le \alpha$.

Now, if $y \nleq \alpha$, $y \in [\alpha)^{\perp}$ then $y = 1 \wedge y$, $1 \in [\alpha)$ and $y \in [\alpha)^{\perp}$.

Suppose that $y \leq \alpha$ and let us prove that y is the greatest lower bound of α and $\alpha \rightarrow y$. Consider c such that $c \leq \alpha$ and $c \leq \alpha \rightarrow y$. Then $\pi_0(c) = \pi_0(y) = \pi_0(\alpha) = 0$, and, for $i \neq 0$, $\pi_i(c) \leq \pi_i(\alpha \rightarrow y) = \pi_i(y)$. So $\pi_i(c) \leq \pi_i(y)$ for every $i \in I$. Thus $c \leq y$ and consequently $y = \alpha \wedge (\alpha \rightarrow y)$.

Uniqueness is immediate.

For condition (2)(iii), we want to prove that for $z \in [\alpha)^{\perp}$, $\alpha \wedge z$ exists. But from Lemma 10 (c), $z = \alpha \rightarrow y$ for some $y \leq \alpha$. Then, by the previous condition, $(\alpha \rightarrow y) \wedge \alpha = y$, so $\alpha \wedge z$ exists in $\boldsymbol{F}_{\mathcal{V}}(X)$. □

In the next theorem we prove that if $\mathcal{V}$ is a subvariety of $\mathcal{HBCK}$ and $\mathcal{V}$ is not k-potent for any $k \in \mathbb{N}$, then $\mathcal{V}$ contains the subvariety $\mathcal{L}$ of Łukasiewicz BCK-algebras.

Observe first that if $\boldsymbol{A} = \boldsymbol{B} \oplus \boldsymbol{C}$ is a subdirectly irreducible algebra in $\mathcal{HBCK}$ generated by a 2-element set $\{g_1, g_2\} \subseteq A$, then $\{g_1, g_2\} \cap C \neq \emptyset$. Suppose that $\{g_1, g_2\} \cap C = \{g_1\}$. Then $\boldsymbol{B} \cong \boldsymbol{C} \cong \mathbf{2}$, the 2-element BCK-algebra, and $\boldsymbol{A} = \mathbf{2} \oplus \mathbf{2}$, which is the 3-element Hilbert chain (and consequently, idempotent). If $\{g_1, g_2\} \subseteq C$, then $\boldsymbol{A} \cong C$ and then $\boldsymbol{A}$ is a Łukasiewicz BCK-chain.

Let $\mathcal{L}_n$ be the subvariety generated by $\boldsymbol{Ł}_n$.

Theorem 12 *Let $\mathcal{V}$ be a subvariety of $\mathcal{HBCK}$. If $\mathcal{V}$ is not k-potent for any $k \in \mathbb{N}$, then $\mathcal{L} \subseteq \mathcal{V}$.*

Proof Suppose that $\mathcal{V}$ is a subvariety of $\mathcal{HBCK}$ that is not k_0-potent for some $k_0 \in \mathbb{N}$. Since (ϵ_k) is an identity in two variables, the 2-generated free algebra $F_{\mathcal{V}}(2)$ is not k_0-potent. Then there exists a 2-generated subdirectly irreducible algebra $\boldsymbol{A} \in \mathcal{V}$ that is not k_0-potent. By the observation preceding this theorem, $\boldsymbol{A}$ is a Łukasiewicz BCK-chain. If $\boldsymbol{A}$ is infinite, then $V(\boldsymbol{A}) = \mathcal{L}$, and hence $\mathcal{L} \subseteq \mathcal{V}$. If $\boldsymbol{A}$ is finite, $\boldsymbol{A} \cong \boldsymbol{Ł}_s$, with $s \geq k_0 + 1$, that is, $\mathcal{L}_s \subseteq \mathcal{V}$, with $s \geq k_0 + 1$.

Suppose now that $\mathcal{V}$ is not k-potent for any $k \in \mathbb{N}$. Then $\mathcal{L}_n \subseteq \mathcal{V}$ for every $n \in \mathbb{N}$. Therefore $V(\boldsymbol{Ł}_n : n \in \mathbb{N}) = \mathcal{L} \subseteq \mathcal{V}$. □

Theorem 13 *If $\mathcal{V}$ is a subvariety of $\mathcal{HBCK}$ and $\mathcal{V}$ is not k-potent for any $k \in \mathbb{N}$, then $\boldsymbol{F}_{\mathcal{V}}(X)$ is indecomposable.*

Proof The case $|X| = 1$ is trivial since then $\boldsymbol{F}_{\mathcal{V}}(X) \cong \boldsymbol{Ł}_1$. Suppose that $|X| > 1$ and suppose that $\boldsymbol{F}_{\mathcal{V}}(X)$ is decomposable. Then there exists α satisfying the conditions of Theorem 6. By Lemma 8, if f is a homomorphism from $\boldsymbol{F}_{\mathcal{V}}(X)$ onto a subdirectly irreducible algebra $\boldsymbol{A}$ in $\mathcal{V}$, then $f(\alpha) = 0$ if and only if $\boldsymbol{A} \cong \boldsymbol{Ł}_1$ and $f(\alpha) = 1$ otherwise. Consider the homomorphism $g : \boldsymbol{F}_{\mathcal{V}}(X) \rightarrow \boldsymbol{C}_{\omega}$ such that for a fixed $x_0 \in X$, $g(x_0) = (0, 1)$ and $g(X \setminus \{x_0\}) = \{(0, 0)\}$. Since $\{(0, 1), (0, 0)\}$ generates $\boldsymbol{C}_{\omega}$, g is onto and $g(\alpha) = (1, 0)$. Consider $\pi : \boldsymbol{C}_{\omega} \rightarrow \boldsymbol{C}_{\omega}/Ł_{\omega} \cong \boldsymbol{Ł}_1$. Then $\pi \circ g : \boldsymbol{F}_{\mathcal{V}}(X) \rightarrow \boldsymbol{Ł}_1$ is onto and $(\pi \circ g)(X) = 0$ and $(\pi \circ g)(\alpha) = 1$, a contradiction. □

Consider now the term $J(x, y) = (((x \rightarrow y) \rightarrow y) \rightarrow x) \rightarrow x$ introduced by Cornish (1981). We define recursively for $l \geq 1$ the terms $J_l(x_1, x_2, \ldots, x_l)$ by $J_1(x_1) = x_1$, $J_2(x_1, x_2) = J(x_1, x_2)$, and $J_{l+1}(x_1, x_2, \ldots, x_{l+1}) = J(J_l(x_1, x_2, \ldots, x_l), x_{l+1})$ (Berman and Blok 2004).

As a consequence of the proof of Berman and Blok (2004, Theorem 2.5) it follows that for $a_1, a_2, \ldots, a_l$ in an algebra $\boldsymbol{A}$ in $\mathcal{HBCK}$, $J_l(a_1, a_2, \ldots, a_l) = J_l(a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(l)})$, for every bijection σ of the set $\{1, 2, \ldots, l\}$. Then for a finite set $S = \{s_1, \ldots, s_n\}$ we write $J(S) = J_n(s_1, \ldots, s_n)$.

Theorem 14 *Let $\mathcal{V}_k$ is a k-potent subvariety of $\mathcal{HBCK}$. If X is a finite set of variables with at least two elements, then $\boldsymbol{F}_{\mathcal{V}_k}(X)$ has a unique non-trivial pair of factor filters $\{F, F^{\perp}\}$ given by*

$$F = \{1, \alpha = J(\{(x \rightarrow y)^{k-1} \rightarrow x,\ x, y \in X\})\}.$$

Proof Let us see that $\pi_0(\alpha) = 0$ and $\pi_i(\alpha) = 1$ for $i \neq 0$. Indeed, for $i = 0$, $\pi_0 : \boldsymbol{F}_{\mathcal{V}_k}(X) \rightarrow \boldsymbol{Ł}_1$ ($Ł_1 = \{e^0 = 1, e = 0\}$) and $\pi_i(x) = 0$ for every $x \in X$. So $(\pi_0(x) \rightarrow \pi_0(y))^{k-1} \rightarrow \pi_0(x) = 0$ for every $x, y \in X$, so $\pi_0(\alpha) = 0$. For $i > 0$, $\pi_i : \boldsymbol{F}_{\mathcal{V}_k}(X) \rightarrow \boldsymbol{A}_i$, where $\boldsymbol{A}_i = \boldsymbol{B}_i \oplus \boldsymbol{C}_i$, and $\boldsymbol{C}_i = \boldsymbol{Ł}_n$, with $n \leq k$. Now, $\pi_i(X)$ generates $\boldsymbol{A}_i$. Then, for $n = 1$ there exists $x \in X$ such that $\pi_i(x) = e$. Thus $(\pi_i(x) \rightarrow \pi_i(y))^{k-1} \rightarrow \pi_i(x) = 1$, and consequently $\pi_i(\alpha) = 1$. If $n > 1$, there exist $x, y \in X$, $x \neq y$, such that $\pi_i(x) = e^d$ and $\pi_i(y) = e^n$, with $(d, n) = 1$. Then $(\pi_i(x) \rightarrow \pi_i(y))^{k-1} \rightarrow \pi_i(x) = 1$ and consequently $\pi_i(\alpha) = 1$.

Then it is easy to check that α satisfies the conditions of Theorem 11. □

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